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A CURIOSITY ABOUT $(-1)^{[e]} + (-1)^{[2e]} + \cdots + (-1)^{[Ne]}$

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ABSTRACT. Let α be an irrational real number; the behaviour of the sum $S_N(\alpha) := (-1)^{[\alpha]} + (-1)^{[2\alpha]} + \cdots + (-1)^{[N\alpha]}$ depends on the continued fraction expansion of $\alpha/2$. Since the continued fraction expansion of $\sqrt{2}/2$ has bounded partial quotients, $S_N(\sqrt{2}) = O\big(\log(N)\big)$ and this bound is best possible. The partial quotients of the continued fraction expansion of e grow slowly and thus $S_N(2e) = O\Big(\frac{\log(N)^2}{\log\log(N)^2}\Big)$, again best possible. The partial quotients of the continued fraction expansion of $e/2$ behave similarly as those of e. Surprisingly enough $S_N(e) = O\Big(\frac{\log(N)}{\log\log(N)}\Big)$.

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1. Introduction

Let α be an irrational real number; we are interested in the asymptotic behaiour of the sum

$$S_N(\alpha) := (-1)^{[\alpha]} + (-1)^{[2\alpha]} + \cdots + (-1)^{[N\alpha]}.$$

The origin of this question seems to go back to [12], where it is remarked that $S_N(\sqrt{2}) = O(\log N)$. More accurate estimates for $S_N(\sqrt{2})$ are available in [5] and were already implicit in [9], where the authors gave an unexpected explicit

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formula[1] for $S_N(\sqrt{2})$ in terms of the continued fraction expansion[2] $\sqrt{2} = [1; \overline{2}]$.

The behaiour of $S_N(\alpha)$ is closely related to the uniform distribution mod 1 of the sequence $(n\alpha/2)_{n\in\mathbb{N}}$. Indeed, $[n\alpha]$ is even if and only if the fractional part $\{n\alpha/2\}$ is in $[0, 1/2)$. Thus,

$$\begin{aligned} S_N(\alpha) &= \big|\{n = 1, \ldots, N \mid [n\alpha] \text{ even }\}\big| - \big|\{n = 1, \ldots, N \mid [n\alpha] \text{ odd }\}\big| \\ &= 2\big|\{n = 1, \ldots, N \mid \{n\alpha/2\} \in [0, 1/2)\}\big| - N \\ &= 2D_N(\alpha/2, 1/2). \end{aligned} \tag{1}$$

Here D_N is the local discrepancy:

$$D_N(\alpha, x) = \big|\{n = 1, \ldots, N \mid \{n\alpha\} \in [0, x)\}\big| - Nx$$

for $\alpha \in \mathbb{R}$ and $x \in [0, 1]$. A lazy way to bound $D_N(\alpha)$ is to put in the picture the global discrepancy[3]

$$\mathcal{D}_N(\alpha) := \sup_{0\le x<y\le 1} \Big|\big|\{n = 1, \ldots, N \mid \{n\alpha\} \in [x, y)\}\big| - N(y - x)\Big|.$$

Thus $|D_N(\alpha, 1/2)| \le \mathcal{D}_N(\alpha)$. For an irrational α, the sequence $(n\alpha)$ is uniform distribution mod 1 by a well known theorem attributed ([3], p. 21) independently to Bohl, Sierpiński and Weyl. This means that $\mathcal{D}_N(\alpha) = o(N)$. More precise estimates for $\mathcal{D}_N$ depend on the diophantine approximation properties of α. We recall that the irrationality exponent $\mu(\alpha)$ of an irrational $\alpha \in \mathbb{R}$ is the infimum (possibly $+\infty$) of the set of positive real numbers μ such that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for all p, $q \in \mathbb{Z}$ with $q > 0$ we have

$$\Big|\alpha - \frac{p}{q}\Big| > \frac{C_\varepsilon}{q^{\mu+\varepsilon}}.$$

It is well known that $\mu(\alpha) \ge 2$ with equality for almost all α. It is also well known that μ is invariant by integral Möbius transformations $\alpha \mapsto \frac{a\alpha+b}{c\alpha+d}$ $(a, b, c, d \in \mathbb{Z}, ad - bc \ne 0)$.

From[4] [3, Theorem 3.2, p. 123] $\mathcal{D}_N(\alpha) = O_\gamma(N^\gamma)$ for any $\gamma > 1 - \frac{1}{\mu(\alpha)-1}$. In particular, if $\mu(\alpha) = 2$ we have $\mathcal{D}_N(\alpha) = O_\gamma(N^\gamma)$ for any $\gamma > 0$. A more precise result holds for irrational numbers α whose continued fraction expansion has bounded partial quotients (and hence irrationality measure 2). In this case we have (see [3, Theorem 3.4, p. 125]) $\mathcal{D}_N(\alpha) = O(\log N)$.

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[1]which can be viewed as an equality between non absolutely convergent Fourier series.

[2]Here and below, $\overline{a_1, \ldots, a_k}$ means $a_1, \ldots, a_k, a_1, \ldots, a_k, \ldots$

[3]Note that some authors, as [3], divide by N in the definition of $\mathcal{D}_N$.

[4]The authors state this result in terms of the *type* of α which is equal to $\mu(\alpha) - 1$.

This last estimate cannot be improved. Indeed the global discrepancy D_N of every infinite sequence $(u_n)_n$ is[5] $\Omega(\log N)$ (see [13, Theorem 1, p. 45]).

Nevertheless, we can construct irrational α such that $|D_N(\alpha,1/2)|$ is as small as we wish. Our first result is:

Theorem 1.1. *Let $\delta\colon \mathbb{N}\to\mathbb{R}^+$ be a function which tends to infinity. Then there exists an irrational number α such that*

$$D_N(\alpha,1/2)=O\big(\delta(N)\big).$$

Equivalently, we can find an irrational α such that[6] $S_n(\alpha)=O\big(\delta(N)\big)$.

By [2, Theorem 8, p. 237], for any irrational α there exists a positive constant $A=A(\alpha)$ such that $|\sum_{n=1}^{N} f(n\alpha)|\ge AN$, where $f(t)=\{t\}-1/2$. By Theorem (1.1) we cannot replace in this statement $\{t\}-1/2$ with $(-1)^{[t]}$, even taking instead of N any function $\delta(N)$ which tends to infinity. See also [10] for a related question.

We then show that for some *classical number* the local discrepancy $D_N(\alpha,1/2)$ can be substantially smaller than $D_N(\alpha)$ and even $o(\log N)$.

Theorem 1.2.

$$\overline{\lim_{N\to+\infty}}\, D_N(e/2)\left(\frac{\log\log N}{\log N}\right)^2=\frac{1}{8} \tag{2}$$

and

$$\overline{\lim_{N\to+\infty}}\, |D_N(e/2,1/2)|\frac{\log\log N}{\log N}=\frac{3}{2}. \tag{3}$$

Let's come back to the sum in the title. The question of providing good bound for $S_N(e)$ goes back to H. Pépin [7], who, in the nice self-contained treatement of this matter [8], already get $S_N(e)=O\big((\log N)^2\big)$. Equations (1) and (2) show that $S_N(e)$ is smaller than what one would expect:

$$S_N(e)=(-1)^{[e]}+(-1)^{[2e]}+\cdots+(-1)^{[ne]}=O\big(\log(N)/\log\log(N)\big). \tag{4}$$

Note that

$$S_N(2e)=(-1)^{[2e]}+(-1)^{[4e]}+\cdots+(-1)^{[2Ne]}=O\Big(\big(\log(N)/\log\log(N)\big)^2\Big)$$

is best possible, by (1) and by (6) of Theorem 1.3 below.

[5]Here Ω is the Landau symbol: if f, g are two functions with $g>0$, then $f=\Omega(g)$, means $f\neq o(g)$.

[6]Note that for any irrational α we have $\overline{\lim}\,|S_n(\alpha)|=+\infty$ ([7, Theorem 1]).

Bounds for $S_N(\alpha)$ are useful to study the convergence of sums of the shape $\sum_n(-1)^{[n\alpha]}u_n$. Let $\Delta u_n = u_{n+1} - u_n$. By partial sommation (as in the solution to [6] proposed by R. Tauraso [14]) we see that such a sum converges if $S_N(\alpha)u_N \to 0$ and $\sum_N S_N(\alpha)\Delta u_N$ converges. By (4) both conditions are satisfied when $\alpha = e$ and $u_n = \frac{\log\log(n+1)}{\log(n+1)^2}$. To get more precise and general results, it might be suitable to make a second partial summation, since the arithmetic mean of $S_N(\alpha)$ behave more regularly.

The gain of the factor $\frac{\log\log N}{\log N}$ in (3) heavily depends on the particular structure of the continued expansion of $e/2$. Let us give a short explication. Both estimates (2) and (3) for the global and local discrepancy of $(ne/2)$ depend on the partial quotients $\{a_n\}_{n\geq 1}$ of the continued fraction expansion of $e/2$. This sequence is unbounded. But in the estimate (3) only the a_n with $n \not\equiv 2 \mod 3$ come in. The corresponding sequence is now bounded. This phenomenon does not occurr if we replace $e/2$ by e, as the following theorem shows.

Theorem 1.3.

$$\overline{\lim_{N\to+\infty}} D_N(e)\left(\frac{\log\log N}{\log N}\right)^2 = \frac{1}{4} \tag{5}$$

and

$$\overline{\lim_{N\to+\infty}} |D_N(e,1/2)|\left(\frac{\log\log N}{\log N}\right)^2 = \frac{1}{4}. \tag{6}$$

Relations (3) and (6) show that the order of growth of $\alpha \mapsto D_N(\alpha,1/2)$ is *not* invariant with respect to Möbius transformations, contrary to what happen for the global discrepancy.

Although our theorems are straightforward applications of known results ([1] and [11]), it seems that they deserve to be remarked.

2. Computations

Proof of Theorem 1.1. The proof is an easy application of [11, Example, p. 1497]. Let $f\colon \mathbb{N} \to \mathbb{N}$ be a function taking odd values and which increases to infinity sufficiently fast. We choose

$$\alpha = \alpha_f = [0; 1, 1, f(1), 1, 1, f(2), \ldots].$$

Let a_j and q_m be the partial quotients and the denominators of the convergents of α. For $N \in \mathbb{N}$ we define $m(N) \in \mathbb{N}$ by the property

$$q_{m(N)} \leq N < q_{m(N)} + 1.$$

Put

$$a_j^+ = \begin{cases} a_j & \text{if } q_{j-1} \text{ is even and } j \text{ is odd;} \\ 0, & \text{otherwise} \end{cases}$$

and

$$a_j^- = \begin{cases} a_j & \text{if } q_{j-1} \text{ and } j \text{ are even;} \\ 0, & \text{otherwise.} \end{cases}$$

Define the following sums:

$$S_m^+ = \frac{1}{4} \sum_{\substack{2|j\le m \\ 2\nmid q_j}} a_{j+1} = \frac{1}{4} \sum_{j=0}^{m} a_{j+1}^+, \qquad S_m^- = \frac{1}{4} \sum_{\substack{2\nmid j\le m \\ 2\nmid q_j}} a_{j+1} = \frac{1}{4} \sum_{j=0}^{m} a_{j+1}^-.$$

Then from [11, Example, p. 1497] we have (as in the deduction of Corollary 1.2 from Theorem 1.1 in [1]):

$$\overline{\lim_{N\to+\infty}} D_N(e/2,1/2)/S_{m(N)}^+ = -\underline{\lim}_{N\to+\infty} D_N(e/2,1/2)/S_{m(N)}^- = 1. \tag{7}$$

From the usual recursive definition of q_m we easily see that[7] q_{j-1} is even if and only if $j \equiv 0 \mod 3$. Thus

$$\{a_j^+\}_{j\ge1} = \{\overline{1,0,0}\}, \quad \{a_j^-\}_{j\ge1} = \{\overline{0,1,0}\}$$

and

$$S_m^+ \sim S_m^- \sim \frac{1}{4} \sum_{k=1}^{[m/3]} 1 \sim \frac{m}{12}. \tag{8}$$

Moreover, from the recursive definition of q_m we have

$$q_m \ge \prod_{j=1}^{[m/3]} f(j).$$

Thus, if f grows sufficiently fast, for $N \in \mathbb{N}$ we have $q_{12[\delta(N)]} \ge N$ and, by definition, $m(N) \le [12\delta(N)]$. By (7) and (8) we have $D_N(\alpha,1/2) = O\big(\delta(N)\big)$ as desired. □

[7] To check this property we can of course reduce modulo 2 all the partial coefficients, thus reduce ourselves to compute the well-known convergents of the golden ratio.

Proof of of Theorem 1.2. To prove (2) we follow the proof of [1, Theorem 3.2(2), p. 286] taking now (cf. (9)) $a_1, a_2, \dots$ be the partial quotients of the continued fraction expansion[8] of $e/2$.

$$e/2 = [1; 2, 1, 3, 1, 1, 1, 3, 3, 3, 1, 3, 1, 3, 5, 3, 1, 5, 1, 3, 7, 3, 1, 7, 1, 3, \dots] . \qquad (9)$$

We easily find

$$\sum_{i=0}^{m} a_{i+1} \sim 2 \sum_{k=1}^{[m/6]} (2k-1) \sim \frac{1}{18} m^2$$

and

$$\sum_{i=0}^{m} \log a_{i+1} \sim 2 \sum_{k=1}^{[m/6]} \log(2k-1) \sim \frac{1}{3} m \log m.$$

Thus

$$\overline{\lim_{N \to +\infty}} D_N(e/2) \left(\frac{\log \log N}{\log N} \right)^2 = \frac{\frac{1}{18}}{4(\frac{1}{3})^2} = \frac{1}{8}.$$

To prove (3) we apply again the formula in [11, Example, p. 1497]. Let a_j and q_m be the partial quotients and the denominators of the convergents of (9). Let $m(N)$, $a_j^{\pm}$ and $S_m^{\pm}$ be as in the the proof of Theorem 1.1. Then

$$\overline{\lim_{N \to +\infty}} D_N(e/2, 1/2)/S^+_{m(N)} = - \underline{\lim_{N \to +\infty}} D_N(e/2, 1/2)/S^-_{m(N)} = 1. \qquad (10)$$

From (9) and from the usual recursive definition of q_m we see (cf. note[7]) that q_{j-1} is even iff $j \equiv 2 \mod 3$. Thus

$$\{a_j^+\}_{j \ge 1} = \{2, \overline{0, 3, 0, 0, 0, 3}\}, \qquad \{a_j^-\}_{j \ge 1} = \{2, \overline{0, 0, 1, 0, 1, 0}\}$$

and

$$S_m^+ \sim \frac{1}{4} \sum_{k=1}^{[m/6]} (3+3) \sim \frac{1}{4} m, \qquad S_m^- \sim \frac{1}{4} \sum_{k=1}^{[m/6]} (1+1) \sim \frac{1}{12} m.$$

Moreover (cf. (9))

$$\log q_m \sim \sum_{i=1}^{m} \log a_i \sim 2 \sum_{k=1}^{[m/6]} \log(2k-1) \sim 2 \frac{m}{6} \log m = \frac{1}{3} m \log m$$

which, by definition of $m(N)$, easily implies $m(N) \sim 3 \frac{\log N}{\log \log N}$. Replacing these estimates in (12) we get

$$\overline{\lim_{N \to +\infty}} D_N(e/2, 1/2)/(\tfrac{3}{4} \tfrac{\log N}{\log \log N}) = - \underline{\lim_{N \to +\infty}} D_N(e/2, 1/2)/(\tfrac{3}{12} \tfrac{\log N}{\log \log N}) = 1.$$

Equation (3) follows. □

[8] which can be easily computed from the well-known Euler continued fraction of e, for example, by known algorithms [4].

Proof of of Theorem 1.3. Equation (5) is a special case of [1, Theorem 3.2(2), p. 286]. The deduction of (6) follows the same lines as that of (3). Let a_j and q_m be the partial quotients and the denominators of the convergents of the continued fraction expansion of e

$$e=[2;1,2,1,1,4,1,1,6,1,1,8,\dots]. \tag{11}$$

Let $m(N)$, $a_j^{\pm}$ and $S_m^{\pm}$ as in the the proof of Theorem 1.1. From [11, Example, p. 1497]:

$$\overline{\lim_{N\to+\infty}}\, D_N(e,1/2)/S^+_{m(N)} = -\underline{\lim}_{N\to+\infty}\, D_N(e,1/2)/S^-_{m(N)}=1. \tag{12}$$

From (11) and from the usual recursive definition of q_m we easily see that q_{j-1} is even iff $j\equiv 0,4 \mod 6$. Thus (cf. (11))

$$\{a_j^+\}_{j\ge1}=\{1,0,1,0,4,0,\ 1,0,1,0,8,0,\ 1,0,1,0,12,0,\dots\};$$

$$\{a_j^-\}_{j\ge1}=\{0,2,0,0,0,0,\ 0,6,0,0,0,0,\ 0,10,0,0,0,0,\dots\}$$

and

$$S_m^+\sim\frac14\sum_{k=1}^{[m/6]}(1+1+4k)\sim\frac{1}{72}m^2,\qquad S_m^-\sim\frac14\sum_{k=1}^{[m/6]}(4k-2)\sim\frac{1}{72}m^2.$$

Moreover (cf. again (11))

$$\log q_m\sim\sum_{i=1}^{m}\log a_i\ \sim\sum_{k=1}^{[m/3]}\log(2k)\sim\frac13 m\log m$$

which implies $m(N)\sim 3\frac{\log N}{\log\log N}$. Replacing these estimates in (12) we get

$$\overline{\lim_{N\to+\infty}}\, D_N(e,1/2)/\big(\tfrac18(\tfrac{\log N}{\log\log N})^2\big) = -\underline{\lim}_{N\to+\infty}\, D_N(e,1/2)/\big(\tfrac18(\tfrac{\log N}{\log\log N})^2\big)=1.$$

Equation (6) follows. □

REFERENCES

[1] BAXA, C.—SCHOISSENGEIER, J.: *Minimum and maximum order of magnitude of the discrepancy of* $(n\alpha)$, Acta Arith. **67** (1994), no. 3, 281–290.

[2] HARDY, G. H.—LITTLEWOOD, J. E.: *The lattice-points of a right-angled triangle. (Second memoir)*, Abh. Math. Semin. Univ. Hamburg **1** (1922), no. 1, 211–248.

[3] KUIPERS, L.—NIEDERREITER, H.: *Uniform Distribution of Sequences*. Dover Books on Mathematics (Paperback) May 12, 2006.

[4] LIARDET, P.—STAMBUL, P.: *Algebraic computations with continued fractions*, J. Number Theory **73** (1988), no. 1, 92–121.

[5] O'BRYANT, K.—REZNICK, B.—SERBINOWSKA, M.: *Almost Alternating Sums*, Amer. Math. Monthly **113** (2006), no. 8, 673–688.
[6] OMARJEE, M.: *Problem 11384*, Amer. Math. Monthly **115** (2008), no. 8, 757.
[7] PÉPIN, H.: *Question et Réponses. Q86*, Revue de mathématiques spéciales, **Octobre 1987**, 116.
[8] PÉPIN, H.: *Question et Réponses. R86*, Revue de mathématiques spéciales, **Janvier 1989**, 229–232.
[9] PERELLI, A.—ZANNIER, U.: *On the parity of* $[n\sqrt{2}]$. Boll. Un. Mat. Ital. A (6), **2** (1983), no. 1, 77–83.
[10] PERELLI, A.—ZANNIER, U.: textitAn Ω result in uniform distribution theory, Quart. J. Math. Oxford Ser. (2) **38** (1987), no. 149, 95–102.
[11] ROÇADAS L.—SCHOISSENGEIER J.: *On the local discrepancy of* $(n\alpha)$*-sequences*, J. Number Theory **131** (2011), no. 8, 1492–1497.
[12] RUDERMAN, H. D.—BORWEIN, D.: *Problems and Solutions: Solutions of Advanced Problems: 6105*, Amer. Math. Monthly **85** (1978), no. 3, 207–208. https://www.jstor.org/stable/2321079?seq=1#metadata_info_tab_contents
[13] SCHMIDT, W. M.: *Irregularities of distribution. VII*, Acta Arith. **21** (1972) 45–50.
[14] TAURASO, R.: *Solution to problem 11384* Amer. Math. Monthly, **115**, October 2008. https://www.mat.uniroma2.it/~tauraso/AMM/AMM11384.pdf

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