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A CLASS OF LITTLEWOOD POLYNOMIALS THAT ARE NOT L^α-FLAT

(WITH AN APPENDIX JOINTLY WITH M. G. NADKARNI[2])

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ABSTRACT. We exhibit a class of Littlewood polynomials that are not L^α-flat for any $\alpha \geq 0$. Indeed, it is shown that the sequence of Littlewood polynomials is not L^α-flat, $\alpha \geq 0$, when the frequency of -1 is not in the interval $\left]\frac{1}{4}, \frac{3}{4}\right[$. We further obtain a generalization of Jensen-Jensen-Hoholdt's result by establishing that the sequence of Littlewood polynomials is not L^α-flat for any $\alpha > 2$ if the frequency of -1 is not $\frac{1}{2}$. Finally, we prove that the sequence of palindromic Littlewood polynomials with even degrees are not L^α-flat for any $\alpha \geq 0$, and we provide a lemma on the existence of c-flat polynomials.

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1. Introduction

The main goal of this paper is to establish that some class of Littlewood polynomials are not L^α-flat, $\alpha \geq 0$. Precisely, we prove that if the sequence of Littlewood polynomials (P_q) is palindromic with even degrees or if the frequency of -1, which occurs as coefficients of those polynomials in (P_q), is not in the interval $\left]\frac{1}{4}, \frac{3}{4}\right[$, then (P_q) are not L^α-flat for any $\alpha \geq 0$.

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We further establish that the sequence of Littlewood polynomials cannot be L^α-flat for $\alpha > 2$ if the frequency of -1 is not $\frac{1}{2}$. This is a strengthening of Theorem 2.1 of [16].

It follows that the search for a sequence of L^α-flat polynomials from the class $\mathcal{L}$ of Littlewood polynomials can be restricted to the subclass of polynomials $P \in \mathcal{L}$ which are not palindromic with even degrees and for which the frequency of -1 is in the interval $\left]\frac{1}{4}, \frac{3}{4}\right[$.

The problem of flat polynomials goes back to Erdős [11], [12] and Newman [24]. Later, Littlewood asked, in his famous paper [20], among several questions, if there are positive absolute constants A and B such that, for arbitrarily large n, one can find a sequence

$$\boldsymbol{\epsilon} = (\epsilon_j)_{j=0}^{n-1} \in \{-1, 1\}^n$$

such that

$$A\sqrt{n} \leq \left|\sum_{j=0}^{n-1} \epsilon_j z^j\right| \leq B\sqrt{n}, \quad \forall\ z \in S^1, \quad \text{where } S^1 \text{ denotes the circle group.}$$

The polynomials of type

$$L_n(z) \stackrel{\text{def}}{=} \sum_{j=0}^{n-1} \epsilon_j z^j$$

are called nowadays Littlewood polynomials or polynomials from the class $\mathcal{L}$. In the modern terminology, Littlewood's question can be reformulated as follows.

Question 1. [[20, Littlewood, 1966] and [21, Problem 19]] Does there exist a sequence of polynomials from the class $\mathcal{L}$ which is flat in the Littlewood sense?

The other, more general question, whether there exists a sequence of trigonometric polynomials

$$K_j(z) = \frac{1}{\sqrt{q_j}}(a_{0,\,j} + a_{1,\,j}z + a_{2,\,j}z^2 + \cdots + a_{q_j-1,\,j}z^{q_j-1}), \tag{2}$$

$$j = 1, 2, \ldots, |a_{k,\,j}| = 1, \quad 0 \leq k \leq q_j - 1,$$

such that

$$\left|K_j(z)\right| \to 1 \quad \text{uniformly as} \quad j \to \infty,$$

was answered affirmatively by J-P. Kahane [19]. Furthermore, Jósef Beck [5] has shown, by applying the random procedure of Kahane, that the sequence

$$K_j, j = 1, 2, \ldots,$$

can be chosen to be flat in the sense of Littlewood with coefficients of

$$\sqrt{q_j}K_j, j = 1, 2, \ldots,$$

chosen from the solutions of $z^{400} = 1$.

Nowadays, the polynomials $K_j, j = 1, \dots,$ such that $\big|K_j(z)\big| \to 1$ uniformly as $j \to \infty$, are called ultraflat polynomials, and the class of polynomials of type (2) is denoted by $\mathcal{G}$.

For more details on the ultraflat polynomials of Kahane, we refer the reader to [27]. Let us also mention that very recently, Bombieri and Bourgain [6] constructed an effective sequence of ultraflat polynomials.

Littlewood's question is also related to the well-know merit factor problem and Turyn-Golay's conjecture [17], arising from digital communications engineering, which states that the merit factor of any binary sequence is bounded.

We remind that the merit factor of a binary sequence

$$\boldsymbol{\epsilon} = (\epsilon_j)_{j=0}^{n-1} \in \{-1, 1\}^n$$

is given by

$$F_n(\boldsymbol{\epsilon}) = \frac{1}{\Big\|P_n\Big\|_4^4 - 1},$$

where

$$P_n(z) = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \epsilon_j z^j, \quad z \in S^1.$$

For a nice account on the merit factor problem, we refer the reader to [7], [18], [14], and for the connection to ergodic theory and spectral theory of dynamical systems to [9].

The problem of flat polynomials has nowadays a long history and there is a large literature on the subject. Moreover, this problem is related to some open problems coming from combinatorics, number theory, digital communication, theory of error codes, complex analysis, spectral theory, ergodic theory and other areas.

To the best of the author's knowledge, the only general result known on flatness in the class $\mathcal{L}$ is due to Saffari and Smith [28]. Unfortunately, the authors in [29] point out that their proof contains a mistake. Therefore, the problem remains open. However, therein, the authors proved that for the palindromic sequence of polynomials from the class $\mathcal{L}$ the L^4 conjecture of Erdős holds (see below). We shall strengthen their result by proving that the palindromic polynomials with even degrees from the class $\mathcal{L}$ are not L^α flat, for $\alpha \geq 0$. This is done by applying Littlewood's criterion [22].

We further exhibit a subclass of Littlewood polynomials which are not L^α--flat, $\alpha > 0$ by establishing one-to-one correspondance between the Littlewood polynomials and the Newman-Bourgain polynomials given by

$$Q_q(z) = \frac{1}{\sqrt{q}} \sum_{j=0}^{q-1} \eta_j z^j, \quad z \in S^1, \text{ where for each } \quad j = 0, \ldots, q-1, \quad \eta_j \in \{0,1\}.$$

Therefore, our main results can be seen as a general results since it reduces the problem of finding flat polynomials in the class $\mathcal{L}$ to a subclass of $\mathcal{L}$. Furthermore, it supports the conjecture mentioned by D. J. Newman in [24] which says that all the analytic trigonometric polynomials P with coefficients ± 1 satisfy

$$\big\|P\big\|_1 < c\big\|P\big\|_2,$$

for some positive constant $c < 1$. Obviously, this conjecture implies the two conjectures of Erdős [10],[11, Problem 22], [12] which states that there is a positive constant d such that for any polynomial P from $\mathcal{L}$ we have

(1) $\|P\|_4 \geq (1+d)\|P\|_2$. ($L^4$ conjecture of Erdős.)

(2) $\|P\|_\infty \geq (1+d)\|P\|_2$. (Ultraflat conjecture of Erdős.)

However, the author in [1] proved that the class of Newman-Bourgain polynomials contain a sequence of L^α-flat polynomials, $0 < \alpha < 2$. This is accomplished by appealing to Singer's construction of the Sidon sets. We refer to [1] for more details.

During the time when this article was under review, P. Balister and al. posted a paper on arXiv.org [4] in which they stated that Littlewood question (Question 1.) has an affirmative answer. Indeed, using Rudin-Shapiro polynomials combined with Spencer's six deviations lemma, the authors constructed a flat polynomials in the Littlewood sense [4]. However, it is easy to see that those polynomials are not L^α-flat, for any $\alpha \geq 0$. [1]

This paper is organized as follows. In Section 2, we give a brief exposition of some basic tools and we state our main results. In Section 3, we prove our first main result in the case that the frequency of -1 is not in $\left[\frac{1}{4}, \frac{3}{4}\right]$. In Section 4, we prove our second main result. Finally, in the appendix, we complete the proof of our first main result.

[1] This follows directly from Theorem 2.4 and Lemma 3.3 in [4]. Indeed, by (2) in Theorem 4.3 and Lemma 3.3 in [4], we get that those polynomials are not almost everywhere flat. For this later notion, we refer to the section on flat polynomials, and for its applications in ergodic theory and dynamical systems, we refer to [2].

2. Basic definitions and tools

Let $\mathrm{d}z$ be the normalized Lebesgue measure on the circle group S^1. As customary, for $f \in L^1(S^1, \mathrm{d}z)$, we define its nth Fourier coefficient by

$$\widehat{f}(n) = \int_{S^1} f(z) z^{-n}\, \mathrm{d}z.$$

A polynomial $f(z) = \sum_{j=0}^{n} a_j z^j$ is palindromic if for any $0 \le k \le n$,

$$\widehat{f}(k) = \widehat{f}(n-k).$$

The L^2-normalized Littlewood polynomials are given by

$$P_q(z) = \frac{1}{\sqrt{q}} \sum_{j=0}^{q-1} \epsilon_j z^j, \quad z \in S^1, \tag{2.1}$$

where for each $j = 0, \ldots, q-1$, $\epsilon_j \in \{+1, -1\}$.

Notice that each sequence $\epsilon = \big(\epsilon_j\big)_{j=0}^{+\infty} \in \{+1,-1\}^{\mathbb{N}}$ can be uniquely associated to a sequence $\eta = \big(\eta_j\big)_{j=0}^{+\infty} \in \{0,1\}^{\mathbb{N}}$ by putting

$$\epsilon_j = 2\eta_j - 1,$$

or

$$\epsilon_j = 1 - 2\eta'_j, \quad \text{with} \quad \eta' = \mathbf{1} - \eta, \quad \mathbf{1} = (1,1,1,\ldots).$$

The previous remark will play a crucial role in our proof. Indeed, if (P_q) is a sequence of L^2-normalized Littlewood polynomials, then

$$\begin{aligned} P_q(z) &= \frac{1}{\sqrt{q}} \sum_{j=0}^{q-1} \epsilon_j z^j \\ &= \frac{2}{\sqrt{q}} \sum_{j=0}^{q-1} \eta_j z^j - \frac{1}{\sqrt{q}} \sum_{j=0}^{q-1} z^j, \qquad (2.2) \\ &= \frac{1}{\sqrt{q}} \sum_{j=0}^{q-1} z^j - \frac{2}{\sqrt{q}} \sum_{j=0}^{q-1} \eta'_j z^j, \quad z \in S^1. \qquad (2.3) \end{aligned}$$

We put

$$Q_q(z) = \frac{1}{\sqrt{q}} \sum_{j=0}^{q-1} \eta_j z^j \quad \text{and} \quad R_q(z) = \frac{1}{\sqrt{q}} \sum_{j=0}^{q-1} \eta'_j z^j, \quad z \in S^1.$$

For a given sequence of L^2-normalized Littlewood polynomials (P_q) (see (2.1)), it is easy to see that for each $q \in \mathbb{N}$ and $j \in \mathbb{N}$, $\widehat{\sqrt{q}P_q}(j)$ is the jth coefficient of the polynomial $\sqrt{q}P_q$. We may further assume without loss of generality in the sequel that the following limit exists

$$\lim_{q \longrightarrow +\infty} \frac{\#\Big\{j \ : \ \widehat{\sqrt{q}P_q}(j) = -1\Big\}}{q} = \mathrm{fr}\,(-1)$$

where $\#E$ denote the cardinality of a set E. $\mathrm{fr}(-1)$ is the frequency of -1 which is also the frequency of 0 for the sequence of polynomials (Q_q). Note that the frequency of 1 are the same for the both sequences of polynomials (P_q) and (Q_q).

Flat polynomials.

For any $\alpha > 0$ or $\alpha = +\infty$, the sequence $\big(P_n(z)\big)$ of analytic trigonometric polynomials of $L^2(S^1, \mathrm{d}z)$ norm 1 is said to be L^α-flat if the sequence $\big(|P_n(z)|\big)$ converges in L^α-norm to the constant function 1 as $n \longrightarrow +\infty$. For $\alpha = 0$, we say that (P_n) is L^α-flat, if the sequence of the Mahler measures $\big(M(P_n)\big)$ converges to 1. We recall that the Mahler measure of a function $f \in L^1(S^1, \mathrm{d}z)$ is defined by

$$M(f) = \|f\|_0 = \lim_{\beta \longrightarrow 0} \|f\|_\beta = \exp\left(\int_{S^1} \log(|f(t)|)\,\mathrm{d}t\right).$$

The sequence $\big(P_n(z)\big)$ is said to be flat in a.e. sense (almost everywhere sense) if the sequence $\big(|P_n(z)|\big)$, converges a.e. to 1 with respect to $\mathrm{d}z$ as $n \longrightarrow +\infty$.

We further say that a sequence $\big(P_n\big)$ of L^2-normalized polynomials from the class $\mathcal{L}$ (or $\mathcal{G}$) is flat in the sense of Littlewood if there exist constants $0 < A < B$ such that for all $z \in S^1$ and for all $n \in \mathbb{N}$ (or at least for sufficiently large $n \in \mathbb{N}$), we have

$$A \leq \big|P_n(z)\big| \leq B.$$

The previous notion of flatness can be extended as follows.

Let $c \in [0,1]$. The sequence $\big(P_n(z)\big)$ of analytic trigonometric polynomials of $L^2(S^1, \mathrm{d}z)$ norm 1 is said to be L^α-c-flat if the sequence $\big(|P_n(z)|\big)$ converges in L^α-norm to the constant function c as $n \longrightarrow +\infty$. The sequence $\big(P_n(z)\big)$ is said to be c-flat in a.e. sense if the sequence $\big(|P_n(z)|\big)$ converges a.e. to c with respect to $\mathrm{d}z$ as $n \longrightarrow +\infty$. Obviously, 0-flat polynomials in a.e. sense and L^α-0-flat polynomials, for any $\alpha \in]0,2[$, exists (see Lemma 3.4).

A formula between Littlewood and Newman-Bourgain Polynomials.

We further assume without loss of generalities that the first and last coefficient of P_q are positive in our definition. This makes the correspondence T defined below one-to-one. Let $\mathcal{NB}$ denote the class of Newman-Bourgain polynomials, i.e., polynomials Q of type

$$\frac{1}{\sqrt{m}}\big(\eta_0+\eta_1 z+\cdots+\eta_{q-2}z^{q-2}+\eta_{q-1}z^{q-1}\big),$$

where

$$\eta_0=\eta_{q-1}=1,\quad \eta_i=0 \text{ or } 1,\quad 1\leq i\leq q-2,\quad \text{and}\quad m=\sum_{i=0}^{q}\eta_i,$$

which is also the number of i with $\eta_i=1$. Note that if P is as in (2.1) and if we put

$$\eta_i=\frac{1}{2}(\epsilon_i+1), 0\leq i\leq q-1,$$

then the polynomial

$$\frac{1}{\sqrt{m}}\big(\eta_0+\eta_1 z+\cdots+\eta_{q-2}z^{q-2}+\eta_{q-1}z^{q-1}\big)$$

is in the class $\mathcal{NB}$, where m is the number of $\eta_i=1$ which is also the number of $\epsilon_i=1$. Let us define one-to-one invertible map T from the class $\mathcal{L}$ to the class $\mathcal{NB}$ by

$$\begin{aligned}\big(T(P)\big)(z) &= T\left(\frac{1}{\sqrt{q}}\Big(\epsilon_0+\epsilon_1 z+\cdots+\epsilon_{q-2}z^{q-2}+\epsilon_{q-1}z^{q-1}\Big)\right)\\ &= \frac{1}{\sqrt{m}}\Big(\eta_0+\eta_1 z+\cdots+\eta_{q-2}z^{q-2}+\eta_{q-1}z^{q-1}\Big),\end{aligned}$$

where $\eta_i=\frac{1}{2}(\epsilon_i+1), 0\leq i\leq q-1$, and m is the number of $\eta_i=1$ which is also the number of $\epsilon_i=1$.

Note

$$T^{-1}\left(\frac{1}{\sqrt{m}}\left(\sum_{i=0}^{q-1}\eta_i z^i\right)\right)=\frac{1}{\sqrt{q}}\left(\sum_{i=0}^{q-1}(2\eta_i-1)z^i\right).$$

Let

$$D(z)=D_q(z)=\frac{1}{\sqrt{q}}\sum_{i=0}^{q-1}z^i.$$

Thus we have that $D(1)=\sqrt{q}$, while for $z\in S^1\setminus\{1\}$,

$$D(z)=\frac{1}{\sqrt{q}}\frac{1-z^q}{1-z}\to 0\quad\text{as}\quad q\to\infty.$$

The formula for polynomials in $\mathcal{L}$ mentioned above is as follows: If P is as in (2.1), then

$$\begin{aligned} P_q(z) &= 2\frac{\sqrt{m}}{\sqrt{q}}\big(T(P_q)\big)(z) - D(z) \\ &= 2\frac{1}{\sqrt{q}}A_q(z) - D_q(z), \end{aligned} \tag{2.4}$$

where m is the number of terms in P with coefficient $+1$, $A(z) = \sqrt{m}\, T(P)(z)$. The proof follows as soon as we write $T(P)(z)$ and $D(z)$ in the right hand side in full form and collect the coefficient of z^i, $0 \leq i \leq q-1$.

Further, we define the one-to-one map S from $\mathcal{L}$ onto $\mathcal{L}$ by

$$S(P) = \frac{1}{\sqrt{q}}\left(\sum_{j=0}^{q-1}(-\epsilon_j)z^j\right),$$

i.e., the polynomial obtained from P by changing the signs of ϵ_j, $j = 0, \dots, q-1$.

Note that the polynomial D and the polynomials in $\mathcal{L}$ and $\mathcal{NB}$ all have $L^2(S^1, \mathrm{d}z)$ norm 1.

Further, we have that if $\big(P_n(z)\big)$ is a.e. flat, then $\big(S(P_n)\big)$ is also a.e. flat.

It is also obvious that the flatness properties are invariant under S. Let us notice that it is a nice exercise to see that the L^4 conjecture and the ultraflat conjecture of Erdős holds in the class of Newman-Bourgain polynomials.

We are now able to state our main results.

Theorem 2.1. *Let (P_q) be a sequence of Littlewood polynomials. Suppose that the frequency of -1 is not in the interval $\left]\frac{1}{4}, \frac{3}{4}\right[$, then (P_q) is not L^α-flat for any $\alpha \geq 0$.*

If we restrict our self to the L^α space with $\alpha > 2$, then we have the following much stronger result

Theorem 2.2. *Let (P_q) be a sequence of Littlewood polynomials. Suppose that the frequency of -1 is not $\frac{1}{2}$. Then, the polynomials (P_q) is not L^α-flat for any $\alpha > 2$. Furthermore,*

$$\lim_{q \longrightarrow +\infty} \Big\| P_q \Big\|_\alpha = +\infty.$$

We state our second main result as follows.

Theorem 2.3. *Let (P_q) be a sequence of Littlewood polynomials. Suppose that each P_q is palindromic with even degree. Then (P_q) is not L^α-flat for any $\alpha \geq 0$.*

3. Proof of Theorem 2.1 when the frequency of -1 is not in $\left[\frac{1}{4}, \frac{3}{4}\right]$.

We start by stating a criterion on the connection between the L^1-flatness and L^α-flatness, for $\alpha > 0$.

Proposition 3.1. *Let $\alpha > 0$ and $\big(P_q(z)\big)_{q\geq 0}$ be a sequence of L^2-normalized polynomials and assume that*

$$\big(P_q(z)\big)_{q\geq 0} \quad \textit{is} \quad L^\alpha\textit{-flat.}$$

Then, there exists a subsequence $\big(P_{q_n}(z)\big)$ which is a.e. flat and L^1-flat. Conversely, assume that

$$\big(P_q(z)\big)_{q\geq 0} \quad \textit{is} \quad L^1\textit{-flat,}$$

then there exists a subsequence $\big(P_{q_n}(z)\big)$ which is a.e. flat and L^α-flat, for each $0 < \alpha < 2$.

For the proof of Proposition 3.1 we need the following tool that is quite useful for proving convergence in L^p when the almost everywhere convergence holds without domination. This tool is based on the notion of uniform integrability. We recall that the sequence $(f_n)_{n\in\mathbb{N}}$ of integrable functions is said to be uniformly integrable if and only if

$$\int_{\{|f_n|>M\}} \big|f_n\big|(x)\,\mathrm{d}\mu(x) \xrightarrow[M\to+\infty]{} 0, \qquad \text{uniformly in } n \in \mathbb{N}.$$

For the other definitions of uniform integrability, we refer to [30, Theorem 16.8]. We further notice that the condition

$$\sup_{n\in\mathbb{N}}\left(\int\left(\big|f_n\big|^{1+\varepsilon}\right)\right) < +\infty,$$

for some ε positive, implies that $\big(f_n\big)$ is uniformly integrable.

Lemma 3.2 (Vitali's convergence theorem)**.** *Let $(X, \mathcal{B}, \mu)$ be a probability space, p a positive number and $\big(f_n\big)$ a sequence in $L^p(X)$ which converges in probability to f. Then, the following are equivalent:*

(i) $(|f_n|^p)_{n\geq 0}$ *is uniformly integrable;*

(ii) $\Big|\Big|f_n - f\Big|\Big|_p \xrightarrow[n\to+\infty]{} 0.$

(iii) $\int_X |f_n|^p\,\mathrm{d}\mu \xrightarrow[n\to+\infty]{} \int_X |f|^p\,\mathrm{d}\mu.$

P r o o f. The classical proof is given for the case $p \geq 1$ (see, for instance [30, p. 165–167]). But it is easy to see that the same arguments are valid by noticing that $L^p(X, \mathcal{B}, \mu)$ equipped with

$$d_p(f,g) = \int |f(x) - g(x)|^p \, \mathrm{d}\mu(x), \ \ f, g \in L^p(X, \mathcal{B}, \mu),$$

is a complete metric space.

Indeed, (ii) $\Longrightarrow$ (iii), since $\Big|\Big|f_n - f\Big|\Big|_p \xrightarrow[n\to+\infty]{} 0$ implies $d_p(f_n, f) \xrightarrow[n\to+\infty]{} 0$, and by the triangle inequality, we have

$$\Big|d_p(f_n, 0) - d_p(0, f)\Big| \leq d_p(f_n, f) \xrightarrow[n\to+\infty]{} 0.$$

(i)$\Longrightarrow$(ii): We start by claiming that $\left(\left|f_n - f_m\right|^p\right)_{n,m\in\mathbb{N}}$ is uniformly integrable. Indeed, let $M > 0$, then

$$\begin{aligned}\int_{\left\{\left|f_n - f_m\right| > 2M\right\}} \left|f_n - f_m\right|^p \mathrm{d}\mu &\leq 2^p \int_{\left\{\left|f_n - f_m\right| > 2M\right\}} \max\{|f_n|, |f_m|\}^p \, \mathrm{d}\mu \\ &\leq \int_{\left\{\left|f_n\right| > M\right\} \cup \left\{\left|f_m\right| > M\right\}} \max\{|f_n|, |f_m|\}^p \, \mathrm{d}\mu,\end{aligned}$$

since for any

$$a, b \in \mathbb{R}, \quad |a - b| \leq 2 \max\{|a|, |b|\}$$

and

$$\left\{\left|f_n - f_m\right| > 2M\right\} \subset \left\{\left|f_n\right| > M\right\} \cup \left\{\left|f_m\right| > M\right\}.$$

Consequently,

$$\begin{aligned}&\int_{\left\{\left|f_n - f_m\right| > 2M\right\}} \left|f_n - f_m\right|^p \mathrm{d}\mu \\ &\leq 2^p \Bigg(\int_{\left\{\left|f_n\right| > M\right\} \cap \left\{\left|f_m\right| > M\right\}} \max\{|f_n|, |f_m|\}^p \, \mathrm{d}\mu \\ &\quad + \int_{\left\{|f_n| > M > |f_m|\right\}} \max\{|f_n|, |f_m|\}^p \, \mathrm{d}\mu + \int_{\left\{|f_m| > M > |f_n|\right\}\}} \max\{|f_n|, |f_m|\}^p \, \mathrm{d}\mu \Bigg) \\ &\leq 2^p \Bigg(\int_{\left\{\left|f_n\right| > M\right\} \cap \left\{\left|f_m\right|\right| > M\}} |f_n|^p \, \mathrm{d}\mu + \int_{\left\{\left|f_n\right| > M\right\} \cap \left\{\left|f_m\right| > M\right\}} |f_m|^p \, \mathrm{d}\mu \Bigg) \\ &\quad + \Bigg(2^p \int_{\left\{|f_n| > M\right\}} |f_n|^p \, \mathrm{d}\mu + \int_{\left\{\left|f_m\right| > M\right\}} |f_m|^p \, \mathrm{d}\mu \Bigg)\end{aligned}$$

Whence

$$\sup_{n,m\in\mathbb{N}}\int_{\left\{\left|f_n-f_m\right|>2M\right\}}\left|f_n-f_m\right|^p\,\mathrm{d}\mu$$

$$\leq 2^{p+1}\left(\sup_{n\in\mathbb{N}}\int_{\left\{\left|f_n\right|>M\right\}}|f_n|^p\,\mathrm{d}\mu+\sup_{m\in\mathbb{N}}\int_{\left\{\left|f_m\right|>M\right\}}|f_m|^p\,\mathrm{d}\mu\right)$$

Letting $M\longrightarrow+\infty$, we get

$$\sup_{n,m\in\mathbb{N}}\int_{\left\{\left|f_n-f_m\right|>2M\right\}}\left|f_n-f_m\right|^p\,\mathrm{d}\mu\longrightarrow 0,$$

and the proof of the claim is complete.
Now, let $\varepsilon>0$ and M sufficiently large such that

$$\int_{\left\{\left|f_n-f_m\right|>2M\right\}}\left|f_n-f_m\right|^p\,\mathrm{d}\mu<\frac{\varepsilon}{3},$$

and

$$\mu\left\{|f_n-f_m|>\left(\tfrac{\varepsilon}{3}\right)^{\frac{1}{p}}\right\}<\frac{\varepsilon}{3.2^pM^p}.$$

Write

$$\int\left|f_n-f_m\right|^p\,\mathrm{d}\mu$$

$$=\int_{\left\{\left|f_n-f_m\right|>2M\right\}}\left|f_n-f_m\right|^p\,\mathrm{d}\mu$$

$$+\int_{\left\{\left|f_n-f_m\right|<2M\right\}}\left|f_n-f_m\right|^p\,\mathrm{d}\mu$$

$$\leq\frac{\varepsilon}{3}+\int_{\left\{\left|f_n-f_m\right|<2M\right\}\cap\left\{\left|f_n-f_m\right|\leq\left(\frac{\varepsilon}{3}\right)^{\frac{1}{p}}\right\}}\left|f_n-f_m\right|^p\,\mathrm{d}\mu$$

$$+\int_{\left\{\left|f_n-f_m\right|<2M\right\}\cap\left\{\left|f_n-f_m\right|>\left(\frac{\varepsilon}{3}\right)^{\frac{1}{p}}\right\}}\left|f_n-f_m\right|^p\,\mathrm{d}\mu$$

$$\leq\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}$$

Letting $n,m\longrightarrow+\infty$ and $\varepsilon\longrightarrow 0$, we see that (f_n) is a Cauchy sequence with respect to d_p. But $L^p(X,\mathcal{A},\mu)$ is complete, then (f_n) converge with respect to d_p to some function $g=f$ a.e., since (f_n) converge in probability to f. Whence

$$\left|\left|f_n-f\right|\right|_p\xrightarrow[n\to+\infty]{}0.$$

As the rest of the proof is similar to the case ($p \geq 1$), we leave it to the reader to verify that (iii)$\Longrightarrow$(i). This finishes the proof of Lemma 3.2. $\square$

We proceed now to the proof of Proposition 3.1.

Proof of Proposition 3.1. Let $\alpha > 0$ and assume that $\big(P_q(z)\big)_{q\geq 0}$ is L^α-flat. Then along a subsequence (q_n) we have that $\big(|P_{q_n}(z)|\big)_{n\geq 0}$ converges a.e. to 1. Whence $\big(P_{q_n}(z)\big)_{n\geq 0}$ is L^1-flat by Vitali's convergence theorem. In the opposite direction, assume that

$$\big(P_q(z)\big)_{q\geq 0} \quad \text{is } L^1\text{-flat},$$

then along a subsequence $(|P_{q_n}(z)|)_{n\geq 0}$ converges a.e. to 1. Again by Vitali's convergence theorem,

$$\big(P_{q_n}(z)\big)_{n\geq 0} \quad \text{is} \quad L^\alpha\text{-flat} \quad \text{for} \quad 0 < \alpha < 2. \qquad \square$$

In the following, we provide a necessary condition for L^1-flatness of a sequence of Littlewood polynomials.

Proposition 3.3. *Let $\big(P_q(z)\big)_{q\geq 0}$ be a sequence of L^2-normalized Littlewood polynomials. Suppose that $\big(P_q(z)\big)_{q\geq 0}$ is L^1-flat polynomials, then the frequency of -1 is in the interval $\big[\frac{1}{4}, \frac{3}{4}\big]$.*

For the proof of Proposition 3.3, we need the following simple lemma.

Lemma 3.4. *The sequence of polynomials $\Big(\frac{1}{\sqrt{q}}\sum_{j=0}^{q-1} z^j\Big)_{q\geq 0}$ is L^α-uniformly integrable, for $\alpha \in]0,2[$.*

Proof. Let $M > 0$, $\beta = \frac{2}{\alpha}$ and β' be such that $\frac{1}{\beta} + \frac{1}{\beta'} = 1$. Then, by Hölder's inequality, we can write

$$\int_{\Big\{\big|\frac{1}{\sqrt{q}}\sum_{j=0}^{q-1} z^j\big|^\alpha > M\Big\}} \left|\frac{1}{\sqrt{q}}\sum_{j=0}^{q-1} z^j\right|^\alpha dz$$

$$\leq \left\|\frac{1}{\sqrt{q}}\sum_{j=0}^{q-1} z^j\right\|_2^{\frac{2}{\beta}} \left(dz\left\{\left|\frac{1}{\sqrt{q}}\sum_{j=0}^{q-1} z^j\right| > \sqrt[\alpha]{M}\right\}\right)^{\frac{1}{\beta'}}$$

$$\leq \left(dz\left\{\left|\frac{1}{\sqrt{q}}\sum_{j=0}^{q-1} z^j\right| > \sqrt[\alpha]{M}\right\}\right)^{\frac{1}{\beta'}},$$

since

$$\left\|\frac{1}{\sqrt{q}}\sum_{j=0}^{q-1} z^j\right\|_2 = 1.$$

Whence, by Markov inequality, we get

$$\mathrm{d}z\left\{\left|\frac{1}{\sqrt{q}}\sum_{j=0}^{q-1} z^j\right| > \sqrt[\alpha]{M}\right\} \leq \frac{1}{\sqrt[\alpha]{M}}\left\|\frac{1}{\sqrt{q}}\sum_{j=0}^{q-1} z^j\right\|_1.$$

This gives

$$\mathrm{d}z\left\{\left|\frac{1}{\sqrt{q}}\sum_{j=0}^{q-1} z^j\right| > \sqrt[\alpha]{M}\right\} \leq \frac{1}{\sqrt[\alpha]{M}},$$

by Cauchy-Schwarz inequality. Letting $M \longrightarrow +\infty$, we conclude that

$$\int_{\left\{\left|\frac{1}{\sqrt{q}}\sum_{j=0}^{q-1} z^j\right|^\alpha > M\right\}}\left|\frac{1}{\sqrt{q}}\sum_{j=0}^{q-1} z^j\right|^\alpha \mathrm{d}z \xrightarrow[M\to+\infty]{} 0.$$

and the proof of the lemma is complete. □

Proof of Proposition 3.3. By (2.3), we have

$$P_q(z) \quad = \quad \frac{1}{\sqrt{q}}\sum_{j=0}^{q-1} z^j - 2R_q(z), \quad \forall z \in S^1.$$

We further have, for any $z \neq 1$,

$$\left|\frac{1}{\sqrt{q}}\sum_{j=0}^{q-1} z^j\right| \xrightarrow[q\to+\infty]{} 0.$$

Hence

$$\left\|\frac{1}{\sqrt{q}}\sum_{j=0}^{q-1} z^j\right\|_1 \xrightarrow[q\to+\infty]{} 0,$$

by Vitali's convergence theorem. Therefore

$$\Big\|\big|P_q(z)\big| - \big|2R_q(z)\big|\Big\|_1 \xrightarrow[q\to+\infty]{} 0.$$

It follows that $\big(P_q(z)\big)_{q\geq 0}$ is L^1-flat if and only if

$$\Big\|\big|R_q(z)\big| - \frac{1}{2}\Big\|_1 \xrightarrow[q\to+\infty]{} 0.$$

Assuming that $\big(P_q(z)\big)_{q\geq 0}$ is L^1-flat. It follows that we have

$$\Big\|R_q(z)\Big\|_1 \xrightarrow[q\to+\infty]{} \frac{1}{2}.$$

Whence, by Cauchy-Schwarz inequality, we can write

$$\left\|R_q(z)\right\|_2 = \sqrt{\frac{\#\left\{j : \widehat{\sqrt{q}R_q}(j) = 1\right\}}{q}} \geq \left\|R_q(z)\right\|_1.$$

Letting $q \longrightarrow +\infty$, we obtain

$$\lim_{q \longrightarrow +\infty} \frac{\#\left\{j : \widehat{\sqrt{q}R_q}(j) = 1\right\}}{q} = \mathrm{fr}(-1) \geq \frac{1}{4}. \tag{3.1}$$

We now apply this arguments again, with R_q replaced by Q_q, to obtain

$$\mathrm{fr}(1) = 1 - \mathrm{fr}(-1) \geq \frac{1}{4}. \tag{3.2}$$

Combining (3.1) with (3.2) completes the proof of Proposition 3.3. □

At this point, we conclude that the proof of the main result (Theorem 2.1), when the frequency of -1 is not in $\left[\frac{1}{4}, \frac{3}{4}\right]$ and $\alpha > 0$ follows easily from Proposition 3.3. To complete the proof for the case that the frequency of -1 is not in $\left[\frac{1}{4}, \frac{3}{4}\right]$, we present the proof of the case $\alpha = 0$.

We proceed by contrapositive. Assume that the sequence (P_q) of L^2-normalized polynomials is L^0-flat. Then, $M(P_q) \xrightarrow[q \to +\infty]{} 1$. But, by Jensen inequality, we have

$$\begin{aligned} M(P_q) &= \exp\left(\int_{S^1} \log(|f(t)|)\,\mathrm{d}t\right) \\ &\leq \left\|P_q\right\|_1 \leq 1. \end{aligned} \tag{3.3}$$

We further have by the Cauchy-Schwartz inequality

$$\begin{aligned} \left\||P_q| - 1\right\|_1 &\leq \left\||P_q|^2 - 1\right\|_1 \\ \left\||P_q|^2 - 1\right\|_1 &\leq \left\||P_q| - 1||_2\right\|_2 \left\||P_q| + 1\right\|_2 \\ &\leq 2\left\||P_q| - 1\right\|_2 \end{aligned} \tag{3.4}$$

It follows, from (3.3) and (3.4), that (P_q) is L^1-flat. By Proposition 3.3, we see that the frequency of -1 lies in the interval $\left[\frac{1}{4}, \frac{3}{4}\right]$ as required.

Now, from Lemma 3.4 it is a simple matter to strengthen Proposition 3.1 as follows.

Proposition 3.5. *Let $\big(P_q(z)\big)_{q\geq 0}$ be a sequence of L^2-normalized Littlewood polynomials. Suppose that $\big(P_q(z)\big)_{q\geq 0}$ is L^α-flat polynomials for some $0<\alpha<2$. Then*

$$\frac{1}{4} \leq \mathrm{fr}\,(-1) \leq \frac{3}{4},$$

Proposition 3.5 is related to the following theorem due to Jensen-Jensen and Høholdt [16].

Theorem 3.6. *Let $\big(P_q(z)\big)_{q\geq 0}$ be a sequence of L^2-normalized Littlewood polynomials. Suppose that*

$$\frac{\#\{j : \widehat{\sqrt{q}P_q}(j) = -1\}}{q} \longrightarrow \mathrm{fr}\,(-1) \quad \textit{as } q \longrightarrow +\infty.$$

If

$$\mathrm{fr}\,(-1) \neq \frac{1}{2}, \quad \textit{then } \Big\|P_q\Big\|_4 \xrightarrow[q\to+\infty]{} +\infty.$$

Obviously, Theorem 3.6 follows immediately form our main result Theorem 2.2. For its proof, we need the following interpolation inequalities due to Marcinkiewicz and Zygmund [31, Theorem 7.5, Chapter X, p.28].

Lemma 3.7. *For $\alpha>1$, $n\geq 1$, and any trigonometric polynomial P of degree $\leq n-1$,*

$$\frac{A_\alpha}{2n}\sum_{j=0}^{2n-1}\big|P(e^{2\pi i\frac{j}{2n}})\big|^\alpha \leq \int_{\mathbb{T}}\Big|P(z)\Big|^\alpha \mathrm{d}z \leq \frac{B_\alpha}{n}\sum_{j=0}^{2n-1}\big|P(e^{2\pi i\frac{j}{2n}})\big|^\alpha, \tag{3.5}$$

where A_α and B_α are independent of n and P.

We are now able to give the proof of Theorem 2.2.

Proof of Theorem 2.2. Let $\beta = \frac{\alpha}{2}$. Then, since $\beta>1$, we can apply Marcinkiewicz-Zygmund inequalities (Lemma 3.5) to get

$$\Big\|\big|P_q\big|^2-1\Big\|_\beta^\beta \geq \frac{A_\beta}{q}\Big|\big|P_q(1)\big|^2-1\Big|^\beta, \tag{3.6}$$

for some $A_\beta>0$. We further have

$$\big|P_q(1)\big|^2 = \Big|\sqrt{q}-2\frac{n_q}{\sqrt{q}}\Big|^2,$$

where n_q is the number of $\eta'_j=1$ which is the number of $\epsilon_j=-1$. This equality is due to the fact that

$$P_q(z) = \frac{1}{\sqrt{q}} \sum_{j=0}^{q-1} z^j - 2R_q(z),$$

and

$$R_q(1) = \frac{n_q}{\sqrt{q}}.$$

From this, we can write

$$\left|P_q(1)\right|^2 = q\left(1 - 2\frac{n_q}{q}\right)^2. \tag{3.7}$$

Whence

$$\begin{aligned}\left\|\left|P_q\right|^2 - 1\right\|_\beta^\beta &\geq \frac{A_\beta}{q}\left|q\left(1 - 2\frac{n_q}{q}\right)^2 - 1\right|^\beta \\ &\geq A_\beta\left|\left(1 - 2\frac{n_q}{q}\right)^2 - \frac{1}{q}\right|^\beta q^{\beta-1}.\end{aligned}$$

Therefore, by the triangle inequality, we can rewrite (3.6) as follows

$$\left(\|P_q\|_\alpha^2 + 1\right)^\beta \geq A_\beta\left|\left(1 - 2\frac{n_q}{q}\right)^2 - \frac{1}{q}\right|^\beta q^{\beta-1}.$$

Letting $q \longrightarrow +\infty$, we conclude that

$$\lim_{q \longrightarrow +\infty} \left(\|P_q\|_\alpha^2 + 1\right)^\beta \geq \left(1 - 2\,\mathrm{fr}(-1)\right)^\alpha \lim_{q \longrightarrow +\infty} q^{\beta-1} = +\infty,$$

since $\mathrm{fr}\,(-1) \neq \frac{1}{2}$. This completes the proof of Theorem 2.2. □

It follows from our proof that if the sequence of polynomials (P_n) from the class $\mathcal{L}$ is flat in the Littlewood sense, then the frequency of -1 is $\frac{1}{2}$. Indeed, assume that the sequence of L^2-normalized Littlewood polynomials (P_q) is flat in the Littlewood sense. Then, there exist $A, B > 0$ such that, for sufficiently large $q \in \mathbb{N}$, we have

$$A < |P_q(z)| < B, \quad \forall z \in S^1.$$

Therefore,

$$A^2 < |P_q(1)|^2 < B^2.$$

This combining with (3.7) gives

$$\frac{A^2}{q} < \left(1 - 2\frac{n_q}{q}\right)^2 < \frac{B^2}{q}.$$

Letting $q \longrightarrow +\infty$, we obtain $\frac{n_q}{q} \longrightarrow \frac{1}{2}$.

4. Proof of Theorem 2.3

The main tool in the proof of our second main result is the following Littlewood's criterion of flatness.

Lemma 4.1 (Littlewood's criterion [22]). *Let*

$$f_n(t) = \sum_{j=1}^{n} a_m \cos(mt + \phi_m)$$

and assume that we have

$$\sum_{m=1}^{n} a_m^2 \leq \frac{K}{n^2} \sum_{m=1}^{n} m^2 a_m^2,$$

for some absolute constant K. Then, for any $\alpha > 0$ there exists a constant $A(k,\alpha)$ such that

$$\|f_n\|_\alpha \leq \big(1 - A(k,\alpha)\big)\|f_n\|_2 \quad \text{if } \alpha < 2;$$

$$\|f_n\|_\alpha \geq \big(1 + A(k,\alpha)\big)\|f_n\|_2 \quad \text{if } \alpha > 2.$$

Notice that we have

$$\big\|f_n'\big\|_2 \leq n\big\|f_n\big\|_2,$$

by Bernstein-Zygmund inequalities [31, Theorem 3.13, Chapter X, p. 11]. Furthermore, the assumption in the Littlewood's criterion says that there is a constant K such that

$$\big\|f_n'\big\|_2 \geq Kn\big\|f_n\big\|_2.$$

We proceed now to prove our second main result.

Let $(P_n(z))$ be a sequence of even degree palindromic polynomials from the class $\mathcal{L}$, where

$$P_n(z) = \sum_{j=0}^{n} \epsilon_j z^j, \quad n = 2, 4, 6 \ldots, z \in S^1.$$

A straightforward computation gives

$$P_n(z) = z^{\frac{n}{2}} L_n(z) - \epsilon_{\frac{n}{2}} z^{\frac{n}{2}}, \quad \forall z \in S^1,$$

where

$$L_n(z) = \sum_{k=0}^{\frac{n}{2}} \epsilon_k \sigma_{\frac{n}{2}-k}(z),$$

and

$$\sigma_l(z) = z^l + \frac{1}{z^l}.$$

Therefore, for any $z \in S^1$, we have

$$L_n(\theta) = \sum_{k=0}^{\frac{n}{2}} a_{\frac{n}{2}-k} \cos(k\theta), \quad a_k = 2\epsilon_k.$$

Applying the Littlewood criterion, it follows that (L_n) is not L^α-flat, $\alpha \geq 0$. We thus conclude that (P_n) is not L^α-flat, $\alpha \geq 0$. This finish the proof of our second main result.

5. Appendix.

Proof of Theorem 2.1 when the frequency of -1 is $1/4$ or $3/4$ (jointly with M. G. Nadkarni).

For the proof of our first main result when the frequency of -1 is equal to $\frac{1}{4}$ or $\frac{3}{4}$, we need some tools from [2].
Let $Q(z) = \frac{1}{\sqrt{m}} \sum_{j=0}^{q-1} \eta_j z^j$ be a polynomial in the class $\mathcal{NB}$, where $m = \sum_{j=0}^{q-1} \eta_j$, which is the number of nonzero terms in Q. Note that $Q(1) = \sqrt{m}$.

$$| Q(z) |^2 = 1 + \sum_{\substack{k=-q-1 \\ k\neq 0}}^{q-1} a_k z^k,$$

where each a_k is a sum of terms of type $\eta_i \eta_j \frac{1}{m}$, $i \neq j$. Note that for each k, $a_{-k} = a_k$. Write

$$L = \sum_{\substack{j=-(q-1) \\ j\neq 0}}^{q-1} a_j = | Q(1) |^2 - 1 = m - 1.$$

Consider the random variables

$$X(k) = z^k - a_k, \quad -(q-1) \leq k \leq q-1$$

with respect to the measure $\nu = \left|Q(z)\right|^2 \mathrm{d}z$. We write

$$m(k,l) = \int_{S^1} X(k)\overline{X(l)}\, \mathrm{d}\nu, \quad -(q-1) \leq k, \quad l \leq q-1, \quad k,l \neq 0$$

and M for the covariance matrix with entries

$$m(k,l), -(q-1) \leq k, \quad l \leq q-1, \quad k,l \neq 0.$$

We call M the covariance matrix associated to $\nu = | Q(z) |^2 \; \mathrm{d}z$.

Since the linear combination of $X(k), -(q-1) \leq k \leq q-1, k \neq 0$, can vanish at no more than a finite set in S^1 and ν is non discrete, the random variables $X(k), -(q-1) \leq k \leq q-1, k \neq 0$, are linearly independent, whence the covariance matrix M is non-singular. M is a $2(q-1) \times 2(q-1)$ positive definite matrix.

Note that

$$m_{i,j} = \int_{S^1} z^{i-j} \, \mathrm{d}\nu - a_i a_j, \quad m_{i,i} = 1 - a_i^2.$$

Let $r(Q) = r > 0$ denote the sum of the entries of the matrix M. Let $C(Q) = C$ sum of the absolute values of the entries of M. Note that $r \leq C$. Also note that since each $\left|m_{i,j}\right| \leq 1$ we have

$$C \leq (2q-1)^2 < 4q^2.$$

We will now consider a sequence $\big(Q_n(z)\big)$ of polynomials from the class $\mathcal{NB}$. The quantities $M(Q_n), r(Q_n), C(Q_n)$, etc. will be now written as M_n, r_n, C_n, etc.

We need the following result from [2, Theorem 5.1].

Lemma 5.1. *If $\big(Q_n(z)\big)$ is an a.e. flat sequence from the class $\mathcal{NB}$, then*

$$\frac{C_n}{m_n^2} \xrightarrow[n\to+\infty]{} +\infty.$$

As a consequence of Lemma 5.1, we have

Corollary 5.2. *If $\big(Q_n\big)$ is an a.e. flat sequence then the ratios $\frac{m_n}{q_n}$, $n = 1, 2, \dots$ converges to zero.*

P r o o f. If not, there is a subsequence over which the ratios $\frac{m_n}{q_n}$, $n = 1, 2, \dots$ converges to a positive constant $c \leq 1$. We may assume without loss of generality that $\big(\frac{m_n}{q_n}\big)$ converges to c. Since $C_n \leq 4q_n^2$, $n = 1, 2, \dots$, we conclude that

$$\frac{C_n}{m_n^2} \leq \frac{4q_n^2}{m_n^2} \xrightarrow[n\to+\infty]{} \frac{4}{c^2} < +\infty,$$

which is a contradiction. The corollary follows. □

Now, we proceed by contradiction to complete the proof of our first main result.

P r o o f. Assume that P_n is $a.e.$ flat and the frequency of -1 is $\frac{1}{4}$. Then, $\big(S(P_n)\big)$ is also flat in $a.e.$ sense. We further have that the frequency of 1 for the sequence $\big(S(P_n)\big)$ is $\frac{1}{4}$, i.e.,

$$\frac{m_n}{q_n} \longrightarrow \frac{1}{4} \text{ as } n \longrightarrow \infty.$$

We thus get, by the formula (2.4),

$$T\big(S(P_n)\big) = (Q_n) \text{ is } a.e. \text{ flat sequence in } \mathcal{NB} \text{ with } \frac{m_n}{q_n} \xrightarrow[n\to\infty]{} \frac{1}{4}$$

which is impossible by Corollary 5.2.

In the same manner, we can see that the same conclusion hold for

$$\lim_{n\longrightarrow+\infty} \frac{m_n}{q_n} = \frac{3}{4}, \text{ by appealing to the formula (2.3).}$$

This completes the proof of our first main result. □

We finish this section by proving the following lemma on the existence of c-flatness. We hope it may find several applications. As customary, for every real number x, $[x]$ denotes its integer part. For any polynomial P and $c \in [0,1]$, we associate to it a polynomial Q defined by

$$Q(z) = P(z) + \sum_{d+1}^{d+[aA^2]} z^j,$$

where d is the degree of P, $A = \|P\|_2$ and $a = \frac{1}{c^2} - 1$. We are now able to state our lemma.

Lemma 5.3 (Kolkata-Workshop, 2019). *Let $c \in [0,1]$ and assume that the sequence of polynomials (P_n) satisfy $\|P_n\|_2 \xrightarrow[n\to+\infty]{} +\infty$. Then, if $\left(\frac{|P_n(z)|}{\|P_n\|_2}\right)_{n\geq1}$ is a.e. flat, then the sequence $\big(Q_n(z)\big)$ is a.e. c-flat.*

P r o o f. We start by computing the L^2-norm of $(Q_n(z))$. We have

$$\begin{aligned}
\|Q_n(z)\|_2^2 &= \int_{S^1} \left| \sum_{j=0}^{d_n-1} a_j z^j + \sum_{j=d_n+1}^{d_n+[aA_n^2]} z^j \right|^2 dz \\
&= \sum_{j=0}^{d_n-1} |a_j|^2 + \sum_{j=d_n+1}^{d_n+[aA_n^2]} 1 \\
&= A_n^2 + [aA_n^2].
\end{aligned}$$

Therefore,

$$\frac{\left|P_n(z)+\sum_{j=d_n+1}^{d_n+[aA_n^2]} z^j\right|}{\sqrt{A_n^2+[aA_n^2]}}$$

converges a.e. to c. Indeed, by the triangle inequalities, we have

$$\left|\left|P_n(z)\right| - \left|\sum_{j=d_n+1}^{d_n+[aA_n^2]} z^j\right|\right| \leq |Q_n(z)| \leq |P_n(z)| + \left|\sum_{j=d_n+1}^{d_n+[aA_n^2]} z^j\right|.$$

We further have

$$\frac{A_n^2+[aA_n^2]}{A_n^2} \xrightarrow[n\to+\infty]{} 1+a,$$

This gives that $\frac{|Q_n(z)|}{\sqrt{A_n^2+[aA_n^2]}}$ converges a.e. to c, since $\frac{|P_n|}{A_n}$ converges a.e. to 1 and

$$\frac{1}{\sqrt{1+a}} = c.$$

This completes the proof of Lemma 5.3. □

Remarks.

1) Formula (2.4) at once shows that if a sequence (P_n) in the class $\mathcal{L}$ is ultraflat, then
   (i) $\lim_{n\to\infty} \frac{m_n}{q_n} = \frac{1}{2}$ and
   (ii) $T(P_n), n=1,2,\ldots$
   converges uniformly to $\frac{1}{\sqrt{2}}$ on compact subsets of $S^1 \setminus \{1\}$.

   It is not known if (i) and (ii) are compatible conditions. However, the numerical evidence from [26] suggest that (i) and (ii) are not compatible.

2) Exploring the limit distribution of the sequence of polynomials from the class $\mathcal{L}$ can be linked to the exploration of the limit distribution of the sequence of polynomials from the class $\mathcal{NB}$ by (2.4). Characterization of the class of distributions which can be a limit distribution of a sequence of polynomials from the class $\mathcal{NB}$ is an open problem. For a very recent work on the subject, we refer to [13] and the references therein.

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REFERENCES

[1] EL ABDALAOUI, E. H.: *Ergodic Banach problem, flat polynomials and Mahler's measures with combinatorics*, preprint 2016, arXiv:1508.06439v5 [math.DS].

[2] EL ABDALAOUI, E. H.—NADKARNI, M.: *Some notes on flat polynomials*, preprint 2014, http://arxiv.org/abs/1402.5457

[3] EL ABDALAOUI, E. H.—NADKARNI, M.: *On flat polynomials with non-negative coefficients*, preprint 2015, https://arxiv.org/abs/1508.00417

[4] BALISTER, P.—BOLLOBÁS, B.—MORRIS, R.—SAHASRABUDHE, J.—TIBA, M.: *Flat Littlewood polynomials exist*, arXiv:1907.09464v1 [math.CA].

[5] BECK, J.: *Flat polynomials on the unit circle - Note on a Problem of Littlewood,* Bull. London Math. Soc., **23** (1991), 269–277.

[6] BOMBIERI, E.—BOURGAIN, J.: *On Kahane's ultraflat polynomials,* J. Eur. Math. Soc. (JEMS) **11** (2009), no. 3, 627–703.

[7] BORWEIN, P.—FERGUSON, R.—KNAUER, J.: *The merit factor problem*, In: *The Merit Factor Problem*. Number Theory and Polynomials, London Math. Soc. Lecture Note Ser., Vol. 352, Cambridge Univ. Press, Cambridge, 2008, 52–70,

[8] CHUNG, K. L.: *A Course in Probability Theory*, 3rd. ed. Academic Press, Inc., San Diego, CA, 2001.

[9] DOWNAROWICZ, T.—LACROIX, Y.: *Merit factors and Morse sequences*, Theoretical Computer Science, (209) (1998), 377–387.

[10] ERDŐS, P.: *An inequality for the maximum of trigonometric polynomials*, Ann. Polon. Math., (12) (1962), 151–154.

[11] ERDŐS, P.: *Some unsolved problems*, Michigan Math. J. **4** (1957), 291–300.

[12] ERDŐS, P.: *Problems and results on polynomials and interpolation.* In: (Proc. NATO Adv. Study Inst., Univ. Durham, Durham, 1979), *Aspects of Contemporary Complex analysis* Academic Press, London-New York, 1980, pp. 383–391,

[13] FUKUYAMA, K.: *A central limit theorem for trigonometric series with bounded gaps*, Probab. Theory Relat. Fields, **149** (2011), 139–148.

[14] HØHOLDT, T.: *The merit factor problem for binary sequences*, In: *Applied Algebra, Algebraic Algorithms and Error-correcting Codes, Lecture Notes in Comput. Sci. Vol. 3857*, Springer-Verlag, Berlin, 2006, pp. 51–59.

[15] JENSEN, H. E.—HØHOLDT, T.: *Determination of the merit factor of Legendre sequences,* IEEE Trans. Inform. Theory, **34** (1988), no. 1, 161–164.

[16] JENSEN, J.—JENSEN, H. E.—HØHOLDT, T.: *The merit factor of binary sequences related to difference sets*, IEEE Trans. Inform. Theory, **37** 1991, no. 3, 617–626.

[17] GOLAY, M. J. E.: *Sieves for low autocorrelation binary sequences*, IEEE Trans. Inform. Theory, **23** (1977), no. 1, pp. 43–51.

[18] JEDWAB, J.: *A survey of the merit factor problem for binary sequences.* In: Third international conference, Seoul, Korea, October 24–28, 2004. (T. Helleseth, ed. et al.), *Sequences and Their Applications SETA–2004*, Revised selected papers. *Lecture Notes in Computer Science* Vol. 3486, Springer-Verlag, Berlin 2005, pp. 30–55.

[19] KAHANE, J-P.: *Sur les polynômes à coefficients unimodulaires*, Bull. London. Math. Soc. **12** (1980), 321–342.

[20] LITTLEWOOD, J. E.: *On polynomials* $\sum^n \pm z^m, \sum^n e^{\alpha_m i} z^m$, $z = e^{\theta i}$, J. London Math. Soc. **41** (1966), 367–376.

[21] LITTLEWOOD, J. E.: *Some Problems in Real and Complex Analysis*. D. C. Heath and Co. Raytheon Education Co., Lexington, Mass. 1968.

[22] LITTLEWOOD, J. E.: *On the mean values of certain trigonometric polynomials*, J. London Math. Soc. **36**, 1961, 307–334.

[23] LITTLEWOOD, J. E.: *On the mean values of certain trigonometric polynomials. II*, Illinois J. Math. **6**, 1962, 1–39.

[24] NEWMAN, D. J.: *Norms of polynomials*, Amer. Math. Monthly, **6**7 (1960), no. 8, pp. 778–779.

[25] NEWMAN, D. J.—BYRNES, J. S.: The L^4 norm of a polynomial with coefficients ± 1, Amer. Math. Monthly **97** (1990), no 1,42–45.

[26] ODLYZKO, A. M.: *Search for ultraflat polynomials with plus and minus one coefficients*, In: (S. Butler ed. et al.) *Connections in Discrete Mathematics*. (A celebration of the work of Ron Graham), Cambridge University Press, Cambridge, 2018, pp. 39–55

[27] QUEFFELEC, H.—SAFFARI, B.: *On Bernstein's inequality and Kahane's ultraflat polynomials,* J. Fourier Anal. Appl. **2** (1996), no. 6, 519–582.

[28] SAFFARI, B.—SMITH, B.: *Inexistence de polynômes ultra-plats de Kahane à coefficients* ± 1*. Preuve de la conjecture d'Erdős. (French) [Nonexistence of ultra-flat Kahane polynomials with coefficients* ± 1*. Proof of the Erdős conjecture],* C. R. Acad. Sci. Paris Sér I Math. **3**06 (1988), no. 16, 695–698.

[29] FREDMAN, M.—SAFFARI, B.—SMITH, B.: *Polynômes réciproques: conjecture d'Erdős en norme* L^4*, taille des autocorrélations et inexistence des codes de Barker. (French) [Self-inversive polynomials:* L^4*-norm Erdős conjecture, size of autocorrelations and nonexistence of Barker codes]* , C. R. Acad. Sci. Paris Sér. I Math. **3**08 (1989), no. 15, 461–464.

[30] SCHILLING, L. R.: *Measures, Integrals and Martingales*. Second edition. Cambridge University Press, Cambridge, 2017.

[31] ZYGMUND, A.: *Trigonometric Series* Vol. I & II. Second edition. Cambridge Univ. Press, Cambridge, 1959.

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