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ON FREIMAN'S $3k-4$ THEOREM

Mario Huicochea

Institute of Physics of the Autonomous University of San Luis Potosí, MÉXICO

ABSTRACT. Let X and Y be nonempty finite subsets of $\mathbb{Z}$ and $X+Y$ its sumset. The structures of X and Y when $r(X,Y):=|X+Y|-|X|-|Y|$ is small have been widely studied; in particular the Generalized Freiman's $3k-4$ Theorem describes X and Y when $r(X,Y)\leq \min\{|X|,|Y|\}-4$. However, not too much is known about X and Y when $r(X,Y)>\min\{|X|,|Y|\}-4$. In this paper we study the structure of X and Y for arbitrary $r(X,Y)$.

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1. Introduction

In this paper $\mathbb{Z}$ and $\mathbb{N}$ are the set of integers and natural numbers, respectively; we consider $0\notin\mathbb{N}$. For any $x,y\in\mathbb{Z}$, we write $[x,y]:=\{z\in\mathbb{Z}:\ x\leq z\leq y\}$. For any nonempty subsets X and Y of $\mathbb{Z}$, write

$$X+Y:=\{x+y:\ x\in X,\, y\in Y\},$$
$$-X:=\{-x:\ x\in X\}.$$

When X and Y are finite, we write

$$r(X,Y):=|X+Y|-|X|-|Y|,$$
$$l(X):=\max X-\min X.$$

We denote by $\mathrm{GCD}(X)$ the greatest common divisor of the elements of X and $\mathrm{GCD}^*(X):=\mathrm{GCD}(X-X)$; note that $\mathrm{GCD}^*(X)=\mathrm{GCD}(X-\{x\})$ for any $x\in X$.

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The relation between $r(X,Y)$ and the structures of X and Y has been widely studied, see [1], [2], [3], [4], [5], [6], [10], [11], [12], [13]. We start recalling a result of I. Ruzsa which bounds $r(X,Y)$.

Theorem 1.1. *Let X and Y be nonempty finite subsets of $\mathbb{Z}$ such that*

$$\mathrm{GCD}^*(X) = 1 \quad \text{and} \quad l(X) \geq 3.$$

If $|Y| > \frac{(l(X)-1)(l(X)-2)}{2}$, then

$$r(X,Y) \geq l(X) - |X|.$$

P r o o f. See [11, Thm. 3.6.1]. □

Let X and Y be nonempty subsets. Write

$$\delta(X,Y) := \begin{cases} 1 & \text{if} \quad \{x\} + X \subseteq Y \text{ for some } x \in \mathbb{Z}; \\ 0, & \text{otherwise.} \end{cases}$$

Freiman's $3k-4$ Theorem is one of the most important inverse theorems in additive number theory. D. Grynkiewicz generalized this theorem. We state a slightly weaker version of his generalization.

Theorem 1.2. *Let X and Y be nonempty finite subsets of $\mathbb{Z}$ such that*

$$\mathrm{GCD}^*(X+Y) = 1.$$

If $r(X,Y) \leq \min\{|X| - \delta(Y,X), |Y| - \delta(X,Y)\} - 3$, then

$$l(X) \leq |X| + r(X,Y), \quad l(Y) \leq |Y| + r(X,Y)$$

and there is $z \in \mathbb{Z}$ such that $[z, z+|X|+|Y|-2] \subseteq X+Y$.

P r o o f. See [5, Thm. 7.1]. □

The assumption $r(X,Y) \leq \min\{|X| - \delta(Y,X), |Y| - \delta(X,Y)\} - 3$ cannot be weakened in Theorem 1.2. For instance, if x, y are positive integers with $y > 3x+1$ and $X = Y = [0,x] \cup [y, y+x]$, then

$$r(X,Y) = 2x - 1 = \min\{|X| - \delta(Y,X), |Y| - \delta(X,Y)\} - 2,$$

but

$$l(X) = l(Y) = y + x.$$

Thus, since $y > 3x+1$,

$$l(X) > |X| + r(X,Y);$$

moreover, if x is fixed and y grows, then $l(X)$ is much larger than $|X| + r(X,Y)$. The purpose of this paper is to obtain similar results to the ones of Theorem 1.2 when we do not have necessarily that

$$r(X,Y) \leq \min\{|X| - \delta(Y,X), |Y| - \delta(X,Y)\} - 3.$$

For nonempty subsets X and Y of $\mathbb{Z}$, we say that Y is X-*disconnected* if there are Y_1 and Y_2 nonempty subsets of Y such that $Y = Y_1 \cup Y_2$ and $(X+Y_1)\cap(X+Y_2) = \emptyset$. If Y is not X-disconnected, we say that it is X-*connected*. The maximal $X-$connected subsets of Y are called the X-*components* of Y and they induce a partition of Y, see [5, p. 16]. Thus, to understand the structure of X and Y, it suffices to study the structure of the X-components of Y. Hence it suffices to study X and Y when $\mathrm{GCD}^*(X) = 1$ and Y is X-connected.

We state the first theorem of this paper.

Theorem 1.3. *Let X and Y be nonempty finite subsets of $\mathbb{Z}$ such that*

$$\mathrm{GCD}^*(X) = 1, \quad |X| \geq 3 \quad \textit{and } Y \textit{ is } X\textit{-connected, and}$$

let $z \in [\min(X+Y) - l(X) + 1, \max(X+Y)]$. Set $I := [\min(X+Y)+1, \max(X+Y) - l(X)]$ and $u := |(X+Y) \cap [z, z+l(X)-1]|$. Then

$$l(Y) \leq \begin{cases} |Y| + \big(l(X)-1\big)\left(\frac{2r(X,Y)-2u+l(X)+3}{|X|-2} + 4\right) & \textit{if} \quad z \in I; \\ |Y| + \big(l(X)-1\big)\left(\frac{4r(X,Y)-4u+l(X)+7}{2(|X|-2)} + 3\right) & \textit{if} \quad z \notin I. \end{cases}$$

An immediate consequence of Theorem 1.3 is that, with the notation and assumption as above, if $[z, z+l(X)-1] \subseteq X+Y$, then

$$l(Y) \leq |Y| + \big(l(X)-1\big)\left(\frac{2r(X,Y) - l(X) + 3}{|X|-2} + 4\right).$$

More generally, if $|(X+Y)\cap[z, z+l(X)-1]|$ is big, the upper bound of Theorem 1.3 is better. The next theorem shows that if $|Y|$ is big enough, then there is $z \in \mathbb{Z}$ such that $|(X+Y)\cap[z, z+l(X)-1]|$ is big.

Theorem 1.4. *Let X and Y be nonempty finite subsets of $\mathbb{Z}$ such that*

$$\mathrm{GCD}^*(X) = 1, \quad |X| \geq 3 \quad \textit{and } Y \textit{ is } X\textit{-connected.}$$

For all $u \in [0, l(X) - |X| + 1]$ such that

$$|Y| > \big(l(X)-u-1\big)\left(\frac{4r(X,Y) + l(X) - u + 3}{2(|X|-2)} + 2\right),$$

there is $z \in \mathbb{Z}$ such that

$$|(X+Y)\cap[z, z+l(X)-1]| \geq l(X) - u \quad \textit{and} \quad r(X,Y) \geq l(X) - |X| - u.$$

Using the previous theorems, we will obtain two inverse results. The first one states that, if X and Y are nonempty finite subsets of $\mathbb{Z}$ with $|X| \geq 3$ and $\mathrm{GCD}^*(X) = 1$, then the number of $X-$connected components of Y is bounded by $r(X,Y)$ and each component is either small or its length is small compared with its size.

Corollary 1.5. *Let X and Y be nonempty finite subsets of $\mathbb{Z}$ with $|X| \geq 3$ and* $\mathrm{GCD}^*(X) = 1$*. Denote by $Y_1, Y_2, \ldots, Y_n$ the X-components of Y.*

i) $r(X, Y) = (n-1)|X| + \sum_{i=1}^{n} r(X, Y_i)$*. In particular*

$$n \leq \frac{r(X,Y)+1}{|X|-1} + 1.$$

ii) *For each $i \in [1, n]$, let u_i be the minimum nonnegative integer u such that*

$$|Y_i| > \big(l(X) - u - 1\big)\left(\frac{4r(X,Y_i) + l(X) - u + 3}{2(|X|-2)} + 2\right).$$

Then

$$|Y_i| \leq (|X|-2)\left(\frac{2r(X,Y_i)+2}{|X|-2} + \frac{5}{2}\right)$$

or

$$l(Y_i) \leq |Y_i| + \big(l(X) - 1\big)\left(\frac{2r(X,Y_i) - l(X) + 2u_i + 3}{|X|-2} + 4\right).$$

Theorem 1.2 gives nontrivial upper bounds of $l(X)$ and $l(Y)$ when $r(X,Y) \leq \min\{|X| - \delta(Y,X), |Y| - \delta(X,Y)\} - 3$ and $\mathrm{GCD}^*(X+Y) = 1$. The next result roughly says that we can find nontrivial upper bounds of $l(X)$ and $l(Y)$ in terms of $|X|$ and $r(X,Y)$ when $|X| \geq 3$, $\mathrm{GCD}^*(X) = 1$, Y is X-connected and $|Y|$ is big compared with $|X|$. In particular we can bound $l(X)$ and $l(Y)$ even if $r(X,Y) > \min\{|X| - \delta(Y,X), |Y| - \delta(X,Y)\} - 3$.

Corollary 1.6. *Let X and Y be nonempty finite subsets of $\mathbb{Z}$ such that $|X| \geq 3$,* $\mathrm{GCD}^*(X) = 1$*, and Y is X-connected. If*

$$|Y| > \big(|X| + r(X,Y)\big)\left(\frac{5(r(X,Y)+1)}{2(|X|-2)} + \frac{5}{2}\right),$$

then

$$l(X) \leq |X| + r(X,Y),$$

$$l(Y) \leq |Y| + \big(|X| + r(X,Y) - 1\big)\left(\frac{2(r(X,Y)+1)}{|X|-2} + 3\right).$$

As we already saw in the example given after Theorem 1.2, the condition that Y is X-connected in Corollary 1.6 is fundamental. Corollary 1.6 can be seen as a partial result in the generalization of Freiman's $3k-4$ Theorem for sets X, Y which do not satisfy that $r(X,Y) \leq \min\{|X| - \delta(Y,X), |Y| - \delta(X,Y)\} - 3$. When $r(X,Y) \leq \min\{|X| - \delta(Y,X), |Y| - \delta(X,Y)\} - 3$, Theorem 1.2 provides better (optimal) bounds of $l(X)$ and $l(Y)$ than Corollary 1.6. Hence we think that even in the case $r(X,Y) > \min\{|X| - \delta(Y,X), |Y| - \delta(X,Y)\} - 3$ our results are not best possible. However, in the last section of this paper, we will give some examples which show that even if our results are not optimal, the bounds we have obtained are far from trivial.

Whenever $\mathrm{GCD}^*(X) > 1$, we can translate and make a dilation by $\frac{1}{\mathrm{GCD}^*(X)}$ to obtain analogous results to the previous statements.

Now we describe roughly the main idea behind the proofs of Theorem 1.3 and Theorem 1.4. Let X and Y be nonempty finite subsets of $\mathbb{Z}$ with $\min X = \min Y = 0$, $\mathrm{GCD}^*(X) = 1$, Y be X-connected and $|Y| \gg |X|$. Then Theorem 1.1 implies that $r(X,Y) \geq l(X) - |X|$. Assume that a is a small nonnegative integer such that $r(X,Y) = l(X) - |X| + a$; then $|X+Y| - |Y| = l(X) + a$. If a is small and $|Y|$ is big, then the former equality implies that we must have that $y + X \subseteq Y$ for many $y \in Y$. Therefore, if we set

$$\Gamma(X) := \{x_1 + x_2 + \ldots + x_n : \ x_1, x_2, \ldots, x_n \in X\},$$

$$\Gamma(\{\max X\} - X) := \{x_1 + x_2 + \ldots + x_n : \ x_1, x_2, \ldots, x_n \in \{\max X\} - X\},$$

then there is $z \in [0, \max Y]$ such that $Y \cap [0,z]$ (resp. $Y \cap [z+1, \max Y]$) is quite similar to $\Gamma(X) \cap [0,z]$ (resp. $(\{\max Y\} - \Gamma(\{\max X\} - X)) \cap [z+1, \max Y]$) up to the defects generated by a. Thus, to understand the structure of Y, it suffices to understand the structure of

$$\Gamma(X) \cap [0,z] \quad \text{and} \quad \big(\{\max Y\} - \Gamma(\{\max X\} - X)\big) \cap [z+1, \max Y]$$

with the defects generated by a; however, this is easier and we may use some combinatorial results and Lev-Smeliansky Theorem to study $\Gamma(X), \Gamma(\{\max X\} - X)$ with the singularities generated by a.

We briefly describe the content of this paper. In Section 2 we study the structure of the subsets Z and W of $\mathbb{Z}$ such that Z and $(Z+W) \setminus W$ are finite, $\min Z = \min W = 0$, $\mathrm{GCD}(Z) = 1$, $|Z| \geq 3$ and $W \cap [z, z + l(Z) - 1] \neq \emptyset$ for all $z \geq 0$. In particular we shall show two results (see Proposition 2.4 and Proposition 2.5 for the precise statements):

i) If Z and W are as above, then the set of nonnegative integers which are not in W is finite and bounded in terms of $l(Z), |Z|$ and $|(Z+W) \setminus W|$.

ii) If Z and W are as above and $x \in \mathbb{Z}$ is such that $|[0,x] \cap W|$ is big enough, then there is $z \in [0,x]$ such that $|(Z+W) \cap [z, z + l(Z) - 1]|$ is big.

In Section 3 an auxiliary statement links the results of the previous section and the main theorems, and then we proceed to prove the main results of this paper. The key point in the proof of Theorem 1.3 is Proposition 2.4. The principal idea in the proof of Theorem 1.4 is Proposition 2.5. In Section 3 we prove Corollary 1.5 and Corollary 1.6. They are straightforward consequences of Theorem 1.3 and Theorem 1.4. At the end of this section, we discuss the sharpness of our results.

2. Structural tools

The purpose of this section is to study the structure of nonempty subsets Z and W of $\mathbb{Z}$ such that Z and $(Z+W)\setminus W$ are finite, $\min Z = \min W = 0$, $\mathrm{GCD}(Z)=1$, $|Z|\geq 3$ and $W\cap[z,z+l(Z)-1]\neq\emptyset$ for all $z\geq 0$. The first tool that we will need is the following theorem of V. Lev and P. Smeliansky.

Theorem 2.1. *Let X and Y be nonempty finite subsets of $\mathbb{Z}$ such that*

$$\mathrm{GCD}^*(Y)=1 \quad \textit{and} \quad l(X)\leq l(Y),$$

and define

$$\delta := \begin{cases} 1 & \textit{if} \;\; l(X)=l(Y); \\ 0, & \textit{otherwise.} \end{cases}$$

Then

$$|X+Y|\geq |X|+\min\{|X|+|Y|-2-\delta, l(Y)\}.$$

Proof. See [10, Thm. 2]. □

We use the previous theorem in the following lemma.

Lemma 2.2. *Let Z and W be nonempty subsets of $\mathbb{Z}$ such that*

$$Z \textit{ is finite,} \quad \min Z=\min W=0 \quad \textit{and} \quad \mathrm{GCD}(Z)=1.$$

For all $z\in\mathbb{Z}$ such that $W\cap[z,z+l(Z)-1]\neq\emptyset$, we have that

$$\begin{aligned}|W\cap[z+l(Z),z+2l(Z)-1]| \geq{}& \min\{|W\cap[z,z+l(Z)-1]|+|Z|-2, l(Z)\} \\ &- \left|\big((Z+W)\setminus W\big)\cap[z,z+2l(Z)-1]\right|.\end{aligned}$$

Proof. On the one hand,

$$Z+\big(W\cap[z,z+l(Z)-1]\big)\subseteq (Z+W)\cap[z,z+2l(Z)-1];$$

therefore

$$\begin{aligned}&\Big(Z+\big(W\cap[z,z+l(Z)-1]\big)\Big)\setminus\big(W\cap[z,z+2l(Z)-1]\big)\\ &=\Big(Z+\big(W\cap[z,z+l(Z)-1]\big)\Big)\setminus W\\ &\subseteq \big((Z+W)\cap[z,z+2l(Z)-1]\big)\setminus W\\ &=\big((Z+W)\setminus W\big)\cap[z,z+2l(Z)-1].\end{aligned}$$

This inclusion implies that

$$\begin{aligned}&\left|Z+\big(W\cap[z,z+l(Z)-1]\big)\right|-|W\cap[z,z+2l(Z)-1]|\\ &\leq \left|\big((Z+W)\setminus W\big)\cap[z,z+2l(Z)-1]\right|,\end{aligned}$$

and hence

$$\begin{aligned}\bigl|Z+\bigl(W\cap[z,z+l(Z)-1]\bigr)\bigr| \leq& |W\cap[z,z+l(Z)-1]| \\ &+|W\cap[z+l(Z),z+2l(Z)-1]| \\ &+\bigl|\bigl((Z+W)\setminus W\bigr)\cap[z,z+2l(Z)-1]\bigr|. \qquad (1)\end{aligned}$$

On the other hand, we have that

$$l\bigl(W\cap[z,z+l(Z)-1]\bigr)\leq l\bigl([z,z+l(Z)-1]\bigr)<l(Z)$$

so, applying Theorem 2.1 to $W\cap[z,z+l(Z)-1]$ and Z, we get that

$$\begin{aligned}&\bigl|Z+\bigl(W\cap[z,z+l(Z)-1]\bigr)\bigr| \qquad (2)\\ &\geq|W\cap[z,z+l(Z)-1]|+\min\{|W\cap[z,z+l(Z)-1]|+|Z|-2,l(Z)\}.\end{aligned}$$

The claim follows from (1) and (2). □

We shall need a combinatorial lemma.

Lemma 2.3. *Let $l,t\in\mathbb{N}$ be such that $t\leq l$, and let $\{r_n\}_{n\in\mathbb{N}}$ and $\{k_n\}_{n\in\mathbb{N}}$ be sequences of nonnegative integers such that $r:=\sum_{n\in\mathbb{N}}r_n$ is finite and $r_n\geq k_nt$ for all $n\in\mathbb{N}$. If $r\geq l$, then*

$$\sum_{n\in\mathbb{N}}\sum_{k=0}^{k_n}\min\{l,r_n-kt\}\leq\left(\frac{2r-l}{2t}+1\right)l.$$

Proof. Insomuch as $r_n\geq k_nt$ for all $n\in\mathbb{N}$,

$$\sum_{n\in\mathbb{N}}\sum_{k=0}^{k_n}\min\{l,r_n-kt\}\leq\sum_{n\in\mathbb{N}}\sum_{k=0}^{\lfloor\frac{r_n}{t}\rfloor}\min\{l,r_n-kt\}. \qquad (3)$$

Now, we show by induction on $\lfloor\frac{r}{t}\rfloor$ that

$$\sum_{n\in\mathbb{N}}\sum_{k=0}^{\lfloor\frac{r_n}{t}\rfloor}\min\{l,r_n-kt\}\leq\sum_{k=0}^{\lfloor\frac{r}{t}\rfloor}\min\{l,r-kt\}. \qquad (4)$$

Since $t\leq l\leq r$, the basis of induction will be $\lfloor\frac{r}{t}\rfloor=1$. If $\lfloor\frac{r_n}{t}\rfloor=0$ for all $n\in\mathbb{N}$, then

$$\sum_{n\in\mathbb{N}}\sum_{k=0}^{\lfloor\frac{r_n}{t}\rfloor}\min\{l,r_n-kt\}=\sum_{n\in\mathbb{N}}r_n=r\leq l+r-t=\sum_{k=0}^{\lfloor\frac{r}{t}\rfloor}\min\{l,r-kt\}.$$

Assume that $\lfloor\frac{r_n}{t}\rfloor>0$ for some $n\in\mathbb{N}$; without loss of generality, assume that $\lfloor\frac{r_1}{t}\rfloor>0$. Since $\lfloor\frac{r}{t}\rfloor=1$, we get that $\lfloor\frac{r_1}{t}\rfloor=1$ and $\lfloor\frac{r_n}{t}\rfloor=0$ for all $n>1$.

Thus

$$\sum_{n\in\mathbb{N}}\sum_{k=0}^{\lfloor\frac{r_n}{t}\rfloor}\min\{l,r_n-kt\}=l+r_1-t+\sum_{n>1}r_n=l+r-t=\sum_{k=0}^{\lfloor\frac{r}{t}\rfloor}\min\{l,r-kt\},$$

and this completes the proof of the basis of induction. Now, we assume that $\lfloor\frac{r}{t}\rfloor>1$ and that (4) holds for all t,r',r such that $\lfloor\frac{r'}{t}\rfloor<\lfloor\frac{r}{t}\rfloor$. If $\lfloor\frac{r_n}{t}\rfloor=0$ for all $n\in\mathbb{N}$, then

$$\begin{aligned}\sum_{n\in\mathbb{N}}\sum_{k=0}^{\lfloor\frac{r_n}{t}\rfloor}\min\{l,r_n-kt\}&=\sum_{n\in\mathbb{N}}r_n\\&=r\\&=\overbrace{t+t+\cdots+t}^{\lfloor\frac{r}{t}\rfloor\text{-times}}+\left(r-t\left\lfloor\frac{r}{t}\right\rfloor\right)\\&\le\left(\sum_{k=0}^{\lfloor\frac{r}{t}\rfloor-1}\min\{l,r-kt\}\right)+\left(r-t\left\lfloor\frac{r}{t}\right\rfloor\right)\\&=\sum_{k=0}^{\lfloor\frac{r}{t}\rfloor}\min\{l,r-kt\},\end{aligned}$$

and this shows (4) in this case. Assume from now on that there is $n\in\mathbb{N}$ such that $\lfloor\frac{r_n}{t}\rfloor>0$; without loss of generality, assume that $\lfloor\frac{r_1}{t}\rfloor>0$. Define

$$r'_n:=\begin{cases}r_n-t & \text{if } n=1,\\ r_n, & \text{otherwise,}\end{cases}$$

and define $r':=\sum_{n\in\mathbb{N}}r'_n$. From the hypothesis of induction,

$$\sum_{n\in\mathbb{N}}\sum_{k=0}^{\lfloor\frac{r'_n}{t}\rfloor}\min\{l,r'_n-kt\}\le\sum_{k=0}^{\lfloor\frac{r'}{t}\rfloor}\min\{l,r'-kt\}.\tag{5}$$

On the other hand,

$$\min\{l,r_1\}\le\min\{l,r\}.\tag{6}$$

Adding (5) and (6), we get (4) and this completes its proof.

We shall prove that

$$\sum_{k=\lfloor\frac{r-l}{t}\rfloor+1}^{\lfloor\frac{r}{t}\rfloor}(r-kt)+\sum_{k=0}^{\lfloor\frac{r-l}{t}\rfloor}l\le l\left(\frac{2r-l}{2t}+1\right)\tag{7}$$

by induction on $\lfloor\frac{r}{t}\rfloor$. Since $t\le l\le r$, we have that $\lfloor\frac{r}{t}\rfloor\ge1$. First assume that $\lfloor\frac{r}{t}\rfloor=1$ so $\lfloor\frac{r-l}{t}\rfloor=0$ (since $r<2t\le t+l$). Moreover, since $t\le l\le r<2t$, we get that

$$r-t \leq l\left(\frac{2r-l}{2t}\right),$$

and therefore

$$\sum_{k=\lfloor\frac{r-l}{t}\rfloor+1}^{\lfloor\frac{r}{t}\rfloor}(r-kt)+\sum_{k=0}^{\lfloor\frac{r-l}{t}\rfloor} l = r-t+l \leq l\left(\frac{2r-l}{2t}+1\right).$$

Assume that $\lfloor\frac{r}{t}\rfloor > 1$ and set $r' := r-t$. Hence by induction

$$\begin{aligned}\sum_{k=\lfloor\frac{r-l}{t}\rfloor+1}^{\lfloor\frac{r}{t}\rfloor}(r-kt)+\sum_{k=0}^{\lfloor\frac{r-l}{t}\rfloor} l &= \left(\sum_{k=\lfloor\frac{r'-l}{t}\rfloor+1}^{\lfloor\frac{r'}{t}\rfloor}(r'-kt)+\sum_{k=0}^{\lfloor\frac{r'-l}{t}\rfloor} l\right)+l\\ &\leq l\left(\frac{2r'-l}{2t}+1\right)+l\\ &= l\left(\frac{2r-l}{2t}+1\right),\end{aligned}$$

and this completes the proof of (7). Now we conclude the proof of the lemma

$$\begin{aligned}\sum_{n\in\mathbb{N}}\sum_{k=0}^{k_n}\min\{l, r_n-kt\} &\leq \sum_{n\in\mathbb{N}}\sum_{k=0}^{\lfloor\frac{r_n}{t}\rfloor}\min\{l, r_n-kt\} && \text{(by (3))}\\ &\leq \sum_{k=0}^{\lfloor\frac{r}{t}\rfloor}\min\{l, r-kt\} && \text{(by (4))}\\ &= \sum_{k=\lfloor\frac{r-l}{t}\rfloor+1}^{\lfloor\frac{r}{t}\rfloor}(r-kt)+\sum_{k=0}^{\lfloor\frac{r-l}{t}\rfloor} l\\ &\leq l\left(\frac{2r-l}{2t}+1\right). && \text{(by (7))}\end{aligned}$$

□

For any $x \in \mathbb{Z}$, set $[x,\infty) := \{z \in \mathbb{Z} :\ z \geq x\}$ and $(-\infty, x] := \{z \in \mathbb{Z} :\ z \leq x\}$.

Proposition 2.4. *Let Z and W be nonempty subsets of $\mathbb{Z}$ such that Z and $(Z+W)\setminus W$ are finite, $\min Z = \min W = 0$, $\mathrm{GCD}(Z) = 1$ and $|Z| \geq 3$, and let $u \in [1, l(Z)-|Z|+2]$. If $|W \cap [z, z+l(Z)-1]| \geq u$ for all $z \in [0,\infty)$, then*

$$|[0,\infty)\setminus W| \leq \left(\frac{2|(Z+W)\setminus W|}{|Z|-2}+\frac{l(Z)-u}{2(|Z|-2)}+1\right)\bigl(l(Z)-u\bigr).$$

P r o o f. For each $n \in \mathbb{N}$, set

$$W_n := W \cap [(n-1)l(Z), nl(Z)-1],$$
$$S_n := \big((Z+W)\setminus W\big) \cap [(n-1)l(Z), (n+1)l(Z)-1].$$

For all $z \in (Z+W)\setminus W$, it is contained in at most two elements of $\{S_n\}_{n\in\mathbb{N}}$ so

$$\sum_{n\in\mathbb{N}} |S_n| \le 2|(Z+W)\setminus W|. \tag{8}$$

Since $(Z+W)\setminus W$ is finite, it is implied by (8) that there is $n_0 \in \mathbb{N}$ such that $S_n = \emptyset$ for all $n > n_0$. For all $n \in \mathbb{N}$, it is assumed that $|W_n| \ge u \ge 1$. In particular W_n is not empty, and hence Lemma 2.2 applied to Z and W_n leads to

$$|W_{n+1}| \ge \min\{|W_n| + |Z| - 2, l(Z)\} - |S_n|, \tag{9}$$

Also, since $|W_n| \ge u$, we have that

$$l(Z) - |W_n| \le l(Z) - u. \tag{10}$$

Define $N = \{n \in \mathbb{N} : \ |W_n| < l(Z) - |Z| + 2\}$. Since $S_n = \emptyset$ for all $n > n_0$ and $|W_{n_0}| \ge u \ge 1$, we obtain from (9) that $|W_n| > l(Z)-1$ for all $n > n_0 + \frac{l(Z)-1}{|Z|-2} + 1$; thus N is finite. Let $m := |(N+\{1\})\setminus N|$ and $p_1, p_2, \dots, p_m, q_1, q_2, \dots, q_m$ be such that $N = \bigcup_{i=1}^m [p_i, q_i]$, and define $N' := \bigcup_{i=1}^m [p_i - 1, q_i]$; assume without loss of generality that $p_1 < p_2 < \cdots < p_m$. For all $n \in \mathbb{N}\setminus N$, we get that $|W_n| + |Z| - 2 \ge l(Z)$ and then (9) yields that

$$l(Z) - |W_{n+1}| \le |S_n|. \tag{11}$$

For all $i \in [1,m]$ and $n \in [p_i, q_i]$, if (9) is applied $n+1-p_i$-times, we get

$$|W_{n+1}| \ge |W_{p_i}| + (n+1-p_i)(|Z|-2) - \sum_{k=p_i}^{n} |S_k|,$$

and therefore

$$\begin{aligned} l(Z) - |W_{n+1}| &\le l(Z) - |W_{p_i}| - (n+1-p_i)(|Z|-2) + \sum_{k=p_i}^{n} |S_k| \\ &\le l(Z) - |W_{p_i}| - (n+1-p_i)(|Z|-2) + \sum_{k=p_i}^{q_i} |S_k|. \end{aligned} \tag{12}$$

We divide the conclusion of the proof into two cases:

- Assume that $1 \in N$ so $p_1 = 1$. We have that for all $n \in [0, q_1]$

$$\begin{aligned} l(Z) - |W_{n+1}| &\le l(Z) - |W_1| - n(|Z|-2) + \sum_{k=1}^{q_1} |S_k| && \text{(by (12))} \\ &\le l(Z) - u - n(|Z|-2) + \sum_{k=1}^{q_1} |S_k|. && \text{(by (10))} \end{aligned} \tag{13}$$

For all $i>1$, note that $p_i-1\in\mathbb{N}\setminus N$; thus for all $n\in[p_i-1,q_i]$, we get that

$$\begin{aligned} l(Z)-|W_{n+1}| &\le l(Z)-|W_{p_i}|-(n+1-p_i)(|Z|-2)+\sum_{k=p_i}^{q_i}|S_k| && \text{(by (12))}\\ &\le -(n+1-p_i)(|Z|-2)+\sum_{k=p_i-1}^{q_i}|S_k|. && \text{(by (11))} \end{aligned} \tag{14}$$

On the one hand, from (10), (13) and (14), we get that

$$\begin{aligned} &\sum_{i=1}^{m}\sum_{n=p_i-1}^{q_i}\big(l(Z)-|W_{n+1}|\big)\le\\ &\quad\left(\sum_{n=0}^{q_1}\min\left\{l(Z)-u-n(|Z|-2)+\sum_{k=1}^{q_1}|S_k|,l(Z)-u\right\}\right)+\\ &\quad\left(\sum_{i=2}^{m}\sum_{n=p_i-1}^{q_i}\min\left\{-(n+1-p_i)(|Z|-2)+\sum_{k=p_i-1}^{q_i}|S_k|,l(Z)-u\right\}\right). \end{aligned} \tag{15}$$

On the other hand, from (10) and (11), we obtain that

$$\sum_{n\in\mathbb{N}\setminus N'}\big(l(Z)-|W_{n+1}|\big)\le\sum_{n\in\mathbb{N}\setminus N'}\min\big\{|S_n|,l(Z)-u\big\}. \tag{16}$$

If we add (15) and (16), we deduce that

$$\begin{aligned} &|[0,\infty)\setminus W|\\ &=\sum_{n\in\mathbb{N}}|[(n-1)l(Z),nl(Z)-1]\setminus W_n|\\ &=\sum_{n\in\mathbb{N}}(l(Z)-|W_n|)\\ &\le\left(\sum_{n=0}^{q_1}\min\left\{l(Z)-u-n(|Z|-2)+\sum_{k=1}^{q_1}|S_k|,l(Z)-u\right\}\right)\\ &\quad+\left(\sum_{i=2}^{m}\sum_{n=p_i-1}^{q_i}\min\left\{-(n+1-p_i)(|Z|-2)+\sum_{k=p_i-1}^{q_i}|S_k|,l(Z)-u\right\}\right)\\ &\quad+\left(\sum_{n\in\mathbb{N}\setminus N'}\min\left\{|S_n|,l(Z)-u\right\}\right). \end{aligned} \tag{17}$$

Label the elements of N' as $N' = \{t_{m+1}, t_{m+2}, t_{m+3} \ldots\}$. We want to apply Lemma 2.3 to

$$r_n = \begin{cases} l(Z) - u + \sum_{k=0}^{q_1} |S_k| & \text{if } n = 1, \\ \sum_{k=p_n-1}^{q_n} |S_k| & \text{if } n \in [2, m], \\ |S_{t_n}| & \text{if } n > m; \end{cases}$$

$$k_n = \begin{cases} q_n - p_n + 1 & \text{if } n \leq m, \\ 0 & \text{if } n > m; \end{cases}$$

$$l = l(Z) - u;$$

$$t = |Z| - 2.$$

On the one hand,

$$l \leq \sum_{n \in \mathbb{N}} r_n = l(Z) - u + \sum_{n \in \mathbb{N}} |S_n| \leq l(Z) - u + 2|(Z + W) \setminus W|,$$

then $\sum_{n \in \mathbb{N}} r_n$ is finite by (8). On the other hand, $l(Z) - |W_n| \geq 0$ for all $n \in \mathbb{N}$; hence (11), (13) and (14) imply that $r_n \geq k_n t$ for all $n \in \mathbb{N}$. Thus the assumptions of Lemma2.3 are satisfied, and we obtain from it and (17) that

$$|[0, \infty) \setminus W| \leq \left(\frac{l(Z) - u + 2\sum_{n \in \mathbb{N}} |S_n|}{2(|Z| - 2)} + 1 \right) \big(l(Z) - u \big). \tag{18}$$

Then

$$\begin{aligned} |[0, \infty) \setminus W| &\leq \left(\frac{l(Z) - u + 2\sum_{n \in \mathbb{N}} |S_n|}{2(|Z| - 2)} + 1 \right) \big(l(Z) - u \big) && \text{(by (18))} \\ &\leq \left(\frac{l(Z) - u}{2(|Z| - 2)} + \frac{2|(Z + W) \setminus W|}{|Z| - 2} + 1 \right) \big(l(Z) - u \big). && \text{(by (8))} \end{aligned}$$

- Assume that $1 \notin N$. For all $i \in [1, m]$, note that $p_i - 1 \in \mathbb{N} \setminus N$; thus for all $n \in [p_i - 1, q_i]$, we get that

$$\begin{aligned} l(Z) - |W_{n+1}| &\leq l(Z) - |W_{p_i}| - (n + 1 - p_i)(|Z| - 2) + \sum_{k=p_i}^{q_i} |S_k| && \text{(by (12))} \\ &\leq -(n + 1 - p_i)(|Z| - 2) + \sum_{k=p_i-1}^{q_i} |S_k|. && \text{(by (11))} \end{aligned} \tag{19}$$

From (10) and (19), we get that

$$\sum_{i=1}^{m}\sum_{n=p_i-1}^{q_i}\big(l(Z)-|W_{n+1}|\big)\le \sum_{i=1}^{m}\sum_{n=p_i-1}^{q_i}\min\Big\{-(n+1-p_i)(|Z|-2)+\sum_{k=p_i-1}^{q_i}|S_k|,l(Z)-u\Big\}. \tag{20}$$

From (10) and (11), we obtain that

$$\sum_{n\in\mathbb{N}\setminus N'}\big(l(Z)-|W_{n+1}|\big)\le\sum_{n\in\mathbb{N}\setminus N'}\min\big\{|S_n|,l(Z)-u\big\}. \tag{21}$$

Also, from (10),

$$l(Z)-|W_1|\le l(Z)-u. \tag{22}$$

If we add (20), (21) and (22), then we have that

$$\begin{aligned}|[0,\infty)\setminus W| &= \sum_{n\in\mathbb{N}}|[(n-1)l(Z),nl(Z)-1]\setminus W_n| = \sum_{n\in\mathbb{N}} l(Z)-|W_n|\\ &\le \left(\sum_{i=1}^{m}\sum_{n=p_i-1}^{q_i}\min\Big\{-(n+1-p_i)(|Z|-2)+\sum_{k=p_i-1}^{q_i}|S_k|,l(Z)-u\Big\}\right)\\ &\quad+\left(\sum_{n\in\mathbb{N}\setminus N'}\min\{|S_n|,l(Z)-u\}\right)+l(Z)-u. \end{aligned} \tag{23}$$

Label the elements of N' as $N'=\{t_{m+2},t_{m+3},t_{m+4},\ldots\}$. We want to apply Lemma 2.3 to

$$r_n=\begin{cases}\sum_{k=p_n-1}^{q_n}|S_k| & \text{if } n\in[1,m],\\ l(Z)-u & \text{if } n=m+1,\\ |S_{t_n}| & \text{if } n>m+1;\end{cases}\qquad k_n=\begin{cases} q_n-p_n+1 & \text{if } n\le m,\\ 0 & \text{if } n>m;\end{cases}$$

$$l=l(Z)-u;$$

$$t=|Z|-2.$$

On the one hand,

$$l\le\sum_{n\in\mathbb{N}}r_n=l(Z)-u+\sum_{n\in\mathbb{N}}|S_n|\le l(Z)-u+2|(Z+W)\setminus W|;$$

then $\sum_{n\in\mathbb{N}}r_n$ is finite by (8). On the other hand, $l(Z)-|W_n|\ge 0$ for all $n\in\mathbb{N}$; hence (11), (19) and (22) imply that $r_n\ge k_nt$ for all $n\in\mathbb{N}$. Thus the assumptions of Lemma (2.3) are satisfied, and we get from it and (23) that

$$|[0,\infty)\setminus W| \le \left(\frac{l(Z)-u+2\sum_{n\in\mathbb{N}}|S_n|}{2(|Z|-2)}+1\right)\big(l(Z)-u\big). \tag{24}$$

Hence

$$\begin{aligned}|[0,\infty)\setminus W| &\le \left(\frac{l(Z)-u+2\sum_{n\in\mathbb{N}}|S_n|}{2(|Z|-2)}+1\right)\big(l(Z)-u\big) && \text{(by (24))}\\ &\le \left(\frac{l(Z)-u}{2(|Z|-2)}+\frac{2|(Z+W)\setminus W|}{|Z|-2}+1\right)\big(l(Z)-u\big). && \text{(by (8))}\end{aligned}$$

□

Proposition 2.5. *Let Z and W be nonempty subsets of $\mathbb{Z}$ such that Z and $(Z+W)\setminus W$ are finite,* $\min Z=\min W=0$, $\mathrm{GCD}(Z)=1$ *and* $|Z|\ge 3$. *Take $u\in[0,l(Z)-|Z|+1]$ and $x\in\mathbb{Z}$ such that*

$$|W\cap[0,x]| > \left(\frac{2|(Z+W)\setminus W|}{|Z|-2}+\frac{l(Z)-u-1}{2(|Z|-2)}+1\right)\big(l(Z)-u-1\big).$$

If $W\cap[z,z+l(Z)-1]\ne\emptyset$ for all $z\in[0,\infty)$, then there is $y\in[0,x]$ such that

$$|(Z+W)\cap[y,y+l(Z)-1]|\ge l(Z)-u.$$

Proof. Since $|W\cap[z,z+l(Z)-1]|\ge 1$ for all $z\in[0,\infty)$, we get from Proposition 2.4 applied to Z and W that

$$|[0,\infty)\setminus W| \le \left(\frac{2|(Z+W)\setminus W|}{|Z|-2}+\frac{l(Z)-1}{2(|Z|-2)}+1\right)\big(l(Z)-1\big).$$

In particular, inasmuch as $W\subseteq Z+W$, note that $[0,\infty)\setminus(Z+W)$ is finite. Set

$$S:=\{w\in\mathbb{Z}:\ |[w,w+l(Z)-1]\cap(Z+W)|\ge l(Z)-u\}.$$

Insomuch as $[0,\infty)\setminus(Z+W)$ is finite, S is not empty. Set $y:=\min S$ and note that it suffices to show that $y\le x$. Notice that

$$|[y-1,y+l(Z)-2]\cap(Z+W)|<l(Z)-u;$$

thus, since $|[y,y+l(Z)-1]\cap(Z+W)|\ge l(Z)-u$, we have that $y-1\in\mathbb{Z}\setminus(Z+W)$. Define

$$W' := \Big(\{y-1\}-\big(\mathbb{Z}\setminus(Z+W)\big)\Big)\cap[0,\infty).$$

Notice that $W'\subseteq[0,\infty)$ and $\min W'=0$ since $y-1\in\mathbb{Z}\setminus(Z+W)$. Now,

$$Z+\Big(\{y-1\}-\big(\mathbb{Z}\setminus(Z+W)\big)\Big)\subseteq\{y-1\}-(\mathbb{Z}\setminus W), \tag{25}$$

and

$$\mathbb{Z}\setminus W=\big(\mathbb{Z}\setminus(Z+W)\big)\cup\Big((\mathbb{Z}\setminus W)\setminus\Big(\mathbb{Z}\setminus\big((Z+W)\setminus W\big)\Big)\Big). \tag{26}$$

Hence

$$\begin{aligned}
Z+W' &= Z+\Big(\Big(\{y-1\}-\big(\mathbb{Z}\setminus(Z+W)\big)\Big)\cap[0,\infty)\Big)\\
&\subseteq \Big(Z+\Big(\{y-1\}-\big(\mathbb{Z}\setminus(Z+W)\big)\Big)\Big)\cap[0,\infty)\\
&\subseteq \big(\{y-1\}-(\mathbb{Z}\setminus W)\big)\cap[0,\infty) && \text{(by (25))}\\
&= \Big(\{y-1\}-\big(\mathbb{Z}\setminus(Z+W)\big)\cup\Big((\mathbb{Z}\setminus W)\setminus\Big(\mathbb{Z}\setminus\big((Z+W)\setminus W\big)\Big)\Big)\Big)\\
&\cap[0,\infty) && \text{(by (26))}\\
&= \Big(\Big(\{y-1\}-\big(\mathbb{Z}\setminus(Z+W)\big)\Big)\cap[0,\infty)\Big)\\
&\cup\Big(\Big(\{y-1\}-\Big((\mathbb{Z}\setminus W)\setminus\Big(\mathbb{Z}\setminus\big((Z+W)\setminus W\big)\Big)\Big)\Big)\cap[0,\infty)\Big)\\
&= W'\cup\Big(\Big(\{y-1\}-\Big((\mathbb{Z}\setminus W)\setminus\Big(\mathbb{Z}\setminus\big((Z+W)\setminus W\big)\Big)\Big)\Big)\cap[0,\infty)\Big). && (27)
\end{aligned}$$

Note that

$$(\mathbb{Z}\setminus W)\setminus\Big(\mathbb{Z}\setminus\big((Z+W)\setminus W\big)\Big)\subseteq(Z+W)\setminus W;$$

hence

$$\begin{aligned}
&\Big(\{y-1\}-\Big((\mathbb{Z}\setminus W)\setminus\Big(\mathbb{Z}\setminus\big((Z+W)\setminus W\big)\Big)\Big)\Big)\cap[0,\infty)\\
&\subseteq\Big(\{y-1\}-\big((Z+W)\setminus W\big)\Big)\cap[0,\infty)\\
&\subseteq\{y-1\}-\big((Z+W)\setminus W\big). && (28)
\end{aligned}$$

From (27) and (28), we get that

$$|(Z+W')\setminus W'|\le|(Z+W)\setminus W|. \tag{29}$$

For all $w\in(-\infty,y-1]$, we have that $|(Z+W)\cap[w,w+l(Z)-1]|<l(Z)-u$ so

$$\big|\big(\mathbb{Z}\setminus(Z+W)\big)\cap[w,w+l(Z)-1]\big|\ge u+1.$$

The previous inequality implies that for all $z\in[0,\infty)$

$$\begin{aligned}
&|W'\cap[z,z+l(Z)-1]|\\
&=\Big|\Big(\Big(\{y-1\}-\big(\mathbb{Z}\setminus(Z+W)\big)\Big)\cap[0,\infty)\Big)\cap[z,z+l(Z)-1]\Big|\\
&=\Big|\Big(\{y-1\}-\big(\mathbb{Z}\setminus(Z+W)\big)\Big)\cap[z,z+l(Z)-1]\Big|\\
&=\big|\big(\mathbb{Z}\setminus(Z+W)\big)\cap[y-1-z,y-2-z+l(Z)]\big|\\
&\ge u+1.
\end{aligned}$$

This means that we can apply Proposition2.4 to Z, W' and $u+1$ to get

$$|[0,\infty)\setminus W'| \leq \left(\frac{2|(Z+W')\setminus W'|}{|Z|-2} + \frac{l(Z)-u-1}{2(|Z|-2)}+1\right)\big(l(Z)-u-1\big),$$

and (29) yields that

$$|[0,\infty)\setminus W'| \leq \left(\frac{2|(Z+W)\setminus W|}{|Z|-2} + \frac{l(Z)-u-1}{2(|Z|-2)}+1\right)\big(l(Z)-u-1\big). \quad (30)$$

Since

$$\begin{aligned}|[0,\infty)\setminus W'| &= \big|\{z\in[0,\infty):\ z\notin\{y-1\}-\big(\mathbb{Z}\setminus(Z+W)\big)\}\big|\\ &= |\{z\in[0,\infty):\ y-1-z\in Z+W\}|\\ &= |\{w\in(-\infty,y-1]:\ w\in Z+W\}|\\ &= |(Z+W)\cap(-\infty,y-1]|\\ &= |(Z+W)\cap[0,y-1]|\,,\end{aligned}$$

we get from (30) that

$$\begin{aligned}|W\cap[0,y-1]| &\leq |(Z+W)\cap[0,y-1]|\\ &= |[0,\infty)\setminus W'|\\ &\leq \left(\frac{2|(Z+W)\setminus W|}{|Z|-2} + \frac{l(Z)-u-1}{2(|Z|-2)}+1\right)\big(l(Z)-u-1\big)\\ &< |W\cap[0,x]|;\end{aligned}$$

this means that $y \leq x$, and it completes the proof. □

Let W be a nonempty subset of $\mathbb{Z}$ such that $\min W = 0$ and $z\in[0,\infty)$. Define

$$\Gamma(W,z) := (W\cap[0,z])\cup[z+1,\infty).$$

The properties of $\Gamma(W,z)$ that will be needed later follow directly from its definition and we establish them in the following remark.

Remark 2.6. Let W and Z be a nonempty subset of $\mathbb{Z}$ such that $\min W = \min Z = 0$ and Z is finite.

i) Let $u\in[1,l(Z)]$ and $z\in[0,\infty)$. If $|W\cap[y,y+l(Z)-1]|\geq u$ for all $y\in[0,z-l(Z)+1]$, then $|\Gamma(W,z)\cap[y,y+l(Z)-1]|\geq u$ for all $y\in[0,\infty)$.

ii) For all $z\in[0,\infty)$,

$$\big(Z+\Gamma(W,z)\big)\setminus\Gamma(W,z) = \big((Z+W)\setminus W\big)\cap[0,z].$$

To conclude this section, we remark the following fact.

Remark 2.7. Let W and Z be a nonempty finite subsets of $\mathbb{Z}$. If there is $z \in [\min W + 1, \max W - l(Z)]$ such that $W \cap [z, z + l(Z) - 1] = \emptyset$, then

$$W_1 := W \cap (-\infty, z-1] \quad \text{and} \quad W_2 := W \cap [z + l(Z), \infty)$$

satisfy that they are not empty, $W = W_1 \cup W_2$ and $(Z + W_1) \cap (Z + W_2) = \emptyset$; in particular, W is $Z-$disconnected.

3. Proof of the main theorems

In this section we complete the proof of the main theorems of this paper. Before we show the main theorems, we prove some auxiliary results.

Lemma 3.1. *Let X and Y be nonempty finite subsets of $\mathbb{Z}$ such that* $\min X = \min Y = 0$ *and $z \in [-l(X)+1, \max(X+Y)]$. Then*

$$\begin{aligned} r(X,Y) =& |(X+Y) \cap [z, z+l(X)-1]| + \big|\big((X+Y) \setminus Y\big) \cap (-\infty, z-1]\big| \\ &+ \big|\big((X+Y) \setminus (\{\max X\} + Y)\big) \cap [z + l(X), \infty)\big| - |X|. \end{aligned}$$

Proof. Set

$$\begin{aligned} U' &:= \big((X+Y) \setminus Y\big) \cap (-\infty, z-1], \\ U'' &:= \big((X+Y) \setminus (\{\max X\} + Y)\big) \cap [z + l(X), \infty). \end{aligned}$$

If $z \in [-l(X), 0]$, then

$$(X+Y) \cap (-\infty, z-1] = U' = Y \cap (-\infty, z-1] = \emptyset,$$

and therefore

$$|(X+Y) \cap (-\infty, z-1]| = |U'| + |Y \cap (-\infty, z-1]| = 0.$$

If $z \in [1, \max(X+Y)]$, then $0 \in (-\infty, z-1]$. Hence

$$|(X+Y) \cap (-\infty, z-1]| = |U'| + |Y \cap (-\infty, z-1]|.$$

Thus, in any case,

$$|(X+Y) \cap (-\infty, z-1]| = |U'| + |Y \cap (-\infty, z-1]|. \tag{31}$$

Proceeding in a symmetric way,

$$|(X+Y) \cap [z + l(X), \infty)| = |U''| + |Y \cap [z, \infty)|. \tag{32}$$

If we add $|(X+Y) \cap [z, z+l(X)-1]|$, (31) and (32), we get

$$|X+Y| = |(X+Y) \cap [z, z+l(X)-1]| + |U'| + |U''| + |Y|,$$

and therefore

$$r(X,Y) = |(X+Y) \cap [z, z+l(X)-1]| + |U'| + |U''| - |X|. \qquad \square$$

To apply the result of the previous section, we will need to apply some affine transformations to our sets. We state the trivial properties of these transformations.

Remark 3.2. Let Z be a nonempty finite subset Z of $\mathbb{Z}$, and set $Z' := Z - \{\min Z\}$ and $Z'' := \{\max Z\} - Z$. Then

i) $\min Z' = \min Z'' = 0$,

ii) $\mathrm{GCD}(Z') = \mathrm{GCD}(Z'') = \mathrm{GCD}^*(Z)$.

Lemma 3.3. *Let X and Y be nonempty finite subsets of $\mathbb{Z}$ such that $\min X = \min Y = 0$ and $z \in [-l(X)+1, \max(X+Y)]$. Set*

$$y' := z - 1, \qquad X' := \{\max X\} - X,$$
$$y'' := \max(X+Y) - \big(z + l(X)\big), \qquad Y' := \{\max Y\} - Y.$$

i) *Assume that $z \in [1, \max(X+Y)]$. Then $\Gamma(Y, y') \neq \emptyset$ and*

$$\big|\big(X + \Gamma(Y,y')\big) \setminus \Gamma(Y,y')\big| = \big|\big((X+Y)\setminus Y\big) \cap (-\infty, y']\big|.$$

Moreover, if Y is X-connected, then $\Gamma(Y,y') \cap [w, w+l(X)-1] \neq \emptyset$ for all $w \in [0,\infty)$.

ii) *Assume that $z \in [-l(X)+1, \max(X+Y)-l(X)]$. Then $\Gamma(Y', y'') \neq \emptyset$ and*

$$\big|\big(X' + \Gamma(Y',y'')\big) \setminus \Gamma(Y',y'')\big| = \big|\big((X+Y)\setminus(\{\max X\}+Y)\big) \cap [z+l(X), \infty)\big|.$$

Moreover, if Y is X-connected, then $\Gamma(Y',y'') \cap [w, w+l(X)-1] \neq \emptyset$ for all $w \in [0,\infty)$.

P r o o f. The proof of i) and ii) are symmetric; we will show only i). Since $z-1 \in [0, \max(X+Y)-1]$, notice that $0 \in Y \cap [0,y'] \subseteq \Gamma(Y,y')$. From Remark 2.6 ii) applied to X, Y and y', we get that

$$\big|\big(X + \Gamma(Y,y')\big) \setminus \Gamma(Y,y')\big| = \big|\big((X+Y)\setminus Y\big) \cap [0, y']\big|. \tag{33}$$

Also we have the trivial equality

$$\big|\big((X+Y)\setminus Y\big) \cap [0, y']\big| = \big|\big((X+Y)\setminus Y\big) \cap (-\infty, z-1]\big|. \tag{34}$$

Thus, from (33) and (34),

$$\big|\big(X + \Gamma(Y,y')\big) \setminus \Gamma(Y,y')\big| = \big|\big((X+Y)\setminus Y\big) \cap (-\infty, z-1]\big|.$$

Finally, if Y is X-connected, Remark 2.7 implies that $Y \cap [v, v+l(X)-1] \neq \emptyset$ for all $v \in [1, \max Y - l(X)]$; therefore $Y \cap [v, v+l(X)-1] \neq \emptyset$ for all $v \in [-l(X)+1, \max Y]$. We get that $|Y \cap [w', w'+l(X)-1]| \geq 1$ for all $w' \in [0, y'-l(X)+1]$. Hence Remark 2.6 i) applied to Y implies that $|\Gamma(Y,y') \cap [w, w+l(X)-1]| \geq 1$ which is equivalent to the claim. $\square$

Now we show Theorem 1.3.

P r o o f. T h e o r e m 1.3. Translating if necessary, we assume that $\min X = \min Y = 0$. Set

$$\begin{aligned} X' &:= \{\max X\} - X, & y'' &:= \max(X+Y) - \big(z + l(X)\big),\\ Y' &:= \{\max Y\} - Y, & U' &:= \big((X+Y)\setminus Y\big) \cap (-\infty, z-1],\\ y' &:= z-1, & U'' &:= \big((X+Y)\setminus(\{\max X\}+Y)\big) \cap [z+l(X), \infty). \end{aligned}$$

Lemma 3.1 yields that

$$|U'| + |U''| = r(X,Y) - |(X+Y)\cap[z, z+l(X)-1]| + |X|. \tag{35}$$

We shall show that if $z \in [1, \max(X+Y)]$, then

$$|[0,y']\setminus Y| \le \big(l(X)-1\big)\left(\frac{2|U'|}{|X|-2} + \frac{l(X)-1}{2(|X|-2)} + 1\right). \tag{36}$$

First note that the definition of $\Gamma(Y,y')$ yields that

$$\Gamma(Y,y') \cap [0,y'] = Y \cap [0,y']$$

so

$$\big([0,\infty)\setminus\Gamma(Y,y')\big) \cap [0,y'] = \big([0,\infty)\setminus Y\big)\cap[0,y']. \tag{37}$$

Hence

$$\begin{aligned} |[0,y']\setminus Y| &\le \big|\big([0,\infty)\setminus Y\big)\cap[0,y']\big|\\ &= \big|\big([0,\infty)\setminus\Gamma(Y,y')\big)\cap[0,y']\big| \qquad \text{(by (37))}\\ &\le \big|\big([0,\infty)\setminus\Gamma(Y,y')\big)\big|. \end{aligned} \tag{38}$$

Insomuch as $z \in [1, \max(X+Y)]$, $\Gamma(Y,y') \ne \emptyset$ by Lemma 3.3 i). Remark 3.1 leads to $\min\Gamma(Y,y') = 0$ and $\mathrm{GCD}^*(X) = 1$. Also, from Lemma 3.3 i), we have

$$\big|\big(X+\Gamma(Y,y')\big)\setminus\Gamma(Y,y')\big| = |U'|. \tag{39}$$

We have that $\big(X+\Gamma(Y,y')\big)\setminus\Gamma(Y,y')$ is finite by (35) and (39). Insomuch as Y is X-connected, Lemma 3.3 i) implies that $|\Gamma(Y,y')\cap[w, w+l(X)-1]| \ge 1$ for all $w \in [0,\infty)$. Thus the assumptions of Propositon 2.4 are satisfied by $X, \Gamma(Y,y')$ and 1, and it implies that

$$|[0,\infty)\setminus\Gamma(Y,y')| \le \big(l(X)-1\big)\left(\frac{2|(X+\Gamma(Y,y'))\setminus\Gamma(Y,y')|}{|X|-2} + \frac{l(X)-1}{2(|X|-2)} + 1\right). \tag{40}$$

Moreover, from (39) and (40), we have that

$$|[0,\infty)\setminus\Gamma(Y,y')| \le \big(l(X)-1\big)\left(\frac{2|U'|}{|X|-2} + \frac{l(X)-1}{2(|X|-2)} + 1\right). \tag{41}$$

Hence (36) is a consequence of (38) and (41).

Proceeding symmetrically, if

$$z \in [-l(X)+1, \quad \max(X+Y)-l(X)],$$

then

$$|[y'+1, \max Y] \setminus Y| \le \big(l(X)-1\big)\left(\frac{2|U''|}{|X|-2} + \frac{l(X)-1}{2(|X|-2)} + 1\right). \tag{42}$$

Set

$$t_1 := \frac{4r(X,Y) - 4|(X+Y)\cap[z, z+l(X)-1]| + l(X) + 7}{2(|X|-2)} + 3\,,$$
$$t_2 := \frac{2r(X,Y) - 2|(X+Y)\cap[z, z+l(X)-1]| + l(X) + 3}{|X|-2} + 4\,.$$

The conclusion of the proof is divided into 3 cases:

- Assume that $z \in [-l(X)+1, 0]$. On the one hand, $z \le 0$ so

$$|[0, y'] \setminus Y| = |U'| = 0. \tag{43}$$

On the other hand, $0 \le \max(X+Y) - l(X)$ so $z \in [0-l(X)+1, \max(X+Y) - l(X)]$. Thus

$$\begin{aligned}
|[0, \max Y] \setminus Y| &\le |[0, y'] \setminus Y| + |[y'+1, \max Y] \setminus Y| \\
&= |[y'+1, \max Y] \setminus Y| && \text{(by (43))} \\
&\le \big(l(X)-1\big)\left(\frac{2|U''|}{|X|-2} + \frac{l(X)-1}{2(|X|-2)} + 1\right) && \text{(by (42))} \\
&= \big(l(X)-1\big)t_1\,. && \text{((35), (43))}
\end{aligned}$$

- Assume that $z \in [\max(X+Y) - l(X)+1, \max(X+Y)]$. On the one hand, $z + l(X) - 1 \ge \max(X+Y)$ so

$$|[y'+1, \max Y] \setminus Y| = |U''| = 0. \tag{44}$$

On the other hand, $z \in [1, \max(X+Y)]$. Thus

$$\begin{aligned}
|[0, \max Y] \setminus Y| &\le |[0, y'] \setminus Y| + |[y'+1, \max Y] \setminus Y| \\
&= |[0, y'] \setminus Y| && \text{(by (44))} \\
&\le \big(l(X)-1\big)\left(\frac{2|U'|}{|X|-2} + \frac{l(X)-1}{2(|X|-2)} + 1\right) && \text{(by (36))} \\
&= \big(l(X)-1\big)t_1\,. && \text{(by (35), (44))}
\end{aligned}$$

- Assume that $z \in [1, \max(X+Y) - l(X)]$. Then $z \in [1, \max(X+Y)]$ and $z \in [-l(X)+1, \max(X+Y) - l(X)]$.

Thus

$$\begin{aligned}|[0,\max Y]\setminus Y| &\le |[0,y']\setminus Y| + |[y'+1,\max Y]\setminus Y|\\ &\le \big(l(X)-1\big)\left(\frac{2(|U'|+|U''|)}{|X|-2}+\frac{l(X)-1}{|X|-2}+2\right) \quad \text{(by (36), (42))}\\ &= \big(l(X)-1\big)t_2\,. \quad \text{(by (35))}\end{aligned}$$

□

Now we prove Theorem 1.4.

P r o o f. T h e o r e m 1.4. Translating if necessary, assume that $\min X = \min Y = 0$. Set

$$\begin{aligned} v &:= \max(X+Y) - l(X),\\ U' &:= \big((X+Y)\setminus Y\big)\cap(-\infty, v-1],\\ U'' &:= \big((X+Y)\setminus(\{\max X\}+Y)\big)\cap[v+l(X),\infty).\end{aligned}$$

First we show that there is $z\in\mathbb{Z}$ such that $|(X+Y)\cap[z,z+l(X)-1]|\ge l(X)-u$. If $|Y\cap[\max Y-l(X)+1,\max Y]|\ge l(X)-u$, then

$$|(X+Y)\cap[v+1,v+l(X)]| \ge |(\{\max X\}+Y)\cap[v+1,v+l(X)]| \ge l(X)-u,$$

and we are done taking $z=v+1$. Hence we assume from now on that

$$|Y\cap[\max Y-l(X)+1,\max Y]| \le l(X)-u-1. \tag{45}$$

Insomuch as

$$|Y| > \left(\frac{2(r(X,Y)+1)}{|X|-2}+\frac{l(X)-u-1}{2(|X|-2)}+2\right)\big(l(X)-u-1\big),$$

we note from (45) that

$$\begin{aligned}|Y\cap[0,l(Y)-l(X)]| &= |Y| - |Y\cap[\max Y - l(X)+1,\max Y]|\\ &> \left(\frac{2(r(X,Y)+1)}{|X|-2}+\frac{l(X)-u-1}{2(|X|-2)}+1\right)\big(l(X)-u-1\big).\end{aligned} \tag{46}$$

Since $v\in[1,\max(X+Y)]$, it is noted that $\Gamma\big(Y,l(Y)-1\big)\ne\emptyset$ by Lemma 3.3 i). Furthermore, the definition of $\Gamma(Y,l(Y)-1)$ yields that

$$\Gamma(Y,l(Y)-1)\cap[0,l(Y)-1] = Y\cap[0,l(Y)-1], \tag{47}$$

and then (46) implies that

$$\begin{aligned}&\big|\Gamma\big(Y,l(Y)-1\big)\cap[0,l(Y)-l(X)]\big|\\ &\qquad > \left(\frac{2(r(X,Y)+1)}{|X|-2}+\frac{l(X)-u-1}{2(|X|-2)}+1\right)\big(l(X)-u-1\big).\end{aligned} \tag{48}$$

Lemma 3.3 i) leads to

$$\Big|\Big(X+\Gamma\big(Y,l(Y)-1\big)\Big)\setminus\Gamma\big(Y,l(Y)-1\big)| = |U'\Big|. \tag{49}$$

From Lemma 3.1,

$$r(X,Y)=|(X+Y)\cap[v,v+l(X)-1]|-|X|+|U'|+|U''|.$$

Thus (49) yields that

$$\Big|\Big(X+\Gamma\big(Y,l(Y)-1\big)\Big)\setminus\Gamma\big(Y,l(Y)-1\big)\Big|\le r(X,Y)+|X|-|(X+Y)\cap[v,v+l(X)-1]|. \tag{50}$$

Since

$$(X+\{\max Y\})\cap[v,v+l(X)]\subseteq(X+Y)\cap[v,v+l(X)],$$

we get that

$$|(X+Y)\cap[v,v+l(X)]|\ge|(X+\{\max Y\})\cap[v,v+l(X)]|=|X|. \tag{51}$$

Moreover, $v+l(X)=\max(X+Y)\in X+Y$, we have from (51)

$$|(X+Y)\cap[v,v+l(X)-1]|\ge|X|-1,$$

and thus (50) yields that

$$\Big|\Big(X+\Gamma\big(Y,l(Y)-1\big)\Big)\setminus\Gamma\big(Y,l(Y)-1\big)\Big|\le r(X,Y)+1. \tag{52}$$

Remark 3.1 leads to $\min\Gamma\big(Y,l(Y)-1\big)=0$ and $\mathrm{GCD}^*(X)=1$. Since Y is X-connected, Lemma 3.3 i) yields that $\big|\Gamma\big(Y,l(Y)-1\big)\cap[w,w+l(X)-1]\big|\ge 1$ for all $w\in[0,\infty)$. Set $t:=\big(l(X)-u-1\big)$. From (48) and (52),

$$\begin{aligned}&\big|\Gamma\big(Y,l(Y)-1\big)\cap[0,l(Y)-l(X)]\big|\\&>\left(\frac{2(r(X,Y)+1)}{|X|-2}+\frac{l(X)-u-1}{2(|X|-2)}+1\right)t\\&\ge\left(\frac{2|(X+\Gamma(Y,l(Y)-1))\setminus\Gamma(Y,l(Y)-1)|}{|X|-2}+\frac{l(X)-u-1}{2(|X|-2)}+1\right)t\,.\end{aligned}$$

Thus the assumptions of Proposition 2.5 are satisfied by $X,\Gamma\big(Y,l(Y)-1\big)$ and $l(Y)-l(X)$. Then Proposition 2.5 implies the existence of $y\in[0,l(Y)-l(X)]$ such that

$$\Big|\Big(X+\Gamma\big(Y,l(Y)-1\big)\Big)\cap[y,y+l(X)-1]\Big|\ge l(X)-u\,. \tag{53}$$

Thus, from (47) and (53), we get

$$|(X+Y)\cap[y,y+l(X)-1]|\ge l(X)-u. \tag{54}$$

Taking $z:=y$, (54) leads to the claim.

For the second claim, we take $z \in \mathbb{Z}$ such that

$$|(X+Y) \cap [z, z+l(X)-1]| \geq l(X) - u\,.$$

Then Lemma 3.1 yields that

$$\begin{aligned} r(X,Y) &= |(X+Y) \cap [z, z+l(X)-1]| - |X| + |U'| + |U''| \\ &\geq |(X+Y) \cap [z, z+l(X)-1]| - |X| \geq l(X) - |X| - u\,. \end{aligned}$$ □

4. Applications and conclusions

We begin this section proving Corollary 1.5.

Proof. Corollary 1.5. First we show i). Note that

$$\begin{aligned} r(X,Y) &= |X+Y| - |X| - |Y| \\ &= \left(\sum_{i=1}^{n} |X+Y_i|\right) - |X| - \left(\sum_{i=1}^{n} |Y_i|\right) \\ &= (n-1)|X| + \sum_{i=1}^{n} (|X+Y_i| - |X| - |Y_i|) \\ &= (n-1)|X| + \sum_{i=1}^{n} r(X,Y_i)\,. \end{aligned} \tag{55}$$

From [13, Lemma 5.3], for all $i \in [1,n]$, we have that

$$r(X,Y_i) \geq -1. \tag{56}$$

Thus, from (55) and (56),

$$r(X,Y) = (n-1)|X| + \sum_{i=1}^{n} r(X,Y_i) \geq (n-1)|X| - n\,,$$

and thereby

$$n \leq \frac{r(X,Y)+1}{|X|-1} + 1.$$

Now we prove ii). If $u_i > l(X) - |X| + 1$, then the definition of u_i yields that

$$|Y_i| \leq \left(\frac{2(r(X,Y_i)+1)}{|X|-2} + \frac{5}{2}\right)(|X|-2)\,.$$

Assume from now on that $u_i \leq l(X) - |X| + 1$. All the assumptions of Theorem 1.4 are satisfied by X, Y_i and u_i so there is $z \in \mathbb{Z}$ such that

$$|(X+Y_i) \cap [z, z+l(X)-1]| \geq l(X) - u_i \geq |X| - 1 \geq 2. \tag{57}$$

Hence (57) implies that $[z, z+l(X)-1]$ intersects $X+Y_i$ and thereby $z \in [\min(X+Y_i) - l(X)+1, \max(X+Y_i)]$. Thus the assumptions of Theorem 1.3 are satisfied by X, Y_i and z, and it implies that if $w_i := |(X+Y_i) \cap [z, z+l(X)-1]|$, then

$$l(Y_i) \leq |Y_i| + \big(l(X) - 1\big)\left(\frac{2r(X, Y_i) - 2w_i + l(X) + 3}{|X| - 2} + 4\right). \tag{58}$$

Then (57) and (58) lead to

$$l(Y_i) \leq |Y_i| + \big(l(X) - 1\big)\left(\frac{2r(X, Y_i) - l(X) + 2u_i + 3}{|X| - 2} + 4\right). \qquad \square$$

We prove Corollary 1.6.

Proof. Corollary 1.6. First we show by contradiction that

$$l(X) \leq |X| + r(X, Y). \tag{59}$$

Assume that $l(X) \geq |X| + r(X, Y) + 1$. By assumption,

$$|Y| > \big(|X| + r(X, Y)\big)\left(\frac{5r(X, Y) + |X| + 3}{2(|X| - 2)} + 2\right). \tag{60}$$

Since $l(X) - |X| - r(X, Y) - 1 \geq 0$, we can apply Theorem 1.4 to X, Y and $l(X) - |X| - r(X, Y) - 1$ by (60), and it yields that

$$r(X, Y) \geq l(X) - |X| - \big(l(X) - |X| - r(X, Y) - 1\big) = r(X, Y) + 1$$

which is impossible. This proves (59).

Now we bound $l(Y)$. The assumption and (59) lead to

$$|Y| > \big(l(X) - 1\big)\left(\frac{4r(X, Y) + l(X) + 3}{2(|X| - 2)} + 2\right).$$

Theorem 1.4 applied to X, Y and 0 implies the existence of $z \in \mathbb{Z}$ such that $|(X + Y) \cap [z, z + l(X) - 1]| = l(X)$. Setting $u := |(X + Y) \cap [z, z + l(X) - 1]|$, Theorem 1.3 yields

$$\begin{aligned} l(Y) &\leq |Y| + \big(l(X) - 1\big)\left(\frac{2r(X, Y) - 2u + l(X) + 3}{|X| - 2} + 4\right) \\ &= |Y| + \big(l(X) - 1\big)\left(\frac{2r(X, Y) - l(X) + 3}{|X| - 2} + 4\right). \end{aligned} \tag{61}$$

We conclude the proof as follows

$$\begin{aligned} l(Y) &\leq |Y| + \big(l(X) - 1\big)\left(\frac{2r(X, Y) - l(X) + 3}{|X| - 2} + 4\right) && \text{(by (61))} \\ &\leq |Y| + \big(l(X) - 1\big)\left(\frac{2(r(X, Y) + 1)}{|X| - 2} + 3\right) \\ &\leq |Y| + \big(|X| + r(X, Y) - 1\big)\left(\frac{2(r(X, Y) + 1)}{|X| - 2} + 3\right). && \text{(by (59))} \end{aligned}$$

$\square$

Now we show that, for arbitrary large values of $r(X,Y)$, the claims of Theorem 1.3 and Theorem 1.4 (therefore also the claims of our corollaries) are not trivial, however we do not know if they are optimal. Let $r, x \in \mathbb{Z}$ be such that $r, x \in [2, \infty)$. Define

$$\begin{aligned}
X_3 &:= \{0\} \cup [rx+1, (r+1)x+1], \\
Y_3 &:= \left(\bigcup_{i=0}^{r-1} \{i(rx+1)\}\right) \cup \left(\bigcup_{i=r}^{2r} [i(rx+1), i(rx+1)+(i-r)x]\right), \\
z_3 &:= 2r(rx+1), \\
X_4 &:= [0,x] \cup \{(r+1)x+1\}, \\
Y_4 &:= \left(\bigcup_{i=0}^{r-1} [i\big((r+1)x+1\big), i\big((r+1)x+1\big)+rx-1]\right), \\
&\quad \cup \left(\bigcup_{i=r}^{2r-1} \left[i\big((r+1)x+1\big), i\big((r+1)x+1\big)+(2r-i)x-1\right]\right).
\end{aligned}$$

On the one hand, if $x \le \frac{3r+1}{r-1}$, then X_3 and Y_3 satisfy the properties of Theorem 1.3 and

$$\begin{aligned}
l(Y_3) =& |Y_3| + \frac{3r+1}{2} rx - 1 \\
\ge& |Y_3| + \frac{r+1}{2}(3rx - x - 3) \\
=& |Y_3| + \big(l(X_3)-1\big) \\
&\times \left(\frac{2r(X_3,Y_3) - 2|(X_3+Y_3)\cap[z_3, z_3+l(X_3)-1]| + l(X_3)}{2(|X_3|-2)}\right).
\end{aligned}$$

On the other hand, X_4 and Y_4 satisfy the properties of Theorem 1.4 and

$$\begin{aligned}
|Y_4| &= r^2x + \frac{r(r+1)}{2}x = rx\left(\frac{3r+1}{2}\right) \\
&\ge \frac{r+1}{2}(3rx - 2x - 5) \\
&= \big(l(X_4)-1\big)\left(\frac{2r(X_4,Y_4) - l(X_4)}{2(|X_4|-2)} - \frac{1}{2}\right),
\end{aligned}$$

but there is not $z_4 \in \mathbb{Z}$ such that $|(X_4+Y_4)\cap[z_4, z_4+l(X_4)-1]| = l(X_4)$. Thus the pair (X_3, Y_3) (resp. (X_4, Y_4)) gives an example which shows that Theorem 1.3 (resp. Theorem 1.4) is not trivial.

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Mario Huicochea
Institute of Physics of the Autonomous University of San Luis Potosí México
Zona Universitaria
78290 San Luis Potosí
S.L.P., MÉXICO
E-mail: dym@cimat.mx