

EXTREME VALUES OF EULER-KRONECKER CONSTANTS

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ABSTRACT. In a family of S_n -fields ($n \leq 5$), we show that except for a density zero set, the lower and upper bounds of the Euler-Kronecker constants are $-(n - 1) \log \log d_K + O(\log \log \log d_K)$ and $\log \log d_K + O(\log \log \log d_K)$, resp., where d_K is the absolute value of the discriminant of a number field K .

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1. Introduction

A number field K of degree n is called an S_n -field if its Galois closure over \mathbb{Q} is an S_n -Galois extension. For an S_n -field K , let $\zeta_K(s)$ be the Dedekind zeta function, with the following Laurent series expansion at $s = 1$:

$$\zeta_K(s) = c_{-1}(s - 1)^{-1} + c_0 + c_1(s - 1) + \dots$$

Then $\gamma_K = c_0/c_{-1}$ is called the Euler-Kronecker constant of K . If $K = \mathbb{Q}$, $\gamma_{\mathbb{Q}}$ is just the Euler constant γ . We have

$$\frac{\zeta'_K(s)}{\zeta_K(s)} = \frac{1}{s - 1} + \gamma_K + b_1(s - 1) + \dots$$

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Let d_K be the absolute value of the discriminant of K . Under GRH, Ihara [4] showed that when n is fixed, for any $\epsilon > 0$,

$$-2(n-1+\epsilon)\log\log d_K \leq \gamma_K \leq 2(1+\epsilon)\log\log d_K.$$

In [1], we showed that under the strong Artin conjecture, GRH, and certain zero-density hypothesis, the upper and lower bounds are

$$-(n-1)\log\log d_K + O(\log\log\log d_K), \quad \log\log d_K + O(\log\log\log d_K), \quad \text{resp.}$$

In this paper, following [2], we show that if $n \leq 5$, except for a density zero set, the above are true upper and lower bounds. More precisely, we prove

THEOREM 1.1. *Let $L(X)$ be the set of S_n -fields ($n \leq 5$) with $X/2 \leq d_K \leq X$. For S_5 -fields, we assume the strong Artin conjecture for $\frac{\zeta_K(s)}{\zeta(s)}$. Then, except for*

$$O\left(X \exp\left(-c' \frac{\log X}{\log\log X} \log\log\log\log X\right)\right)$$

fields for some constant $c' > 0$,

$$-(n-1)\log\log d_K + O(\log\log\log d_K) \leq \gamma_K \leq \log\log d_K + O(\log\log\log d_K).$$

Next, we construct an infinite family of S_n -fields with extreme values.

THEOREM 1.2. *Let $n \leq 5$. For $n = 5$, we assume the strong Artin conjecture.*

- 1) *Let $L_l(X)$ be the set of S_n -fields K of signature (r_1, r_2) with $X/2 \leq d_K \leq X$ for which*

$$\gamma_K = -(n-1)\log\log d_K + O(\log\log\log d_K).$$

Then

$$|L_l(X)| \geq A(r_2)X \exp\left(-c'_1 \frac{\log X}{\log\log X}\right) \quad \text{for some } c'_1 > 0.$$

- 2) *Let $L_u(X)$ be the set of S_n -fields K of signature (r_1, r_2) with $X/2 \leq d_K \leq X$ for which*

$$\gamma_K = \log\log d_K + O(\log\log\log d_K).$$

Then

$$|L_u(X)| \geq A(r_2)X \exp\left(-c'_1 \frac{\log X}{\log\log X}\right) \quad \text{for some } c'_1 > 0,$$

where $A(r_2)$ is a constant which occurs in Theorem 2.1.

2. Counting number fields with local conditions

Let K be an S_n -field of signature (r_1, r_2) for $n \leq 5$. We recall counting S_n -fields with finitely many local conditions: Let $\mathcal{S} = (\mathcal{L}\mathcal{C}_p)$ be a finite set of local conditions: $\mathcal{L}\mathcal{C}_p = \mathcal{S}_{p,C}$ means that p is unramified and the conjugacy class of Frob_p is C . Define $|\mathcal{S}_{p,C}| = \frac{|C|}{|S_n|(1+f(p))}$ for some positive valued function $f(p)$ which satisfies $f(p) = O(\frac{1}{p})$. More explicitly [2], we have

$$\begin{aligned} f(p) &= p^{-1} + p^{-2} && \text{if } n = 3; \\ f(p) &= p^{-1} + 2p^{-2} + p^{-3} && \text{if } n = 4; \\ f(p) &= p^{-1} + 2p^{-2} + 2p^{-3} + p^{-4} && \text{if } n = 5. \end{aligned}$$

There are also several splitting types of ramified primes, which are denoted by r_1, r_2, \dots, r_w : $\mathcal{L}\mathcal{C}_p = \mathcal{S}_{p,r_j}$ means that p is ramified and its splitting type is r_j . We assume that there are positive valued functions $c_1(p), c_2(p), \dots, c_w(p)$ with $\sum_{i=1}^w c_i(p) = f(p)$ and define $|\mathcal{S}_{p,r_i}| = \frac{c_i(p)}{1+f(p)}$. We define the local condition $\mathcal{L}\mathcal{C}_p = \mathcal{S}_{p,r}$ which means that p is ramified, i.e, $r = r_j$ for some j . Define $|\mathcal{S}_{p,r}| = \frac{f(p)}{1+f(p)}$. Let $|\mathcal{S}| = \prod_p |\mathcal{L}\mathcal{C}_p|$.

Let $L_n(X)^{(r_2)}$ be the set of S_n -fields K of signature (r_1, r_2) with $X/2 < d_K < X$, and let $L_n(X; \mathcal{S})^{(r_2)}$ be the set of S_n -fields K of signature (r_1, r_2) with $X/2 < d_K < X$ and the local conditions \mathcal{S} . It is assumed that we pick up only one number field K from n conjugate number fields. Then we have

THEOREM 2.1. *Let $n \leq 5$. For some positive constants $\delta < 1$ and κ ,*

$$\begin{aligned} |L_n(X)^{(r_2)}| &= A(r_2)X + O(X^\delta), \\ |L_n(X; \mathcal{S})^{(r_2)}| &= |\mathcal{S}|A(r_2)X + O\left(\left(\prod_{p \in \mathcal{S}} p\right)^\kappa X^\delta\right), \end{aligned} \tag{2.1}$$

where the implied constant is uniformly bounded for p and local conditions at p , and see [3] for the precise value of $A(r_2)$.

Here we can control all primes only up to $c \log X$, where $c < \frac{1-\delta}{\kappa}$. To ease the notations, throughout the article, we denote $L_n(X)^{(r_2)}$ by $L(X)$ if there is no danger of confusion.

If \widehat{K} is the Galois closure of K over \mathbb{Q} , then we have $\zeta_K(s) = \zeta(s)L(s, \rho)$, where ρ is the $(n-1)$ -dimensional standard representation of S_n . Then

$$\gamma_K = \gamma + \frac{L'}{L}(1, \rho).$$

This leads us to the study of $\frac{L'}{L}(1, \rho)$. By abuse of notation, we denote $L(X)$ as a set of L -functions $L(s, \rho)$. Here we need care in order to ensure one to one correspondence between two sets. Two number fields K_1 and K_2 are said to be arithmetically equivalent if $\zeta_{K_1}(s) = \zeta_{K_2}(s)$. If two number fields K_1 and K_2 are conjugate, then they are arithmetically equivalent. The converse is not always true. A number field K_1 is called arithmetically solitary if $\zeta_{K_1}(s) = \zeta_{K_2}(s)$ implies that K_1 and K_2 are conjugate. It is known that S_n -fields and A_n -fields are arithmetically solitary. See [6, Chap. II]. Hence if $K_1, K_2 \in L(X)$ are not conjugate, then $\zeta_{K_1}(s) \neq \zeta_{K_2}(s)$. So if we set

$$\zeta_{K_1}(s) = \zeta(s)L(s, \rho_1) \quad \text{and} \quad \zeta_{K_2}(s) = \zeta(s)L(s, \rho_2),$$

then $L(s, \rho_1) \neq L(s, \rho_2)$, and $L(s, \rho) \in L(X)$ makes sense.

3. Proofs of the theorems

Let

$$L(s, \rho) = \sum_{n=1}^{\infty} \lambda_{\rho}(n)n^{-s} = \prod_p \prod_{i=1}^{n-1} \left(1 - \frac{\alpha_i(p)}{p^s}\right)^{-1}.$$

Then we have

$$-\frac{L'}{L}(s, \rho) = \sum_{n=1}^{\infty} \Lambda(n)a_{\rho}(n)n^{-s}, \quad a_{\rho}(p^k) = \sum_{j=1}^d \alpha_j(p)^k.$$

Recall [3, Proposition 2.1]:

PROPOSITION 3.1. *Suppose $L(s, \rho)$ is entire and is zero free in the rectangle $[\alpha, 1] \times [-x, x]$. Then*

$$-\frac{L'}{L}(1, \rho) = \sum_{n < x} \frac{a_{\rho}(n)\Lambda(n)}{n} + O_{\alpha} \left(\frac{(\log d_K)(\log x) + (\log x)^2}{x^{1-\frac{\alpha+1}{2}}} \right).$$

By setting $x = (\log d_K)^{\beta}$ with $\beta(1 - \frac{\alpha+1}{2}) > 3$, we have

$$\begin{aligned} -\frac{L'}{L}(1, \rho) &= \sum_{n < (\log d_K)^{\beta}} \frac{a_{\rho}(n)\Lambda(n)}{n} + O \left(\frac{1}{\log d_K} \right) \\ &= \sum_{p < (\log d_K)^{\beta}} \frac{a_{\rho}(p)\log p}{p} + O(1). \end{aligned}$$

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We prove

PROPOSITION 3.2. *Let $n \leq 5$, and $x = (\log X)^\beta$, $y = c_1 \log X$ with $\beta > 1$ and $0 < c_1 < \frac{1-\delta}{\kappa}$. Then except for $O(X \exp(-c' \frac{\log X}{\log \log X} \log \log \log X))$ L -functions in $L(X)$ for some constant $c' > 0$, L -functions in $L(X)$ satisfy*

$$\left| \sum_{y < p < x} \frac{a_\rho(p) \log p}{p} \right| \leq \log \log \log X.$$

By applying Kowalski-Michel zero density theorem [7] to the family $L(X)$ (See [2] for the details), we may assume that every L -function in $L(X)$ outside the exceptional set in Proposition 3.2 has the desired zero free region of the form in Proposition 3.1. Then outside the exceptional set,

$$-\frac{L'}{L}(1, \rho) = \sum_{p < c_1 \log d_K} \frac{a_\rho(p) \log p}{p} + O(\log \log \log X).$$

Now, Theorem 1.1 follows immediately by observing that

$$-1 \leq a_\rho(p) \leq n - 1, \quad \text{and} \quad \sum_{p \leq x} \frac{\log p}{p} = \log x + O(1).$$

In order to prove Proposition 3.2, we imitate the proof of Proposition 4.2 in [2]. Namely, we prove

PROPOSITION 3.3. *Let $n \leq 5$, and $x = (\log X)^\beta$ and $y = c_1 \log X$. Then for $r \leq c_2 \frac{\log X}{\log \log X}$ with a sufficiently small c_2 ,*

$$\sum_{L(s, \rho) \in L(X)} \left(\sum_{y < p < x} \frac{a_\rho(p) \log p}{p} \right)^{2r} \ll r (4(\beta + 1)(n - 1)^2 N_n^2)^r \frac{(2r)!}{r!} \left(\frac{\log x}{y} \right)^r X,$$

where N_n is the number of splitting types in S_n -fields, and the implied constant depends only on n .

By Stirling's formula,

$$r (4(\beta + 1)(n - 1)^2 N_n^2)^r \frac{(2r)!}{r!} \leq r \sqrt{2} \left(\frac{16(\beta + 1)(n - 1)^2 N_n^2}{e} \right)^r r^r.$$

Here $N_2 = 3, N_3 = 5, N_4 = 11, N_5 = 17$. (cf. [2]).

Proof. By the multinomial formula, the left hand side is

$$\sum_{L(s,\rho) \in L(X)} \sum_{u=1}^{2r} \frac{1}{u!} \sum_{r_1, \dots, r_u}^{(1)} \frac{(2r)!}{r_1! \cdots r_u!} \sum_{p_1, \dots, p_u}^{(2)} \frac{a_\rho(p_1)^{r_1} \cdots a_\rho(p_u)^{r_u}}{p_1^{r_1} \cdots p_u^{r_u}} \times (\log p_1)^{r_1} \cdots (\log p_u)^{r_u}, \quad (3.1)$$

where $\sum_{r_1, \dots, r_u}^{(1)}$ means that the sum is over the ordered u -tuples (r_1, \dots, r_u) of positive integers such that $r_1 + \cdots + r_u = 2r$, and $\sum_{p_1, \dots, p_u}^{(2)}$ means the sum over the u -tuples (p_1, \dots, p_u) of distinct primes such that $y < p_i < x$ for each i . Each ordered u -tuple (r_1, \dots, r_u) gives a composition of $2r$. Here a composition means an ordered partition. We write that formula (3.1):

$$(3.1) = \sum_{u=1}^{2r} \sum_{r_1, \dots, r_u}^{(1)} \frac{(2r)!}{r_1! \cdots r_u!} \frac{1}{u!} \sum_{p_1, \dots, p_u}^{(2)} \frac{(\log p_1)^{r_1} \cdots (\log p_u)^{r_u}}{p_1^{r_1} \cdots p_u^{r_u}} \times \left(\sum_{L(s,\rho) \in L(X)} a_\rho(p_1)^{r_1} \cdots a_\rho(p_u)^{r_u} \right).$$

We show that for any composition $r_1 + r_2 + \cdots + r_u = 2r$,

$$\frac{(2r)!}{r_1! \cdots r_u!} \frac{1}{u!} \sum_{p_1, \dots, p_u}^{(2)} \frac{(\log p_1)^{r_1} \cdots (\log p_u)^{r_u}}{p_1^{r_1} \cdots p_u^{r_u}} \times \left(\sum_{L(s,\rho) \in L(X)} a_\rho(p_1)^{r_1} \cdots a_\rho(p_u)^{r_u} \right) \ll ((n-1)N_n)^{2r} X \frac{(2r)!}{r!} \frac{(\beta+1)^r (\log x)^r}{y^r}. \quad (3.2)$$

Since the number of compositions of $2r$ is 2^{2r-1} , it implies that the formula (3.1):

$$(3.1) \ll r(4(\beta+1)(n-1)^2 N_n^2)^r \frac{(2r)!}{r!} \frac{(\log x)^r}{y^r} X.$$

First, we consider compositions with $r_i \geq 2$ for all i . Then by using the trivial bound,

$$\begin{aligned} & \sum_{p_1, \dots, p_u}^{(2)} \frac{(\log p_1)^{r_1} \cdots (\log p_u)^{r_u}}{p_1^{r_1} \cdots p_u^{r_u}} \left(\sum_{L(s,\rho) \in L(X)} a_\rho(p_1)^{r_1} \cdots a_\rho(p_u)^{r_u} \right) \\ & \ll d^{2r} X \left(\sum_{y < p_1 < x} \frac{(\log p_1)^{r_1}}{p_1^{r_1}} \right) \cdots \left(\sum_{y < p_u < x} \frac{(\log p_u)^{r_u}}{p_u^{r_u}} \right) \\ & \ll d^{2r} X 2^{2r} \left(\frac{\log x}{y} \right)^{2r-u}. \end{aligned}$$

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where we used the fact that for $r \geq 2$, $\sum_{y < p < x} \frac{(\log p)^r}{p^r} \leq \frac{r}{r-1} \left(\frac{\log x}{y}\right)^{r-1}$.

In [2], we showed that for any r_1, \dots, r_u such that $r_1 + \dots + r_u = 2r$, and $r_i \geq 2$ for all i ,

$$\frac{1}{u! r_1! \cdots r_u!} \left(\frac{\log y}{y}\right)^{r-u} \leq \frac{1}{r!}.$$

By using the fact that $\frac{\log x}{\log y} \leq \beta + 1$ for sufficiently large X , we have (3.2) in this case.

Next, suppose $r_i = 1$ for some i . We may assume that

$$\begin{aligned} r_1 + \dots + r_m + r_{m+1} + \dots + r_u &= 2r, \\ r_1 = \dots = r_m &= 1, \quad \text{and} \quad r_{m+1} > 1, \dots, r_u > 1. \end{aligned}$$

Consider

$$\sum_{L(s, \rho) \in L(X)} a_\rho(p_1)^{r_1} \cdots a_\rho(p_u)^{r_u}. \tag{3.3}$$

Let N be the number of conjugacy classes of G . Recall that there are w ramified splitting types so that $N_n = N + w$. Partition the sum $\sum_{L(s, \rho) \in L(X)}$ into N_n^u sums, namely, given $(\mathcal{S}_1, \dots, \mathcal{S}_u)$, where \mathcal{S}_i is either $\mathcal{S}_{p_i, C}$ or \mathcal{S}_{p_i, r_j} , we consider the set of $L(s, \rho) \in L(X)$ with the local conditions \mathcal{S}_i for each i . Note that in each such partition, $a_\rho(p_1)^{r_1} \cdots a_\rho(p_u)^{r_u}$ remains a constant.

Suppose one of p_1, \dots, p_m is unramified, say p_1 . Consider $N(N + w)^{u-1}$ such partitions in (3.3). Fix the splitting types of p_2, \dots, p_u and let Frob_{p_1} runs through the conjugacy classes of G . Let $L(X; p_2, \dots, p_u)$ be the set of $L(s, \rho) \in L(X)$ with the fixed splitting types. Then the sum of such N partitions is

$$\sum_C a_\rho(p_1) a_\rho(p_2) \cdots a_\rho(p_u)^{r_u} \sum_{L(s, \rho) \in L(X; p_2, \dots, p_u)} 1.$$

By (2.1),

$$\sum_{L(s, \rho) \in L(X; p_2, \dots, p_u)} 1 = \frac{|C|}{|G|(1 + f(p_1))} A(p_2, \dots, p_u) X + O((p_1 \cdots p_u)^\kappa X^\delta),$$

for some constant $A(p_2, \dots, p_u)$. Let χ_ρ be the character of ρ . Then $a_\rho(p) = \chi_\rho(g)$, where $g = \text{Frob}_p$. By orthogonality of characters,

$$\sum_C |C| a_\rho(p_1) = \sum_{g \in G} \chi_\rho(g) = 0.$$

Hence the above sum is $O(N(p_1 \cdots p_u)^\kappa X^\delta)$. By the trivial bound,

$$|a_\rho(p_2) \cdots a_\rho(p_u)^{r_u}| \leq (n - 1)^{2r}.$$

By using

$$\sum_{y < p < x} p^a (\log p)^b \ll (\log x)^{b-1} x^{a+1} \quad \text{for } b \geq 1,$$

the contribution from these partitions to (3.2) is

$$\begin{aligned} &\ll N(n-1)^{2r} X^\delta \frac{(2r)!}{r_1! \cdots r_u!} \frac{1}{u!} \prod_{i=1}^m \left(\sum_{y < p_i < x} p_i^{\kappa-1} \log p_i \right) \\ &\quad \times \prod_{i=m+1}^u \left(\sum_{y < p_i < x} p_i^{\kappa-r_i} (\log p_i)^{r_i} \right) \\ &\ll N(n-1)^{2r} X^\delta \frac{(2r)!}{r_1! \cdots r_u!} \frac{1}{u!} x^{u\kappa+u-2r} (\log x)^{2r-u} \\ &\ll N(n-1)^{2r} X^\delta \frac{(2r)!}{r!} x^{u\kappa+u-2r+1} y^{m+r-u+1}. \end{aligned}$$

Here we used [2, Lemma 4.3] for the last inequality. If we choose c_2 sufficiently small, $X^\delta (\log X)^{2r\kappa\beta+1} \ll X^{\frac{2^{2r}(\log x)^r}{y^r}}$. This verifies (3.2).

Now suppose that p_1, \dots, p_m are all ramified. Then by (2.1), the number of elements in the set of $L(s, \rho) \in L(X)$ with the local condition $\mathcal{S}_{p_i, r}$ for $i = 1, \dots, m$, is

$$\prod_{i=1}^m \frac{f(p_i)}{1+f(p_i)} A(r_2) X + O((p_1 \cdots p_m)^\kappa X^\delta),$$

Note that $\frac{f(p)}{1+f(p)} \ll \frac{1}{p}$. By the trivial bound,

$$|a_\rho(p_1)^{r_1} \cdots a_\rho(p_u)^{r_u}| \leq (n-1)^u \leq (n-1)^{2r}.$$

Hence the main term contributes to (3.2)

$$\begin{aligned} &X(n-1)^{2r} \sum_{p_1, \dots, p_u}^{(2)} \frac{(\log p_1) \cdots (\log p_m) (\log p_{m+1})^{r_{m+1}} \cdots (\log p_u)^{r_u}}{p_1^2 \cdots p_m^2 p_{m+1}^{r_{m+1}} \cdots p_u^{r_u}} \\ &\ll X(n-1)^{2r} \prod_{i=1}^m \left(\sum_{y < p_i < x} p_i^{-2} (\log p_i) \right) \prod_{i=m+1}^u \left(\sum_{y < p_i < x} p_i^{-r_i} (\log p_i)^{r_i} \right) \\ &\ll X(n-1)^{2r} y^{-m} \left(\frac{\log x}{y} \right)^{2r-u}. \end{aligned}$$

As above, (3.2) is verified in this case.

The contribution of the error term $O((p_1 \cdots p_m)^\kappa X^\delta)$ to (3.2) is similar to the case when p_1 is unramified. \square

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Now take $y = c_1 \log X$, and $r = c_2 \frac{\log X}{\log \log X}$. Then from Proposition 3.3, the number of $L(s, \rho) \in L(X)$ such that $|\sum_{y < p < x} \frac{a_\rho(p) \log p}{p}| > \log \log \log X$, is

$$\ll X \exp\left(-c' \frac{\log X}{\log \log X} \log \log \log \log X\right), \quad (3.4)$$

for some $c' > 0$. This proves Proposition 3.2.

Now, let $\mathcal{S} = (S_{p,C})_{p \leq y}$ be the set of local conditions such that for every prime $p \leq y$, $\text{Frob}_p = 1$, i.e., $a_\rho(p) = n - 1$. Then

$$|L(X, \mathcal{S})| = A(r_2)X \prod_{p \leq y} \frac{\frac{1}{|S_n|}}{1 + f(p)} + O\left(\left(\prod_{p \leq y} p\right)^\gamma X^\delta\right).$$

Here,

$$\begin{aligned} \log \prod_{p \leq y} \frac{\frac{1}{|S_n|}}{1 + f(p)} &= -(\log |S_n|) \sum_{p \leq y} 1 - \sum_{p \leq y} \log(1 + f(p)), \\ \sum_{p \leq y} 1 &= \frac{y}{\log y} + O\left(\frac{y}{(\log y)^2}\right), \end{aligned}$$

and

$$\sum_{p \leq y} \log(1 + f(p)) = \log \log y + O(1).$$

Hence,

$$|L(X, \mathcal{S})| \gg X \exp\left(-c'_1 \frac{\log X}{\log \log X}\right) \text{ for some } c'_1 > 0.$$

This is larger than (3.4). Also we may assume that almost all L -functions in $L(X, \mathcal{S})$ have the desired zero-free region of the form in Proposition 3.1. Hence

$$\begin{aligned} -\frac{L'}{L}(1, \rho) &= (n - 1) \sum_{p \leq y} \frac{\log p}{p} + O(\log \log \log d_K) \\ &= (n - 1) \log \log d_K + O(\log \log \log d_K). \end{aligned}$$

This proves Theorem 1.2-1).

Similarly, consider C , the conjugacy class of $(1 \dots n)$. Then $a_\rho(p) = -1$ if $\text{Frob}_p \in C$. Let $\mathcal{S} = (S_{p,C})_{p \leq y}$ be the set of local conditions such that for every prime $p \leq y$, $\text{Frob}_p \in C$. Then

$$|L(X, \mathcal{S})| = A(r_2)X \prod_{p \leq y} \frac{\frac{n}{|S_n|}}{1 + f(p)} + O\left(\left(\prod_{p \leq y} p\right)^\gamma X^\delta\right).$$

As above, we can show that $|L(X, \mathcal{S})| \gg X \exp\left(-c'_1 \frac{\log X}{\log \log X}\right)$ for some $c'_1 > 0$, and

$$\begin{aligned} -\frac{L'}{L}(1, \rho) &= -\sum_{p \leq y} \frac{\log p}{p} + O(\log \log \log d_K) \\ &= -\log \log |d_K| + O(\log \log \log d_K). \end{aligned}$$

This proves Theorem 1.2–2).

REMARK 3.5. In [3], we have shown that the average of k th moments of $-\frac{L'}{L}(1, \rho)$ is a constant, i.e.,

$$\frac{1}{|L(X)|} \sum_{L(s, \rho) \in L(X)} \left(-\frac{L'}{L}(1, \rho)\right)^k = C_k + O\left(\frac{1}{\log X}\right).$$

REMARK 3.6. If we take $r = 1$ in Proposition 3.3, then the number of $L(s, \rho) \in L(X)$ such that $\left|\sum_{y < p < x} \frac{a_\rho(p) \log p}{p}\right| > \log \log \log X$, is $O\left(\frac{X}{(\log X)(\log \log X)}\right)$, which is larger than $X \exp\left(-c'_1 \frac{\log X}{\log \log X}\right)$. This is the reason why we need to take $r = c_2 \frac{\log X}{\log \log X}$ in Proposition 3.3 in order to prove Theorem 1.2.

4. Logarithmic derivatives inside the critical strip

As in Proposition 3.1, we can show that for $0 < a < \frac{1}{4}$, if $L(s, \rho)$ is entire and is zero free in the rectangle $[1 - 2a, 1] \times [-(\log d_K)^\beta, (\log d_K)^\beta]$,

$$-\frac{L'}{L}(1 - a, \rho) = \sum_{p < (\log d_K)^\beta} \frac{a_\rho(p) \log p}{p^{1-a}} + O(1). \quad (4.1)$$

By using the fact that $\sum_{y < p < x} \frac{(\log p)^r}{p^{(1-a)r}} \ll \left(\frac{\log x}{y^{1-a}}\right)^{r-1}$, as in Proposition 3.3, we can show

PROPOSITION 4.1. *Let $n \leq 5$, and $x = (\log X)^\beta$ and $y = c_1 \log X$. Then for $r \leq c_2 \frac{\log X}{\log \log X}$ with a sufficiently small c_2 ,*

$$\sum_{L(s, \rho) \in L(X)} \left(\sum_{y < p < x} \frac{a_\rho(p) \log p}{p^{1-a}}\right)^{2r} \ll r (4(\beta + 1)(n - 1)^2 N_n^2)^r \frac{(2r)!}{r!} \left(\frac{\log x}{y^{1-a}}\right)^r X.$$

This implies

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PROPOSITION 4.2. *Let $n \leq 5$, and $x = (\log X)^\beta$, $y = c_1 \log X$ with $\beta > 1$ and $0 < c_1 < \frac{1-\delta}{\kappa}$. Let $0 < a < \frac{1}{4}$. Then except for $O\left(X^{1-c'}\right)$ L -functions in $L(X)$ for some constant $c' > 0$, L -functions in $L(X)$ satisfy*

$$\left| \sum_{y < p < x} \frac{a_\rho(p) \log p}{p^{1-a}} \right| \leq (\log X)^{\frac{3}{4}a}.$$

By applying Kowalski-Michel zero density theorem [7] to the family $L(X)$, we can show that every L -function in $L(X)$ outside $O(X^{1-\delta})$ set has the desired zero free region $[1-2a, 1] \times [-(\log d_K)^\beta, (\log d_K)^\beta]$ for sufficiently small a . Therefore, except for $O(X^{1-c'})$ L -functions in $L(X)$ for $c' > 0$, for sufficiently small a ,

$$-\frac{L'}{L}(1-a, \rho) \ll (\log d_K)^a.$$

Also from (4.1), we can see that there are infinitely many S_n -fields with

$$-\frac{L'}{L}(1-a, \rho) = A(\log d_K)^a + O\left((\log d_K)^{\frac{3}{4}a}\right),$$

and infinitely many S_n -fields with

$$-\frac{L'}{L}(1-a, \rho) = -B(n-1)(\log d_K)^a + O\left((\log d_K)^{\frac{3}{4}a}\right)$$

for some absolute constants $A, B > 0$.

Note that under GRH, we only obtain the bound ([5, p.115])

$$-\frac{L'}{L}(1-a, \rho) \ll (\log d_K)^{2a}.$$

REMARK 4.2. We would like to make corrections on [2]. They do not affect the results of the paper:

- 1) In Theorem 1.2 and 1.3, $\frac{\log X}{\log \log X}$ should be replaced by $c'_1 \frac{\log X}{\log \log X}$ for some $c'_1 > 0$. Or state them as in Theorem 1.2 of this paper.
- 2) In Proposition 4.2, $2r$ should be multiplied on the right side. Also the implied constant depends only on d .

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REFERENCES

- [1] CHO, P. J.—KIM, H. H.: *Logarithmic derivatives of Artin L-functions*, Compos. Math. **149** (2013), no. 4, 568–586.
- [2] CHO, P. J.—KIM, H. H.: *Extreme residues of Dedekind zeta functions*, Math. Proc. Cambridge Philos. Soc. **163** (2017), no. 2, 369–380.
- [3] CHO, P. J.—KIM, H. H.: *Moments of logarithmic derivatives of L-functions*, J. Number Theory **183** (2018), 40–61.
- [4] IHARA, Y.: *On the Euler-Kronecker constants of global fields and primes with small norms*, In: *Algebraic Geometry and Number Theory, Progress in Mathematics, Vol. 253*, Birkhäuser, Boston, MA, 2006, pp. 407–451.
- [5] IWANIEC, H.—KOWALSKI, E.: *Analytic Number Theory*. In: *Amer. Math. Soc. Colloq. Publ. Vol. 53*, American Mathematical Society, Providence, RI, 2004.
- [6] KLINGEN, N.: *Arithmetical Similarities*. In: *Prime decomposition and finite group theory*. In: *Oxford Mathematical Monographs*. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1998.
- [7] KOWALSKI, E.—MICHEL, P.: *Zeros of families of automorphic L-functions close to 1*, Pacific J. Math. **207** (2002), no. 2, 411–431.

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