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# ON THE DISCREPANCY OF RANDOM WALKS ON THE CIRCLE

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ABSTRACT. Let $X_1, X_2, \dots$ be i.i.d. absolutely continuous random variables, let $S_k = \sum_{j=1}^k X_j \pmod 1$ and let $D_N^*$ denote the star discrepancy of the sequence $(S_k)_{1\le k\le N}$. We determine the limit distribution of $\sqrt{N} D_N^*$ and the weak limit of the sequence $\sqrt{N}(F_N(t)-t)$ in the Skorohod space $D[0,1]$, where $F_N(t)$ denotes the empirical distribution function of the sequence $(S_k)_{1\le k\le N}$.

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## 1. Introduction

Let $X_1, X_2, \dots$ be i.i.d. absolutely continuous random variables and let $S_k = \sum_{j=1}^k X_j \pmod 1$. Then $S_k, k = 0, 1, \dots$ is a random walk on the circle and by a classical result of Lévy [6], the distribution of $S_k$ converges weakly to the uniform distribution on $(0, 1)$. Schatte [8] proved that the speed of convergence



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is exponential, and letting

$$D_N^* := \sup_{0\le a<1} \left| \frac{1}{N} \sum_{k=1}^{N} I_{(0,a)}(S_k) - a \right| \qquad (N = 1, 2, \dots)$$

denote the star discrepancy of the sequence $(S_k)_{1\le k\le N}$, he also proved in [9] the law of the iterated logarithm

$$\limsup_{N\to\infty} \sqrt{\frac{N}{\log\log N}} D_N^* = \gamma \quad \text{a.s.}, \tag{1}$$

where

$$\gamma = \sup_{x\in[0,1)} \sigma^2(x)$$

with

$$\sigma^2(x) = x - x^2 + 2\sum_{j=1}^{\infty} \big(\mathbb{E}I_{(0,x)}(U) I_{(0,x)}(U + X_j) - x^2\big). \tag{2}$$

Here $U$ is a uniform (0, 1) random variable independent of the sequence $(X_n)_{n\ge1}$ and for $0 \le a < b \le 1$, $I_{(a,b)}$ denotes the indicator function of $(a, b)$, extended with period 1. Letting

$$F_N(t) = F_N(t, \omega) = \frac{1}{N} \sum_{k=1}^{N} I_{(0,t)}(S_k) \qquad (0 \le t \le 1) \tag{3}$$

denote the empirical distribution function of the first $N$ terms of the sequence $(S_n)_{n\ge1}$, Berkes and Raseta [2] proved a Strassen type functional LIL for $F_N(t)$, yielding precise asymptotics for several functionals of the empirical process. The purpose of the present paper is to prove the following result, determining the limit distribution of $\sqrt{N}D_N^*$.

**Theorem 1.** *Let $X_1, X_2, \dots$ be i.i.d. random variables and assume $X_1$ is bounded with bounded density. Let*

$$\Gamma(s, t) = s(1 - t) + \sum_{k=1}^{\infty} \mathbb{E}f_s(U) f_t(U + S_k) + \sum_{k=1}^{\infty} \mathbb{E}f_t(U) f_s(U + S_k), \tag{4}$$

*where $U$ is a $U(0, 1)$ variable independent of $(X_n)_{n\in\mathbb{N}}$ and $f_s = I_{(0,s)} - s$. Then the series in (4) are absolutely convergent and*

$$\sqrt{N} D_N^* \xrightarrow{d} \sup_{0\le t\le1} |K(t)|, \tag{5}$$

*where $K(s)$ a mean zero Gaussian process with covariance function $\Gamma(s, t)$.*

Actually, Theorem 1 will be deduced from a more general functional result describing the weak limit behaviour of the empirical distribution function $F_N(t)$.

**Theorem 2.** *Under the conditions of Theorem 1 we have*

$$\sqrt{N}\big(F_N(t)-t\big) \overset{D[0,1]}{\longrightarrow} K(t) \quad \text{as} \quad N \to \infty. \tag{6}$$

Relation (6) expresses weak convergence in the Skorohod space $D[0,1]$, see Billingsley [4] for basic definitions and facts for weak convergence of probability measures on metric spaces.

By a classical result of Donsker [5], if $X_1, X_2, \ldots$ are i.i.d. random variables with distribution function $F$ and $F_N$ denotes the empirical distribution function of the sample $(X_1, \ldots, X_N)$, then

$$\sqrt{N}\big(F_N(t)-F(t)\big) \overset{D[0,1]}{\longrightarrow} B\big(F(t)\big)$$

where $B$ is Brownian bridge. Note the substantial difference caused by considering mod 1 sums in the present case.

If $X_1$ has a lattice distribution, the situation changes essentially. For example, in [1] it is shown that if $\alpha$ is irrational and $X_1$ takes the values $\alpha$ and $2\alpha$ with probability $1/2-1/2$, then up to logarithmic factors, the order of magnitude of $D_N^*$ is

$$O\left(N^{-1/2}\right) \quad \text{or} \quad O\left(N^{-1/\gamma}\right)$$

according as $\gamma<2$ or $\gamma>2$, where $\gamma$ is the Diophantine rank of $\alpha$, i.e., the supremum of numbers $c$ such that $|\alpha-p/q|<q^{-c-1}$ holds for infinitely many fractions $p/q$. The asymptotic distribution of $D_N^*$ in this case remains open.

## 2. Proofs

The proof of our theorems uses, similarly to that of the functional LIL in [2], a traditional blocking argument combined with a coupling lemma of Schatte, see Lemma 1 below. The substantial new difficulty is to prove the tightness of the sequence $\sqrt{N}(F_N(t)-t)$, since the standard maximal inequalities (e.g., Billingsley's inequalities in [4], Section 2.12) are not applicable here. We circumvent this difficulty by proving a Chernoff type exponential bound (Lemma 6) for the considered partial sums which, combined with the chaining method of Philipp [7], yields the desired fluctuation inequality (Lemma 7).

**Lemma 1.** *Let*
$$\ell \geq 1 \quad \text{and} \quad I_1, I_2, \ldots$$
*be disjoint blocks of integers with $\geq \ell$ integers between consecutive blocks. Then there exists a sequence $\delta_1, \delta_2, \ldots$ of random variables such that*
$$|\delta_n| \leq Ce^{-\lambda \ell}$$
*with some positive constants $C, \lambda$ and the random vectors*
$$\{S_i, i \in I_1\}, \quad \{S_i - \delta_1, i \in I_2\}, \ldots, \{S_i - \delta_{n-1}, i \in I_n\}, \ldots$$
*are independent and have, except for the first one, uniformly distributed components.*

Proof. For the proof, see [2]. The uniformity statement is implicit in the proof; see also Lemma 4.3 of [3]. □

In what follows, $C, \lambda, \gamma, \gamma' \ldots$ will denote positive constants, possibly different at different places, depending (at most) on the distribution of $X_1$. The relation $\ll$ will mean the same as the big $O$ notation, with a constant depending on the distribution of $X_1$.

Let $\mathcal{F}$ denote the class of functions $f$ of the form
$$f = I_{(a,b)} - (b-a) \quad (0 \leq a < b \leq 1),$$
extended with period 1. For $f \in \mathcal{F}$ we put
$$A^{(f)} := \|f\|^2 + 2 \sum_{k=1}^{\infty} \mathbb{E} f(U) f(U + S_k), \tag{7}$$
where $U$ is a uniform $(0,1)$ random variable, independent of $(X_j)_{j \geq 1}$ and $\|f\|$ denotes the $L^2(0,1)$ norm of $f$. Put further
$$\widetilde{m}_k = \sum_{j=1}^{k} \left\lfloor j^{1/2} \right\rfloor, \quad \widehat{m}_k = \sum_{j=1}^{k} \left\lfloor j^{1/4} \right\rfloor$$
and let $m_k = \widetilde{m}_k + \widehat{m}_k$. Using Lemma 1 we can construct sequences $(\Delta_k)_{k \in \mathbb{N}}$, $(\Pi_k)_{k \in \mathbb{N}}$ of random variables such that $\Delta_0 = 0$, $\Pi_0 = 0$,
$$|\Delta_k| \leq Ce^{-\lambda k^{1/4}}, \qquad |\Pi_k| \leq Ce^{-\lambda \sqrt{k}} \tag{8}$$
and
$$T_k^{(f)} := \sum_{j=m_{k-1}+1}^{m_{k-1}+\lfloor \sqrt{k} \rfloor} f(S_j - \Delta_{k-1}), \qquad k = 1, 2, \ldots,$$
$$T_k^{*(f)} = \sum_{j=m_{k-1}+\lfloor \sqrt{k} \rfloor + 1}^{m_k} f(S_j - \Pi_{k-1}), \qquad k = 1, 2, \ldots$$
are sequences of independent random variables.

Since $\int_0^1 f(x)\,\mathrm{d}x = 0$ for $f \in \mathcal{F}$, the uniformity statement in Lemma 1 implies that

$$\mathbb{E}T_k^{(f)} = \mathbb{E}T_k^{*(f)} = 0 \quad \text{for } k \geq 2.$$

The following asymptotic estimates for the variances of $T_k^{(f)}$ and $T_k^{*(f)}$ are from [2].

**Lemma 2.** *For $f \in \mathcal{F}$ we have*

$$\sum_{k=1}^{n} \mathbb{V}\mathrm{ar}\left(T_k^{(f)}\right) \sim A^{(f)} \widetilde{m}_n\,, \qquad \sum_{k=1}^{n} \mathbb{V}\mathrm{ar}\left(T_k^{*(f)}\right) \sim A^{(f)} \widehat{m}_n\,,$$

*where $A^{(f)}$ is defined by* (7)*.*

Since

$$\mathrm{Cov}\left(T_k^{(f)}, T_k^{(g)}\right) = \frac{1}{4}\left(\mathbb{V}\mathrm{ar}\left(T_k^{(f+g)}\right) - \mathbb{V}\mathrm{ar}\left(T_k^{(f-g)}\right)\right),$$

Lemma 2 implies

$$\sum_{k=1}^{n} \mathrm{Cov}\left(T_k^{(f)},\, T_k^{(g)}\right) \sim \frac{1}{4}\left(A^{(f+g)} - A^{(f-g)}\right)\widetilde{m}_n \tag{9}$$

and

$$\sum_{k=1}^{n} \mathrm{Cov}\left(T_k^{*(f)}, T_k^{*(g)}\right) \sim \frac{1}{4}\left(A^{(f+g)} - A^{(f-g)}\right)\widehat{m}_n. \tag{10}$$

From (7) it follows that

$$A^{(f+g)} - A^{(f-g)} = 4\langle f, g\rangle + 4\sum_{k=1}^{\infty} \mathbb{E}f(U)g(U+S_k) + 4\sum_{k=1}^{\infty} \mathbb{E}g(U)f(U+S_k). \tag{11}$$

**Lemma 3.** *Let $f \in \mathcal{F}$, $h > 0$ and let $\xi$ be a random variable with $|\xi| < h$. Then for any $n \geq 1$ we have*

$$\mathbb{E}|f(S_n + \xi) - f(S_n)|^2 \leq Ch.$$

P r o o f. Since $X_1$ is bounded with bounded density, Theorem 1 of [8] implies that the sums $S_n = \sum_{k=1}^{n} X_k \pmod 1$ have a uniformly bounded density and thus

$$\mathbb{P}(S_n \in J) \leq C|J| \quad \text{for any interval } J. \tag{12}$$

Now if $f = I_{(a,b)} - (b-a)$, then $|f(S_n+\xi) - f(S_n)| = |I_{(a,b)}(S_n+\xi) - I_{(a,b)}(S_n)|$ is different from 0 only if one of $S_n + \xi$ and $S_n$ lies in $(a, b)$ and the other does not, which, in view of $|\xi| < h$, implies that $S_n$ lies closer to the boundary of $(a, b)$ than $h$, i.e., $S_n \in (a, a+h)$ or $S_n \in (b-h, b)$. Since $|f(S_n + \xi) - f(S_n)| \leq 2$, Lemma 3 follows from (12). $\square$

**Lemma 4.** *For $f \in \mathcal{F}$ and any $M \geq 0$, $N \geq 1$ we have*

$$\mathbb{E}\left(\sum_{k=M+1}^{M+N} f(S_k)\right)^2 \leq C\|f\|N. \tag{13}$$

P r o o f. We first show

$$|\mathbb{E}f(S_k)f(S_\ell)| \leq Ce^{-\lambda(\ell-k)}\|f\| \qquad (k<\ell). \tag{14}$$

Indeed, by the proof of Lemma 1 in [2], there exists a r.v. $\Delta$ with $|\Delta| \leq Ce^{-\lambda(\ell-k)}$ such that $S_\ell - \Delta$ is a uniform r.v. independent of $S_k$. Hence

$$\mathbb{E}f(S_\ell - \Delta) = \int_0^1 f(t)\,\mathrm{d}t = 0$$

and thus

$$\mathbb{E}f(S_k)f(S_\ell - \Delta) = \mathbb{E}f(S_k)\mathbb{E}f(S_\ell - \Delta) = 0. \tag{15}$$

On the other hand,

$$\begin{aligned}
&|\mathbb{E}f(S_k)f(S_\ell) - \mathbb{E}f(S_k)f(S_\ell - \Delta)| \\
&\leq \mathbb{E}\big(|f(S_k)|\,|f(S_\ell) - f(S_\ell - \Delta)|\big) \\
&\leq \big(\mathbb{E}f^2(S_k)\big)^{1/2}\big(\mathbb{E}|f(S_\ell) - f(S_\ell - \Delta)|^2\big)^{1/2}.
\end{aligned} \tag{16}$$

Using (12) we get

$$\mathbb{E}f^2(S_k) \leq C\int_0^1 f^2(t)\,\mathrm{d}t = C\|f\|^2. \tag{17}$$

Also, $|\Delta| \leq Ce^{-\lambda(\ell-k)}$ and Lemma 3 imply

$$\mathbb{E}|f(S_\ell) - f(S_\ell - \Delta)|^2 \leq Ce^{-\lambda(\ell-k)} \tag{18}$$

which, together with (16)–(18), gives

$$|\mathbb{E}f(S_k)f(S_\ell) - \mathbb{E}f(S_k)f(S_\ell - \Delta)| \leq Ce^{-\lambda(\ell-k)}.$$

Thus using (15) we get (14). Now by (14)

$$\left|\sum_{M+1\leq k<\ell\leq M+N} \mathbb{E}f(S_k)f(S_\ell)\right| \leq CN\|f\|\sum_{\ell\geq 1} e^{-\lambda\ell} \leq CN\|f\|$$

which, together with (17), completes the proof of Lemma 4. □

Let $0 < t_1 < \cdots < t_r \le 1$ and put

$$\mathbf{Y}_k = \big(f_{(0,t_1)}(S_k), f_{(0,t_2)}(S_k), \ldots, f_{(0,t_r)}(S_k)\big),$$

where

$$f_{(a,b)} = I_{(a,b)} - (b-a),$$

with the indicator $I_{(a,b)}$ extended with period 1, as before.

**Lemma 5.** *We have*

$$N^{-1/2} \sum_{k=1}^{N} \mathbf{Y}_k \xrightarrow{d} N(\mathbf{0}, \mathbf{\Sigma}), \tag{19}$$

*where*

$$\mathbf{\Sigma} = \big(\Gamma(t_i, t_j)\big)_{1 \le i,j \le r}.$$

P r o o f. Let

$$\mathbf{T}_k = \left(T_k^{(f_{(0,t_1)})}, \ldots, T_k^{(f_{(0,t_r)})}\right),$$

$$\mathbf{T}_k^* = \left(T_k^{*(f_{(0,t_1)})}, \ldots, T_k^{*(f_{(0,t_r)})}\right),$$

and let $\mathbf{\Sigma_k}$ denote the covariance matrix of the vector $\mathbf{T}_k$. From (9), (10) and (11) it follows that

$$m_n^{-1}\left(\mathbf{\Sigma_1} + \cdots + \mathbf{\Sigma_n}\right) \longrightarrow \mathbf{\Sigma}.$$

Clearly,

$$|\mathbf{T}_k| \le C_r k^{1/2} = o\left(m_k^{1/2}\right),$$

where $C_r$ is a constant depending on $r$, showing that the sequence $(\mathbf{T}_k)_{k\ge 1}$ of independent, mean $\mathbf{0}$ random vectors satisfies the Lindeberg condition and thus

$$m_n^{-1/2} \sum_{k=1}^{n} \mathbf{T}_k \xrightarrow{d} N(\mathbf{0}, \mathbf{\Sigma}). \tag{20}$$

A similar statement holds for the sequence $(\mathbf{T}_k^*)_{k\ge1}$, implying that

$$\left|\sum_{k=1}^{n} \mathbf{T}_k^*\right| = O_P(\widehat{m}_n) = o_P\left(m_n^{1/2}\right), \tag{21}$$

and consequently,

$$m_n^{-1/2} \sum_{k=1}^{n} (\mathbf{T}_k + \mathbf{T}_k^*) \xrightarrow{d} N(\mathbf{0}, \mathbf{\Sigma}). \tag{22}$$

Now, using (8) and Lemma 3 we get

$$\left\| T_k^{(f)} - \sum_{j=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor} f(S_j) \right\| \ll \sqrt{k} e^{-\lambda k^{1/4}}$$

and

$$\left\| T_k^{*(f)} - \sum_{j=m_{k-1}+\lfloor\sqrt{k}\rfloor+1}^{m_k} f(S_j) \right\| \ll k^{1/4} e^{-\lambda k^{1/2}},$$

and thus

$$\left\| \sum_{k=1}^{m_n} \mathbf{Y}_k - \sum_{k=1}^{n} (\mathbf{T}_k + \mathbf{T}_\mathbf{k}^*) \right\| = O(1).$$

Together with (22) this shows that (19) holds for the indices $N = m_n$. To get (19) for all $N$, observe that $m_k \sim ck^{3/2}$ and thus for $m_k \le N < m_{k+1}$ we have

$$\left| \sum_{j=1}^{N} \mathbf{Y}_j - \sum_{j=1}^{m_k} \mathbf{Y}_j \right| = O(m_{k+1} - m_k) = O\big(k^{1/2}\big) = O\big(m_k^{1/3}\big) = O\big(N^{1/3}\big).$$

This completes the proof of Lemma 5. $\square$

**Lemma 6.** *For $f \in \mathcal{F}$, any $N \ge 1$, $t \ge 1$ and $\|f\| \ge \frac{1}{5} N^{-5/18}$ we have*

$$\mathbb{P}\left\{ \left| \sum_{k=1}^{N} f(S_k) \right| \ge t \|f\|^{1/4} \sqrt{N} \right\} \ll \exp\left(-Ct\|f\|^{-7/20}\right) + t^{-2} \exp\left(-CN^{1/3}\right). \tag{23}$$

**Remark.** The constants $1/5, 5/18, 1/4, 7/20, 1/3$ in (23) are not sharp and the inequality could be easily improved. However, the present form of Lemma 6 will suffice for the chaining argument in Lemma 7.

P r o o f. Put

$$\psi(n) = \sup_{0 \le x \le 1} |\mathbb{P}(S_n \le x) - x|.$$

By Theorem 1 of [8] we have

$$\psi(n) \le Ce^{-\gamma n} \qquad (n \ge 1)$$

for some constant $\gamma > 0$. Divide the interval $[1, N]$ into subintervals $I_1, \dots, I_L$, with $L \sim N^{2/3}$, where each interval $I_\nu$ contains $\sim N^{1/3}$ terms. We set

$$\sum_{k=1}^{N} f(S_k) = \eta_1 + \dots + \eta_L, \quad \text{where} \quad \eta_\nu = \sum_{k \in I_\nu} f(S_k).$$

We deal with the sums $\sum \eta_{2j}$ and $\sum \eta_{2j+1}$, separately. Since there is a separation $\sim N^{1/3}$ between the even block sums $\eta_{2j}$, we can apply Lemma 1 to get

$$\eta_{2j} = \eta_{2j}^{*} + \eta_{2j}^{**},$$

where

$$\eta_{2j}^{*} = \sum_{k\in I_{2j}} f(S_k - \Delta_j),$$

$$\eta_{2j}^{**} = \sum_{k\in I_{2j}} \big(f(S_k) - f(S_k - \Delta_j)\big).$$

Here the $\Delta_j$ are r.v.'s with $|\Delta_j| \le \psi(N^{1/3}) \le C\exp(-\gamma N^{1/3})$ and the r.v.'s $\eta_{2j}^{*}$ $j = 1, 2, \ldots$ are independent. Conditionally on $\Delta_j$, the distribution of $S_k$ in a term of $\eta_{2j}^{**}$ is the same as the (unconditional) distribution of an $S_{k_1} + c$ with $k_1 < k$ and a constant $c$ and thus by Lemma 3, the $L_2$ norm of each summand in $\eta_{2j}^{**}$ is $\le C\psi^{1/2}(N^{1/3}) \le C\exp(-\gamma N^{1/3})$ and thus for $\|f\| \ge N^{-1}$ we have

$$\begin{aligned}\|\eta_{2j}^{**}\| &\le CN\exp\big(-\gamma N^{1/3}\big) \le C\|f\|N^2\exp\big(-\gamma N^{1/3}\big)\\ &\le C\|f\|\exp\big(-\gamma' N^{1/3}\big).\end{aligned} \tag{24}$$

Thus

$$\Big\|\sum \eta_{2j}^{**}\Big\| \le C\|f\|\exp\big(-\gamma'' N^{1/3}\big)$$

and therefore by the Markov inequality

$$\begin{aligned}\mathbb{P}\left(\left|\sum \eta_{2j}^{**}\right| \ge t\|f\|^{1/4}\sqrt{N}\right) &\le Ct^{-2}\|f\|^{-1/2}N^{-1}\|f\|^2\exp\big(-2\gamma'' N^{1/3}\big)\\ &\le Ct^{-2}\exp\big(-2\gamma'' N^{1/3}\big).\end{aligned}$$

Let now $|\lambda| \le dN^{-1/3}$ with a sufficiently small constant $d > 0$. Then $|\lambda\eta_{2j}^{*}| \le 1/2$ for all $N$ and thus using $e^x \le 1 + x + x^2$ for $|x| \le 1/2$ we get, using $E\eta_{2j}^{*} = 0$ for $j \ge 2$,

$$\begin{aligned}\mathbb{E}\left(\exp\lambda\left(\sum_j \eta_{2j}^{*}\right)\right) &= \prod_j \mathbb{E}\big(e^{\lambda\eta_{2j}^{*}}\big) \le \prod_j \mathbb{E}\big(1+\lambda\eta_{2j}^{*}+\lambda^2\eta_{2j}^{*2}\big)\\ &= \prod_j\big(1+\lambda^2\mathbb{E}\eta_{2j}^{*2}\big) \le \exp\left(\lambda^2\sum_j \mathbb{E}\eta_{2j}^{*2}\right).\end{aligned} \tag{25}$$

Here, and in the rest of the proof of the lemma, the sums and products are extended for $j \ge 2$. By Lemma 4

$$\|\eta_{2j}\| \le C\|f\|^{1/2}N^{1/6},$$

which, together with (24) and the Minkowski inequality, implies

$$\|\eta_{2j}^{*}\| \le C\|f\|^{1/2}N^{1/6}.$$

Thus the last expression in (25) cannot exceed

$$\exp\left(\lambda^2 C\|f\| \sum_j N^{1/3}\right) \le \exp\left(\lambda^2 C\|f\|N\right).$$

We choose now

$$\lambda = \frac{d}{2} N^{-1/2} \|f\|^{-3/5}$$

with the number $d$ introduced before and note that by $\|f\| \ge \frac{1}{5} N^{-5/18}$ we have

$$|\lambda| \le dN^{-1/3}.$$

Thus using the Markov inequality, we get

$$\begin{aligned}
\mathbb{P}\left\{\left|\sum_j \eta_{2j}^*\right| \ge t\|f\|^{1/4}\sqrt{N}\right\} &\le 2\exp\left\{-\lambda t\|f\|^{1/4}\sqrt{N} + \lambda^2 C\|f\|N\right\} \\
&= 2\exp\left(-\|f\|^{-7/20} t + C\|f\|^{-1/5}\right) \\
&\le 2\exp\left(-C\|f\|^{-7/20} t\right).
\end{aligned}$$

Recall that the sum here is extended for $j \ge 2$. However, the term corresponding to $j = 1$ is $O(N^{1/3})$ and since $\|f\|^{1/4}\sqrt{N} \ge N^{0.4}$ for $N \ge N_0$ by the assumptions of the lemma, the last chain of estimates remains valid by including the term $j = 1$ in the sum in the first probability and changing $t$ to $2t$. A similar argument applies for the odd blocks $\eta_{2j+1}^*$ (note that $\mathbb{E}\eta_1^*$ can be different from 0, but this causes no problem), completing the proof of Lemma 6. $\square$

**Lemma 7.** *For any $N \ge 1$, $0 < \delta < 1$ we have*

$$\mathbb{P}\left(\sup_{0\le a\le\delta}\left|\sum_{k=1}^N (I_{(0,a)}(S_k) - a)\right| \gg \delta^{1/8}\sqrt{N} + N^{4/9}\right) \ll \delta^4 + N^{-2}.$$

P r o o f. For any $h \ge 1$, $1 \le j \le 2^h$ let $\varphi_h^{(j)}$ denote the indicator function of the interval $[(j-1)2^{-h}, j2^{-h})$ and put

$$F(N, j, h) = \left|\sum_{k=1}^N (\varphi_h^{(j)}(S_k) - 2^{-h})\right|.$$

Clearly, $\|\varphi_h^{(j)}\| = 2^{-h/2}$.

We observe that if $0 \le a \le 1$ has the dyadic expansion

$$a = \sum_{j=1}^{\infty} \varepsilon_j 2^{-j}, \qquad \varepsilon_j = 0, 1$$

and $H \ge 1$ is an arbitrary integer, then $g_a = I_{(0,a)}$ satisfies

$$\sum_{h=1}^{H} \varrho_h(x) \le g_a(x) \le \sum_{h=1}^{H} \varrho_h(x) + \sigma_H(x), \tag{26}$$

where $\varrho_h$ is the indicator function of

$$\left[ \sum_{j=1}^{h-1} \varepsilon_j 2^{-j}, \sum_{j=1}^{h} \varepsilon_j 2^{-j} \right)$$

and $\sigma_H$ is the indicator function of

$$\left[ \sum_{j=1}^{H} \varepsilon_j 2^{-j}, \sum_{j=1}^{H} \varepsilon_j 2^{-j} + 2^{-H} \right).$$

For $\varepsilon_h = 0$, clearly, $\varrho_h \equiv 0$ and thus (26) remains valid if in the sums we keep only those terms where $\varepsilon_h = 1$. Also, for $\varepsilon_h = 1$, $\varrho_h$ coincides with one of the $\varphi_h^{(j)}$ and $\sigma_H$ also coincides with some of the $\varphi_H^{(j)}$. It follows that

$$\begin{aligned} g_a(x) - a \ &\le \sum_{1 \le h \le H}^{*} \big(\varrho_h(x) - \varepsilon_h 2^{-h}\big) \\ &\qquad + \big(\sigma_H(x) - 2^{-H}\big) + 2^{-H} \end{aligned}$$

and

$$g_a(x) - a \ \ge \sum_{1 \le h \le H}^{*} \big(\varrho_h(x) - \varepsilon_h 2^{-h}\big) - 2^{-H},$$

where $\sum^*$ means that the summation is extended only for those $h$ such that $\varepsilon_h = 1$. Setting $x = S_k$ and summing for $1 \le k \le N$ we get

$$\begin{aligned} \sum_{k \le N} \big(g_a(S_k) - a\big) \ &\le \sum_{k \le N} \sum_{1 \le h \le H}^{*} \big(\varrho_h(S_k) - \varepsilon_h 2^{-h}\big) \\ &\qquad + \sum_{k \le N} \big(\sigma_H(S_k) - 2^{-H}\big) + N 2^{-H} \end{aligned}$$

and

$$\sum_{k \le N} \big(g_a(S_k) - a\big) \ \ge \sum_{k \le N} \sum_{1 \le h \le H}^{*} \big(\varrho_h(S_k) - \varepsilon_h 2^{-h}\big) - N 2^{-H}.$$

Hence it follows that for any $N \ge 1$, $H \ge 1$ there exist suitable integers $1 \le j_h \le 2^h$, $1 \le h \le H$ such that

$$\begin{aligned}\left|\sum_{k\le N}(g_a(S_k)-a)\right| &\le 2\sum_{h\le H}\left|\sum_{k\le N}\varphi_h^{(j_h)}(S_k)-2^{-h}\right|+N2^{-H}\\ &=2\sum_{h\le H}F(N,j_h,h)+N2^{-H}. \end{aligned}\tag{27}$$

Introduce the events

$$\begin{aligned}G(N,j,h)&=\left\{F(N,j,h)\ge 2^{-h/8}\sqrt{N}\right\},\\ G_N&=\bigcup_{A\le h\le B\log_2 N}\ \bigcup_{j\le 2^h}G(N,j,h)\quad\text{with}\quad A=\log_2\frac{1}{a},\quad B=\frac{5}{9}.\end{aligned}$$

For $h \le B\log_2 N$ we have

$$\|\varphi_j^{(h)}-2^{-h}\|\ge 2^{-h/2}-2^{-h}\ge 2^{-h/2}(1-1/\sqrt{2})\ge\frac{1}{5}N^{-5/18}$$

and thus applying (23) with $t=1$, we get

$$\mathbb{P}\big(G(N,h,j)\big)\ll\exp\big(-C2^{7h/40}\big)+N^{-3},$$

and consequently,

$$\mathbb{P}(G_N)\ll\sum_{h=A}^{\infty}2^h\exp\big(-C2^{7h/40}\big)+N^{-3}\sum_{h\le B\log_2 N}2^h. \tag{28}$$

Clearly, the second term on the right-hand side of (28) is $\ll N^{-2}$. On the other hand, the terms of the first sum in (28) decrease superexponentially and thus the sum can be bounded by a constant times its first term, i.e., the sum is

$$\ll 2^A\exp(-C2^{7A/40})\ll 2^{-4A}\ll a^4.$$

Hence

$$\mathbb{P}(G_N)\ll a^4+N^{-2}.$$

Note that when breaking the interval $(0,a)$ into dyadic intervals of length $2^{-h}$ we automatically have $h\ge\log\frac{1}{a}=A$ and thus choosing

$$H=[B\log_2 N],$$

it follows that with the exception of a set with probability $\ll a^4+N^{-2}$, for any $0<a\le\delta$ the expression in the second line of (27) is

$$\begin{aligned}\ll\sum_{A\le h\le H}2^{-h/8}\sqrt{N}+N2^{-H}\ll 2^{-A/8}\sqrt{N}+N^{4/9}\ll\\ a^{1/8}\sqrt{N}+N^{4/9}\ll\delta^{1/8}\sqrt{N}+N^{4/9}.\end{aligned}$$

This proves Lemma 7. □

Lemma 5 implies the convergence of the finite dimensional distributions of the sequence $\sqrt{N}(F_N(t)-t)$ in (5) to those of $K$ and to prove Theorem 2 it remains to prove the tightness of the sequence in $D[0,1]$. To this end, fix $\varepsilon > 0$ and choose $h$ so that $2^{-h} \le \varepsilon < 2^{-(h-1)}$. Note that for $j = 0, 1, \ldots, 2^h - 1$ we have

$$\mathbb{P}\left(\sup_{0\le a\le 2^{-h}}\left|\sum_{k=1}^{N}\bigl(f_{j2^{-h}+a}(S_k)-f_{j2^{-h}}(S_k)\bigr)\right| \gg 2^{-h/8}\sqrt{N}+N^{4/9}\right) \ll 2^{-4h}+N^{-2}. \quad (29)$$

For $j = 0$ relation (29) is identical with Lemma 7 and for $j = 1, 2, \ldots$ the proof is the same. It follows that

$$\mathbb{P}\left(\max_{0\le j\le 2^h-1}\sup_{0\le a\le 2^{-h}}\left|\sum_{k=1}^{N}\bigl(f_{j2^{-h}+a}(S_k)-f_{j2^{-h}}(S_k)\bigr)\right| \gg 2^{-h/8}\sqrt{N}+N^{4/9}\right) \ll 2^{-3h}+2^hN^{-2}. \quad (30)$$

Then (30) implies that with the exception of a set with probability

$$\ll 2^{-3h}+2^hN^{-2} \ll \varepsilon^3 + N^{-2}\varepsilon^{-1}$$

the fluctuation of the process $\sqrt{N}(F_N(t)-t)$ over any subinterval of $(0,1)$ with length $\le \varepsilon$ is

$$\ll \varepsilon^{1/8}+N^{-1/18}.$$

By Theorem 15.5 of Billingsley [4, p. 127], the sequence $\sqrt{N}(F_N(t)-t)$ is tight in $D[0,1]$. This completes the proof of Theorem 2; Theorem 1 follows immediately from Theorem 2.

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