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THE INEQUALITY OF ERDŐS-TURÁN-KOKSMA IN THE TERMS OF THE FUNCTIONS OF THE SYSTEM $\Gamma_{\mathcal{B}_s}$

Tsvetelina Petrova

South West University "Neofit Rilski", Blagoevgrad, BULGARIA

ABSTRACT. In the present paper the author uses the function system $\Gamma_{\mathcal{B}_s}$ constructed in Cantor bases to show upper bounds of the extreme and star discrepancy of an arbitrary net in the terms of the trigonometric sum of this net with respect to the functions of this system. The obtained estimations are inequalities of the type of Erdős-Turán-Koksma. These inequalities are very suitable for studying of nets constructed in the same Cantor system.

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1. Introduction

Let $s \geq 1$ be a fixed integer and will denote the dimension through the paper. The inequality of Erdős-Turán-Koksma gives an upper bound of the discrepancy of an arbitrary net in $[0,1)^s$ in the terms of certain trigonometric sums. Results of this direction was presented in the monograph of Drmota and Tichy [2], see Kuipers and Niederreiter [9] for its more general form, also Niederreiter [11] for versions adapted to certain sequences of rational numbers. These inequalities are an appropriate tool to establish the uniformly distribution of point sets. Very well approach to construct low$-$discrepancy sequence is based on using $b-$adic arithmetic, see Hellekalek and Niederreiter [7]. To study these classes

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of sequences and net usually are used some orthonormal function systems, constructed in the same base b.

With big success the functions of the trigonometric system are used. So, for an arbitrary integer k an a real $x \in [0,1)$ the function $e_k(x)$ is defined as

$$e_k(x) = e^{2\pi \mathbf{i} kx}.$$

For arbitrary vectors $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{Z}^s$ and $\mathbf{x} = (x_1, \ldots, x_s) \in [0,1)^s$ we define the function $e_{\mathbf{k}}(\mathbf{x}) = \prod_{j=1}^{s} e_{k_j}(x_j)$. The set $\mathcal{T}_s = \{e_{\mathbf{k}}(\mathbf{x}) : \mathbf{k} \in \mathbb{Z}_0^s,\ \mathbf{x} \in [0,1)^s\}$ is called trigonometric function system. The Walsh function system in base b, as some their generalizations, as the system of Walsh functions over finite groups, see Larcher, Niedrreiter and Schmid [10], the function system constructed by using the $b-$adic arithmetics, see Hellekalek and Niederreiter [7], and the multiplicative system or Vilenkin function system, see Vilenkin [12], are a perfect tool for investigation of the uniformly distribution of sequences.

Following Chrestenson [1] we will present the definition of the Walsh functions. So, let $b \geq 2$ be a fixed integer and will denote the base in our work.

Definition 1.1. For an arbitrary non-negative integer k and a real $x \in [0,1)$ with the $b-$adic representations

$$k = \sum_{i=0}^{\nu} k_i b^i \quad \text{and} \quad x = \sum_{i=0}^{\infty} x_i b^{-i-1},$$

where $k_i, x_i \in \{0, 1, \ldots, b-1\}, k_\nu \neq 0$ and for infinitely many values of i, $x_i \neq b-1$, the corresponding kth Walsh function ${}_b\,\mathrm{wal}_k : [0,1) \to \mathbb{C}$ is defined as ${}_b\,\mathrm{wal}_k(x) = e^{\frac{2\pi \mathbf{i}}{b}(k_0 x_0 + \cdots + k_\nu x_\nu)}$.

Let us signify $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For arbitrary vectors $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ and $\mathbf{x} = (x_1, \ldots, x_s) \in [0,1)^s$ the multidimensional Walsh function in base b is defined as ${}_b\,\mathrm{wal}_{\mathbf{k}}(\mathbf{x}) = \prod_{j=1}^{s} {}_b\,\mathrm{wal}_{k_j}(x_j)$. The system

$$\mathcal{W}(b) = \left\{ {}_b \underset{\mathbf{k}}{\mathrm{wal}}(\mathbf{x}) : \mathbf{k} \in \mathbb{N}_0^s,\ \mathbf{x} \in [0,1)^s \right\}$$

is called Walsh function system in base b.

In the case when $b = 2$ the system $\mathcal{W}(2)$ is the system of the Walsh functions in base $b = 2$, see Walsh [13].

Quite recently, Hellekalek [6], Hellekalek and Niederreiter [7] use the $b-$adic arithmetics to construct the system $\Gamma_{\mathbf{b}}$, where $\mathbf{b} = (b_1, \ldots, b_s)$ is a vector of not necessarily distinct integers $b_j \geq 2$. Some applications of the system $\Gamma_{\mathbf{b}}$ to the theory of the uniformly distributed sequences are shown.

The different kinds of the discrepancy are quantitative measures for studying the quality of the distribution of the points of nets and sequences. The details are as following: Let $\mathcal{J}$ and $\mathcal{J}^*$ denote the families of subintervals of $[0,1)^s$ of the form

$$J = \prod_{j=1}^{s} [u_j, v_j), \quad \text{and} \quad J^* = \prod_{j=1}^{s} [0, v_j),$$

where for $1 \leq j \leq s, 0 \leq u_j < v_j \leq 1$.

Let $P_N = \{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}$ be an arbitrary net of $N \geq 1$ points in $[0,1)^s$. For an arbitrary subinterval $J \in \mathcal{J}$ let us denote $A(J;N) = \#\{\mathbf{x}_n : 0 \leq n \leq N-1, \ \mathbf{x}_n \in J\}$. Also, for an arbitrary subinterval $J^* \in \mathcal{J}^*$ the quantity $A(J^*;N)$ has the same sense. Let λ_s denote the Lebesgue measure on $[0,1)^s$. We will give the next definition:

Definition 1.2. The extreme discrepancy $D(P_N)$ and the star discrepancy $D^*(P_N)$ of the net P_N are defined, respectively, as

$$D(P_N) = \sup_{J \in \mathcal{J}} \left| \frac{A(J;N)}{N} - \lambda_s(J) \right| \quad \text{and} \quad D^*(P_N) = \sup_{J^* \in \mathcal{J}^*} \left| \frac{A(J^*;N)}{N} - \lambda_s(J^*) \right|.$$

The inequality of Koksma-Hlawka shows the importance of the discrepancy to the theory and the practice of the Monte Carlo and the Quasi-Monte Carlo integration. So, for functions defined on $[0,1)^s$ with bounded variation $V(f)$ in sense of Hardy and Krausse the inequality

$$\left| \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n) - \int_{[0,1]^s} f(\mathbf{x}) \dot{\mathbf{x}} \right| \leq V(f) \cdot D^*(P_N)$$

holds, here $D^*(P_N)$ is the star discrepancy of the net $P_N = \{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}$ of the nodes of the integration.

The original inequality of Erdős-Turán-Koksma, see Erdős-Turán [3] and Koksma [8], gives upper bounds of the extreme and the star discrepancy of nets in $[0,1)^s$ in the terms of the trigonometric sums with respect to the functions of the system $\mathcal{T}_s$.

Niederreiter [11] considers special kind of nets and gives the form of this inequality in the terms of the functions of the trigonometric function system. So, let $M \geq 2$ be an arbitrary integer. Let us consider the net $\mathcal{P}_N = \{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}$, where for $0 \leq n \leq N-1$ $\mathbf{y}_n \in \mathbb{Z}^s$ and $\mathbf{x}_n = \left\{\frac{\mathbf{y}_n}{M}\right\}$ is the fractional part of $\frac{\mathbf{y}_n}{M}$. Then, the inequality holds

$$D(\mathcal{P}_N) \leq 1 - \left(1 - \frac{2}{M}\right)^s + \sum_{\mathbf{k} \in C_s^*(M)} \frac{1}{r(\mathbf{k}, M)} \left| \frac{1}{N} \sum_{n=0}^{N-1} e_{\mathbf{k}}(\mathbf{x}_n) \right|,$$

where $C_s(M) = \left(-\frac{M}{2}, \frac{M}{2}\right)^s \bigcap \mathbb{Z}^s$, $C_s^*(M) = C_s(M)\backslash\{\mathbf{0}\}$, for and arbitrary vector $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{Z}^s$ the coefficient $r(\mathbf{k}, M) = \prod_{j=1}^{s} r(k_j, M)$ and

$$r(k_j, M) = \begin{cases} 1 & \text{if} \quad k_j = 0, \\ M \sin \pi \frac{|k_j|}{M} & \text{if} \quad k_j \neq 0. \end{cases}$$

Hellekalek [5] shows the form of inequality of Erdős-Turán-Koksma in the terms of the functions from Walsh functions system. Also Hellekalek [6] used the functions from the system $\Gamma_{\mathbf{p}}$, where $\mathbf{p} = (p_1, \ldots, p_s)$ is a vector of prime numbers, to obtain an upper bound of the discrepancy in the terms of the functions of this system.

The rest of the paper is organised in the following manner: In Section 2 we give some preliminary results which will be fruitful to present our main results. In Section 3 the main results of the paper are presented. They are estimations of the extreme and the star discrepancy and are inequalities of the type of the inequality of Erdős-Turán-Koksma. In Section 4 we give the proofs of the main results.

2. Preliminary results

We need to introduce some significations and to present four lemmas, where the Fourier's analysis of the local discrepancy will be developed. In this way, to a large degree we will reduce the weight of the formulation of the presented theorem.

We will give the concept of the so-called $B-$adic Cantor system, or a system with variable bases. For this purpose, let $B = \{b_0, b_1, b_2, \ldots : b_i \geq 2 \text{ for } i \geq 0\}$ be a sequence of integers. The generalized powers are defined in the following recursive way: $B_0 = 1$ and for $i \geq 0$ $B_{i+1} = B_i \cdot b_i$. An arbitrary integer $k \geq 0$ has a unique $B-$adic representation of the form $k = \sum_{i=0}^{\nu} k_i B_i$, where for $i \geq 0$ $k_i \in \{0, 1, \ldots, b_i - 1\}$ and $k_\nu \neq 0$. An arbitrary real $x \in [0, 1)$ has a $B-$adic representation of the form $x = \sum_{i=0}^{\infty} \frac{x_i}{B_{i+1}}$, where for $i \geq 0$ $x_i \in \{0, 1, \ldots, b_i - 1\}$. Under the assumption that for infinitely many values of i we have that $x_i \neq b_i - 1$ the representation of x is unique. Let us denote $k(0) = 0$, $x(0) = 0$ and for each integer $g \geq 1$ let us define $k(g) = \sum_{i=0}^{g-1} k_i B_i$ and $x(g) = \sum_{i=0}^{g-1} \frac{x_i}{B_{i+1}}$.

We will present the concept of the multidimensional Cantor systems. So, let for $1 \leq j \leq s$, $B_j = \left\{b_0^{(j)}, b_1^{(j)}, b_2^{(j)}, \ldots : b_i^{(j)} \geq 2 \text{ for } i \geq 0\right\}$ be given sequences of bases. Let us denote $\mathcal{B}_s = (B_1, \ldots, B_s)$. Let $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ and $\mathbf{x} = (x_1, \ldots, x_s) \in [0, 1)^s$ be arbitrary vectors. For $1 \leq j \leq s$ the coordinates k_j and x_j we present in the corresponding B_j-adic number system.

In this sense, we will call that the vectors $\mathbf{k}$ and $\mathbf{x}$ are presented in the $\mathcal{B}_s-$adic Cantor system.

Following Hellekalek and Niederreiter [7] we will give the concept of a function system constructed in $\mathcal{B}_s-$adic Cantor system.

Definition 2.1. For an arbitrary integer $k \geq 0$ and a real number $x \in [0,1)$ which have the $B-$adic representations of the form $k = \sum_{i=0}^{\nu} k_i B_i$ and $x = \sum_{i=0}^{\infty} \frac{x_i}{B_{i+1}}$, where for $i \geq 0$ $k_i, x_i \in \{0,1,\ldots,b_i-1\}, k_\nu \neq 0$ and for infinitely many i we have that $x_i \neq b_i - 1$, the kth function ${}_B\gamma_k : [0,1) \to \mathbb{C}$ is defined as

$$ {}_B\gamma_k(x) = e^{2\pi i \left(\frac{k_0}{B_1}+\frac{k_1}{B_2}+\cdots+\frac{k_\nu}{B_{\nu+1}}\right)(x_0B_0+x_1B_1+\cdots)}. $$

Now, we will give the concept of the multidimensional version of the above functions.

Definition 2.2. For an arbitrary vector $\mathbf{k}=(k_1,\ldots,k_s) \in \mathbb{N}_0^s$ the $\mathbf{k}$-the function ${}_{\mathcal{B}_s}\gamma_{\mathbf{k}} : [0,1)^s \to \mathbb{C}$ is defined as

$$ {}_{\mathcal{B}_s}\gamma_{\mathbf{k}}(\mathbf{x}) = \prod_{j=1}^{s} {}_{B_j}\gamma_{k_j}(x_j), \quad \mathbf{x} = (x_1,\ldots,x_s) \in [0,1)^s. $$

The set

$$ \Gamma_{\mathcal{B}_s} = \{{}_{\mathcal{B}_s}\gamma_{\mathbf{k}}(\mathbf{x}) : \mathbf{k} \in \mathbb{N}_0^s, \ \ \mathbf{x} \in [0,1)^s\} $$

we will call $\mathcal{B}_s-$adic function system.

Let us make the following choice of the sequences of bases: For $1 \leq j \leq s$ let $B_j = \{b_j, b_j, \ldots : b_j \geq 2 \ \text{ for } \ j \geq 2\}$. Let us denote $\mathbf{b} = (B_1,\ldots,B_s)$. Then, from the system $\Gamma_{\mathcal{B}_s}$ the system $\Gamma_{\mathbf{b}}$, which was introduced by Hellekalek and Niederraiter [7], is obtained.

For $1 \leq j \leq s$ let p_j be an arbitrary prime number and to denote $\mathbf{p} = (p_1,\ldots,p_s)$. For $1 \leq j \leq s$ let us define the set of bases $B_j = \{p_j, p_j, \ldots\}$. Then, from the system $\Gamma_{\mathcal{B}_s}$ the system $\Gamma_{\mathbf{p}}$, which was introduced by Hellekalek [6], is obtained.

Grozdanov and Petrova [4] use the functions of the system $\Gamma_{\mathcal{B}_s}$ to solve some problems of the theory of the Quasi-Monte Carlo integration and the uniformly distributed sequence. They prove the next statement:

Proposition 2.3. *The system $\Gamma_{\mathcal{B}_s}$ is a complete orthonormal basis of the space $L_2([0,1]^2)$.*

For every integrable in sense of Riemann on $[0,1]^s$ function $f(\mathbf{x})$ and for an arbitrary vector $\mathbf{k} \in \mathbb{N}_0^s$ the Fourier's coefficients of $f(\mathbf{x})$ is defined as

$$ {}_{\Gamma_{\mathcal{B}_s}}\widehat{f}(\mathbf{k}) = \int_{[0,1]^s} f(\mathbf{x}) {}_{\mathcal{B}_s}\overline{\gamma}_{\mathbf{k}}(\mathbf{x})\, \mathrm{d}\mathbf{x}. $$

Lemma 2.4. *Let β, $0 \le \beta < 1$ be an arbitrary fixed real and have the $B-$adic representation $\beta = 0.\beta_0\beta_1\beta_2\ldots$ Let us denote $I = [0,\beta)$. For $x \in [0,1)$ let us defined the function $f_I(x) = 1_I(x) - \lambda(I)$, where $1_I(x)$ is the characteristic function on the interval I. Then, we have the next:*

i) *Trivially ${}_{\Gamma_B}\widehat{f}_I(0) = 0$;*

ii) *Let $k \ge 1$ be an arbitrary fixed integer and have the $B-$adic representation $k = k_gB_g + k_{g-1}B_{g-1} + \cdots + k_0B_0$, where $g \ge 0$, for $0 \le i \le g$ $k_i \in \{0,1,\ldots,b_i-1\}$ and $k_g \ne 0$. Then, the equalities hold*

$$
{}_{\Gamma_B}\widehat{\widehat{f}}_I(k) =
$$
$$
\begin{cases} \frac{1}{B_{g+1}}\,{}_B\gamma_k\big(\beta(g)\big)\left\{\frac{e^{2\pi\mathbf{i}\frac{k_g\beta_g}{b_g}}-1}{e^{2\pi\mathbf{i}\frac{k_g}{b_g}}-1} + e^{2\pi\mathbf{i}\frac{k_g}{b_g}}B_{g+1}\big(\beta-\beta(g+1)\big)\right\} & \text{if} \quad \beta_g \ne 0, \\ {}_B\gamma_k\big(\beta(g)\big)\big(\beta-\beta(g+1)\big) & \text{if} \quad \beta_g = 0; \end{cases}
$$

iii) *The inequality $\left|{}_{\Gamma_B}\widehat{f}_I(k)\right| \le \frac{1}{B_{g+1}} \cdot \frac{1}{\sin\pi\frac{k_g}{b_g}}$ holds.*

Lemma 2.5. *Let $\alpha \ge 1$ be a fixed integer. For an arbitrary integer b such that $0 < b \le B_\alpha$ let I be the interval of the form $I = \left[0, \frac{b}{B_\alpha}\right)$. We define the function $f_I(x) = 1_I(x) - \lambda(I)$, $x \in [0,1)$, where $1_I(x)$ is the characteristic function on the interval I. Then, we have the following:*

i) *Trivially we have that ${}_{\Gamma_B}\widehat{f}_I(0) = 0$;*

ii) *For each integers g and k such that $0 \le g \le \alpha-1$, $B_g \le k \le B_{g+1}-1$ and k is of the form $k = k_gB_g + k_{g-1}B_{g-1} + \cdots + k_0$, where $k_g \ne 0$, the inequality holds*

$$
\left|{}_{\Gamma_B}\widehat{f}_I(k)\right| \le \frac{1}{B_{g+1}} \cdot \frac{1}{\sin\pi\frac{k_g}{b_g}};
$$

iii) *For each integer $k \ge B_\alpha$ the equality holds ${}_{\Gamma_B}\widehat{f}_I(k) = 0$;*

iv) *For each integer $k \ge 0$ let us define the function*

$$
\rho^*_{\Gamma_B(k)} = \begin{cases} 1 & \text{if} \quad k = 0, \\ \frac{1}{B_{g+1}} \cdot \frac{1}{\sin\pi\frac{k_g}{b_g}} & \text{if} \quad k_gB_g \le k \le (k_g+1)B_g - 1, \\ & \quad 0 \le g \le \alpha-1, \\ & \quad k_g \in \{1,2,\ldots,b_g-1\}, \\ 0 & \text{if} \quad k \ge B_g. \end{cases}
$$

Then, the inequality $\left|{}_{\Gamma_B}\widehat{f}_I(k)\right| \le \rho^*_{\Gamma_B}(k)$ holds.

v) *If* $\alpha = 1$ *and* $g = 0$ *the result of the statement* ii) *can be improved as: For each integer* k, $B_0 \le k \le B_1 - 1$ *the equality* $\left|{}_{\Gamma_B}\widehat{f}_I(k)\right| = \frac{1}{B_1}\frac{\left|\sin\pi\frac{bk}{b_0}\right|}{\sin\pi\frac{k}{b_0}}$ *holds.*

Lemma 2.6. *Let* $\alpha \ge 1$ *be an arbitrary and fixed integer. For arbitrary integers* a *and* b *such that* $0 \le a < b \le B_\alpha$ *let* I *be the interval of the form* $I = \left[\frac{a}{B_\alpha}, \frac{b}{B_\alpha}\right)$. *Let us define the function* $f_I(x) = 1_I(x) - \lambda(I), x \in [0,1)$, *where* $1_I(x)$ *is the characteristic function on the interval* I. *Then the following holds:*

i) *Trivially we have that* ${}_{\Gamma_B}\widehat{f}_I(0) = 0$;

ii) *For each integers* g *and* k *such that* $0 \le g \le \alpha - 1$, $B_g \le k \le B_{g+1} - 1$ *and* k *is of the form* $k = k_g B_g + k_{g-1}B_{g-1} + \cdots + k_0$, *where* $k_g \ne 0$, *the inequality holds*

$$|{}_{\Gamma_B}\widehat{f}_I(k)| \le \frac{1}{B_{g+1}} \cdot \frac{2}{\sin\pi\frac{k_g}{b_g}};$$

iii) *For each integer* $k \ge B_\alpha$ *the equality* ${}_{\Gamma_B}\widehat{f}_I(k) = 0$ *holds;*

iv) *For each integer* $k \ge 0$ *let us define the function*

$$\rho_{\Gamma_B(k)} = \begin{cases} 1 & if \quad k = 0, \\ \frac{1}{B_{g+1}} \cdot \frac{2}{\sin\pi\frac{k_g}{b_g}} & if \quad k_g B_g \le k \le (k_g+1)B_g - 1, \\ & \qquad 0 \le g \le \alpha - 1, \\ & \qquad k_g \in \{1, 2, \ldots, b_g - 1\}, \\ 0 & if \quad k \ge B_g. \end{cases}$$

Then, the inequality $|{}_{\Gamma_B}\widehat{f}_I(k)| \le \rho_{\Gamma_b}(k)$ *holds.*

v) *If* $\alpha = 1$ *and* $g = 0$, *the result of* ii) *can be improved in the following way: For each integer* k *such that* $B_0 \le k \le B_1 - 1$ *the equality*

$$\left|{}_{\Gamma_B}\widehat{f}_I(k)\right| = \frac{1}{B_1}\frac{\left|\sin\pi\frac{b-a}{b_0}k\right|}{\sin\pi\frac{k}{b_0}}$$

holds.

Lemma 2.7. *Let* $\alpha = (\alpha_1, \ldots, \alpha_s)$, *where for* $1 \le j \le s$, $\alpha_j \ge 0$ *are integers, be a given vector. Let*

$$G = \prod_{j=1}^{s}\left[\frac{a_j}{B^{(j)}_{\alpha_j}}, \frac{b_j}{B^{(j)}_{\alpha_j}}\right) \qquad and \qquad G^* = \prod_{j=1}^{s}\left[0, \frac{b_j}{B^{(j)}_{\alpha_j}}\right),$$

where for $1 \le j \le s$ $0 \le a_j < b_j \le B^{(j)}_{\alpha_j}$, *are subintervals of* $[0,1)^s$. *Let us define the functions* $f_G(\mathbf{x}) = 1_G(\mathbf{x}) - \lambda_s(G)$, $f_{G^*}(\mathbf{x}) = 1_{G^*}(\mathbf{x}) - \lambda_s(G^*)$, $\mathbf{x} \in [0,1)^s$, *where* $1_G(\mathbf{x})$ *and* $1_{G^*}(\mathbf{x})$ *are the characteristic functions on intervals* G *and* G^*.

Let us define the sets

$$\Delta(\alpha) = \left\{ \mathbf{k} = (k_1, \dots k_s) \in \mathbb{N}_0^s, \quad 0 \le k_j \le B^{(j)}_{\alpha_j}, \quad 1 \le j \le s \right\}$$

and

$$\Delta^*(\alpha) = \Delta(\alpha) \backslash \{\mathbf{0}\}.$$

Then, the following statements hold:

i) *If* $\mathbf{k} = \mathbf{0}$, *then* ${}_{\Gamma_{\mathcal{B}_s}}\widehat{f}_G(\mathbf{0}) = 0$ *and* ${}_{\Gamma_{\mathcal{B}_s}}\widehat{f}_{G^*}(\mathbf{0}) = 0$;

ii) *For each vector* $\mathbf{k} \in \left(\mathbb{N}_0^s \setminus \Delta^*(\alpha)\right)$ *the equalities* ${}_{\Gamma_{\mathcal{B}_s}}\widehat{f}_G(\mathbf{k}) = 0$ *and* ${}_{\Gamma_{\mathcal{B}_s}}\widehat{f}_{G^*}(\mathbf{k}) = 0$ *hold;*

iii) *The functions* $f_G(\mathbf{x})$ *and* $f_{G^*}(\mathbf{x})$ *have the following representations as polynomials over the system* $\Gamma_{\mathcal{B}_s}$:

$$f_G(\mathbf{x}) = \sum_{\mathbf{k} \in \Delta^*(\alpha)} {}_{\Gamma_{\mathcal{B}_s}}\widehat{f}_G(\mathbf{k})_{\mathcal{B}_s}\gamma_{\mathbf{k}}(\mathbf{x}), \ \forall \ \ \mathbf{x} \in [0,1)^s$$

and

$$f_{G^*}(\mathbf{x}) = \sum_{\mathbf{k} \in \Delta^*(\alpha)} {}_{\Gamma_{\mathcal{B}_s}}\widehat{f}_{G^*}(\mathbf{k})_{\mathcal{B}_s}\gamma_{\mathbf{k}}(\mathbf{x}), \ \forall \ \ \mathbf{x} \in [0,1)^s;$$

iv) *For each vector* $\mathbf{k} = (k_1, \dots, k_s) \in \mathbb{N}_0^s$ *let us define the coefficients*

$$\rho_{\Gamma_{\mathcal{B}_s}}(\mathbf{k}) = \prod_{j=1}^{s} \rho_{\Gamma_{\mathcal{B}_j}}(k_j) \qquad \textit{and} \qquad \rho^*_{\Gamma_{\mathcal{B}_s}}(\mathbf{k}) = \prod_{j=1}^{s} \rho^*_{\Gamma_{\mathcal{B}_j}}(k_j),$$

where for $1 \le j \le s$ *the coefficients* $\rho_{\Gamma_{\mathcal{B}_j}}(k_j)$ *and* $\rho^*_{\Gamma_{\mathcal{B}_j}}(k_j)$ *are defined respectively in the conditions of Lemma 2.6* iv) *and in Lemma 2.5* iv). *Then, the inequalities hold*

$$\left| {}_{\Gamma_{\mathcal{B}_s}}\widehat{f}_G(\mathbf{k}) \right| \le \rho_{\Gamma_{\mathcal{B}_s}}(\mathbf{k}) \qquad \textit{and} \qquad \left| {}_{\Gamma_{\mathcal{B}_s}}\widehat{f}_{G^*}(\mathbf{k}) \right| \le \rho^*_{\Gamma_{\mathcal{B}_s}}(\mathbf{k}).$$

3. The main results

In the next theorem we will give upper bounds of the extreme and the star discrepancy of an arbitrary net in the terms of the trigonometric sum of the net with respect to the functions of the system $\Gamma_{\mathcal{B}_s}$.

Theorem 3.1. *Let $M \geq 2$ be an arbitrary integer. For $1 \leq j \leq s$ let the numbers $\alpha_j \geq 1$ be defined by the condition $B^{(j)}_{\alpha_j} = \min_{\alpha \geq 1}\{B^{(j)}_{\alpha} : B^{(j)}_{\alpha} \geq M\}$ and to define the vector $\alpha = (\alpha_1, \ldots, \alpha_s)$. Let the sets $\Delta(\alpha)$ and $\Delta^*(\alpha)$ which have been defined in Lemma 2.7 correspond to the vector α.*

Let $P_N = \{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}$ be an arbitrary net of N points in $[0,1)^s$. Then, the following holds:

i) ***Raw form of the inequality of Erdős-Turán-Koksma.***
The extreme discrepancy and the star discrepancy of the net P_N satisfy the inequalities

$$D(P_N) \leq 1 - \left(1 - \frac{2}{M}\right)^s + \sup_{J \in \mathcal{J}} \sum_{\mathbf{k} \in \Delta^*(\alpha)} \left|{}_{\Gamma_{\mathcal{B}_s}}\widehat{f}_{G(J)}(\mathbf{k})\right| \cdot S({}_{\mathcal{B}_s}\gamma_{\mathbf{k}}; P_N)|,$$

and

$$D^*(P_N) \leq 1 - \left(1 - \frac{1}{M}\right)^s + \sup_{J^* \in \mathcal{J}^*} \sum_{\mathbf{k} \in \Delta^*(\alpha)} \left|{}_{\Gamma_{\mathcal{B}_s}}\widehat{f}_{G(J^*)}(\mathbf{k})\right| \cdot |S({}_{\mathcal{B}_s}\gamma_{\mathbf{k}}; P_N)|,$$

where $S({}_{\mathcal{B}_s}\gamma_{\mathbf{k}}; P_N) = \frac{1}{N}\sum_{n=0}^{N-1} {}_{\mathcal{B}_s}\gamma_{\mathbf{k}}(\mathbf{x}_n)$ denotes the trigonometric sum of the net P_N with respect to the function ${}_{\mathcal{B}_s}\gamma_{\mathbf{k}}$. The sets $G(J)$ and $G(J^*)$ will be defined in Step 2 of the proof of the Theorem;

ii) ***Refined form of the inequality of Erdős-Turán-Koksma.***
Let the sequences of bases $B_1, \ldots, B_s$ be limited from above, i.e., there exists an absolute constant $q \geq 2$, such that for $1 \leq j \leq s$ and each $i \geq 0$ we have $b^{(j)}_i \leq q$. Let us to define the integer $r > 0$ by the condition that for each j such that $1 \leq j \leq s$ we have $q^{\alpha_j} < r \cdot M$. Let us denote

$$B_{\Gamma_{\mathcal{B}_s}}(\alpha) = \max_{\mathbf{k} \in \Delta^*(\alpha)} |S({}_{\mathcal{B}_s}\gamma_{\mathbf{k}}; P_N)|.$$

Then, the inequalities holds

$$D(P_N) \leq 1 + \left(1 - \frac{2}{M}\right)^s + B_{\Gamma_{\mathcal{B}_s}}(\alpha)\left\{\left[1 + \left(\frac{4}{\pi} + \frac{4}{5\log q}\right](\log r + \log M)\right]^s - 1\right\}.$$

and

$$D^*(P_N) \leq 1 - \left(1 - \frac{1}{M}\right)^s + B_{\Gamma_{\mathcal{B}_s}}(\alpha)\left\{\left[1 + \left(\frac{2}{\pi} + \frac{2}{5\log q}\right)(\log r + \log M)\right]^s - 1\right\}.$$

Corollary 3.2. *Let in Theorem 3.1 we replace the system $\Gamma_{\mathcal{B}_s}$ with the system $\Gamma_{\mathbf{p}}$. Let $M = p^\alpha$ for some integer α. Let $P_N = \{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}$ be an arbitraty net. Then, we obtain the corresponding upper bound of the extreme discrepancy in the terms of the functions of the system Γ_p. The last result have been obtained by Hellekalek* [6].

4. Proof of the preliminary results

P r o o f o f L e m m a 2.4. Let $k \geq 1$ be an arbitrary and fixed integer. First we will obtain an useful formula for the function ${}_B\gamma_k(x)$. For this purpose, let the integer $k \geq 1$ and the real $x \in [0,1)$ have the $B-$adic representations $k = k_g B_g + k_{g-1}B_{g-1} + \cdots + k_1 B_1 + k_0$, where for $0 \leq i \leq g \;\; k_i \in \{0,1,\ldots,b_i-1\}, \;\; k_g \neq 0$ and $x = \sum_{i=0}^{\infty} \frac{x_i}{B_{i+1}}$, where for $i \geq 0 \;\; x_i \in \{0,1,\ldots,b_i-1\}$ and for infinitely many values of i we have $x_i \neq b_i - 1$. Then, we have that

$$\begin{aligned}
{}_B\gamma_k(x) &= e^{2\pi \mathbf{i}\left(\frac{k_0}{B_1}+\ldots+\frac{k_{g-1}}{B_g}+\frac{k_g}{B_{g+1}}\right)\cdot(x_0+x_1B_1+\cdots+x_gB_g)} \\
&= e^{2\pi \mathbf{i}\left(\frac{k_0}{B_1}+\cdots+\frac{k_g}{B_{g+1}}\right)\cdot(x_0+x_1B_1+\cdots+x_{g-1}B_{g-1})} \\
&\quad \times e^{2\pi \mathbf{i}\left(\frac{k_0}{B_1}+\cdots+\frac{k_{g-1}}{B_g}\right)\cdot x_gB_g} \cdot e^{2\pi \mathbf{i}\frac{k_g}{B_{g+1}}x_gB_g} \\
&= {}_B\gamma_k\big(x(g)\big) \cdot e^{2\pi \mathbf{i}\frac{k_g}{b_g}x_g},
\end{aligned}$$

i.e., the equality ${}_B\gamma_k(x) = {}_B\gamma_k\big(x(g)\big) \cdot e^{2\pi i \frac{k_g}{b_g}x_g}$ holds.

For each integer $k \geq 0$ the equality holds

$${}_{\Gamma_B}\widehat{\overline{f}}_I(k) = \int_0^1 [1_I(x) - \lambda(I)]_B\gamma_k(x)\,\mathrm{d}x.$$

i) Let $k = 0$. Then, we have that

$${}_{\Gamma_B}\widehat{\overline{f}}_I(0) = \int_0^1 [1_I(x) - \lambda(I)]\,\mathrm{d}x = \lambda(I) - \lambda(I) = 0.$$

ii) Let $k \geq 1$. Then, the following holds

$$\begin{aligned}
{}_{\Gamma_B}\widehat{\overline{f}}_I(k) &= \int_0^1 1_I(x)_B\gamma_k(x)\,\mathrm{d}x = \int_0^\beta {}_B\gamma_k(x)\,\mathrm{d}x \\
&= \int_0^{\beta(g)} {}_B\gamma_k(x)\,\mathrm{d}x + \int_{\beta(g)}^{\beta} {}_B\gamma_k(x)\,\mathrm{d}x.
\end{aligned} \tag{1}$$

Let us use the presentation $\beta(g) = 0.\beta_0\beta_1\ldots\beta_{g-1} = \frac{p}{B_g}$ for some integer p such that $0 \leq p \leq B_g - 1$. Hence, we have that

$$\int_0^{\beta(g)} {}_B\gamma_k(x)\,\mathrm{d}x = \sum_{\alpha=0}^{p-1}\sum_{\mu=0}^{b_g-1}\int_{\frac{\alpha}{B_g}+\frac{\mu}{B_{g+1}}}^{\frac{\alpha}{B_g}+\frac{\mu+1}{B_{g+1}}} {}_B\gamma_k(x).$$

For $x \in \left[\frac{\alpha}{B_g} + \frac{\mu}{B_{g+1}}, \frac{\alpha}{B_g} + \frac{\mu+1}{B_{g+1}}\right)$ we have that ${}_B\gamma_k(x) = {}_B\gamma_k\left(\frac{\alpha}{B_g}\right) \cdot e^{2\pi\mathbf{i}\frac{k_g}{b_g}\mu}$ and obtain

$$\int_0^{\beta(g)} {}_B\gamma_k(x)\,\mathrm{d}x = \frac{1}{B_{g+1}} \sum_{\alpha=0}^{p-1} {}_B\gamma_k\left(\frac{\alpha}{B_g}\right) \sum_{\mu=0}^{b_g-1} e^{2\pi\mathbf{i}\frac{k_g}{b_g}\mu} = 0. \tag{2}$$

From (1) and (2) we obtain that

$$_{\Gamma_B}\overline{\hat{f}}_I(k) = \int_{\beta(g)}^{\beta} {}_B\gamma_k(x)\,\mathrm{d}x.$$

Let us assume that $\beta_g \neq 0$.

$$\begin{aligned}
_{\Gamma_B}\overline{\hat{f}}_I(k) &= \int_{\beta(g)}^{\beta} {}_B\gamma_k(x)\,\mathrm{d}x \\
&= \sum_{\mu=0}^{\beta_g-1} \int_{\beta(g)+\frac{\mu}{B_{g+1}}}^{\beta(g)+\frac{\mu+1}{B_{g+1}}} {}_B\gamma_k(x)\,\mathrm{d}x + \int_{\beta(g+1)}^{\beta} {}_B\gamma_k(x)\,\mathrm{d}x \\
&= \frac{1}{B_{g+1}} {}_B\gamma_k\big(\beta(g)\big) \sum_{\mu=0}^{\beta_g-1} e^{2\pi\mathbf{i}k_g\frac{\mu}{b_g}} \\
&\quad + {}_B\gamma_k\big(\beta(g+1)\big) \cdot \big(\beta - \beta(g+1)\big) \\
&= \frac{1}{B_{g+1}} {}_B\gamma_k\big(\beta(g)\big) \frac{e^{2\pi\mathbf{i}\frac{k_g\beta_g}{b_g}} - 1}{e^{2\pi\mathbf{i}\frac{k_g}{b_g}} - 1} \\
&\quad + {}_B\gamma_k\big(\beta(g+1)\big) \cdot \big(\beta - \beta(g+1)\big).
\end{aligned}$$

Let us assume that $\beta_g = 0$. Then we have the equality

$$_{\Gamma_B}\overline{\hat{f}}_I(k) = \int_{\beta(g)}^{\beta} {}_B\gamma_k(x)\,\mathrm{d}x = {}_B\gamma_k\big(\beta(g)\big)\big(\beta - \beta(g+1)\big).$$

Thus the statement ii) of the Lemma is proved. We note the fact that the both formulas of ii) can be combined into one, namely,

$$\begin{aligned}
_{\Gamma_B}\overline{\hat{f}}_I(k) &= \frac{1}{B_{g+1}} {}_B\gamma_k\big(\beta(g)\big) \frac{e^{2\pi\mathbf{i}\frac{k_g\beta_g}{b_g}} - 1}{e^{2\pi\mathbf{i}\frac{k_g}{b_g}} - 1} + {}_B\gamma_k\big(\beta(g+1)\big) \cdot \big(\beta - \beta(g+1)\big) \\
&= {}_B\gamma_k\big(\beta(g)\big) \left[\frac{1}{B_{g+1}} \cdot \frac{e^{2\pi\mathbf{i}\frac{k_g\beta_g}{b_g}} - 1}{e^{2\pi\mathbf{i}\frac{k_g}{b_g}} - 1} + e^{2\pi\mathbf{i}\frac{k_g\beta_g}{b_g}} \cdot \big(\beta - \beta(g+1)\big)\right].
\end{aligned}$$

iii) From the statement ii) of the Lemma we obtain that

$$\left|{}_{\Gamma_B}\widehat{\overline{f}}_I(k)\right| = \left|{}_B\gamma_k\big(\beta(g)\big)\right| \cdot \left|\frac{1}{B_{g+1}} \cdot \frac{e^{2\pi\mathbf{i}\frac{k_g\beta_g}{b_g}}-1}{e^{2\pi\mathbf{i}\frac{k_g}{b_g}}-1} + e^{2\pi\mathbf{i}\frac{k_g\beta_g}{b_g}} \cdot \big(\beta-\beta(g+1)\big)\right|$$

$$= \frac{1}{B_{g+1}}\left|\frac{e^{2\pi\mathbf{i}\frac{k_g\beta_g}{b_g}}-1}{e^{2\pi\mathbf{i}\frac{k_g}{b_g}}-1} + e^{2\pi\mathbf{i}\frac{k_g\beta_g}{b_g}} B_{g+1}\big(\beta-\beta(g+1)\big)\right|.$$

Let us denote $C = e^{2\pi\mathbf{i}\frac{k_g}{b_g}}$ and $\delta = B_{g+1}\big(\beta-\beta(g+1)\big)$. It is obviously that $0 \leq \delta \leq 1$. Thus we obtain that

$$\begin{aligned}
\left|{}_{\Gamma_B}\widehat{\overline{f}}_I(k)\right| &= \frac{1}{B_{g+1}} \cdot \left|\frac{C^{\beta_g}-1}{C-1} + C^{\beta_g}\cdot\delta\right| \\
&= \frac{1}{B_{g+1}} \cdot \left|C^{\beta_g}\left(\frac{1}{C-1}+\delta\right) - \frac{1}{C-1}\right| \\
&\leq \frac{1}{B_{g+1}}\left\{|C^{B_g}|\cdot\left|\frac{1}{C-1}+\delta\right| + \frac{1}{|C-1|}\right\} \\
&= \frac{1}{B_{g+1}}\left\{\frac{|1-\delta+\delta C|}{|C-1|} + \frac{1}{|C-1|}\right\} \\
&\leq \frac{1}{B_{g+1}}\left\{\frac{1-\delta+\delta}{|C-1|} + \frac{1}{|C-1|}\right\} \\
&= \frac{1}{B_{g+1}} \cdot \frac{2}{|C-1|}. \qquad (3)
\end{aligned}$$

For an arbitrary real t the equality $e^{\mathbf{i}t}-1 = 2\sin\frac{t}{2}\left(-\sin\frac{t}{2}+\mathbf{i}\cos\frac{t}{2}\right)$ gives us that

$$|e^{\mathbf{i}t}-1| = 2\cdot\left|\sin\frac{t}{2}\right|\left|-\sin\frac{t}{2}+\mathbf{i}\cos\frac{t}{2}\right| = 2\cdot\left|\sin\frac{t}{2}\right|.$$

In the case when $t = 2\pi\frac{k_g}{b_g}$ we obtain that $\left|e^{2\pi\mathbf{i}\frac{k_g}{b_g}}-1\right| = 2\sin\pi\frac{k_g}{b_g}$. From the last equality and (3) we obtain that

$$\left|{}_{\Gamma_B}\widehat{\overline{f}}_I(k)\right| = \frac{1}{B_{g+1}} \cdot \frac{1}{\sin\pi\frac{k_g}{b_g}}.$$

Lemma 2.4 is finally proved. □

Proof of Lemma 2.5. For each integer $k \geq 0$ we have that

$$ {}_{\Gamma_B}\widehat{f}_I(k) = \int_0^1 1_I(x)_B\overline{\gamma}_k(x)\,\mathrm{d}x - \lambda(I)\int_0^1 {}_B\overline{\gamma}_k(x)\,\mathrm{d}x. \tag{4} $$

i) In the case when $k = 0$ we directly obtain the ${}_{\Gamma_B}\widehat{f}_I(0) = 0$.

ii) Let us assume that $0 \leq g \leq \alpha - 1$. In Lemma 2.4 iii) was proved the equality

$$ \left|{}_{\Gamma_B}\widehat{f}_I(k)\right| = \frac{1}{B_{g+1}} \cdot \frac{1}{\sin \pi \frac{k_g}{b_g}}. $$

iii) Let $k \geq B_\alpha$. Then, there exists unique integer $g \geq \alpha$ such that $B_g \leq k \leq B_{g+1} - 1$. Let us assume that $g \geq \alpha + 1$. Then, the presentation $\frac{b}{B_\alpha} = \frac{h}{B_g}$, where $h = b \cdot b_\alpha \dots b_{g-1}$, gives us that

$$ {}_{\Gamma_B}\widehat{f}_{[0,\frac{b}{B_\alpha})}(k) = \int_0^{\frac{b}{B_\alpha}} {}_B\overline{\gamma}_k(x)\,\mathrm{d}x = \int_0^{\frac{h}{B_g}} {}_B\overline{\gamma}_k(x)\,\mathrm{d}x = \sum_{p=0}^{h-1}\int_{\frac{p}{B_g}}^{\frac{p+1}{B_g}} {}_B\overline{\gamma}_k(x)\,\mathrm{d}x = 0. \tag{5} $$

Let us assume that $g = \alpha$. In this case we have that

$$ {}_{\Gamma_B}\widehat{f}_{[0,\frac{b}{B_\alpha}]}(k) = \int_0^{\frac{b}{B_\alpha}} {}_B\overline{\gamma}_k(x)\,\mathrm{d}x = \sum_{p=0}^{b-1}\int_{\frac{p}{B_g}}^{\frac{p+1}{B_g}} {}_B\overline{\gamma}_k(x)\,\mathrm{d}x = 0. \tag{6} $$

From (4), (5) and (6) we obtain that for any integer $k \geq B_\alpha$ the equality ${}_{\Gamma_B}\widehat{f}_{[0,\frac{b}{B_\alpha})}(k) = 0$ holds.

iv) This statement is a direct consequence of i), ii) and iii).

v) In the case when $\alpha = 1$ and $g = 0$ we have that $1 \leq k \leq b_0 - 1$. We have that ${}_B\gamma_k(x) = e^{2\pi \mathbf{i}\frac{x_0 k}{b_0}}$ and therefore

$$ {}_{\Gamma_B}\overline{\widehat{f}}_{\left[0,\frac{b}{B_1}\right)}(k) = \int_0^{\frac{b}{B_1}} {}_B\gamma_k(x)\,\mathrm{d}x = \sum_{\mu=0}^{b-1}\int_{\frac{\mu}{b_0}}^{\frac{\mu+1}{b_0}} {}_B\gamma_k(x)\,\mathrm{d}x $$

$$ = \sum_{\mu=0}^{b-1}\int_{\frac{\mu}{b_0}}^{\frac{\mu+1}{b_0}} e^{2\pi\mathbf{i}\frac{\mu k}{b_0}}\,\mathrm{d}x = \frac{1}{b_0}\sum_{\mu=0}^{b-1} e^{2\pi\mathbf{i}\frac{\mu k}{b_0}} = \frac{1}{b_0}\,\frac{e^{2\pi\mathbf{i}\frac{bk}{b_0}} - 1}{e^{2\pi\mathbf{i}\frac{k_0}{b_0}} - 1}. $$

Hence, we obtain that

$$ \left|{}_{\Gamma_b}\widehat{f}_{\left[0,\frac{b}{B_1}\right]}(k)\right| = \frac{1}{b_0}\,\frac{\left|\sin\pi\frac{bk}{b_0}\right|}{\sin\pi\frac{k}{b_0}}. $$ □

P r o o f o f L e m m a 2.6. For every $x \in [0,1)$ we have that

$$f_I(x) = f_{[0,\frac{b}{B_\alpha})}(x) - f_{[0,\frac{a}{B_\alpha})}(x)$$

and for each integer $k \geq 0$ we obtain that

$$\Gamma_B\widehat{f}_I(k) = \Gamma_B\widehat{f}_{[0,\frac{b}{B_\alpha})}(k) - \Gamma_B\widehat{f}_{[0,\frac{a}{B_\alpha})}(k). \tag{7}$$

i) Let us assume that $k = 0$. Then, according to (7) we obtain that

$$\Gamma_B\widehat{f}_I(0) = \Gamma_B\widehat{f}_{[0,\frac{b}{B_\alpha})}(0) - \Gamma_B\widehat{f}_{[0,\frac{a}{B_\alpha})}(0) = 0.$$

ii) Let us assume that $0 \leq g \leq \alpha - 1$. According to (7) and the result ii) of Lemma 2.5 we obtain

$$\left|\Gamma_B\widehat{f}_I(k)\right| \leq \left|\Gamma_B\widehat{f}_{[0,\frac{b}{B_\alpha})}(k)\right| + \left|\Gamma_B\widehat{f}_{[0,\frac{a}{B_\alpha})}(k)\right| \leq \frac{1}{B_{g+1}} \cdot \frac{2}{\sin\pi\frac{k_g}{b_g}}.$$

iii) Let $k \geq B_\alpha$. Then, there exists unique integer $g \geq \alpha$ such that $B_g \leq k \leq B_{g+1} - 1$. Let us assume that $g \geq \alpha + 1$. We have that $\frac{a}{B_\alpha} = \frac{h}{B_g}$, where $h = a \cdot b_\alpha \dots b_{g-1}$. Therefore we obtain that

$$\Gamma_B\widehat{f}_{[0,\frac{b}{B_\alpha})}(k) = \int_0^{\frac{a}{B_\alpha}} {}_B\overline{\gamma}_k(x)\,\mathrm{d}x = \int_0^{\frac{h}{B_g}} {}_B\overline{\gamma}_k(x)\,\mathrm{d}x = \sum_{p=0}^{h-1}\int_{\frac{p}{B_g}}^{\frac{p+1}{B_g}} {}_B\overline{\gamma}_k(x)\,\mathrm{d}x = 0.$$

Let us assume that $g = \alpha$. We obtain that

$$\Gamma_B\widehat{f}_{[0,\frac{a}{B_\alpha})}(k) = \int_0^{\frac{a}{B_g}} {}_B\overline{\gamma}_k(x)\,\mathrm{d}x = \sum_{h=0}^{a-1}\int_{\frac{h}{B_g}}^{\frac{h+1}{B_g}} {}_B\gamma_k(x)\,\mathrm{d}x = 0.$$

By analogy we can prove that $\Gamma_B\widehat{f}_{[0,\frac{b}{B_\alpha})}(k) = 0$. By using the equalities $\Gamma_B\widehat{f}_{[0,\frac{a}{B_\alpha})}(k) = 0$ and $\Gamma_B\widehat{f}_{[0,\frac{b}{B_\alpha})}(k) = 0$ and the presentation (7) we obtain that $\Gamma_B\widehat{f}_I(k) = 0$.

iv) This statement is a direct consequence of ii) and iii) of the Lemma.

v) Let $\alpha = 1$. Then we have that $B_0 \leq k < B_1$ and obtain that

$$\int_0^1 1_{\left[0,\frac{b}{B_1}\right)}(x)\,{}_B\overline{\gamma}_k(x)\,\mathrm{d}x = \int_0^{\frac{b}{b_0}} {}_B\overline{\gamma}_k(x)\,\mathrm{d}x = \sum_{\mu=0}^{b-1}\int_{\frac{\mu}{b_0}}^{\frac{\mu+1}{b_0}} {}_B\overline{\gamma}_k(x)\,\mathrm{d}x$$

$$= \frac{1}{b_0} \cdot \sum_{\mu=0}^{b-1} e^{2\pi\mathbf{i}\frac{k}{b_0}\mu} = \frac{1}{B_1} \cdot \frac{e^{2\pi\mathbf{i}\frac{bk}{b_0}} - 1}{e^{2\pi\mathbf{i}\frac{k}{b_0}} - 1}. \tag{8}$$

By analogue we can prove that

$$\int_0^1 1_{\left[0,\frac{a}{B_1}\right)}(x)_B\overline{\gamma}_k(x)\,\mathrm{d}x = \frac{1}{B_1}\cdot\frac{e^{2\pi\mathbf{i}\frac{ak}{b_0}}-1}{e^{2\pi\mathbf{i}\frac{k}{b_0}}-1}. \tag{9}$$

Then, from (7), (8) and (9) we obtain that

$$_{\Gamma_B}\widehat{f}_I(k) = \frac{1}{B_1}\cdot\frac{e^{2\pi\mathbf{i}b\frac{k}{b_0}}-e^{2\pi\mathbf{i}a\frac{k}{b_0}}}{e^{2\pi\mathbf{i}\frac{k}{b_0}}-1}. \tag{10}$$

Let t_1 and t_2 be arbitrary real numbers. We will use the presentations

$$\begin{aligned} e^{\mathbf{i}t_1}-e^{\mathbf{i}t_2} &= (\cos t_1-\cos t_2)+\mathbf{i}(\sin t_1-\sin t_2)\\ &= -2\sin\frac{t_1+t_2}{2}\cdot\sin\frac{t_1-t_2}{2}\\ &\quad + 2\mathbf{i}\cos\frac{t_1+t_2}{2}\cdot\sin\frac{t_1-t_2}{2}\\ &= 2\cdot\sin\frac{t_1-t_2}{2}\left(-\sin\frac{t_1+t_2}{2}+\mathbf{i}\cos\frac{t_1+t_2}{2}\right). \end{aligned}$$

Hence, we obtain that

$$\left|e^{\mathbf{i}t_1}-e^{\mathbf{i}t_2}\right| = 2\left|\sin\frac{t_1-t_2}{2}\right|\left|-\sin\frac{t_1+t_2}{2}+\mathbf{i}\cos\frac{t_1+t_2}{2}\right| = 2\left|\sin\frac{t_1-t_2}{2}\right|.$$

By using the above explanation, from 10 we obtain that

$$\left|_{\Gamma_B}\widehat{f}_I(k)\right| = \frac{1}{B_1}\cdot\frac{\left|\sin\pi\frac{(b-a)k}{b_0}\right|}{\sin\pi\frac{k}{b_0}}.$$

With this the Lemma is finally proved. □

P r o o f o f L e m m a 2.7. Let for $1\le j\le s$ us denote

$$G_j = \left[\frac{a_j}{B^{(j)}_{\alpha_j}}, \frac{b_j}{B^{(j)}_{\alpha_j}}\right) \quad\text{and}\quad G^*_j = \left[0, \frac{b_j}{B^{(j)}_{\alpha_j}}\right).$$

Then, the presentations hold

$$G = \prod_{j=1}^{s} G_j \quad\text{and}\quad G^* = \prod_{j=1}^{s} G^*_j.$$

The functions $f_G(\mathbf{x})$ and $f_{G^*}(\mathbf{x})$ are of the form

$$f_G(\mathbf{x}) = \prod_{j=1}^{s} 1_{G_j}(\mathbf{x}) - \prod_{j=1}^{s}\frac{b_j-a_j}{B^{(j)}_{\alpha_j}} \quad\text{and}\quad f_{G^*}(\mathbf{x}) = \prod_{j=1}^{s} 1_{G^*_j}(\mathbf{x}) - \prod_{j=1}^{s}\frac{b_j}{B^{(j)}_{\alpha_j}}.$$

For each vector $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ we will calculate the Fourier's coefficients

$$
\begin{aligned}
{}_{\Gamma_{\mathcal{B}_s}}\widehat{f}_G(\mathbf{k}) &= \int_{[0,1]^s} [1_G(\mathbf{x}) - \lambda_s(G)]\,{}_{\Gamma_{\mathcal{B}_s}}\overline{\gamma}_{\mathbf{k}}(\mathbf{x})\, \mathrm{d}x \\
&= \prod_{j=1}^{s} \int_0^1 1_{G_j}(x_j)\,{}_{\Gamma_{B_j}}\overline{\gamma}_{k_j}(x_j)\, \mathrm{d}x_j - \prod_{j=1}^{s} \frac{b_j - a_j}{B_{\alpha_j}^{(j)}} \int_0^1 {}_{\Gamma_{B_j}}\overline{\gamma}_{k_j}(x_j)\, \mathrm{d}x_j \qquad (11)
\end{aligned}
$$

and

$$
{}_{\Gamma_{\mathcal{B}_s}}\widehat{f}_{G^*}(\mathbf{k}) = \prod_{j=1}^{s} \int_0^1 1_{G_j^*}(x_j)\,{}_{\Gamma_{B_j}}\overline{\gamma}_{k_j}(x_j)\, \mathrm{d}x_j - \prod_{j=1}^{s} \frac{b_j}{B_{\alpha_j}^{(i)}} \int_0^1 {}_{\Gamma_{B_j}}\overline{\gamma}_{k_j}(x_j)\, \mathrm{d}x_j. \qquad (12)
$$

i) Let us assume that $\mathbf{k} = \mathbf{0}$. By using the equalities (11) and (12) we obtain that

$$
\begin{aligned}
{}_{\Gamma_{\mathcal{B}_s}}\overline{\widehat{f}}_G(\mathbf{0}) &= \prod_{j=1}^{s} \int_0^1 1_{G_j}(x_j)\ \mathrm{d}x_j - \prod_{j=1}^{s} \frac{b_j - a_j}{B_{\alpha_j}^{(j)}} \\
&= \prod_{j=1}^{s} \frac{b_j - a_j}{B_{\alpha_j}^{(j)}} - \prod_{j=1}^{s} \frac{b_j - a_j}{B_{\alpha_j}^{(j)}} = 0
\end{aligned}
$$

and by analogy ${}_{\Gamma_{\mathcal{B}_s}}\widehat{f}_{G^*}(\mathbf{0}) = 0$.

ii) Let us assume that $\mathbf{k} \neq \mathbf{0}$. Then, there exists at less one index δ, $1 \leq \delta \leq s$ such that $k_\delta \neq 0$ and hence $\int_0^1 {}_{B_\delta}\gamma_{k_\delta}(x_\delta)\ \mathrm{d}x_\delta = 0$. From(11) and (12) we obtain that

$$
{}_{\Gamma_{\mathcal{B}_s}}\overline{\widehat{f}}_G(\mathbf{k}) = \prod_{j=1}^{s} \int_0^1 1_{G_j}(x_j)\,{}_{\Gamma_{B_j}}\gamma_{k_j}(x_j)\ \mathrm{d}x_j = \prod_{j=1}^{s} {}_{\Gamma_{B_j}}\overline{\widehat{1}}_{G_j}(k_j).
$$

The Fourier's coefficient ${}_{\Gamma_{\mathcal{B}_s}}\overline{\widehat{f}}_{G^*}(\mathbf{k})$ of the function $f_{G^*}(\mathbf{x})$ can be calculated by a similar way.
So, we got the following results:

$$
{}_{\mathcal{B}_s}\overline{\widehat{f}}_G(\mathbf{k}) = \prod_{j=1}^{s} {}_{\Gamma_{B_j}}\overline{\widehat{1}}_{G_j}(k_j) \quad \text{and} \quad {}_{\mathcal{B}_s}\overline{\widehat{f}}_{G^*}(\mathbf{k}) = \prod_{j=1}^{s} {}_{\Gamma_{B_j}}\overline{\widehat{1}}_{G_j^*}(k_j). \qquad (13)
$$

Now, let us assume that $\mathbf{k} \in \left(\mathbb{N}_0^s \backslash \Delta^*(\alpha)\right)$ and $\mathbf{k} \neq \mathbf{0}$. This means that there exists at less one index δ, $1 \leq \delta \leq s$ such that $k_\delta \geq B_{\alpha_\delta}(\delta)$. Then, according to Lemma 2.6 iii) and Lemma 2.5 iii) we have that

$$
{}_{\Gamma_{B_\delta}}\widehat{1}_{G_\delta}(k_\delta) = 0 \quad \text{and} \quad {}_{\Gamma_{B_\delta}}\widehat{1}_{G_\delta^*}(k_\delta) = 0
$$

and from the equalities (13) we obtain that

$$
{}_{\Gamma_{\mathcal{B}_s}}\widehat{f}_G(\mathbf{k}) = 0 \quad \text{and} \quad {}_{\Gamma_{\mathcal{B}_s}}\widehat{f}_{G^*}(\mathbf{k}) = 0.
$$

iii) The statement iii) is a direct consequence of i) and ii) of the Lemma.

iv) According to the equality (13) and the statements of Lemma 2.5 iv) and Lemma 2.6 iv) we obtain

$$\left|{}_{\Gamma_{\mathcal{B}_s}}\widehat{\overline{\overline{f}}}_G(\mathbf{k})\right| = \prod_{j=1}^{s}\left|{}_{\Gamma_{B_j}}\widehat{\overline{1}}_{G_j}(k_j)\right| \leq \prod_{j=1}^{s}\rho_{\Gamma_{B_j}}(k_j) = \rho_{\Gamma_{\mathcal{B}_s}}(\mathbf{k})$$

and

$$\left|{}_{\Gamma_{\mathcal{B}_s}}\widehat{\overline{\overline{f}}}_{G^*}(\mathbf{k})\right| = \prod_{j=1}^{s}\left|{}_{\Gamma_{B_j}}\widehat{\overline{\overline{f}}}_{G_j^*}(k_j)\right| \leq \prod_{j=1}^{s}\rho^*_{\Gamma_{B_j}}(k_j) = \rho^*_{\Gamma_{\mathcal{B}_s}}(\mathbf{k}).$$

The Lemma is finally proved. □

5. Proofs of the main results

P r o o f o f T h e o r e m 3.1. Following Hellekalek [5] and [6] the proof will be realized into two steps - discretization and estimation of the discrete discrepancy.

Step 1. (Discretization)**.** Let for $1 \leq j \leq s$ the generalized powers $B^{(j)}_{\alpha_j}$ are defined in the condition of the Theorem. Let $J = \prod_{j=1}^{s}[u_j, v_j)$, where for $1 \leq j \leq s$ $0 \leq u_j < v_j \leq 1$, be an arbitrary subinterval of $[0,1)^s$. For $1 \leq j \leq s$ let us define the numbers

$$a'_j = \min\left\{a : 0 \leq a < B^{(j)}_{\alpha_j},\ u_j \leq \frac{a}{B^{(j)}_{\alpha_j}}\right\},$$

$$a''_j = \max\left\{a : 0 \leq a < B^{(j)}_{\alpha_j},\ \frac{a}{B^{(j)}_{\alpha_j}} \leq u_j\right\},$$

$$b'_j = \max\left\{b : 0 \leq b \leq B^{(j)}_{\alpha_j},\ \frac{b}{B^{(j)}_{\alpha_j}} \leq v_j\right\}$$

and

$$b''_j = \min\left\{b : 0 \leq b \leq B^{(j)}_{\alpha_j},\ v_j \leq \frac{b}{B^{(j)}_{\alpha_j}}\right\}$$

and to construct the intervals

$$\underline{J} = \prod_{j=1}^{s}\left[\frac{a'_j}{B^{(j)}_{\alpha_j}}, \frac{b'_j}{B^{(j)}_{\alpha_j}}\right) \quad \text{and} \quad \overline{J} = \prod_{j=1}^{s}\left[\frac{a''_j}{B^{(j)}_{\alpha_j}}, \frac{b''_j}{B^{(j)}_{\alpha_j}}\right).$$

Let $G(J)$ denote one of the both of intervals $\underline{J}$ and $\overline{J}$. We note the fact that the interval $G(J)$ satisfies the conditions of Lemma 2.7.

To estimate the quantity $\lambda_s(\overline{J}) - \lambda_s(\underline{J})$ we will use the result of Niederreiter [11, Lemma 3.9]. Let for $1 \le j \le s$ $t_j = \frac{b''_j - a''_j}{B^{(j)}_{\alpha_j}}$ and $u_j = \frac{b'_j - a'_j}{B^{(j)}_{\alpha_j}}$. Then, we have that

$$|t_j - u_j| = \frac{b''_j - a''_j}{B^{(j)}_{\alpha_j}} - \frac{b'_j - a'_j}{B^{(j)}_{\alpha_j}} = \frac{b''_j - b'_j}{B^{(j)}_{\alpha_j}} - \frac{a'_j - a''_j}{B^{(j)}_{\alpha_j}} \le \frac{2}{B^{(j)}_{\alpha_j}} \le \frac{2}{M}.$$

Hence, we obtain that

$$\lambda_s(\overline{J}) - \lambda_s(\underline{J}) \le 1 - \left(1 - \frac{2}{M}\right)^s. \tag{14}$$

Step 2. (Estimation of the discrete discrepancy). The inclusions $\underline{J} \subseteq J \subseteq \overline{J}$ give us that the inequalities $A(\underline{J}; N) \le A(J; N) \le A(\overline{J}; N)$ hold. For the quantities $N^{-1} \cdot A(J; N)$ and $\lambda_s(J)$ we will consider the both cases:

$$N^{-1} \cdot A(J; N) \ge \lambda_s(J) \quad \text{and} \quad N^{-1} \cdot A(J; N) < \lambda_s(J).$$

This will permit us to prove the inequality

$$\begin{aligned} \left|N^{-1} \cdot A(J; N) - \lambda_s(J)\right| &\le \lambda_s(\overline{J}) - \lambda_s(\underline{J}) + \\ &\max\left\{\left|N^{-1} \cdot A(\underline{J}; N) - \lambda_s(\underline{J})\right|, \left|N^{-1} \cdot A(\overline{J}; N) - \lambda_s(\overline{J})\right|\right\}. \end{aligned} \tag{15}$$

By using the statements of Lemma 2.7 the quantity

$$\begin{aligned} R_N\big(G(J)\big) &= N^{-1} A\big(G(J); N\big) - \lambda_s\big(G(J)\big) \\ &= N^{-1} \sum_{n=0}^{N-1} \left[1_{G(J)}(\mathbf{x}_n) - \lambda_s\big(G(J)\big)\right] \\ &= N^{-1} \sum_{n=0}^{N-1} f_{G(J)}(\mathbf{x}_n) \end{aligned}$$

has a development as a Fourier's series of the form

$$\begin{aligned} R_N\big(G(J)\big) &= N^{-1} \sum_{n=0}^{N-1} \sum_{\mathbf{k} \in \Delta^*(\alpha)} {}_{\Gamma_{\mathcal{B}_s}}\widehat{f}_{G(J)}(\mathbf{k})_{\mathcal{B}_s}\gamma_{\mathbf{k}}(\mathbf{x}_n) \\ &= \sum_{\mathbf{k} \in \Delta^*(\alpha)} {}_{\Gamma_{\mathcal{B}_s}}\widehat{f}_{G(J)}(\mathbf{k}) \left[N^{-1} \sum_{n=0}^{N-1} {}_{\mathcal{B}_s}\gamma_{\mathbf{k}}(\mathbf{x}_n)\right] \\ &= \sum_{\mathbf{k} \in \Delta^*(\alpha)} {}_{\Gamma_{\mathcal{B}_s}}\widehat{f}_{G(J)}(\mathbf{k}) \cdot S({}_{\mathcal{B}_s}\gamma_{\mathbf{k}}; P_N). \end{aligned}$$

Hence, we obtain that

$$\sup_{J\in\mathcal{J}}\left|R_N\big(G((J)\big)\right| \leq \sup_{J\in\mathcal{J}} \sum_{\mathbf{k}\in\Delta^*(\alpha)} \left|{}_{\Gamma_{\mathcal{B}_s}}\widehat{f}_{G(J)}(\mathbf{k})\right| \cdot \left|S({}_{\mathcal{B}_s}\gamma_{\mathbf{k}}; P_N)\right|. \tag{16}$$

From (14), (15) and (16) we obtain that

$$D(P_N) \leq 1-\left(1-\frac{2}{M}\right)^s + \sup_{J\in\mathcal{J}} \sum_{\mathbf{k}\in\Delta^*(\alpha)} \left|{}_{\Gamma_{\mathcal{B}_s}}\widehat{f}_{G(J)}(\mathbf{k})\right| \cdot \left|S({}_{\mathcal{B}_s}\gamma_{\mathbf{k}}; P_N)\right|.$$

ii) According to part i) of the Theorem, Lemma 2.7 iii) and the introduced quantity $B_{\Gamma_{\mathcal{B}_s}}(\alpha)$ we obtain that

$$\begin{aligned} D(P_N) &\leq 1-\left(1-\frac{2}{M}\right)^s + B_{\Gamma_{\mathcal{B}_s}}(\alpha) \sum_{\mathbf{k}\in\Delta^*(\alpha)} \left|{}_{\Gamma_{\mathcal{B}_s}}\widehat{f}_{G(J)}(\mathbf{k})\right| \\ &\leq 1-\left(1-\frac{2}{M}\right)^s + B_{\Gamma_{\mathcal{B}_s}}(\alpha) \sum_{\mathbf{k}\in\Delta^*(\alpha)} \rho_{\Gamma_{\mathcal{B}_s}}(\mathbf{k}). \end{aligned} \tag{17}$$

We will use the presentation

$$\begin{aligned} \sum_{\mathbf{k}\in\Delta^*(\alpha)} \rho_{\,\Gamma_{\mathcal{B}_s}}(\mathbf{k}) &= \sum_{\mathbf{k}\in\Delta(\alpha)} \rho_{\,\Gamma_{\mathcal{B}_s}}(\mathbf{k}) - 1 \\ &= \prod_{j=1}^{s}\left[1+\sum_{k_j=1}^{B^{(j)}_{\alpha_j}-1} \rho_{\,\Gamma_{B_j}}(k_j)\right]-1. \end{aligned} \tag{18}$$

On other side, according to Lemma 2.6 iv) for an arbitrary fixed integer $\alpha \geq 1$ we have

$$\begin{aligned} \sum_{k=1}^{B_\alpha-1}\rho_{\Gamma_B}(k) &\leq 2\sum_{g=0}^{\alpha-1}\frac{1}{B_{g+1}}\sum_{k_g=1}^{b_g-1}\frac{1}{\sin\pi\frac{k_g}{b_g}}\sum_{k=k_gB_g}^{(k_g+1)B_g-1}1 \\ &= 2\sum_{g=0}^{\alpha-1}\frac{1}{b_g}\sum_{k_g=1}^{b_g-1}\frac{1}{\sin\pi\frac{k_g}{b_g}}. \end{aligned} \tag{19}$$

Niederreiter [11] proved the next result: For each integer $b \geq 2$ the inequality holds

$$\sum_{k=1}^{b-1}\frac{1}{\sin\pi\frac{k}{b}} \leq \frac{2}{\pi}b\cdot\log b+\frac{2}{5}b. \tag{20}$$

Hence, from (19) and (20) we obtain that

$$\sum_{k=1}^{B_\alpha-1}\rho_{\Gamma_B}(k) \leq 2\left(\frac{2}{\pi}\log q+\frac{2}{5}\right)\cdot\alpha. \tag{21}$$

From (18) and (21) we obtain that

$$\sum_{\mathbf{k}\in\Delta^*(\alpha)}\rho_{\Gamma_B}(\mathbf{k}) \leq \prod_{j=1}^{s}\left[1+\left(\frac{4}{\pi}\log q+\frac{4}{5}\right)\alpha_j\right]-1. \tag{22}$$

We will use that for $1 \leq j \leq s$ the inequalities $M \leq B^{(j)}_{\alpha_j} \leq q^{\alpha_j}$ holds. The number r introduced in condition of the Theorem gives us that for $1 \leq j \leq s$, $q^{\alpha_j} < r\cdot M$. Hence, for $1 \leq j \leq s$ the inequality $\alpha_j \leq \frac{\log r+\log M}{\log q}$ holds and from (22) we obtain

$$\sum_{\mathbf{k}\in\Delta^*(\alpha}\rho_{\Gamma_B}(\mathbf{k}) \leq \left[1+\left(\frac{4}{5}+\frac{4}{5\log q}\right)(\log r+\log M)\right]^s-1. \qquad \square$$

From (17) and the above inequality we prove the statement ii) of the Theorem.

P r o o f. Now, we will prove the upper bound of the star discrepancy $D^*(P_N)$ on the net P_N. The proof again will be divided into two parts: discretization and estimation of the discrete discrepancy.

Step 1. (Discretization)**.** Let for $1 \leq j \leq s$ the generalized powers $B^{(j)}_{\alpha_j}$ are defined in the condition of the Theorem. Let $J^* = \prod_{j=1}^{s}[0, v_j)$, where for $1 \leq j \leq s$ we have $0 \leq v_j \leq 1$, be an arbitrary subinterval of $[0,1)^s$. For $1 \leq j \leq s$ let us define the numbers $a_j = \max\{a : 0 \leq a < B^{(j)}_{\alpha_j}, \frac{a}{B^{(j)}_{\alpha_j}} \leq v_j\}$ and to construct the intervals

$$\underline{J}^* = \prod_{j=1}^{s}\left[0, \frac{a_j}{B^{(j)}_{\alpha_j}}\right) \qquad \text{and} \qquad \overline{J}^* = \prod_{j=1}^{s}\left[0, \frac{a_j+1}{B^{(j)}_{\alpha_j}}\right).$$

Let $G(J^*)$ means one of the both intervals $\underline{J}^*$ and $\overline{J}^*$.

According to Niederreiter [12, Lemma 3.9] we can prove that

$$\lambda_s(\overline{J}^*)-\lambda_s(\underline{J}^*) \leq 1-\left(1-\frac{1}{M}\right)^s. \tag{23}$$

By using the statements of Lemma 2.7 the quantity

$$\begin{aligned} R_N\big(G(J^*)\big) &= N^{-1}A(G(J^*);N)-\lambda_s\big(G(J^*)\big) \\ &= N^{-1}\sum_{n=0}^{N-1}\left[1_{G(J^*)}(\mathbf{x}_n)-\lambda_s\big(G(J^*)\big)\right] \end{aligned}$$

has a development as a Fourier‘s series of the form

$$\begin{aligned} R_N\big(G(J^*)\big) &= N^{-1}\sum_{n=0}^{N-1}\sum_{\mathbf{k}\in\Delta^*(\alpha)} {}_{\Gamma_{\mathcal{B}_\alpha}}\widehat{f}_{G(J^*)}(\mathbf{k})_{\mathcal{B}_s}\gamma_{\mathbf{k}}(\mathbf{x}_n) \\ &= \sum_{\mathbf{k}\in\Delta^*(\alpha)} {}_{\Gamma_{\mathcal{B}_s}}\widehat{f}_{G(J^*)}(\mathbf{k})\left[N^{-1}\sum_{n=0}^{N-1} {}_{\mathcal{B}_s}\gamma_{\mathbf{k}}(\mathbf{x}_n)\right] \\ &= \sum_{\mathbf{k}\in\Delta^*(\alpha)} {}_{\Gamma_{\mathcal{B}_s}}\widehat{f}_{G(J^*)}(\mathbf{k})\cdot S({}_{\mathcal{B}_s}\gamma_{\mathbf{k}};P_N). \end{aligned}$$

Hence, we obtain that

$$\sup_{J^*\subseteq\mathcal{J}^*}\big|R_N\big(G(J^*)\big)\big| \le \sup_{J^*\subseteq\mathcal{J}^*}\sum_{\mathbf{k}\in\Delta^*(\alpha)}|{}_{\Gamma_{\mathcal{B}_s}}\widehat{f}_{G(J^*)}(\mathbf{k})|\cdot|S({}_{\mathcal{B}_s}\gamma_{\mathbf{k}};P_N)|. \tag{24}$$

From (23) and (24) we obtain the raw form of the inequality of Erdős-Turán--Koksma for the star discrepancy.

Step 2. (estimation of the discrete discrepancy)**.** By using a technique which is similar to one for obtaining of the refined form of the inequality of Erdős-Turán-Koksma for the extreme discrepancy, we can prove the corresponding refined form of this inequality for the star discrepancy. □

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Tsvetelina Petrova
Department of Mathematics
Faculty of Natural Science and Mathematics
University of South West University
"Neofit Rilski"
Ivan Michailov str. 66
2700 Blagoevgrad
BULGARIA
E-mail: pcvetelina488@gmail.com