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UNIFORM DISTRIBUTION OF THE WEIGHTED SUM-OF-DIGITS FUNCTIONS

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Abstract. The higher-dimensional generalization of the weighted q-adic sum-of-digits functions $s_{q,\boldsymbol{\gamma}}(n)$, $n = 0, 1, 2, \dots$, covers several important cases of sequences investigated in the theory of uniformly distributed sequences, e.g., d-dimensional van der Corput-Halton or d-dimensional Kronecker sequences. We prove a necessary and sufficient condition for the higher-dimensional weighted q-adic sum-of-digits functions to be uniformly distributed modulo one in terms of a trigonometric product. As applications of our condition we prove some upper estimates of the extreme discrepancies of such sequences, and that the existence of distribution function $g(x) = x$ implies the uniform distribution modulo one of the weighted q-adic sum-of-digits function $s_{q,\boldsymbol{\gamma}}(n)$, $n = 0, 1, 2, \dots$ We also prove the uniform distribution modulo one of related sequences $h_1 s_{q,\boldsymbol{\gamma}}(n) + h_2 s_{q,\boldsymbol{\gamma}}(n+1)$, where h_1 and h_2 are integers such that $h_1 + h_2 \neq 0$ and that the akin two-dimensional sequence $\big(s_{q,\boldsymbol{\gamma}}(n), s_{q,\boldsymbol{\gamma}}(n+1)\big)$ cannot be uniformly distributed modulo one if $q \geq 3$. The properties of the two-dimensional sequence $\big(s_{q,\boldsymbol{\gamma}}(n), s_{q,\boldsymbol{\gamma}}(n+1)\big)$, $n = 0, 1, 2, \dots$, will be instrumental in the proofs of the

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final section, where we show how the growth properties of the sequence of weights influence the distribution of values of the weighted sum-of-digits function which in turn imply a new property of the van der Corput sequence.

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1. Introduction

Let $d \in \mathbb{N}$ be a fixed positive integer.[1] Given a real number x, let $\lfloor x \rfloor$ denote the integral part of x, and $\{x\} = x - \lfloor x \rfloor$ be the fractional part of x or the residue of x modulo one in symbols $x \bmod 1$. Finally, let $\|x\| = \min\big(\{x\}, 1 - \{x\}\big)$ be the distance to nearest integer function.

A sequence of the d-dimensional vectors $\vec{x}_n$, $n = 0, 1, 2, \dots$, in $\mathbb{R}^d$ is said to be **uniformly distributed mod 1** (abbreviated to u.d. mod 1) if

$$\lim_{N\to\infty} \frac{A([\vec{a}, \vec{b}); N; \vec{x}_n \bmod 1)}{N} = \prod_{j=1}^{d} (b_j - a_j) \tag{1}$$

for all intervals $[\vec{a}, \vec{b}) \subseteq [0,1)^d$ with $\vec{a} = (a_1, \dots, a_d)$ and $\vec{b} = (b_1, \dots, b_d)$. Here, $A(I; N; \vec{x}_n)$ denotes the number of elements, out of the first N elements of the sequence $\vec{x}_n$, $n = 0, 1, 2, \dots$, that lie in set $I \subseteq \mathbb{R}^d$. If $\vec{x}_n \in [0,1)^d$ for each n, we simply say that $\vec{x}_n$ is uniformly distributed (abbreviated to u.d.).

In (1) we can obviously restrict the definition to intervals of the form $[0, \vec{x})$ where $\vec{x} = (x^{(1)}, x^{(2)}, \dots, x^{(d)}) \in [0,1)^d$. If there exists a strictly increasing sequence of positive integers $\mathcal{N} = (N_1 < N_1 < N_3 < \dots)$ such that

$$\lim_{j\to\infty} \frac{A([\vec{0}, \vec{x}); N_j; \vec{x}_n \bmod 1)}{N_j} = \prod_{j=1}^{d} x^{(j)} \tag{2}$$

for all $\vec{x} \in [0,1)^d$ then the sequence $\vec{x}_n$, $n = 0, 1, 2, \dots$, is called **$\mathcal{N}$-almost u.d. mod** 1.

Weyl's criterion (see, e.g. [11, p. 48]) says that the sequence $\vec{x}_n$, $n = 0, 1, 2, \dots$, in $\mathbb{R}^d$ is u.d. mod 1 if and only if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i \langle \vec{h}, \vec{x}_n \rangle} = 0, \tag{3}$$

[1] In what follows, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$ will always denote the set of positive, non-negative or of the all integers, and $\mathbb{R}$ the set of real numbers.

where $\langle \vec{x}, \vec{y} \rangle$ stands for the standard inner product in $\mathbb{R}^d$, and it is $\mathcal{N}$-almost u.d. mod 1 if and only if

$$\lim_{j\to\infty} \frac{1}{N_j} \sum_{n=0}^{N_j-1} e^{2\pi \mathrm{i}\langle \vec{h}, \vec{x}_n \rangle} = 0, \tag{4}$$

in both cases for every lattice point $\vec{h} \in \mathbb{Z}^d$, $\vec{h} \neq \vec{0}$.

The theory of uniform distribution (abbreviated also to u.d.) is not restricted only to the interval $[0,1)$. A more general case can be considered: If $[a,b) \subset \mathbb{R}$ is a non-degenerate (finite) interval, the sequence x_n, $n = 0,1,2,\dots$, of real numbers from $[a,b)$ is said to be **u.d. with respect to interval** $[a,b)$ if for any subinterval $[c,d)$ of $[a,b)$ we have

$$\lim_{n\to\infty} \frac{A([c,d);N;x_n)}{N} = \frac{d-c}{b-a}. \tag{5}$$

In what follows let

$$\boldsymbol{\gamma} = (\gamma_0, \gamma_1, \gamma_2, \dots) \tag{6}$$

always denote a sequence of real numbers, and q a fixed positive integer greater or equal to 2. For a non-negative integer n with base q representation

$$n = n_0 + n_1 q + n_2 q^2 + \cdots + n_\ell q^\ell + \cdots, \tag{7}$$

with n_j in the set of q-adic digits $\{0,1,\dots,q-1\}$ for $j \geq 0$ we have $n_\ell \neq 0$ for $\ell = \lfloor \log_q n \rfloor$ and $n_j = 0$ for $j > \ell$. **The weighted q-adic sum-of-digits function** is defined by the equation

$$s_{q,\boldsymbol{\gamma}}(n) = \gamma_0 n_0 + \gamma_1 n_1 + \gamma_2 n_2 + \cdots + \gamma_\ell n_\ell. \tag{8}$$

A special case of this definition is presented by the q-adic van der Corput sequence [11, p. 127]

$$\phi_q(n) = \frac{n_0}{q} + \frac{n_1}{q^2} + \frac{n_2}{q^3} + \cdots + \frac{n_\ell}{q^{\ell+1}},$$

where $\gamma_i = q^{-i-1}$ for all $i \in \mathbb{N}_0$. The q-adic van der Corput sequence is a well known prototype of a u.d. sequence.

To consider a d-dimensional generalization with $d > 1$, let $(q_1, q_2, \dots, q_d)$ be a d-tuple of positive integers greater or equal to 2 and

$$\mathit{\Gamma} = \begin{pmatrix} \boldsymbol{\gamma}^{(1)} \\ \boldsymbol{\gamma}^{(2)} \\ \vdots \\ \boldsymbol{\gamma}^{(d)} \end{pmatrix} = \begin{pmatrix} \gamma_0^{(1)} & \gamma_1^{(1)} & \gamma_2^{(1)} & \cdots \\ \gamma_0^{(2)} & \gamma_1^{(2)} & \gamma_2^{(2)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ \gamma_0^{(d)} & \gamma_1^{(d)} & \gamma_2^{(d)} & \cdots \end{pmatrix} \tag{9}$$

be a $d \times \infty$-matrix over $\mathbb{R}$ and with $\vec{\gamma}_j = \big(\gamma_j^{(1)}, \gamma_j^{(2)}, \dots, \gamma_j^{(d)}\big)$ transposed in the $(j+1)$st column, $j = 0, 1, 2, \dots$ If $n = \sum_{j=0}^{\infty} n_j^{(i)} q_i^j$ is the q_i-adic representation of n, and $n^{(i)} = \big(n_0^{(i)}, n_1^{(i)}, n_2^{(i)}, \dots\big)$ is the (infinite) row vector of the digits of the q_i-adic representation of n, then

$$s_{q_i,\boldsymbol{\gamma}^{(i)}}(n) = \big\langle \boldsymbol{\gamma}^{(i)}, n^{(i)} \big\rangle.$$

For every $n \in \mathbb{N}_0$ define

$$s_{q_1,\dots,q_d,\Gamma}(n) = \big(s_{q_1,\boldsymbol{\gamma}^{(1)}}(n), s_{q_2,\boldsymbol{\gamma}^{(2)}}(n), \dots, s_{q_d,\boldsymbol{\gamma}^{(d)}}(n)\big). \tag{10}$$

If $q_1 = q_2 = \cdots = q_d = q$, we write

$$s_{q,\Gamma}(n) = \big(s_{q,\boldsymbol{\gamma}^{(1)}}(n), s_{q,\boldsymbol{\gamma}^{(2)}}(n), \dots, s_{q,\boldsymbol{\gamma}^{(d)}}(n)\big). \tag{11}$$

Let us mention several related examples:

Example 1. If q_i's are pairwise coprime integers greater or equal to 2 and $\gamma_j^{(i)} = q_i^{-j-1}$ for all $i = 1, \dots, d$ and $j = 0, 1, 2, \dots$, then the resulting sequence (10) is the d-**dimensional van der Corput-Halton sequence** which is u.d. in $[0,1)^d$.

Example 2. If $\gamma_j^{(i)} = q_i^j \alpha_i$ for all $i \in \{1, \dots, d\}$ and all $j \in \mathbb{N}$, we obtain the d-**dimensional Kronecker sequence** which is u.d. mod 1 in $[0,1)^d$ if and only if $1, \alpha_1, \dots, \alpha_d$ are linearly independent over $\mathbb{Q}$ (see e.g. [5]). If $d = 1$ this sequence is u.d. if and only if α_1 is irrational.

Example 3. The following example is of a "hybrid type" (cf. [16, p. 377]). Let $\big(x^{(1)}(n), \dots, x^{(s)}(n)\big)$, $n = 0, 1, 2, \dots$, be an s-dimensional van der Corput-Halton sequence and $\big(y^{(1)}(n), \dots, y^{(t)}(n)\big)$, $n = 0, 1, 2, \dots$, be a t-dimensional Kronecker sequence such that $s \geq 1$ and $t \geq 1$. Then the so-called **Halton--Kronecker sequence** given by $\big(x^{(1)}(n), \dots, x^{(s)}(n), y^{(1)}(n), \dots, y^{(t)}(n)\big)$, $n = 0, 1, 2, \dots$, is u.d. mod 1 in $[0,1)^{s+t}$ if and only if its Kronecker part is u.d. mod 1 in $[0,1)^t$.

Example 4. Let d be a positive integer and let $s_q^{(d)}(n) = \sum_{j=0}^{\infty} n_j^d$ denote the sum of the dth powers of the q-adic digits of the positive integer n. If $\theta \in \mathbb{R}$ then sequences of the form $\theta s_q^{(d)}(n)$ with n running over $\mathbb{N}_0$ or over the set of prime numbers were studied by several authors. We get a special case of the weighted q-adic sum-of-digits functions in the case where the weights form a constant sequence $\gamma_j = \theta$ for all $j = 0, 1, 2, \dots$, and the exponent $d = 1$, that is $s_{q,\boldsymbol{\gamma}}(n) = \theta s_q^{(1)}(n)$. M. Mendès France [15] proved that sequence $\theta s_q^{(1)}(n)$, $n = 0, 1, 2, \dots$, is u.d. mod 1 if and only if θ is irrational. This result was later reproved by J. Coquet who proved that for every $k \in \mathbb{N}$ the sequence $\theta s_q^{(k)}(n)$, $n = 0, 1, 2, \ldots$, is u.d. mod 1 if and only if θ is irrational.[2] Ch. Mauduit and J. Rivat [14, Théorème 2] proved that $\theta s_q^{(1)}(n)$ is u.d. mod 1 when θ is irrational and n runs through the prime numbers only. R. F. Tichy and G. Turnwald [21] proved estimates for the discrepancy of the sequence $\alpha s_q^{(d)}(n)$, $n = 0, 1, 2, \ldots$, for irrational α of finite approximation type η.[3]

Example 5. If the q_i's are pairwise coprime bases and the coordinates of $\vec{\gamma}_j$, $j = 0, 1, 2, \ldots$, are constant sequences, i.e., $\vec{\gamma}_j = (\alpha_1, \ldots, \alpha_d)$ for all $j \in \mathbb{N}_0$, then (10) is u.d. in $[0,1)^d$ if and only if $\alpha_1, \ldots, \alpha_d$ are irrational numbers [3].

In [17] (c.f. also [19, 1.22]) F. Pillichshammer proposed the following general problem:

Open question. *Let $q_1, \ldots, q_d$ be a d-tuple of pairwise coprime integers greater or equal to 2. Under which conditions on the weight sequences forming Γ is the sequence*

$$s_{q_1,\ldots,q_d,\Gamma}(n), \quad n = 0, 1, 2, \ldots, \tag{12}$$

u.d. mod 1*?*

In the same paper F. Pillichshammer proved the following result[4] when all the bases coincide.

Proposition 1 ([17, Theorem 1]). *Let the base $q \in \mathbb{N}$ be at least 2. The sequence $s_{q,\Gamma}(n)$ is u.d.* mod 1 *if and only if for every integral vector $\vec{h} \in \mathbb{Z}^d \setminus \{\vec{0}\}$ one of the following conditions is fulfilled: either*

$$\sum_{\substack{k=0 \\ \langle \vec{h}, \vec{\gamma}_k \rangle q \notin \mathbb{Z}}}^{\infty} \|\langle \vec{h}, \vec{\gamma}_k \rangle\|^2 = \infty, \tag{13}$$

or, there exists a non-negative integer k with

$$\langle \vec{h}, \vec{\gamma}_k \rangle \notin \mathbb{Z} \quad \text{and} \quad \langle \vec{h}, \vec{\gamma}_k \rangle q \in \mathbb{Z}. \tag{14}$$

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[2] Coquet [1] also proved that if the real sequence $\lambda(n)$, $n = 0, 1, 2, \ldots$, is well distributed mod 1, that is, if for every subinterval $[a, b) \subset [0, 1)$, we have $\lim_{N\to\infty} \frac{A([a,b);N;x_{n+k} \bmod 1)}{N} = b - a$ uniformly in $k = 0, 1, 2, \ldots$, then so is sequence $\lambda(s_q^{(1)}(n))$, $n = 0, 1, 2, \ldots$ He claims that (1) the converse is also true, (2) if $\lambda(s_q^{(1)}(n))$, $n = 0, 1, 2, \ldots$, is u.d. mod 1, then it also is the sequence $\lambda(n)$, $n = 0, 1, 2, \ldots$, and that (3) if a q-additive sequence is u.d. mod 1, then it is well distributed.

[3] The irrational number α is of type η if given $\epsilon > 0$, $h^{\eta+\epsilon}\|h\alpha\| \geq c(\alpha, \epsilon)$ for all positive integers h and $c(\alpha, \epsilon)$ a positive constant.

[4] A forerunner of this result can be found in [13] and a generalization in [9].

Note, that the q-adic van der Corput sequence, where $d = 1$, satisfies (14). To see this take $k = s$ where s is the maximal exponent such that q^{s+1} divides h. Kronecker sequences and Drmota-Larcher's result mentioned in Example 5 in the one-dimensional case fulfill (13) due to irrationality of weights.

R.Hofer [8] proved a related sufficient condition on the weight sequences which gives a partial answer to Pillichshammer's question. Her condition for the u.d. mod 1 of the sequence (12) requires the divergence of the series

$$\sum_{i=0}^{\infty} \left\| h \left(\gamma_{2i+1}^{(j)} - q_j \gamma_{2i}^{(j)} \right) \right\|^2$$

for each dimension $j \in \{1, \dots, d\}$ and every non-zero integer h. Unfortunately, her sufficient condition is not necessary. Her sufficient condition generalized that proved in [3] for sequences mentioned in Example 5. It also does not cover some prototype classes of u.d. sequences as the d-dimensional Kronecker sequences or d-dimensional van der Corput-Halton ones.

R.Hofer *et al.* [9] proposed a further generalization of the one-dimensional weighted q-adic sum-of-digits function, called generalized weighted digit-block--counting function. Their conditions guaranteeing its u.d. mod 1 generalize (13) and (14) but are rather technical.

2. Outline of the paper

In Theorem 1 of Section 3 we replace conditions (13) and (14) with one involving a trigonometric product. More precisely, sequence $s_{q,\Gamma}(n)$, $n = 0, 1, 2, \dots$, is u.d. mod 1 if and only if

$$\lim_{N\to\infty} \prod_{\substack{j=0 \\ \langle \vec{h}, \vec{\gamma}_j \rangle \notin \mathbb{Z}}}^{N-1} \frac{|\sin \pi q \langle \vec{h}, \vec{\gamma}_j \rangle|}{q |\sin \pi \langle \vec{h}, \vec{\gamma}_j \rangle|} = 0$$

for every integral vector $\vec{h} \in \mathbb{Z}^d \setminus \{\vec{0}\}$.

In Section 4, Theorem 2 gives an upper bound for the discrepancy of the sequences $s_{q,\Gamma}(n)$ mod 1, $n = 0, 1, 2, \dots$, based on this trigonometric product. Example 7 applies Theorem 2 to the case of a weight sequence $\gamma_n = \{n\alpha\}$, $n = 0, 1, 2, \dots$, where α is an irrational number.

In Theorem 3 of Section 5 we prove that for the u.d. mod 1 of the weighted sum-of-digits function $s_{q,\boldsymbol{\gamma}}(n)$, $n = 0, 1, 2, \dots$, it is necessary and sufficient to possess the distribution function $g(x) = x$.

In Section 6, in Theorems 4 and 5 we prove that the u.d. of the sequence $s_{q,\gamma}(n)$, $n \in \mathbb{N}_0$, pulls along also the u.d. of the sequence $\big(h_1 s_{q,\gamma}(n) + h_2 s_{q,\gamma}(n+1)\big) \bmod 1$, $n \in \mathbb{N}_0$, for all $h_1, h_2 \in \mathbb{Z}$ satisfying $h_1 + h_2 \neq 0$, but not that of the two-dimensional sequence

$$\big(s_{q,\gamma}(n), s_{q,\gamma}(n+1)\big) \bmod 1, \quad n = 1, 2, \dots \tag{15}$$

In the final Section 7, in Theorem 5 we work out a new property of the q-adic van der Corput sequence. Namely, if the weights γ_λ, $\lambda = 0, 1, 2, \dots$, satisfy conditions

$$\gamma_0 \geq \gamma_1 \geq \dots \geq 0,$$
$$(q-1)\sum_{i=0}^{\infty} \gamma_i = 1,$$

then the q-adic van der Corput's sequence is the only u.d. sequence modulo one meeting these two conditions.

3. A trigonometric criterion

The trigonometric factors appear in our criterion via the following well-known equations

$$\frac{1}{q}\left|\sum_{n=0}^{q-1} e^{2\pi i n x}\right| = \begin{cases} \frac{|\sin \pi q x|}{q|\sin \pi x|}, & \text{if } x \notin \mathbb{Z}, \\ 1, & \text{if } x \in \mathbb{Z}. \end{cases} \tag{16}$$

Since the points $e^{2\pi i 0 x}$, $e^{2\pi i 1 x}$,...,$e^{2\pi i (q-1) x}$ lie on the unit circle, the triangular inequality yields the inequality $\frac{1}{q}\left|\sum_{n=0}^{q-1} e^{2\pi i n x}\right| \leq 1$, or

$$\frac{|\sin \pi q x|}{q|\sin \pi x|} \leq 1 \quad \forall q \in \mathbb{N}. \tag{17}$$

If moreover we have $e^{2\pi i n x} \neq 1$ for some $n \in \{0, 1, \dots, q-1\}$ and $q \geq 2$, then $\frac{1}{q}\sum_{n=0}^{q-1} e^{2\pi i n x}$ lies strictly inside the unit circle, that is

$$\frac{|\sin \pi q x|}{q|\sin \pi x|} < 1 \tag{18}$$

for all $q \in \mathbb{N}$, $q \geq 2$, and for all $x \in (0,1)$. On the other hand, if $x \in \mathbb{Z}$, then arguing with the so-called removable singularity technique, we can put

$$\frac{|\sin \pi q x|}{q|\sin \pi x|} = 1. \tag{19}$$

The next criterion we were not able to trace in the literature. Its proof can also be based on Theorem 1 of [2][5] or from its generalization Theorem C (ii) of [10] using (16). We offer here a direct and more transparent proof of its statement.

Theorem 1. *Let $q \geq 2$ be an integer and Γ be the $d \times \infty$-matrix of real weights* (9). *Then the sequence $s_{q,\Gamma}(n)$, $n = 0, 1, 2, \dots$, is u.d.* mod 1 *if and only if for every integral vector $\vec{h} \in \mathbb{Z}^d \setminus \{\vec{0}\}$ we have*

$$\lim_{N\to\infty} \prod_{\substack{j=0 \\ \langle \vec{h}, \vec{\gamma}_j \rangle \notin \mathbb{Z}}}^{N-1} \frac{|\sin \pi q \langle \vec{h}, \vec{\gamma}_j \rangle|}{q |\sin \pi \langle \vec{h}, \vec{\gamma}_j \rangle|} = 0. \tag{20}$$

Proof. Let $\mathcal{Q} = (q^0 < q^1 < q^2 < \cdots)$. In this part of the proof we show that relation (20) is equivalent to the $\mathcal{Q}$-almost u.d. mod 1 of the sequence $s_{q,\Gamma}(n)$, $n = 0, 1, 2, \dots$ To prove this, we use Weyl's sum for q^N terms with n represented in form

$$n = n_0 + n_1 q + \cdots + n_{N-1} q^{N-1}, \quad n_i \in \{0, 1, \dots, q-1\} \tag{21}$$

and with an arbitrary integral vector $\vec{h} \in \mathbb{Z}^d \setminus \{\vec{0}\}$.

Then

$$\begin{aligned}
&\frac{1}{q^N} \sum_{n=0}^{q^N-1} e^{2\pi \mathrm{i} \langle \vec{h}, s_{q,\Gamma}(n) \rangle} = \frac{1}{q^N} \sum_{n=0}^{q^N-1} e^{2\pi \mathrm{i} \sum_{j=1}^d h_j s_{\gamma^{(j)}}(n)} \\
&= \frac{1}{q^N} \sum_{n_0=0}^{q-1} \cdots \sum_{n_{N-1}=0}^{q-1} e^{2\pi \mathrm{i} \sum_{j=1}^d h_j \sum_{i=0}^{N-1} n_i \gamma_i^{(j)}} \\
&= \frac{1}{q^N} \sum_{n_0=0}^{q-1} \cdots \sum_{n_{N-1}=0}^{q-1} e^{2\pi \mathrm{i} n_0 \sum_{j=1}^d h_j \gamma_0^{(j)}} \cdots e^{2\pi \mathrm{i} n_{N-1} \sum_{j=1}^d h_j \gamma_{N-1}^{(j)}} \\
&= \frac{1}{q^N} \left(\sum_{n_0=0}^{q-1} \prod_{j=1}^d e^{2\pi \mathrm{i} n_0 h_j \gamma_0^{(j)}} \right) \cdots \left(\sum_{n_{N-1}=0}^{q-1} \prod_{j=1}^d e^{2\pi \mathrm{i} n_{N-1} h_j \gamma_{N-1}^{(j)}} \right) \\
&= \left(\frac{1}{q} \sum_{n_0=0}^{q-1} e^{2\pi \mathrm{i} n_0 \langle \vec{h}, \vec{\gamma}_0 \rangle} \right) \cdots \left(\frac{1}{q} \sum_{n_{N-1}=0}^{q-1} e^{2\pi \mathrm{i} n_{N-1} \langle \vec{h}, \vec{\gamma}_{N-1} \rangle} \right).
\end{aligned} \tag{22}$$

Relation (16) implies that the sums in the parentheses where the scalar product in the summands is an integer are equal to 1. Application of the first part of the

[5]The paper contains small inaccuracies. For instance, the paper quoted in footnote 4 on p. 292 appeared in 1963, or the proof of Proposition 5 implies a strict inequality in its statement.

same relation (16) to the remaining sums in parentheses of (22) yields a product containing only factors with non-integral scalar products

$$\left|\frac{1}{q^N}\sum_{n=0}^{q^N-1} e^{2\pi i\langle\vec{h},s_{q,\Gamma}(n)\rangle}\right| = \prod_{\substack{j=0\\ \langle\vec{h},\vec{\gamma}_j\rangle\notin\mathbb{Z}}}^{N-1} \frac{|\sin\pi q\langle\vec{h},\vec{\gamma}_j\rangle|}{q|\sin\pi\langle\vec{h},\vec{\gamma}_j\rangle|}. \tag{23}$$

Weyl's criterion (4) immediately finishes this part of the proof.

Now we prove that the $\mathcal{Q}$-almost u.d. mod 1 of the sequence $s_{q,\Gamma}(n)$, that is the validity of (20), implies its u.d. mod 1.

Given a positive integer M let

$$q^N \le M < q^{N+1} \text{ and } M = \sum_{j=0}^{N} m_j q^j \text{ with } m_j \in \{0,1,\dots,q-1\} \text{ for each } j.$$

Split the set of all

$$n\text{'s with } 0 \le n = \sum_{j=0}^{N} n_j q^j \le M-1, \text{ where } n_j \in \{0,1,\dots,q-1\},\ j=0,1,\dots,N,$$

into the following disjoint subsets

$$\begin{aligned}
A_1 &= \big\{0\le n\le M-1:\ n_0\in\{0,1,\dots,q-1\},\dots,n_{N-1}\in\{0,1,\dots,q-1\},\\
&\qquad n_N\in\{0,1,\dots,m_N-1\}\big\},\\
A_2 &= \big\{0\le n\le M-1:\ n_0\in\{0,1,\dots,q-1\},\dots,n_{N-2}\in\{0,1,\dots,q-1\},\\
&\qquad n_{N-1}\in\{0,1,\dots,m_{N-1}-1\}, n_N=m_N\big\},\\
&\ \vdots\\
A_j &= \big\{0\le n\le M-1:\ n_0\in\{0,1,\dots,q-1\},\dots,n_{N-j}\in\{0,1,\dots,q-1\},\\
&\qquad n_{N-j+1}\in\{0,1,\dots,m_{N-j+1}-1\}, n_{N-j+2}=m_{N-j+2},\dots,n_N=m_N\big\},\\
&\ \vdots\\
A_N &= \big\{0\le n\le M-1:\ n_0\in\{0,1,\dots,q-1\}, n_1\in\{0,1,\dots,m_1-1\},\\
&\qquad n_2=m_2,\dots,n_{N-1}=m_{N-1},n_N=m_N\big\},\\
A_{N+1} &= \big\{0\le n\le M-1:\ n_0\in\{0,1,\dots,m_0-1\}, n_1=m_1,n_2=m_2,\dots,\\
&\qquad n_{N-1}=m_{N-1},n_N=m_N\big\}.
\end{aligned}$$

For the cardinalities $|A_j|$ of sets A_j, $j=1,\dots,N+1$, we have

$$|A_j| = m_{N-j+1}\cdot q^{N-j+1}, \quad j=1,2,\dots,N+1, \tag{24}$$

where if $m_{N-j+1}=0$, then $|A_j|=0$.

Let $\vec{h} \in \mathbb{Z}^d \setminus \{\vec{0}\}$ be an arbitrary integral vector. Then

$$\sum_{n\in A_j} e^{2\pi\mathrm{i}\langle\vec{h},s_{q,\Gamma}(n)\rangle} = \left(\sum_{n_0=0}^{q-1} e^{2\pi\mathrm{i}n_0\langle\vec{h},\vec{\gamma}_0\rangle}\right)\cdots\left(\sum_{n_{N-j}=0}^{q-1} e^{2\pi\mathrm{i}n_{N-j}\langle\vec{h},\vec{\gamma}_{N-j}\rangle}\right)\times$$
$$\left(\sum_{n_{N-j+1}=0}^{m_{N-j+1}-1} e^{2\pi\mathrm{i}n_{N-j+1}\langle\vec{h},\vec{\gamma}_{N-j+1}\rangle}\right) e^{2\pi\mathrm{i}n_{N-j+2}\langle\vec{h},\vec{\gamma}_{N-j+2}\rangle}\cdots$$
$$e^{2\pi\mathrm{i}n_{N-1}\langle\vec{h},\vec{\gamma}_{N-1}\rangle}.$$

Taking into account (24) and again applying (16) to every expression in parentheses we obtain

$$\frac{1}{|A_j|}\left|\sum_{n\in A_j} e^{2\pi\mathrm{i}\langle\vec{h},s_{q,\Gamma}(n)\rangle}\right| =$$
$$\left(\prod_{\substack{t=0\\ \langle\vec{h},\vec{\gamma}_t\rangle\notin\mathbb{Z}}}^{N-j} \frac{|\sin\pi q\langle\vec{h},\vec{\gamma}_t\rangle|}{q|\sin\pi\langle\vec{h},\vec{\gamma}_t\rangle|}\right)\frac{|\sin\pi m_{N-j+1}\langle\vec{h},\vec{\gamma}_{N-j+1}\rangle|}{m_{N-j+1}|\sin\pi\langle\vec{h},\vec{\gamma}_{N-j+1}\rangle|}. \qquad (25)$$

The second factor appears in (25) only if $\langle\vec{h},\vec{\gamma}_{N-j+1}\rangle \notin \mathbb{Z}$, otherwise, it is equal to 1. In any case it is ≤ 1 due to (17).

Let k be an integer such that $1 \leq k \leq N$, then

$$|A_{k+1}\cup A_{k+2}\cup\cdots\cup A_{N+1}| = q^{N-k}m_{N-k} + q^{N-k-1}m_{N-k-1} + \cdots + m_0$$
$$\leq q\frac{q^{N-k+1}-1}{q-1} = O\left(q^{N-k+1}\right), \qquad (26)$$

since $m_i < q$ for each i.

Returning back to the estimates of Weyl's sum, compute

$$\frac{1}{M}\left|\sum_{n=0}^{M-1} e^{2\pi\mathrm{i}\langle\vec{h},s_{q,\Gamma}(n)\rangle}\right| \leq \sum_{j=1}^{k}\frac{|A_j|}{M}\frac{1}{|A_j|}\left|\sum_{n\in A_j} e^{2\pi\mathrm{i}\langle\vec{h},s_{q,\Gamma}(n)\rangle}\right| + O\left(\frac{q^{N-k+1}}{M}\right)$$
$$\leq \sum_{j=1}^{k} q^{-j+2}\prod_{\substack{t=0\\ \langle\vec{h},\vec{\gamma}_t\rangle\notin\mathbb{Z}}}^{N-j}\frac{|\sin\pi q\langle\vec{h},\vec{\gamma}_t\rangle|}{q|\sin\pi\langle\vec{h},\vec{\gamma}_t\rangle|} + O\left(\frac{1}{q^{k-1}}\right) \qquad (27)$$

because $|A_j|/M \le q^{-j+2}$ and $q^N < M$. The last estimate holds for an arbitrary k such that $1 \le k \le N$. Our assumption that

$$\prod_{\substack{t=0\\ \langle \vec{h},\vec{\gamma}_t\rangle\notin\mathbb{Z}}}^{N-1} \frac{|\sin \pi q\langle \vec{h},\vec{\gamma}_t\rangle|}{q|\sin \pi\langle \vec{h},\vec{\gamma}_t\rangle|} \to 0$$

for $N \to \infty$ implies that for a fixed j we also have

$$\prod_{\substack{t=0\\ \langle \vec{h},\vec{\gamma}_t\rangle\notin\mathbb{Z}}}^{N-j} \frac{|\sin \pi q\langle \vec{h},\vec{\gamma}_t\rangle|}{q|\sin \pi\langle \vec{h},\vec{\gamma}_t\rangle|} \to 0 \quad \text{as } N \to \infty.$$

Fixing k and letting $N \to \infty$ we obtain from(27)

$$\frac{1}{M}\left|\sum_{n=0}^{M-1} e^{2\pi \mathrm{i}\langle \vec{h}, s_{q,\Gamma}(n)\rangle}\right| \le O\left(\frac{1}{q^{k-1}}\right).$$

Since the above inequality holds for every fixed $k \in \mathbb{N}$ and all sufficiently large $M \in \mathbb{N}$, we have

$$\limsup_{M\to\infty} \frac{1}{M}\left|\sum_{n=0}^{M-1} e^{2\pi \mathrm{i}\langle \vec{h}, s_{q,\Gamma}(n)\rangle}\right| \le O\left(\frac{1}{q^{k-1}}\right).$$

Consequently,

$$\lim_{M\to\infty} \frac{1}{M}\left|\sum_{n=0}^{M-1} e^{2\pi \mathrm{i}\langle \vec{h}, s_{q,\Gamma}(n)\rangle}\right| = 0.$$

and the sufficiency of (20) follows. □

There are several simple situations where the series (13) diverges. For instance, if the sequence of elements $\|\langle \vec{h},\vec{\gamma}_k\rangle\|$, $k = 0,1,2,\dots$, satisfying condition $\langle \vec{h},\vec{\gamma}_k\rangle q \notin \mathbb{Z}$ has a non-vanishing limit point. More explicitly, if the sequence $\langle \vec{h},\vec{\gamma}_k\rangle \bmod 1$, $k = 0,1,2,\dots$, has a limit point lying in $(0,1)$, then (18) implies that the right-hand (23) tends to zero. This argument has the following simple consequences:

Corollary 1. (i) *If a sequence* $\langle \vec{h},\vec{\gamma}_k\rangle \bmod 1$, $k = 0,1,2,\dots$, *has a limit point in the open interval* $(0,1)$ *for every non-zero integer* $\vec{h}$, *then sequence* $s_{q,\Gamma}(n) \bmod 1$, $n = 0,1,2,\dots$, *is u.d.*
(ii) *If a sequence* $\langle \vec{h},\vec{\gamma}_k\rangle \bmod 1$, $k = 0,1,2,\dots$, *has* mod 1 *different* lim inf *and* lim sup *for every non-zero integral vector* $\vec{h}$, *then sequence* $s_{q,\Gamma}(n) \bmod 1$, $n = 0,1,2,\dots$, *is u.d.*

Similarly, if in the one-dimensional case the sequence $\boldsymbol{\gamma}$ has an irrational limit point modulo one. This limit point trivially lies in the interior of the unit interval, and consequently such sequence $s_{q,\boldsymbol{\gamma}}(n) \bmod 1$, $n = 0, 1, 2, \dots$, is u.d.[6] A related d-dimensional problem was also considered by F. Pillichshammer's [17, Example 2]. The next example covers these cases:

Example 6. Assume that the sequence of the d-dimensional vectors of weights $\vec{\gamma}_j = \left(\gamma_j^{(1)}, \gamma_j^{(2)}, \dots, \gamma_j^{(d)}\right)$ modulo one has a limit point $\vec{\rho} = \left(\rho^{(1)}, \rho^{(2)}, \dots, \rho^{(d)}\right)$ for which the elements of the set $\left\{1, \rho^{(1)}, \rho^{(2)}, \dots, \rho^{(d)}\right\}$ are linearly independent over $\mathbb{Q}$. Then the sequence $s_{q,\Gamma}(n)$ is u.d. mod 1.

To see this, let n_k, $k = 0, 1, \dots$, be such that

$$\vec{\gamma}_{n_k} = \left(\gamma_{n_k}^{(1)}, \gamma_{n_k}^{(2)}, \dots, \gamma_{n_k}^{(d)}\right) \bmod 1 \to \vec{\rho} = \left(\rho^{(1)}, \rho^{(2)}, \dots, \rho^{(d)}\right) \quad \text{for } k \to \infty.$$

Then for every $\vec{h} \in \mathbb{Z}^d \setminus \{\vec{0}\}$ the number $\langle \vec{h}, \vec{\rho} \rangle$ is irrational and (18) implies that the number

$$\frac{|\sin \pi q \langle \vec{h}, \vec{\rho} \rangle|}{q|\sin \pi \langle \vec{h}, \vec{\rho} \rangle|},$$

say $= \xi$, is strictly less than 1. Then for all sufficiently large k we have

$$\frac{|\sin \pi q \langle \vec{h}, \vec{\gamma}_{n_k} \rangle|}{q|\sin \pi \langle \vec{h}, \vec{\gamma}_{n_k} \rangle|} < \frac{1}{2}(1 + \xi) < 1,$$

and consequently also

$$\prod_{\substack{n=0 \\ \langle \vec{h}, \vec{\gamma}_n \rangle \notin \mathbb{Z}}}^{N-1} \frac{|\sin \pi q \langle \vec{h}, \vec{\gamma}_n \rangle|}{q|\sin \pi \langle \vec{h}, \vec{\gamma}_n \rangle|} \to 0,$$

that is, $s_{q,\Gamma}(n)$, $n = 0, 1, 2, \dots$ is u.d. mod 1.

Remark 1. Notice that Theorem 1 implies the "if" part of Pillichshammer's condition (14). Namely, if for every integral vector $\vec{h} \in \mathbb{Z}^d \setminus \{\vec{0}\}$ there exists positive integer k such that $\langle \vec{h}, \vec{\gamma}_k \rangle \notin \mathbb{Z}$ and $\langle \vec{h}, \vec{\gamma}_k \rangle q \in \mathbb{Z}$, then the product (20) vanishes.

4. Discrepancy estimates

The quality of distribution modulo one of a sequence $\vec{x}_n$, $n = 0, 1, 2, \dots$, in $\mathbb{R}^d$ is measured by quantities called discrepancies. In the theory of the uniform distribution, the knowledge of a suitable discrepancy estimate of a sequence

[6]This is a generalization of Amer. Math. Monthly Problem 6542 [1987, 386] proposed by A. M. Odlyzko.

distribution is more valuable than the plain information of its uniform distribution. From the variety of the discrepancy notions, the **(extreme) discrepancy** of the finite sequence $\vec{x}_0$, $\vec{x}_1, \dots, \vec{x}_{M-1}$ of real numbers is defined by

$$D_M(\vec{x}_n) = D_M(\vec{x}_0, \dots, \vec{x}_{M-1}) = \sup_{I \subseteq [0,1)^d} \left| \frac{A\,(I; M; \vec{x}_n \bmod 1)}{M} - \lambda_d(I) \right|,$$

where $I = [\alpha_1, \beta_1) \times \cdots \times [\alpha_d, \beta_d)$ stands for a rectangle with sides parallel to axes and $\lambda_d(I)$ denotes its d-dimensional volume. A sequence of points $\vec{x}_n$, $n = 0, 1, 2, \dots$, in the d-dimensional space $\mathbb{R}^d$ is u.d. mod 1 if and only if $D_M(\vec{x}_0, \dots, \vec{x}_{M-1}) \to 0$ for M approaching $+\infty$.

One of the fundamental tools used in proofs devoted to bounding the discrepancy of sequences is the classical Erdős-Turán inequality (cf. [12, Theorem 2.5]) and variants thereof [20, p. 1-44]) or their generalizations (cf. [12, p. 116] or [20, p. 1-63]). One of them, the Erdős-Turán-Koksma inequality gives an upper bound for the discrepancy of a sequence in the d-dimensional unit cube $[0,1]^d$ in terms of exponential sums:

Lemma 1 (Erdős-Turán-Koksma's inequality)**.** [7] *Let $\vec{x}_0$, $\vec{x}_2, \dots, \vec{x}_{M-1}$ be points in the d–dimensional unit cube $[0,1)^d$ and H be an arbitrary positive integer. Then for discrepancy $D_M(\vec{x}_n)$ we have*

$$D_M(\vec{x}_n) \le C_d \left(\frac{1}{H} + \sum_{0 < \|\vec{h}\|_\infty \le H} \frac{1}{r(\vec{h})} \left| \frac{1}{M} \sum_{n=0}^{M-1} e^{2\pi i \langle \vec{h}, \vec{x}_n \rangle} \right| \right), \tag{28}$$

where $r(\vec{h}) = \prod_{i=1}^d \max(1, |h_i|)$, $\|\vec{h}\|_\infty = \max\limits_{1 \le i \le d} |h_i|$, $\vec{h} = (h_1, \dots, h_d) \in \mathbb{Z}^d$ and the constant C_d only depends on the dimension d.

Inserting the estimate (23) and (27) which are part of the proof of Theorem 1 we can reformulate (28) as follows:

Theorem 2. *Let $q \ge 2, N, M$ be positive integers such that $q^N \le M < q^{N+1}$. Let Γ be the $d \times \infty$-matrix of real weights* (9)*. Then for the discrepancy of the sequence*

$$s_{q,\Gamma}(n) \mod 1, \quad n = 0, 1, 2, \dots, M-1,$$

we have

[7] For a general form of the involved absolute constants consult [6].

$$D_M\big(s_{q,\Gamma}(n) \bmod 1\big) \le C_d\left(\frac{1}{H} + \sum_{0<\|\vec{h}\|_\infty\le H} \frac{1}{r(\vec{h})}\left(\sum_{j=1}^{k} q^{-j+2} \prod_{\substack{t=0\\ \langle\vec{h},\vec{\gamma}_t\rangle\notin\mathbb{Z}}}^{N-j} \frac{|\sin\pi q\langle\vec{h},\vec{\gamma}_t\rangle|}{q|\sin\pi\langle\vec{h},\vec{\gamma}_t\rangle|} + O\Big(\frac{1}{q^{k-1}}\Big)\right)\right). \tag{29}$$

for every integer k satisfying $1 \le k \le N$.

Example 7. Consider the one-dimensional application of Theorem 2 in the case where the weight sequence $\boldsymbol{\gamma}$ is given by relation

$$\gamma_n = \{n\alpha\}, \quad n = 0, 1, 2, \dots, \tag{30}$$

where α is an irrational number.

Note that the weight sequence $\gamma_n = \{\alpha n\}$, $n = 0, 1, 2, \dots$, is u.d. modulo one and hence, if $\overline{\gamma} \in (0, 1)$ then there exists a subsequence of weights γ_{n_k}, $k = 1, 2, 3, \dots$, converging to $\overline{\gamma}$, and for any positive integer h we have $h\gamma_{n_k} \to h\overline{\gamma}$. There also exist $\varepsilon = \varepsilon(h) > 0$ and $0 < \delta = \delta(h, \varepsilon) < 1$ such that

$$\gamma_{n_k} \in \Big(\overline{\gamma} - \frac{\varepsilon}{h}, \overline{\gamma} + \frac{\varepsilon}{h}\Big) \text{ implies } \frac{|\sin\pi qh\gamma_{n_k}|}{q|\sin\pi h\gamma_{n_k}|} < \delta.$$

This implication remains valid even for every γ_n satisfying

$$\gamma_n \in \Big(\overline{\gamma} - \frac{\varepsilon}{h}, \overline{\gamma} + \frac{\varepsilon}{h}\Big) \tag{31}$$

when replacing γ_{n_k} in the above inequality.

Now, given an $H \in \mathbb{N}$, let I be an interval of the form $|I| = \{k_1\alpha\}$ with $k_1 \in \mathbb{N}$ such that the inclusion

$$I \subset \Big(\overline{\gamma} - \frac{\varepsilon}{h}, \overline{\gamma} + \frac{\varepsilon}{h}\Big)$$

holds for every $h = 1, 2, \dots, H$. Denote $\delta_0(\varepsilon) = \max_{h=1,2,\dots,H} \delta(h, \varepsilon)$. Relation (31) implies the implication

$$\gamma_n \in I \Rightarrow \frac{|\sin\pi qh\gamma_n|}{q|\sin\pi h\gamma_n|} < \delta_0(\varepsilon) \tag{32}$$

for all $\gamma_n \in I$ (not only for those with $\gamma_{n_k} \in I$). If we denote

$$A(a, b) = \#\{a \le n \le b; \gamma_n \in I\},$$

then for *every* γ_n (not only for $\gamma_n \in I$ due to (17)) we have

$$\prod_{a\le n\le b} \frac{|\sin\pi qh\gamma_n|}{|\sin\pi h\gamma_n|} \le \delta_0(\varepsilon)^{A(a,b)}. \tag{33}$$

Notice that we simultaneously have $h\gamma_n \notin \mathbb{Z}$.

Let I be an interval such that $|I| = \{k_1\alpha\}$ with $k_1 \in \mathbb{N}$. Then by the result of E. Hecke [7] (cf. also [18]) we have

$$(b-a)|I| - 2k_1 \le \#\big\{a < n \le b; \{n\alpha\} \in I\big\} \le (b-a)|I| + 2k_1 \tag{34}$$

for every $a < b$, and $A(a,b) > (b-a)\{k_1\alpha\} - 2k_1$.

If $q^N \le M < q^{N+1}$ and integer k satisfies $1 \le k \le N$, then relations (29) implies

$$D_M\big(s_{q,\boldsymbol{\gamma}}(n) \bmod 1\big) \le C_1\left(\frac{1}{H} + \log eH\left(\sum_{j=1}^{k} \frac{1}{q^{j-2}} \prod_{i=0}^{N-j} \frac{|\sin \pi q h \gamma_i|}{q|\sin \pi h \gamma_i|} + O\Big(\frac{1}{q^{k-1}}\Big)\right)\right)$$

and using (34) we obtain

$$D_M\big(s_{q,\boldsymbol{\gamma}}(n) \bmod 1\big) \le C_1\left(\frac{1}{H} + \log eH\left(\sum_{j=1}^{k} \frac{1}{q^{j-2}} \delta_0^{(N-j-1)\{k_1\alpha\}-2k_1} + O\Big(\frac{1}{q^{k-1}}\Big)\right)\right).$$

This discrepancy estimate implies that the sum-of-digits function $s_{q,\boldsymbol{\gamma}}(n)$, $n = 0, 1, \dots,$ with weight sequence $\gamma_n = \{n\alpha\}$, $n = 0, 1, 2, \dots,$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is u.d. modulo one. Nevertheless, the mere fact that such a sequence $s_{q,\boldsymbol{\gamma}}(n)$, $n = 0, 1, \dots,$ is u.d. modulo one also follows from (13).

5. Distribution functions of $s_{q,\boldsymbol{\gamma}}(n) \bmod 1$

Due to some inconsistencies in the definitions of distribution functions let us recall some definitions for reader's convenience.

Let x_n, $n = 0, 1, 2, \dots,$ be a sequence of numbers taken from the unit interval $[0, 1)$. Then

$$F_N(x) = \begin{cases} \dfrac{A\big([0,x), N, x_n\big)}{N} & \text{if } x \in [0,1), \\ 1 & \text{if } x = 1. \end{cases}$$

is the so-called **step distribution function** of the finite sequence $x_0, \dots, x_{N-1}$ over $[0, 1)$. In general, a function $g : [0, 1] \to [0, 1]$ is called a **distribution function** if:

(i) $g(x)$ is non-decreasing,

(ii) $g(0) = 0$ and $g(1) = 1$.

We shall identify any two d.f.s g_1, g_2 if $g_1(x) = g_2(x)$ for every their common continuity point $x \in [0,1]$, or equivalently, if $g_1(x) = g_2(x)$ a.e. on $[0,1]$.

Distribution function $g(x)$ is called a **distribution function of sequence** x_n, $n=0,1,2,\dots$, if there exists an increasing sequence of positive integers $N_1, N_2, \dots$ such that[8]

$$\lim_{k\to\infty} F_{N_k}(x) = g(x) \quad \text{a.e.} \quad \text{on } [0,1].$$

Finally, distribution function $g(x)$ is called an **asymptotic distribution function** of the sequence x_n, $n = 0,1,2,\dots$, if $\lim_{N\to\infty} F_N(x) = g(x)$ a.e. on $[0,1]$. Sequence x_n, $n=0,1,2,\dots$, is u.d. in $[0,1]$ if and only if $g(x) = x$ is its asymptotic distribution function.

Before stating the main result of this part, we prove the following two lemmas:

Lemma 2. *Let $A \le B$ be positive integers and $\alpha_0, \alpha_1, \dots, \alpha_{B-1}$ be complex numbers of magnitude not exceeding 1. Then*

$$\left| \frac{1}{A}\left|\sum_{j=0}^{A-1} \alpha_j\right| - \frac{1}{B}\left|\sum_{j=0}^{B-1} \alpha_j\right| \right| \le 2\left|1 - \frac{A}{B}\right|. \tag{35}$$

P r o o f. The triangle inequality yields

$$\left| \left|\sum_{j=0}^{A-1} \alpha_j\right| - \left|\sum_{j=0}^{B-1} \alpha_j\right| \right| \le \left|\sum_{j=0}^{A-1} \alpha_j - \sum_{j=0}^{B-1} \alpha_j\right| \le |A - B|.$$

Consequently,

$$\left| \frac{1}{B}\left|\sum_{j=0}^{B-1} \alpha_j\right| - \frac{1}{A}\left|\sum_{j=0}^{A-1} \alpha_j\right| + \left(1 - \frac{A}{B}\right)\frac{1}{A}\left|\sum_{j=0}^{A-1} \alpha_j\right| \right| =$$

$$\left| \frac{1}{B}\left|\sum_{j=0}^{B-1} \alpha_j\right| - \frac{1}{B}\left|\sum_{j=0}^{A-1} \alpha_j\right| \right| \le \frac{1}{B}|B - A| = \left|1 - \frac{A}{B}\right|. \tag{36}$$

Taking into account that an inequality $|X - Y + T| \le |Z|$ implies $|X - Y| \le |T| + |Z|$, inequality (36) concludes the proof of the Lemma. □

[8]Note that in this case the a.e. convergence (**almost everywhere convergence**) is the same as the **weak convergence** since every monotone function is almost everywhere continuous.

Lemma 3. *Let f be a q-additive function and k_1, k_2, N be positive integers such that $k_1 \le k_2$. Then*

$$\left| \left| \sum_{n=0}^{k_2 q^N - 1} e^{2\pi i h f(n)} \right| - \left| \sum_{n=0}^{k_1 q^N - 1} e^{2\pi i h f(n)} \right| \right| \le |k_2 - k_1| \left| \sum_{n=0}^{q^N - 1} e^{2\pi i h f(n)} \right|.$$

Proof. The q-additivity of $f(n)$ implies that for arbitrary positive integers k and N we have

$$\sum_{n=0}^{kq^N - 1} e^{2\pi i h f(n)} = \sum_{j=0}^{k-1} \sum_{n=0}^{q^N - 1} e^{2\pi i h f(n)} \cdot e^{2\pi i h f(j q^N)}.$$

Consequently,

$$\left| \sum_{n=0}^{kq^N - 1} e^{2\pi i h f(n)} \right| = \left| \sum_{n=0}^{q^N - 1} e^{2\pi i h f(n)} \right| \cdot \left| \sum_{j=0}^{k-1} e^{2\pi i h f(j q^N)} \right|. \tag{37}$$

Then assuming $k_1 \le k_2$ we have

$$\begin{aligned} \left| \sum_{n=0}^{k_2 q^N - 1} e^{2\pi i h f(n)} \right| &= \left| \sum_{n=0}^{q^N - 1} e^{2\pi i h f(n)} \right| \cdot \left| \sum_{j=0}^{k_2-1} e^{2\pi i h f(j q^N)} \right| \\ &\le \left| \sum_{n=0}^{q^N - 1} e^{2\pi i h f(n)} \right| \cdot \left(\left| \sum_{j=0}^{k_1-1} e^{2\pi i h f(j q^N)} \right| + (k_2 - k_1) \right) \\ &= \left| \sum_{n=0}^{k_1 q^N - 1} e^{2\pi i h f(n)} \right| + (k_2 - k_1) \left| \sum_{n=0}^{q^N - 1} e^{2\pi i h f(n)} \right|, \end{aligned}$$

where we used (37) at the beginning and in the last line from the right to the left. □

Introducing factor $\frac{1}{k_2 q^N}$ in the previous proof we obtain

Corollary 2. *Let k_1, k_2, N be positive integers such that $k_1 \le k_2$. Then*

$$\frac{1}{k_2 q^N} \left| \sum_{n=0}^{k_2 q^N - 1} e^{2\pi i h s_{q,\gamma}(n)} \right| \le \frac{k_1}{k_2} \cdot \frac{1}{k_1 q^N} \left| \sum_{n=0}^{k_1 q^N - 1} e^{2\pi i h s_{q,\gamma}(n)} \right| + \left(1 - \frac{k_1}{k_2} \right) \frac{1}{q^N} \left| \sum_{n=0}^{q^N - 1} e^{2\pi i h s_{q,\gamma}(n)} \right|.$$

Theorem 3. *If the function $g(x) = x$, $x \in [0,1]$, is a distribution function of the sequence $s_{q,\gamma}(n)$ mod 1, $n = 0, 1, 2, \dots$, then this sequence is u.d., i.e., $g(x) = x$ is its asymptotic d.f.*

P r o o f. By Weyl's criterion function $g(x) = x$ is a distribution function of the sequence $s_{q,\boldsymbol{\gamma}}(n) \bmod 1$ if and only if there exists an increasing integral sequence M_j, $j = 0, 1, \dots$, such that

$$\frac{1}{M_j}\left|\sum_{n=0}^{M_j-1} e^{2\pi i h s_{q,\boldsymbol{\gamma}}(n)}\right| \to 0$$

for j running to ∞ for every $h \in \mathbb{Z} \setminus \{0\}$.

Fix an arbitrary and sufficiently large integer N_0. Given a sufficiently large j, let k_j be the uniquely determined integer for which

$$k_j q^{N_0} \le M_j < (k_j + 1) q^{N_0}. \tag{38}$$

Clearly, if $M_j \to \infty$ then $k_j \to \infty$, too. Lemma 2 and (38) yield

$$\left|\frac{1}{M_j}\left|\sum_{n=0}^{M_j-1} e^{2\pi i h s_{q,\boldsymbol{\gamma}}(n)}\right| - \frac{1}{k_j q^{N_0}}\left|\sum_{n=0}^{k_j q^{N_0}-1} e^{2\pi i h s_{q,\boldsymbol{\gamma}}(n)}\right|\right| < \frac{2}{k_j + 1},$$

and thus for $M_j \to \infty$ we obtain

$$\frac{1}{k_j q^{N_0}}\left|\sum_{n=0}^{k_j q^{N_0}-1} e^{2\pi i h s_{q,\boldsymbol{\gamma}}(n)}\right| \to 0 \quad \text{as } j \to \infty. \tag{39}$$

Now suppose on the contrary, that sequence $s_{q,\boldsymbol{\gamma}}(n) \bmod 1$ is not u.d. (i.e., $g(x) = x$ is not its asymptotic distribution function). Since the sequence of products on the right-hand side of (23) is decreasing and all its elements lie in $[0, 1]$, Theorem 1 and (23) show that sequence $s_{q,\boldsymbol{\gamma}}(n) \bmod 1$ is not u.d., if and only if there exist an integer h and a positive δ such that

$$\frac{1}{q^N}\left|\sum_{n=0}^{q^N-1} e^{2\pi i h(s_{q,\boldsymbol{\gamma}}(n))}\right| = \prod_{\substack{i=0\\ h\gamma_i \notin \mathbb{Z}}}^{N-1} \frac{|\sin \pi q h \gamma_i|}{q|\sin \pi h \gamma_i|} \to \delta \tag{40}$$

if $N \to \infty$. Given k_j, define N by inequalities

$$q^{N-1-N_0} \le k_j < q^{N-N_0}. \tag{41}$$

If we denote $k_2 = q^{N-N_0}$, then (40) implies

$$\lim_{N\to\infty} \frac{1}{q^N}\left|\sum_{n=0}^{q^N-1} e^{2\pi i h(s_{q,\boldsymbol{\gamma}}(n))}\right| = \lim_{N\to\infty} \frac{1}{k_2 q^{N_0}}\left|\sum_{n=0}^{k_2 q^{N_0}-1} e^{2\pi i h s_{q,\boldsymbol{\gamma}}(n)}\right| = \delta. \tag{42}$$

Inserting k_j for k_1 and q^{N-N_0} for k_2 in Corollary 2 we obtain the inequality

$$\frac{1}{q^N}\left|\sum_{n=0}^{q^N-1} e^{2\pi i h s_{q,\gamma}(n)}\right| \le \frac{k_j}{q^{N-N_0}}\cdot\frac{1}{k_j q^{N_0}}\left|\sum_{n=0}^{k_j q^{N_0}-1} e^{2\pi i h s_{q,\gamma}(n)}\right| + \left(1-\frac{k_j}{q^{N-N_0}}\right)\frac{1}{q^{N_0}}\left|\sum_{n=0}^{q^{N_0}-1} e^{2\pi i h s_{q,\gamma}(n)}\right|.$$

If $M_j \to \infty$, then also $k_j \to \infty$ and $N \to \infty$. Inequalities (41) imply

$$\frac{k_j}{q^{N-N_0}} \ge \frac{q^{N-1-N_0}}{q^{N-N_0}} = \frac{1}{q}. \tag{43}$$

Letting $N \to \infty$ relations (39) and (43) lead to a contradiction

$$\delta \le \left(1-\frac{1}{q}\right)\delta$$

and the proof is finished. □

6. Uniform distribution of $\big(h_1 s_{q,\gamma}(n) + h_2 s_{q,\gamma}(n+1)\big)$ mod 1

In this section we shall study two related sequences from the point of view of their u.d. behaviour. Namely, the one-dimensional sequence

$$\big(h_1 s_{q,\gamma}(n) + h_2 s_{q,\gamma}(n+1)\big) \mod 1, \qquad n = 0,1,2,\dots, \tag{44}$$

and the two-dimensional one

$$\big(h_1 s_{q,\gamma}(n),\ h_2 s_{q,\gamma}(n+1)\big) \mod 1, \qquad n = 0,1,2,\dots, \tag{45}$$

where h_1 and h_2 are integers. It turns out that they behave quite differently.

Theorem 4. *If the sequence $s_{q,\gamma}(n)$ mod 1, $n = 0,1,2,\dots$, is u.d. then the sequence (44) is also u.d. for every pair of integers (h_1,h_2) such that $h_1+h_2 \ne 0$.*

P r o o f. We shall again employ Weyl's criterion and as in the proof of Theorem 1 also this one will be divided into two steps.

1°. In the first step we prove that the sequence $\big(h_1 s_{q,\gamma}(n)+h_2 s_{q,\gamma}(n+1)\big)$ mod 1 is almost u.d. with respect to the sequence of indices q^N, $N = 1,2,\dots$ provided $h_1, h_2 \in \mathbb{Z}$ and $h_1 + h_2 \ne 0$.

Given a sufficiently large N, divide the set of all n, $0 \le n \le q^N - 1$, written in the form

$$n = n_0 + n_1 q + n_2 + \dots + n_{N-1} q^{N-1}, \quad \text{where} \quad 0 \le n_i \le q - 1, \tag{46}$$

into the following disjoint subsets

$$\begin{aligned}
P_0 &= \big\{0 \le n \le q^N - 1;\ n_0 \in \{0, 1, \dots, q-2\},\ \ n_1 \in \{0, 1, \dots, q-1\}, \dots \\
&\quad \dots, n_{N-1} \in \{0, 1, \dots, q-1\}\big\}; \\
P_1 &= \big\{1 \le n \le q^N - 1;\ n_0 = q - 1,\ \ n_1 \in \{0, 1, \dots, q-2\}, \dots \\
&\quad \dots, n_{N-1} \in \{0, 1, \dots, q-1\}\big\}; \\
&\ \ \vdots \\
P_t &= \big\{1 \le n \le q^N - 1;\ \ n_0 = q - 1, n_1 = q - 1, \dots, n_{t-1} = q - 1, \\
&\quad n_t \in \{0, 1, \dots, q-2\},\ \ n_{t+1} \in \{0, 1, \dots, q-1\}, \dots \\
&\quad \dots, n_{N-1} \in \{0, 1, \dots, q-1\}\big\}; \\
&\ \ \vdots \\
P_{N-1} &= \big\{1 \le n \le q^N - 1;\ \ n_0 = q - 1, n_1 = q - 1, \dots, n_{N-2} = q - 1, \\
&\quad n_{N-1} \in \{0, 1, \dots, q-2\}\big\} \\
P_N &= \big\{1 \le n \le q^N - 1; n_0 = q - 1,\ \ n_1 = q - 1, \dots, n_{N-2} = q - 1, \\
&\quad n_{N-1} = q - 1\big\}.
\end{aligned}$$

Here

$$|P_t| = \begin{cases} (q-1)q^{N-1-t}, & t = 0, \dots, N-1, \\ 1, & t = N. \end{cases}$$

Now we split Weyl's sum[9]

$$\frac{1}{q^N} \sum_{n=0}^{q^N - 1} e^{2\pi i (h_1 s_{q,\gamma}(n) + h_2 s_{q,\gamma}(n+1))}$$

into $N + 1$ subsums

$$\frac{1}{q^N} \sum_{n \in P_t} e^{2\pi i (h_1 s_{q,\gamma}(n) + h_2 s_{q,\gamma}(n+1))} \quad \text{for } t = 0, 1, \dots, N.$$

To determine a suitable interrelationship between t and n we shall employ an idea developed in [4]. For every n with $0 \le n < q^N - 1$, there exists uniquely determined $\tau = \tau(n)$, $0 \le \tau \le N - 1$, such that

$$n = (q-1) + (q-1)q + \dots + (q-1)q^{\tau-1} + n_\tau q^\tau + \dots + n_{N-1} q^{N-1}, \tag{47}$$

[9]The constant h is dissolved in both h_1 and h_2.

i.e., τ is the least index such that $n_0 = n_1 = \cdots = n_{\tau-1} = q-1$ and $n_\tau < q-1$. If $n = q^N - 1$, put $\tau = N$.

Thus if $\tau(n) = t$, that is if $n \in P_t$, then

$$n = (q-1)(q^0 + q^1 + \cdots + q^{t-1}) + n_t q^t + \cdots + n_{N-1} q^{N-1},$$

$$s_{q,\boldsymbol{\gamma}}(n) = (q-1)(\gamma_0 + \gamma_1 + \cdots + \gamma_{t-1}) + \gamma_t n_t + \cdots + \gamma_{N-1} n_{N-1}, \tag{48}$$

$$n+1 = 0 + 0q + \cdots + 0q^{t-1} + (n_t+1)q^t + \cdots + n_{N-1} q^{N-1}, \tag{49}$$

$$s_{q,\boldsymbol{\gamma}}(n+1) = 0.(\gamma_0 + \cdots + \gamma_{t-1}) + (n_t+1)\gamma_t + n_{t+1}\gamma_{t+1} + \cdots + n_{N-1}\gamma_{N-1}, \tag{50}$$

$$s_{q,\boldsymbol{\gamma}}(n+1) = s_{q,\vec{\gamma}}(n) - (q-1)(\gamma_0 + \gamma_1 + \cdots + \gamma_{t-1}) + \gamma_t, \tag{51}$$

and

$$\begin{aligned} & h_1 s_{q,\boldsymbol{\gamma}}(n) + h_2 s_{q,\boldsymbol{\gamma}}(n+1) \\ &= (h_1+h_2) s_{q,\boldsymbol{\gamma}}(n) + h_2\big(\gamma_t - (q-1)(\gamma_0 + \gamma_1 + \cdots + \gamma_{t-1})\big) \\ &= (h_1+h_2)\big((q-1)(\gamma_0 + \gamma_1 + \cdots + \gamma_{t-1})\big) \\ &\quad + h_2\big(\gamma_t - (q-1)(\gamma_0 + \gamma_1 + \cdots + \gamma_{t-1})\big) \\ &\quad + (h_1+h_2)(\gamma_t n_t + \gamma_{t+1} n_{t+1} + \cdots + \gamma_{N-1} n_{N-1}). \end{aligned}$$

Then

$$\begin{aligned} & \frac{1}{q^N} \sum_{n \in P_t} e^{2\pi i (h_1 s_{q,\boldsymbol{\gamma}}(n) + h_2 s_{q,\boldsymbol{\gamma}}(n+1))} \\ &= \frac{1}{q^N} e^{2\pi i ((h_1+h_2)(q-1)(\gamma_0+\gamma_1+\cdots+\gamma_{t-1}) + h_2(\gamma_t - (q-1)(\gamma_0+\gamma_1+\cdots+\gamma_{t-1})))} \\ &\quad \times \sum_{n_t=0}^{q-2} \sum_{n_{t+1}=0}^{q-1} \cdots \sum_{n_{N-1}=0}^{q-1} e^{2\pi i ((h_1+h_2)(\gamma_t n_t + \gamma_{t+1} n_{t+1} + \cdots + \gamma_{N-1} n_{N-1}))}. \end{aligned} \tag{52}$$

and using (16) we obtain for $t = 0, 1, \ldots, N-1$ that

$$\begin{aligned} & \left| \frac{1}{q^N} \sum_{n \in P_t} e^{2\pi i (h_1 s_{q,\boldsymbol{\gamma}}(n) + h_2 s_{q,\boldsymbol{\gamma}}(n+1))} \right| \\ &= \frac{1}{q^{N-(N-t)}} \cdot \frac{q-1}{q} \cdot \left| \frac{1}{q-1} \sum_{n_t=0}^{q-2} e^{2\pi i (h_1+h_2)\gamma_t n_t} \right| \\ &\quad \times \left| \frac{1}{q} \sum_{n_{t+1}=0}^{q-1} e^{2\pi i (h_1+h_2)\gamma_{t+1} n_{t+1}} \right| \cdots \left| \frac{1}{q} \sum_{n_{N-1}=0}^{q-1} e^{2\pi i (h_1+h_2)\gamma_{N-1} n_{N-1}} \right| \\ &= \frac{1}{q^t} \cdot \frac{q-1}{q} \cdot \frac{|\sin \pi (q-1)(h_1+h_2)\gamma_t|}{(q-1)|\sin \pi (h_1+h_2)\gamma_t|} \prod_{\substack{j=t+1 \\ (h_1+h_2)\gamma_j \notin \mathbb{Z}}}^{N-1} \frac{|\sin \pi q (h_1+h_2)\gamma_j|}{q|\sin \pi (h_1+h_2)\gamma_j|}, \end{aligned}$$

and
$$\left|\frac{1}{q^N}\sum_{n\in P_N} e^{2\pi i(h_1 s_{q,\boldsymbol{\gamma}}(n)+h_2 s_{q,\boldsymbol{\gamma}}(n+1))}\right| = \frac{1}{q^N}.$$

Consequently,
$$\left|\frac{1}{q^N}\sum_{n=0}^{q^N-1} e^{2\pi i(h_1 s_{q,\boldsymbol{\gamma}}(n)+h_2 s_{q,\boldsymbol{\gamma}}(n+1))}\right|$$
$$= \left|\frac{1}{q^N}\sum_{t=0}^{N}\sum_{n\in P_t} e^{2\pi i(h_1 s_{q,\boldsymbol{\gamma}}(n)+h_2 s_{q,\boldsymbol{\gamma}}(n+1))}\right|$$
$$\le \frac{q-1}{q}\sum_{t=0}^{N-1}\frac{1}{q^t}\prod_{\substack{j=t+1\\(h_1+h_2)\gamma_j\notin\mathbb{Z}}}^{N-1}\frac{|\sin \pi q(h_1+h_2)\gamma_j|}{q|\sin \pi(h_1+h_2)\gamma_j|}+\frac{1}{q^N}. \tag{53}$$

Now, split the last sum into two ones choosing an integer k between 0 and N and obtain
$$\le \frac{q-1}{q}\sum_{t=0}^{k-1}\frac{1}{q^t}\prod_{\substack{j=t+1\\(h_1+h_2)\gamma_j\notin\mathbb{Z}}}^{N-1}\frac{|\sin \pi q(h_1+h_2)\gamma_j|}{q|\sin \pi(h_1+h_2)\gamma_j|}+\frac{q-1}{q}\sum_{t=k}^{N}\frac{1}{q^t}+\frac{1}{q^N}. \tag{54}$$

Assumption that $s_{q,\boldsymbol{\gamma}}(n) \bmod 1$, $n = 0, 1, 2, \dots$, is u.d. implies via Theorem 1 that for $h_1 + h_2 \neq 0$ we have
$$\prod_{\substack{j=0\\(h_1+h_2)\gamma_j\notin\mathbb{Z}}}^{N-1}\frac{|\sin \pi q(h_1+h_2)\gamma_j|}{q|\sin \pi(h_1+h_2)\gamma_j|}\to 0$$
as $N \to \infty$ and thus for a fixed k and every $t = 0, \dots, k-1$ we also have
$$\prod_{\substack{j=t+1\\(h_1+h_2)\gamma_j\notin\mathbb{Z}}}^{N-1}\frac{|\sin \pi q(h_1+h_2)\gamma_j|}{q|\sin \pi(h_1+h_2)\gamma_j|}\to 0$$
as $N \to \infty$. In other words, fixing k and letting $N \to \infty$, the right hand side of (54) reduces to
$$\frac{1}{q^k} \quad \text{as } N \to \infty.$$

The limit passage $k \to \infty$ then implies according to Weyl's criterion that sequence $h_1 s_{q,\boldsymbol{\gamma}}(n) + h_2 s_{q,\boldsymbol{\gamma}}(n+1) \bmod 1$ is almost u.d. with respect to the sequence of indexes q^N, $N = 1, 2, \dots$

2^0. Now, we prove that the almost u.d. of $\big(h_1 s_{q,\boldsymbol{\gamma}}(n) + h_2 s_{q,\boldsymbol{\gamma}}(n+1)\big) \bmod 1$ with respect to the sequence of indices q^N, $N = 1, 2, \dots$, implies its u.d. provided $s_{q,\boldsymbol{\gamma}}(n) \bmod 1$ is u.d.

Take an arbitrarily large integer M and squeeze it in the following way

$$kq^N \le M < (k+1)q^N,$$

where k and N are suitable integers which will be specified later. Then according to Weyl's criterion it is sufficient to prove that

$$\frac{1}{M}\left|\sum_{n=0}^{M-1} e^{2\pi i\left(h_1 s_{q,\boldsymbol{\gamma}}(n)+h_2 s_{q,\boldsymbol{\gamma}}(n+1)\right)}\right| \to 0 \qquad \text{as } M \to \infty. \tag{55}$$

To do this, (35) implies

$$\left|\frac{1}{M}\left|\sum_{n=0}^{M-1} e^{2\pi i\left(h_1 s_{q,\boldsymbol{\gamma}}(n)+h_2 s_{q,\boldsymbol{\gamma}}(n+1)\right)}\right| - \frac{1}{kq^N}\left|\sum_{n=0}^{kq^N-1} e^{2\pi i\left(h_1 s_{q,\boldsymbol{\gamma}}(n)+h_2 s_{q,\boldsymbol{\gamma}}(n+1)\right)}\right|\right| \le 2\left|1-\frac{kq^N}{M}\right| \le 2\left|1-\frac{kq^N}{(k+1)q^N}\right| = \frac{2}{k+1}. \tag{56}$$

Since the weighted q-adic sum-of-digits function is q-additive, that is

$$s_{q,\boldsymbol{\gamma}}\big(b+jq^N\big) = s_{q,\boldsymbol{\gamma}}(b) + s_{q,\boldsymbol{\gamma}}\big(jq^N\big) \quad \text{for } 0 \le b < q^N \text{ and } j \in \mathbb{N}_0,$$

we have for $0 \le n \le q^N - 2$ and $j \in \mathbb{N}_0$ that

$$e^{2\pi i(h_1 s_{q,\boldsymbol{\gamma}}(n+jq^N)+h_2 s_{q,\boldsymbol{\gamma}}(n+1+jq^N))} = e^{2\pi i(h_1 s_{q,\boldsymbol{\gamma}}(n)+h_2 s_{q,\boldsymbol{\gamma}}(n+1))} \cdot e^{2\pi i(h_1 s_{q,\boldsymbol{\gamma}}(jq^N)+h_2 s_{q,\boldsymbol{\gamma}}(jq^N))}.$$

Thus

$$\begin{aligned}
&\frac{1}{kq^N}\sum_{n=0}^{kq^N-1} e^{2\pi i(h_1 s_{q,\boldsymbol{\gamma}}(n)+h_2 s_{q,\boldsymbol{\gamma}}(n+1))} \\
&= \frac{1}{kq^N}\sum_{j=0}^{k-1}\sum_{n=0}^{q^N-1} e^{2\pi i(h_1 s_{q,\boldsymbol{\gamma}}(n+jq^N)+h_2 s_{q,\boldsymbol{\gamma}}(n+1+jq^N))} \\
&= \frac{1}{kq^N}\sum_{j=0}^{k-1}\left(\sum_{n=0}^{q^N-2} e^{2\pi i(h_1 s_{q,\boldsymbol{\gamma}}(n+jq^N)+h_2 s_{q,\boldsymbol{\gamma}}(n+1+jq^N))} + e^{2\pi i(h_1 s_{q,\boldsymbol{\gamma}}(q^N-1+jq^N)+h_2 s_{q,\boldsymbol{\gamma}}(q^N+jq^N))}\right).
\end{aligned}$$

For the absolute value we then obtain

$$\begin{aligned}
&\frac{1}{kq^N}\sum_{n=0}^{kq^N-1}\left|e^{2\pi i(h_1 s_{q,\gamma}(n)+h_2 s_{q,\gamma}(n+1))}\right| \\
&\le \frac{1}{kq^N}\sum_{j=0}^{k-1}\left(\left|\sum_{n=0}^{q^N-2} e^{2\pi i(h_1 s_{q,\gamma}(n+jq^N)+h_2 s_{q,\gamma}(n+1+jq^N))}\right|+1\right) \\
&= \frac{1}{q^N}\left|\sum_{n=0}^{q^N-2} e^{2\pi i(h_1 s_{q,\gamma}(n)+h_2 s_{q,\gamma}(n+1))}\right|+\frac{1}{q^N}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\frac{1}{q^N}\left|\sum_{n=0}^{q^N-2} e^{2\pi i(h_1 s_{q,\gamma}(n)+h_2 s_{q,\gamma}(n+1))}\right| \\
&= \frac{1}{q^N}\left|\sum_{n=0}^{q^N-1} e^{2\pi i(h_1 s_{q,\gamma}(n)+h_2 s_{q,\gamma}(n+1))}\right. \\
&\qquad \left. - e^{2\pi i(h_1 s_{q,\gamma}(q^N-1+jq^N)+h_2 s_{q,\gamma}(q^N+jq^N))}\right| \\
&\le \frac{1}{q^N}\left|\sum_{n=0}^{q^N-1} e^{2\pi i(h_1 s_{q,\gamma}(n)\,+\,h_2 s_{q,\gamma}(n+1))}\right|+\frac{1}{q^N}.
\end{aligned}$$

Finally,

$$\begin{aligned}
&\frac{1}{kq^N}\left|\sum_{n=0}^{kq^N-1} e^{2\pi i(h_1 s_{q,\gamma}(n)\,+\,h_2 s_{q,\gamma}(n+1))}\right| \\
&\le \frac{1}{q^N}\left|\sum_{n=0}^{q^N-1} e^{2\pi i(h_1 s_{q,\gamma}(n)+h_2 s_{q,\gamma}(n+1))}\right|+\frac{2}{q^N}. \qquad (57)
\end{aligned}$$

Along the same arguments to those accompanying (53) we can prove that

$$\frac{1}{q^N}\left|\sum_{n=0}^{q^N-1} e^{2\pi i(h_1 s_{q,\gamma}(n)+h_2 s_{q,\gamma}(n+1))}\right| \to 0 \quad \text{as } N\to\infty$$

and (57) implies that also

$$\frac{1}{kq^N}\left|\sum_{n=0}^{kq^N-1} e^{2\pi i(h_1 s_{q,\gamma}(n)+h_2 s_{q,\gamma}(n+1))}\right| \to 0 \quad \text{as } N\to\infty.$$

and (55) follows. □

Weyl's criterion shows a connection between the distribution properties of the two-dimensional sequence

$$\big(s_{q,\boldsymbol{\gamma}}(n), s_{q,\boldsymbol{\gamma}}(n+1)\big) \bmod 1, \quad n = 0,1,2,\dots,$$

and sequence

$$h_1 s_{q,\boldsymbol{\gamma}}(n) + h_2 s_{q,\boldsymbol{\gamma}}(n+1) \bmod 1, \quad n = 0,1,2,\dots$$

More precisely, the sequence

$$\big(s_{q,\boldsymbol{\gamma}}(n), s_{q,\boldsymbol{\gamma}}(n+1)\big) \bmod 1, \quad n = 0,1,2,\dots,$$

is u.d. if and only if

$$\big(h_1 s_{q,\boldsymbol{\gamma}}(n) + h_2 s_{q,\boldsymbol{\gamma}}(n+1)\big) \bmod 1, \quad n = 0,1,2,\dots,$$

is u.d. for all couples of integers $(h_1,h_2) \neq (0,0)$. The trap is in the condition $h_1 + h_2 \neq 0$ as the next theorem shows. Some results on the distribution of the sequence $\big(s_{q,\boldsymbol{\gamma}}(n), s_{q,\boldsymbol{\gamma}}(n+1)\big) \bmod 1$, $n = 0,1,2,\dots$, will be proved also in the next section.

Theorem 5. *Given an integer $q > 2$ and a sequence of real weights γ_i, $i = 0,1,2,\dots$, the two-dimensional sequence $\big(s_{q,\boldsymbol{\gamma}}(n), s_{q,\boldsymbol{\gamma}}(n+1)\big) \bmod 1$, $n = 0,1,2,\dots$, cannot be u.d.*

Proof. If h is a non-zero integer, then (52) implies

$$\begin{aligned}
&\frac{1}{q^N}\sum_{n\in P_t} e^{2\pi i (h s_{q,\boldsymbol{\gamma}}(n) - h s_{q,\boldsymbol{\gamma}}(n+1))}\\
&= \frac{1}{q^N} e^{2\pi i(-h(\gamma_t - (q-1)(\gamma_0+\gamma_1+\cdots+\gamma_{t-1})))} \sum_{n_t \le q-2, n_{t+1}\le q-1,\dots,n_{N-1}\le q-1} 1\\
&= \frac{1}{q^N} e^{2\pi i(-h(\gamma_t - (q-1)(\gamma_0+\gamma_1+\cdots+\gamma_{t-1})))}(q-1)q^{N-t-1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left|\frac{1}{q^N}\sum_{n=0}^{q^N-1} e^{2\pi i (h s_{q,\boldsymbol{\gamma}}(n) - h s_{q,\boldsymbol{\gamma}}(n+1))}\right|\\
&= \frac{q-1}{q}\left|\sum_{t=0}^{N}\frac{1}{q^t}\, e^{2\pi i(-h(\gamma_t - (q-1)(\gamma_0+\gamma_1+\cdots+\gamma_{t-1})))}\right|\\
&\ge \frac{q-1}{q}\left(1 - \sum_{t=1}^{N}\frac{1}{q^t}\right) > 1 - \frac{2}{q}.
\end{aligned}$$

and Weyl's criterion confirms the statement of Theorem 5. □

7. A new property of the van der Corput sequence

In this section we study the one-dimensional sum-of-digits functions $s_{q,\boldsymbol{\gamma}}(n)$ with weights γ_j satisfying a summation criterion

$$(q-1)\sum_{j=0}^{\infty}\gamma_j = S \tag{58}$$

and a mnonoticity criterion

$$\gamma_0 \ge \gamma_1 \ge \gamma_2 \ge \cdots > 0. \tag{59}$$

In Proposition 5 we prove that if $S = 1$ then the q-adic van der Corput sequence is the only uniformly distributed sequence of this form.

Notice that condition (59) is not a restriction in general, for Proposition 1 has the following simple consequence

Corollary 3. *Given a sequence of weights $\boldsymbol{\gamma} = \{\gamma_0, \gamma_1, \dots\}$ and a permutation π of the set $\mathbb{N}_0$ of non-negative integers, then sequence $s_{q,\boldsymbol{\gamma}}(n)$, $n = 0, 1, 2, \dots$, and sequence $s_{q,\boldsymbol{\gamma}_\pi}(n)$, $n = 0, 1, 2, \dots$, are simultaneously u.d.* mod 1 *or simultaneously not u.d.* mod 1*, where $\boldsymbol{\gamma}_\pi = \{\gamma_{\pi(0)}, \gamma_{\pi(1)}, \gamma_{(2)}, \dots\}$.*

Condition (59) is, as expected, not necessary for a sequence $s_{q,\boldsymbol{\gamma}}(n)$, $n = 0, 1, 2, \dots$, to be u.d. mod 1 as the following example shows: Let θ be an irrational number and let $\gamma_j = \theta$ for infinitely many indices j. Then $s_{q,\boldsymbol{\gamma}}(n)$, $n = 0, 1, 2, \dots$, is u.d. mod 1. Another construction is given in the following example:

Example 8. Assume that the sequence of weights $\boldsymbol{\gamma}$ contains an infinite subsequence of the form C_j/q^j, $j = 0, 1, 2, \dots$, in an arbitrary order and with possible repetitions of each its term, where C_j's are integers coprime to q. Then relation (14) shows that this sequence $s_{q,\boldsymbol{\gamma}}(n)$, $n = 0, 1, 2, \dots$, is u.d. mod 1.

In the proofs of the theorems in this section we employ some features of a technique developed in [4] for two dimensional sequences $\big(s_{q,\boldsymbol{\gamma}}(n), s_{q,\boldsymbol{\gamma}}(n+1)\big)$, $n = 0, 1, 2, \dots$

Proposition 2. *Let $\boldsymbol{\gamma}$ be a non-increasing sequence of positive real weights, i.e.,* (59) *holds. Then for each $n = 0, 1, 2, \dots$, represented in the form* (47)*, the point $\big(s_{q,\boldsymbol{\gamma}}(n), s_{q,\boldsymbol{\gamma}}(n+1)\big)$ lies on the diagonal of the two-dimensional interval*

$$\begin{aligned} I_{\tau-1} = {} & [(q-1)(\gamma_0 + \cdots + \gamma_{\tau-1}), (q-1)(\gamma_0 + \gamma_1 + \cdots) - \gamma_\tau] \\ & \times [\gamma_\tau, (q-1)(\gamma_\tau + \gamma_{\tau+1} + \cdots)], \end{aligned} \tag{60}$$

where $\tau = \tau(n)$ is defined in (47)*.*

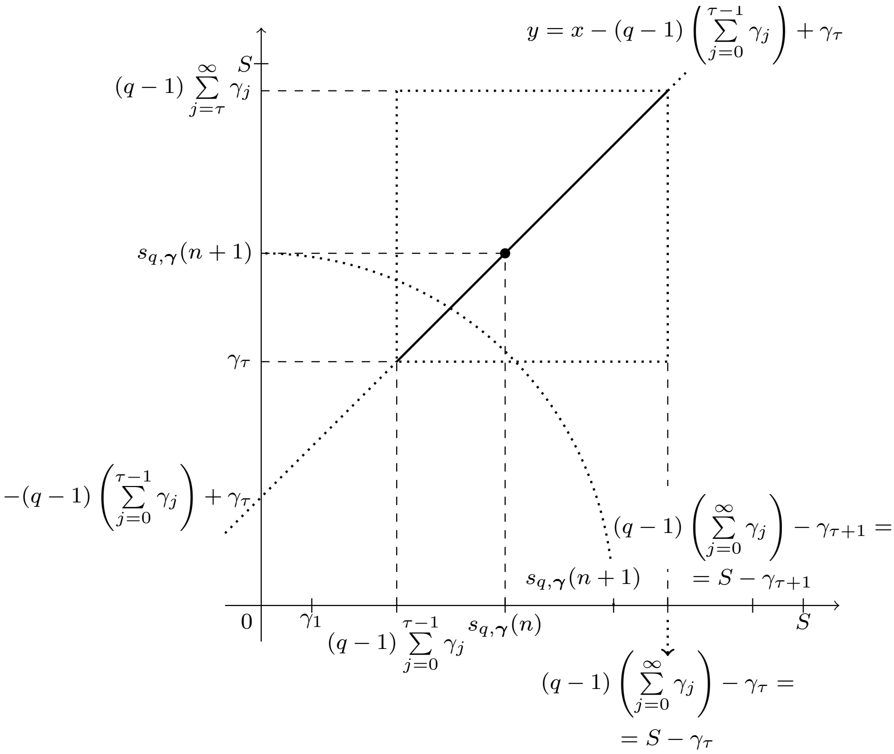

FIGURE 1. Position of point $\big(s_{q,\boldsymbol{\gamma}}(n), s_{q,\boldsymbol{\gamma}}(n+1)\big)$.

Here

$$\begin{aligned} I_{-1} &= [0,(q-1)(\gamma_0+\gamma_1+\cdots)-\gamma_0]\times[\gamma_0,(q-1)(\gamma_0+\gamma_1+\cdots)] \\ &= [0,1-\gamma_0]\times[\gamma_0,1] \end{aligned} \tag{61}$$

if $S = 1$ in (58). Also note that

$$(q-1)\sum_{j=0}^{\infty}\gamma_j - \gamma_\tau \ge (q-1)\sum_{j=0}^{\tau-1}\gamma_j + \sum_{j=\tau+1}^{\infty}\gamma_j > (q-1)\sum_{j=0}^{\tau-1}\gamma_j,$$

since the γ_i's are positive, and thus the intervals $I_{\tau-1}$, $\tau = 0,1,2,\ldots$, are non-degenerate.

Proof of Proposition 2: In other words, assuming n is written in the form (47), the point

$$\big(s_{q,\boldsymbol{\gamma}}(n), s_{q,\boldsymbol{\gamma}}(n+1)\big), \qquad n = 0,1,2,\ldots,$$

lies on the line segment (cf. Fig. 1) with equation

$$y = x - (q-1)(\gamma_0 + \cdots + \gamma_{\tau-1}) + \gamma_\tau \tag{62}$$

with abscissas

$$x \in [(q-1)(\gamma_0 + \cdots + \gamma_{\tau-1}), (q-1)(\gamma_0 + \gamma_1 + \cdots) - \gamma_\tau]$$

and ordinates

$$y \in [\gamma_\tau, (q-1)(\gamma_\tau + \gamma_{\tau+1} + \cdots)].$$

The equation for the line segment follows from (51), and the interval for the x-coordinate follows from (48). To see this note that

$$s_{q,\boldsymbol{\gamma}}(n) \geq (q-1)(\gamma_0 + \cdots + \gamma_{\tau-1}) + 0 \cdot \gamma_\tau + 0 \cdot \gamma_{\tau+1} + \cdots$$

$$\begin{aligned} s_{q,\boldsymbol{\gamma}}(n) &\leq (q-1)(\gamma_0 + \cdots + \gamma_{\tau-1}) + (q-2) \cdot \gamma_\tau + (q-1)(\gamma_{\tau+1} + \gamma_{\tau+2} + \cdots) \\ &= (q-1)(\gamma_0 + \gamma_1 + \cdots) - \gamma_\tau. \end{aligned} \tag{63}$$

Similarly, (50) implies

$$s_{q,\boldsymbol{\gamma}}(n+1) \geq 0.(\gamma_0 + \gamma + \cdots + \gamma_{\tau-1}) + (0+1)\gamma_\tau + 0 \cdot \gamma_{\tau+1} + \cdots + 0 \cdot \gamma_\ell = \gamma_\tau$$

$$s_{q,\boldsymbol{\gamma}}(n+1) \leq (q-2+1)\gamma_\tau + (q-1)\gamma_{\tau+1} + \cdots + (q-1)\gamma_\ell + \cdots ,$$

and Theorem is proved. □

Proposition 3. *Let $\boldsymbol{\gamma}$ be a sequence of positive real numbers satisfying conditions* (58) *and* (59). *Let there exists $\lambda = 0, 1, 2, \ldots$ such that*

$$(q-1)(\gamma_{\lambda+2} + \gamma_{\lambda+3} + \cdots) < \gamma_{\lambda+1}. \tag{64}$$

Then the interval

$$J = \left((q-1)\sum_{j=0}^{\infty} \gamma_j - \gamma_{\lambda+1}, (q-1)\sum_{j=0}^{\lambda+1} \gamma_j \right)$$

with positive volume does not contain an element of the form $s_{q,\boldsymbol{\gamma}}(n)$.

P r o o f. In this case,

$$\begin{aligned} (q-1)\sum_{j=0}^{\infty} \gamma_j - \gamma_{\lambda+1} &= (q-1)\sum_{j=0}^{\lambda} \gamma_j + (q-1)\sum_{j=\lambda+1}^{\infty} \gamma_j - \gamma_{\lambda+1} \\ &< (q-1)\sum_{j=0}^{\lambda} \gamma_j < (q-1)\sum_{j=0}^{\lambda+1} \gamma_j, \end{aligned} \tag{65}$$

that is, the intervals I_λ and $I_{\lambda+1}$ are non-overlapping (cf. Fig. 2).

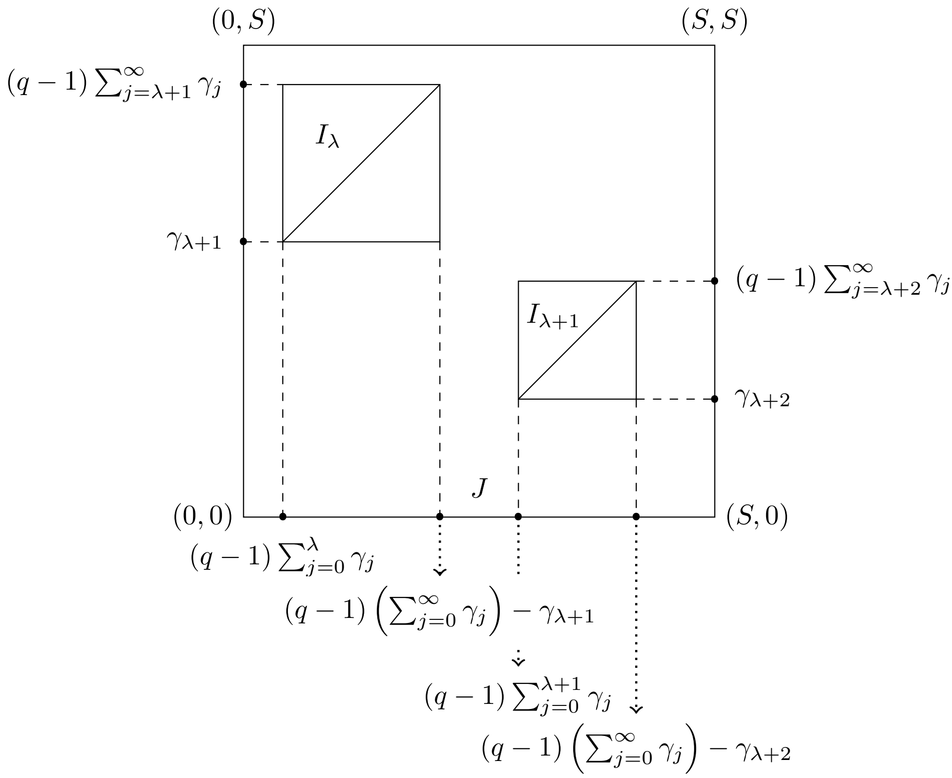

FIGURE 2. Non-overlapping intervals I_λ and $I_{\lambda+1}$.

Let n be of form (47). If $\tau > \lambda + 1$, then

$$s_{q,\boldsymbol{\gamma}}(n) \geq (q-1)(\gamma_0 + \cdots + \gamma_\lambda + \cdots + \gamma_{\tau-1}) \geq (q-1)(\gamma_0 + \cdots + \gamma_{\lambda+1}),$$

and $s_{q,\boldsymbol{\gamma}}(n) \notin J$. If $\tau \leq \lambda + 1$, then due to the choice of τ we have $n_\tau \leq q-2$, and therefore

$$\begin{aligned} s_{q,\boldsymbol{\gamma}}(n) &\leq (q-1)(\gamma_0 + \cdots + \gamma_{\tau-1}) + (q-1)\gamma_\tau - \gamma_\tau + n_{\tau+1}\gamma_{\tau+1} + \cdots + n_\ell \gamma_\ell \\ &\leq (q-1)(\gamma_0 + \gamma_1 + \cdots) - \gamma_\tau \leq (q-1)(\gamma_0 + \gamma_1 + \cdots) - \gamma_{\lambda+1}. \end{aligned}$$

Again, $s_{q,\boldsymbol{\gamma}}(n) \notin J$. Since interval J has a positive length, sequence $s_{q,\boldsymbol{\gamma}}(n)$, $n = 0, 1, 2, \ldots$, cannot be u.d. with respect to interval $[0, S]$. □

Proposition 4. *Let γ be a sequence of positive real numbers satisfying conditions* (58) *and* (59)*. Let there exits a $\lambda = 0, 1, 2, \dots$ such that*

$$(q-1)(\gamma_{\lambda+2} + \gamma_{\lambda+3} + \cdots) > \gamma_{\lambda+1}. \tag{66}$$

Then sequence $s_{q,\gamma}(n)$, $n = 0, 1, 2, \dots$, is not u.d. in the interval $[0, S]$.

Proof. If there exits a $\lambda \in \mathbb{N}_0$ such that $(q-1)(\gamma_{\lambda+2} + \gamma_{\lambda+3} + \cdots) > \gamma_{\lambda+1}$, then

$$(q-1)\sum_{j=0}^{\lambda+1}\gamma_j < (q-1)\left(\sum_{j=0}^{\infty}\gamma_j\right) - \gamma_{\lambda+1},$$

i.e., the x-projections of two-dimensional intervals I_λ and $I_{\lambda+1}$ overlap in interval (cf. Fig. 3)

$$J_1 = \left[(q-1)\sum_{j=0}^{\lambda+1}\gamma_j, (q-1)\left(\sum_{j=0}^{\infty}\gamma_j\right) - \gamma_{\lambda+1}\right].$$

If $s_{q,\gamma}(n) \in J_1$, then $\tau(n) \geq \lambda + 2$. If there exits a $\lambda \in \mathbb{N}_0$ such that

$$(q-1)(\gamma_{\lambda+2} + \gamma_{\lambda+3} + \cdots) > \gamma_{\lambda+1},$$

then

$$(q-1)\sum_{j=0}^{\lambda+1}\gamma_j < (q-1)\left(\sum_{j=0}^{\infty}\gamma_j\right) - \gamma_{\lambda+1},$$

i.e., the x-projections of the two-dimensional intervals I_λ and $I_{\lambda+1}$ overlap in interval (as for instance in cf. Fig. 3)

$$J_1 = \left[(q-1)\sum_{j=0}^{\lambda+1}\gamma_j, (q-1)\left(\sum_{j=0}^{\infty}\gamma_j\right) - \gamma_{\lambda+1}\right].$$

Given an interval $J \subset [0, 1)$, define

$$\begin{aligned} A_N(J) &= \#\{s_{q,\gamma}(n) \in J;\, n \leq N\}, \\ B_N(J) &= \#\{s_{q,\gamma}(n+1) \in J;\, n \leq N\}. \end{aligned}$$

Let intervals J_2 and J_3 be defined as indicated in Fig. 3. Then

$$A_N(J_1) = B_N(J_2) + B_N(J_3)$$

while

$$|J_1| = |J_2| = |J_3|.$$

This contradicts with the u.d. of sequence $s_{q,\vec{\gamma}}(n)$, since in this case we simultaneously have

$$\frac{A_N(J_1)}{N} \to |J_1|,$$
$$\frac{B_N(J_2)}{N} \to |J_2|,$$
$$\frac{B_N(J_3)}{N} \to |J_3|.$$

□

In the above Propositions, we studied weights γ_λ such that

$$(q-1)(\gamma_{\lambda+2}+\gamma_{\lambda+3}+\cdots) < \gamma_{\lambda+1} \quad \text{and} \quad (q-1)(\gamma_{\lambda+2}+\gamma_{\lambda+3}+\cdots) > \gamma_{\lambda+1}$$

for some λ. The remaining case $(q-1)(\gamma_{\lambda+2}+\gamma_{\lambda+3}+\cdots) = \gamma_{\lambda+1}$ is handled in the next proposition.

Proposition 5. *Let $\boldsymbol{\gamma}$ be a sequence of positive real numbers such that for every $\lambda = 0, 1, 2, \ldots$ we have*

$$(q-1)(\gamma_{\lambda+2}+\gamma_{\lambda+3}+\cdots) = \gamma_{\lambda+1}. \tag{67}$$

If $\boldsymbol{\gamma}$ satisfies conditions (58) *and* (59)*, and $S = 1$ then sequence $s_{q,\boldsymbol{\gamma}}(n)$, $n = 0, 1, 2, \ldots$, is the q-adic van der Corput sequence. Consequently, if $\boldsymbol{\gamma}$ satisfies conditions* (58), (59) *and $S = 1$ then every uniformly distributed $\boldsymbol{\gamma}$-weighted q-adic sum-of-digits function $s_{q,\boldsymbol{\gamma}}(n)$, $n = 0, 1, 2, \ldots$, is the q-adic van der Corput sequence.*

Proof. If for every $\lambda \in \mathbb{N}$ we have $(q-1)(\gamma_{\lambda+2}+\gamma_{\lambda+3}+\cdots) = \gamma_{\lambda+1}$, then the assumptions

$$(q-1)\sum_{j=0}^{\infty}\gamma_j = 1 \quad \text{and} \quad (q-1)\sum_{j=1}^{\infty}\gamma_j = \gamma_0$$

imply that $(q-1)\gamma_0 + \gamma_0 = 1$, that is $\gamma_0 = q^{-1}$. The induction on n yields

$$\gamma_n = (q-1)\sum_{j=n+1}^{\infty}\gamma_j = (q-1)\left(\sum_{j=n}^{\infty}\gamma_j\right) - (q-1)\gamma_n = \gamma_{n-1} - (q-1)\gamma_n,$$

i.e., $\gamma_n = q^{-n-1}$. Proposition is proved. □

If $S = 1$ and (67) holds, then the fact that sequence $s_{q,\boldsymbol{\gamma}}(n)$, $n = 0, 1, 2, \ldots$, is q-adic van der Corput's sequence, also follows from Pillichshammer's Proposition 1. Namely, if $S = 1$ the above reasoning shows that all the weights must be rational numbers. Along similar lines there follows that if taken in the irreducible forms, their denominator must be positive powers of q, that is $\gamma_k = c_k \cdot q^{-\lambda_k}$,

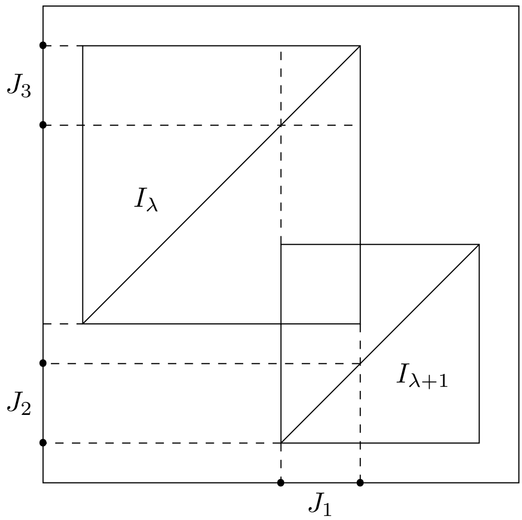

FIGURE 3. Overlapping intervals I_λ and $I_{\lambda+1}$.

where c_k is coprime to q for every k. Given a $k \in \mathbb{N}$ and taking $h = q^k$, condition (14) shows that at least λ_k equals $k-1$. In other words, the set of λ_k's exhausts all the $\mathbb{N}$. If at least one c_k would be > 1, or at least one λ_k appears twice, then the left hand side of (58) would be greater than 1, a contradiction which proves that starting $s_{q,\vec{\gamma}}(n)$, $n = 0, 1, 2, \dots$ is actually the q-adic van der Corput sequence.

Notice the following result complementing the above results and following directly from Pillichshammer's Proposition 1: Let the weight sequence γ_n, $n = 0, 1, 2, \dots$ satisfy the condition $\sum_{n=0}^{\infty} ||\gamma_n||^2 < \infty$. Then:

- If $\sum\limits_{n=0}^{\infty} ||\gamma_n|| < \frac{1}{q-1}$, then $s_{q,\gamma}(n)$ is not u.d.
- If $\sum\limits_{n=0}^{\infty} ||\gamma_n|| = \frac{1}{q-1}$, then $s_{q,\gamma}(n)$ is u.d. if and only if it is a permutation of the van der Corput sequence.
- If $\sum\limits_{n=0}^{\infty} ||\gamma_n|| > \frac{1}{q-1}$, then $s_{q,\gamma}(n)$ is u.d. if and only if it contains terms of the form $\frac{n_k}{q^k}$, where $n_k \in \mathbb{N}$ is coprime with q for all $k = 1, 2, 3, \dots$

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