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IN RESIDUE CLASSES OF THE BASE-q AND
OSTROWSKI DIGITAL SUMS

DIVYUM SHARMA

Department of Pure Mathematics, University of Waterloo, Ontario, Canada.

ABSTRACT. Let q be an integer greater than or equal to 2, and let $S_q(n)$ denote the sum of digits of n in base q. For

$$\alpha = [0; \overline{1, m}], \ m \geq 2,$$

let $S_\alpha(n)$ denote the sum of digits in the Ostrowski α-representation of n. Let $m_1, m_2 \geq 2$ be integers with

$$\gcd(q-1, m_1) = \gcd(m, m_2) = 1.$$

We prove that there exists $\delta > 0$ such that for all integers r_1, r_2,

$$|\{0 \leq n < N : S_q(n) \equiv r_1 \ (\mathrm{mod} \ m_1), \ S_\alpha(n) \equiv r_2 \ (\mathrm{mod} \ m_2)\}|$$
$$= \frac{N}{m_1 m_2} + O(N^{1-\delta}).$$

The asymptotic relation implied by this equality was proved by Coquet, Rhin & Toffin and the equality was proved for the case $\alpha = [\ \overline{1}\ ]$ by Spiegelhofer.

Communicated by Werner Georg Nowak

1. Introduction

Let q be an integer greater than or equal to 2, and let $S_q(n)$ denote the sum of digits of n in base q. Much effort has been made to understand the behaviour of this function. Bush [9] studied the asymptotic behaviour of its mean value. The distribution of the values of this function has also been investigated [15]. There has been a keen interest in the sum of digits of primes [18] and polynomials [4]. (We also refer to [5], [12] and the references there for other related results.)

Throughout this paper, q, q_1, q_2, m_1, m_2 denote integers greater than or equal to 2. Gel′fond [13] proved that if m_1 is coprime to $q-1$, then the function $S_q(n)$ is uniformly distributed modulo m_1. Further, he conjectured that if

$$\gcd(q_1, q_2) = \gcd(m_1, q_1 - 1) = \gcd(m_2, q_2 - 1) = 1,$$

then there exists $\delta = \delta(q_1, q_2, m_1, m_2) > 0$ such that

$$|\{0 \le n < N : S_{q_1}(n) \equiv r_1 \pmod{m_1},\ S_{q_2}(n) \equiv r_2 \pmod{m_2}\}|$$
$$= \frac{N}{m_1 m_2} + O(N^{1-\delta})$$

for all integers r_1, r_2. The asymptotic relation implied in this conjecture was proved by Bésineau [7]; while the conjecture was proved in its full strength by Kim [16] (See also [5] for some recent improvements of Kim's result).

In this paper we are concerned with the above problem for the sum of digits functions of the base-q and *Ostrowski representation* of integers. In 1922, Ostrowski [19] discovered a numeration system based on continued fractions. He showed that the sequence of the denominators of the convergents to the simple continued fraction expansion of an irrational number forms the basis for a numeration system. More precisely, he proved the following result.

THEOREM 1.1. [1, Theorem 3.9.1] *Let α be an irrational real number having continued fraction expansion $[a_0; a_1, \ldots]$. Let $(q_i)_{i \ge 0}$ be the sequence of the denominators of the convergents to the continued fraction expansion. Then every non-negative integer n can be expressed uniquely as*

$$n = \sum_{0 \le i \le \ell} b_i q_i, \tag{1}$$

where the b_i's are integers satisfying

(i) $0 \le b_0 < a_1$.

(ii) $0 \le b_i \le a_{i+1}$ *for $i \ge 1$.*

(iii) *For $i \ge 1$, if $b_i = a_{i+1}$, then $b_{i-1} = 0$.*

In fact, the three conditions above are equivalent to the inequality

$$b_0 q_0 + b_1 q_1 + \cdots + b_i q_i < q_{i+1}. \tag{2}$$

Note that Condition (iii) states that the relation $q_{i+1} = a_{i+1} q_i + q_{i-1}$ cannot be used to replace a linear combination of summands with another summand. The expression given in (1) is called the *Ostrowski α-representation* of n.

See [6] for a survey on the connections between the Ostrowski numeration systems and combinatorics of words, and [3] for a study of ergodic and topological-dynamical properties of various dynamical systems associated to Ostrowski α--representations. We refer to [8] for an analysis of the asymptotic average of the number of non-zero terms required in these representations.

If the Ostrowski α-representation of a positive integer n is given by

$$n = \sum_{0 \leq i \leq \ell} b_i(n) q_i, \tag{3}$$

let

$$S_\alpha(n) = \sum_{0 \leq i \leq \ell} b_i(n)$$

be the sum of digits. In [10], Coquet, Rhin & Toffin studied the relation between the functions $S_q(n)$ and $S_\alpha(n)$. They proved the following theorem.

THEOREM. Let q be an integer greater than or equal to 2 and let α be an irrational real number. The sequence $n \to x S_q(n) + y S_\alpha(n)$ is uniformly distributed modulo 1 if and only if at least one of x and y is irrational.

In [20], Spiegelhofer considered the case when

$$\alpha = \frac{1 + \sqrt{5}}{2} = [\overline{1}].$$

(Note that in this case, the sequence (q_i) is the sequence of Fibonacci numbers and that every non-negative integer can be uniquely expressed as a sum of non-consecutive Fibonacci numbers. This representation is known as the *Zeckendorf representation* of integers (see [23]).) He proved that if $\theta \in \mathbb{R}$ and $\gamma \in \mathbb{R} \setminus \mathbb{Z}$, then

$$\sum_{n < N} e\big(\theta S_q(n) + \gamma S_\alpha(n)\big) = O(N^{1-\delta})$$

for some $\delta > 0$. (Throughout this paper, $e(x)$ denotes $\exp(2\pi i x)$.) As a consequence, he obtained

THEOREM. [20, Corollary 5.3] Let $\alpha = (1 + \sqrt{5})/2$ and let $q, m_1, m_2 \geq 2$ be integers with $\gcd(m_1, q - 1) = 1$. There exists $\delta > 0$ such that for all integers r_1, r_2,

$$|\{0 \leq n < N : S_q(n) \equiv r_1 \ (\mathrm{mod}\ m_1), \ S_\alpha(n) \equiv r_2 \ (\mathrm{mod}\ m_2)\}|$$
$$= \frac{N}{m_1 m_2} + O(N^{1-\delta}).$$

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In [11], Coquet, Rhin & Toffin gave three sufficient conditions for the set

$$\{n \in \mathbb{N} : S_q(n) \equiv r_1 \ (\mathrm{mod} \ m_1), \ S_\alpha(n) \equiv r_2 \ (\mathrm{mod} \ m_2)\}$$

to have asymptotic density equal to $1/(m_1 m_2)$. One of these conditions is that the sequence (q_k) be lacunary and $\gcd(a_k, m_2)$ be equal to one for infinitely many indices k. Note that this condition is satisfied for

$$\alpha = [0; \overline{1, m}] = \frac{-m + \sqrt{m^2 + 4m}}{2}, \ m \geq 2$$

as

$$\frac{q_{k+1}}{q_k} \geq \begin{cases} 1 + \frac{q_{k-1}}{q_k} \geq 1 + \frac{1}{m+1} & \text{if } a_{k+1} = 1, \\ m + \frac{q_{k-1}}{q_k} \geq m + \frac{1}{2} & \text{if } a_{k+1} = m. \end{cases}$$

For these values of α, we improve the above asymptotic estimate to an estimate with error term $O(N^{1-\delta})$. Let $\|x\| = \min_{j \in \mathbb{Z}} |x - j|$. We prove

THEOREM 1.2. *Let $q \geq 2$ be an integer and let*

$$\alpha = [0; \overline{1, m}], \ m \geq 2.$$

Let $\theta, \gamma \in \mathbb{R}$ with $\|m\gamma\| \neq 0$. Then there exists $\delta > 0$ such that

$$\sum_{n < N} e\big(\theta S_q(n) + \gamma S_\alpha(n)\big) = O(N^{1-\delta}).$$

As a consequence, we obtain

COROLLARY 1.3. *Let q and α be as in Theorem 1.2 and let $m_1, m_2 \geq 2$ be integers with $\gcd(q-1, m_1) = \gcd(m, m_2) = 1$. There exists $\delta > 0$ such that for all integers r_1, r_2,*

$$|\{0 \leq n < N : S_q(n) \equiv r_1 \ (\mathrm{mod} \ m_1), \ S_\alpha(n) \equiv r_2 \ (\mathrm{mod} \ m_2)\}|$$

$$= \frac{N}{m_1 m_2} + O(N^{1-\delta}).$$

The proof relies on Weyl's and van der Corput's method. In Section 2, we introduce some notation and record some preliminary lemmas. Following [20], we obtain a characterization of integers with the same initial digits in their Ostrowski α-representations in Section 3. We then use it to obtain an analogue of inverse discrete Fourier transform in this case and also derive a uniform upper bound for the *Fourier coefficients*. Finally, we prove Theorem 1.2 and Corollary 1.3 in Section 4.

2. Preliminaries

Let $q, m, \alpha, \theta, \gamma$ be as in the statement of Theorem 1.2. Let $\lceil x \rceil$ denote the smallest integer greater than or equal to x and $\{x\}$ denote the fractional part of x. Recall that the Vinogradov symbol $f \ll F$ means that there is a constant c such that the inequality $|f| \leq cF$ holds. The implied constant c can depend on q, m, γ. We will repeatedly use the fact that for real numbers x, y and a positive integer a, we have $\|x + y\| \leq \|x\| + \|y\|$ and hence $\|ax\| \leq a\|x\|$. Since

$$\alpha = [0; \overline{1, m}],$$

we have $q_0 = q_1 = 1$ and

$$q_i = \begin{cases} mq_{i-1} + q_{i-2} & \text{if } i \text{ is even,} \\ q_{i-1} + q_{i-2} & \text{if } i \text{ is odd.} \end{cases}$$

Let $d = m^2 + 4m$ and

$$\varphi = \frac{m + 2 + \sqrt{d}}{2}. \tag{4}$$

Then

$$q_{2\ell} = \frac{m + \sqrt{d}}{2\sqrt{d}} \varphi^\ell - \frac{m - \sqrt{d}}{2\sqrt{d}} \varphi^{-\ell}, \tag{5}$$

$$q_{2\ell+1} = \frac{1}{\sqrt{d}} \varphi^{\ell+1} - \frac{1}{\sqrt{d}} \varphi^{-\ell-1}.$$

By Theorem 1.1, the digits in the α-representation (3) of a positive integer n satisfy
$$b_{2\ell}(n) \leq 1 \text{ and } b_{2\ell+1}(n) \leq m$$

for all non-negative integers ℓ. Given an integer $k \geq 1$, let $t_\alpha(n; k)$ denote the truncation of the sum in (3) after k digits, i.e.,

$$t_\alpha(n; k) = \sum_{0 \leq i \leq k-1} b_i(n) q_i$$

and let $S_{\alpha,k}(n)$ denote the sum of the first k digits, i.e.,

$$S_{\alpha,k}(n) = \sum_{0 \leq i \leq k-1} b_i(n).$$

By (2),

$$t_\alpha(n; k) < q_k. \tag{6}$$

Next, let

$$S_{q,t}(n) = S_q\big(n(\bmod q^t)\big).$$

Let $G_t(\ell) = G_t(\ell, \theta)$ denote the discrete Fourier coefficients of the function $e\big(\theta S_q(n)\big)$, i.e.,

$$G_t(\ell, \theta) = \frac{1}{q^t} \sum_{u < q^t} e\big(\theta S_q(u) - \ell u q^{-t}\big).$$

Then

$$e\big(\theta S_{q,t}(n)\big) = \sum_{\ell < q^t} e(\ell n q^{-t}) G_t(\ell, \theta), \tag{7}$$

$$e\big(-\theta S_{q,t}(n)\big) = \sum_{\ell < q^t} e(\ell n q^{-t}) \overline{G_t(-\ell, \theta)}. \tag{8}$$

Note that, by Parseval's identity,

$$\sum_{\ell < q^t} |G_t(\ell)|^2 = 1. \tag{9}$$

For negative integers n, we define

$$S_q(n) = S_\alpha(n) = 0.$$

We now list some results needed in the proof. The following is an elementary lemma on exponential sums.

LEMMA 2.1. *Let $x \in \mathbb{R}$ and $N, R \geq 0$. Then*

(i) [17, Lemma1]

$$\left| \sum_{n < N} e(nx) \right| \leq \min\left(N, \frac{1}{2\|x\|} \right).$$

(ii)

$$\sum_{|r| < R} (R - |r|) e(rx) = \left| \sum_{r < R} e(rx) \right|^2.$$

We now record an estimate from [20], which is proved using a discrepancy estimate for the sequence $(n\varphi)$, where φ has bounded partial quotients. We use it for φ as given in (4).

LEMMA 2.2. [20, Lemma 5.8] *Let I be a finite interval in $\mathbb{Z}$. Let K and a be real numbers with $K \geq 1$. Then*

$$\sum_{h \in I} \min\left(K, \frac{1}{\|a + h\varphi\|^2} \right) \ll \sqrt{K}\lambda(I) + K \ln \lambda(I).$$

(Here, λ denotes the Lebesgue measure on $\mathbb{R}$.)

Next, we state the version of the Weyl-van der Corput inequality that will be used later.

LEMMA 2.3. [14, Lemma 2.5] *Let $z_0, \ldots, z_{N-1}$ be complex numbers. For all positive integers R, we have*

$$\left| \sum_{n=0}^{N-1} z_n \right|^2 \leq \frac{N+R}{R} \sum_{|r|<R} \left(1 - \frac{|r|}{R}\right) \left| \sum_{\substack{0 \leq n < N \\ 0 \leq n+r < N}} \overline{z}_n z_{n+r} \right|.$$

The following lemma states that for *most* integers n, the representations of n and $n+r$ may differ at digits corresponding to the first few base elements only.

LEMMA 2.4. *Let N, r, k, t be non-negative integers with $k \geq 2$ and let θ, γ be real numbers. Then*

(i) $\left| \left\{ n < N : e\big(\theta S_q(n+r)\big)\overline{e\big(\theta S_q(n)\big)} \neq e\big(\theta S_{q,t}(n+r)\big)\overline{e\big(\theta S_{q,t}(n)\big)} \right\} \right|$

$$\leq \frac{Nr}{q^t} + r.$$

(ii) $\left| \left\{ n < N : e\big(\gamma S_\alpha(n+r)\big)\overline{e\big(\gamma S_\alpha(n)\big)} \neq e\big(\gamma S_{\alpha,k}(n+r)\big)\overline{e\big(\gamma S_{\alpha,k}(n)\big)} \right\} \right|$

$$\leq \frac{Nr}{q_{k-1}}.$$

Proof.
See [20, Lemma 1.17] for a proof of (i) and [21, Lemma 2.6] for a proof of (ii). □

3. Lemmas

We first derive a characterization (Corollary 3.3) of integers n with the same value of $t_\alpha(n; k)$, for a given k. Later, we use this to obtain the *discrete Fourier transform* for the function $e\big(\gamma S_{\alpha,k}(n)\big)$. This is analogous to [20, Proposition 5.7 & Proposition 5.4].

LEMMA 3.1. *Let $k \geq 2$ be an integer. Let $p_k(n) = (-1)^k n\varphi$, where φ is as in (4). Define*

$$A_k^{(1)} = \begin{cases} \left[\frac{m-\sqrt{d}}{2\varphi^{k_0}}, \frac{1}{\varphi^{k_0}} \right) & \text{if } k = 2k_0, \quad k_0 \in \mathbb{N}, \\ \left[\frac{-1}{\varphi^{k_0+1}}, \frac{-m+\sqrt{d}}{2\varphi^{k_0}} \right) & \text{if } k = 2k_0 + 1, \ k_0 \in \mathbb{N}, \end{cases}$$

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$$A_k^{(2)} = \begin{cases} \left[\frac{m-\sqrt{d}}{2\varphi^{k_0}}, \frac{1}{\varphi^{k_0+1}} \right) & \text{if } k = 2k_0, \quad k_0 \in \mathbb{N}, \\ \left[\frac{-1}{\varphi^{k_0+1}}, \frac{-m-1+\sqrt{d}}{\varphi^{k_0}} \right) & \text{if } k = 2k_0+1, \ k_0 \in \mathbb{N}, \end{cases}$$

and

$$R_k(u) = p_k(u) + \begin{cases} A_k^{(1)} & \text{if } 0 \le u < q_{k-1}, \\ A_k^{(2)} & \text{if } q_{k-1} \le u < q_k. \end{cases}$$

Then

$$p_k(n) \in R_k\big(t_\alpha(n;k)\big) + \mathbb{Z}.$$

P r o o f. By (5) and the definition of $t_\alpha(n;k)$,

$$n\varphi - t_\alpha(n;k)\varphi$$

$$= \sum_{\ell \ge \lceil k/2 \rceil} b_{2\ell}(n)\varphi \left(\frac{m+\sqrt{d}}{2\sqrt{d}}\varphi^\ell - \frac{m-\sqrt{d}}{2\sqrt{d}}\varphi^{-\ell} \right)$$

$$+ \sum_{\ell \ge \lceil (k-1)/2 \rceil} b_{2\ell+1}(n)\varphi \left(\frac{\varphi^{\ell+1}}{\sqrt{d}} - \frac{\varphi^{-\ell-1}}{\sqrt{d}} \right)$$

$$= \sum_{\ell \ge \lceil k/2 \rceil} b_{2\ell}(n)q_{2\ell+2}$$

$$+ \frac{m-\sqrt{d}}{2\sqrt{d}} \sum_{\ell \ge \lceil k/2 \rceil} b_{2\ell}(n)(\varphi^{-\ell-1} - \varphi^{-\ell+1})$$

$$+ \sum_{\ell \ge \lceil (k-1)/2 \rceil} b_{2\ell+1}(n)q_{2\ell+3}$$

$$+ \frac{1}{\sqrt{d}} \sum_{\ell \ge \lceil (k-1)/2 \rceil} b_{2\ell+1}(n)(\varphi^{-\ell-2} - \varphi^{-\ell})$$

$$= \sum_{\ell \ge \lceil k/2 \rceil} b_{2\ell}(n)q_{2\ell+2}$$

$$+ \sum_{\ell \ge \lceil (k-1)/2 \rceil} b_{2\ell+1}(n)q_{2\ell+3}$$

$$+ \frac{(-m+\sqrt{d})}{2} \sum_{\ell \ge \lceil k/2 \rceil} \frac{b_{2\ell}(n)}{\varphi^\ell} - \sum_{\ell \ge \lceil (k-1)/2 \rceil} \frac{b_{2\ell+1}(n)}{\varphi^{\ell+1}}.$$

Note that the first two terms in the above expression are integers.

We first consider the case when k is even. Write $k = 2k_0$, $k_0 \in \mathbb{N}$. Now

$$\sum_{\ell \geq \lceil (k-1)/2 \rceil} \frac{b_{2\ell+1}(n)}{\varphi^{\ell+1}} \leq m \sum_{\ell \geq k_0} \frac{1}{\varphi^{\ell+1}} = \frac{-m + \sqrt{d}}{2\varphi^{k_0}}$$

since

$$\frac{1}{1 - \varphi^{-1}} = \frac{2}{-m + \sqrt{d}}.$$

Suppose that $b_{k-1}(n) = 0$. Then

$$\frac{(-m + \sqrt{d})}{2} \sum_{\ell \geq \lceil k/2 \rceil} \frac{b_{2\ell}(n)}{\varphi^\ell} \leq \frac{(-m + \sqrt{d})}{2} \sum_{\ell \geq k_0} \frac{1}{\varphi^\ell} = \frac{1}{\varphi^{k_0}}.$$

If $b_{k-1}(n) \neq 0$, then by condition (iii) of Theorem 1.1, $b_k(n) \neq 1$. Hence $b_k(n) = 0$ and

$$\frac{(-m + \sqrt{d})}{2} \sum_{\ell \geq \lceil k/2 \rceil} \frac{b_{2\ell}(n)}{\varphi^\ell} \leq \frac{(-m + \sqrt{d})}{2} \sum_{\ell \geq k_0 + 1} \frac{1}{\varphi^\ell} = \frac{1}{\varphi^{k_0+1}}.$$

Further, note that if $b_{k-1}(n) = 0$, then $t_\alpha(n; k) = t_\alpha(n; k-1)$ and hence by (6),

$$t_\alpha(n; k) = t_\alpha(n; k-1) < q_{k-1}.$$

If $b_{k-1}(n) \neq 0$, then

$$t_\alpha(n; k) = \sum_{0 \leq i \leq k-1} b_i(n) q_i \geq q_{k-1}.$$

This proves the lemma when k is even. Next, we consider the case when k is odd. Write $k = 2k_0 + 1$, $k_0 \in \mathbb{N}$. Then

$$n(-\varphi) - t_\alpha(n; k)(-\varphi)$$
$$\equiv \frac{(m - \sqrt{d})}{2} \sum_{\ell \geq k_0 + 1} \frac{b_{2\ell}(n)}{\varphi^\ell} + \sum_{\ell \geq k_0} \frac{b_{2\ell+1}(n)}{\varphi^{\ell+1}} \pmod 1.$$

Since

$$\sum_{\ell \geq k_0 + 1} \frac{b_{2\ell}(n)}{\varphi^\ell} \leq \frac{1}{\varphi^{k_0+1}} \frac{2}{-m + \sqrt{d}},$$

we get

$$\frac{(m - \sqrt{d})}{2} \sum_{\ell \geq k_0 + 1} \frac{b_{2\ell}(n)}{\varphi^\ell} \geq -\frac{1}{\varphi^{k_0+1}}.$$

Suppose that $b_{k-1}(n) = 0$. Then

$$\sum_{\ell \geq k_0} \frac{b_{2\ell+1}(n)}{\varphi^{\ell+1}} \leq \frac{m}{\varphi^{k_0+1}} \frac{2}{-m + \sqrt{d}} = \frac{-m + \sqrt{d}}{2\varphi^{k_0}}.$$

If $b_{k-1}(n) \neq 0$, then by condition (iii) of Theorem 1.1, $b_k(n) \neq m$. Hence

$$\sum_{\ell \geq k_0} \frac{b_{2\ell+1}(n)}{\varphi^{\ell+1}} \leq \frac{m-1}{\varphi^{k_0+1}} + m \sum_{\ell \geq k_0+1} \frac{1}{\varphi^{\ell+1}}$$

$$= \frac{m-1}{\varphi^{k_0+1}} + \frac{-m+\sqrt{d}}{2\varphi^{k_0+1}}$$

$$= \frac{-m-1+\sqrt{d}}{\varphi^{k_0}}.$$

As before, this proves the lemma when k is odd. $\qquad\square$

LEMMA 3.2. *Fix an integer $k \geq 2$. The sets*

$$R_k(u) + \mathbb{Z}, \ 0 \leq u < q_k,$$

form a partition of $\mathbb{R}$.

P r o o f. For each integer u with $0 \leq u < q_k$, let

$$\tilde{R}_k(u) = R_k(u) \bmod 1.$$

This set is the union of at most two intervals. Further, the sum of the measures of the sets $\tilde{R}_k(u)$ is 1. To prove this, we first consider the case when k is even. Write $k = 2k_0$. By Lemma 3.1, if $0 \leq u < q_{k-1}$, then the measure of $R_k(u)$

$$\frac{1}{\varphi^{k_0}} - \frac{m-\sqrt{d}}{2\varphi^{k_0}} = \frac{2-m+\sqrt{d}}{2\varphi^{k_0}}.$$

Further, the measure of $\tilde{R}_k(u)$ is equal to the measure of $R_k(u)$ as

$$\frac{1}{\varphi^{k_0}} - \frac{m-\sqrt{d}}{2\varphi^{k_0}} < 1.$$

Similarly, if $q_{k-1} \leq u < q_k$, then the measure of $\tilde{R}_k(u)$ is

$$\frac{1}{\varphi^{k_0+1}} - \frac{m-\sqrt{d}}{2\varphi^{k_0}} = \frac{1}{2\varphi^{k_0}}\left(2\varphi^{-1} - m + \sqrt{d}\right)$$

$$= \frac{1}{2\varphi^{k_0}}\left(m+2-\sqrt{d}-m+\sqrt{d}\right)$$

$$= \frac{1}{\varphi^{k_0}}.$$

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Hence, the sum of measures of the sets $\tilde{R}_k(u)$ is

$$\frac{2-m+\sqrt{d}}{2\varphi^{k_0}}q_{k-1} + \frac{1}{\varphi^{k_0}}(q_k - q_{k-1})$$

$$= \frac{2-m+\sqrt{d}}{2\varphi^{k_0}}q_{k-1} + \frac{1}{\varphi^{k_0}}\big((m-1)q_{k-1} + q_{k-2}\big)$$

$$= \frac{m+\sqrt{d}}{2\varphi^{k_0}}q_{k-1} + \frac{1}{\varphi^{k_0}}q_{k-2}$$

$$= \frac{m+\sqrt{d}}{2\varphi^{k_0}}\left(\frac{1}{\sqrt{d}}\varphi^{k_0} - \frac{1}{\sqrt{d}}\varphi^{-k_0}\right)$$

$$\quad + \frac{1}{\varphi^{k_0}}\left(\frac{m+\sqrt{d}}{2\sqrt{d}}\varphi^{k_0-1} - \frac{m-\sqrt{d}}{2\sqrt{d}}\varphi^{-(k_0-1)}\right)$$

$$= \frac{m+\sqrt{d}}{2\sqrt{d}}(1+\varphi^{-1})$$

$$\quad - \frac{1}{\varphi^{2k_0}}\left(\frac{m+\sqrt{d}}{2\sqrt{d}} + \frac{m-\sqrt{d}}{2\sqrt{d}}\varphi\right)$$

$$= 1 - 0 = 1.$$

A similar calculation shows that the sum of measures also equals one when k is odd. Now we show that the sets $\tilde{R}_k(u)$ cover the interval $[0,1)$. If not, pick

$$x \in [0,1) \setminus \bigcup_{0 \le u < q_k} \tilde{R}_k(u).$$

Then, there exists $\epsilon > 0$ such that the sets $[x, x+\epsilon]$ and $\bigcup \tilde{R}_k(u)$ are disjoint. Since φ is irrational, the sequence $(\{p_k(n)\})$ is dense in $[0,1)$ by Kronecker's theorem (see, for example, [2, Theorem 7.7]). Hence there is an integer n_0 such that $\{p_k(n_0)\} \in [x, x+\epsilon]$. Therefore

$$\{p_k(n_0)\} \notin \bigcup_{0 \le u < q_k} \tilde{R}_k(u).$$

This contradicts Lemma 3.1. Thus the interval $[0,1)$ is the union of the sets $\tilde{R}_k(u)$.

Finally, we show that these sets are disjoint. If not, there exist x, v, w with $v \ne w$, such that

$$x \in \tilde{R}_k(v) \cap \tilde{R}_k(w).$$

Then there is an $\epsilon > 0$ such that

$$\lambda\big(\tilde{R}_k(v) \cap \tilde{R}_k(w)\big) \ge \epsilon.$$

Thus

$$1 = \lambda \left(\bigcup_{0 \le u < q_k} \tilde{R}_k(u) \right)$$

$$= \lambda \left(\left(\tilde{R}_k(v) \setminus \left(\tilde{R}_k(v) \cap \tilde{R}_k(w) \right) \right) \cup \bigcup_{u \ne v} \tilde{R}_k(u) \right)$$

$$\le \sum_{0 \le u < q_k} \lambda \left(\tilde{R}_k(u) \right) - \epsilon = 1 - \epsilon,$$

which is a contradiction. Therefore the sets $\tilde{R}_k(u)$ must be disjoint. $\qquad \square$

As an immediate consequence of Lemmas 3.1 and 3.2, we get

COROLLARY 3.3. *Let $n \ge 0$, $k \ge 2$ and $0 \le u < q_k$. Then*

$$t_\alpha(n; k) = u$$

if and only if

$$(-1)^k n\varphi \in R_k(u) + \mathbb{Z}.$$

We now present an inversion formula as in (7) for the function $e\big(\gamma S_{\alpha,k}(n)\big)$.

LEMMA 3.4. *Let $\gamma \in \mathbb{R}$ and $h, n \in \mathbb{Z}$ with $n \ge 0$. Let H, k be positive integers with $k \ge 2$. Define*

$$M_k^{(1)}(h, \gamma) = \sum_{0 \le u < q_{k-1}} e\big(\gamma S_\alpha(u) - h p_k(u)\big),$$

$$M_k^{(2)}(h, \gamma) = \sum_{q_{k-1} \le u < q_k} e\big(\gamma S_\alpha(u) - h p_k(u)\big),$$

where $p_k(u)$ is as defined in Lemma 3.1. For $|h| \le H$, there exist complex numbers $b_H^{(1)}(h), b_H^{(2)}(h), c_H^{(1)}(h)$ and $c_H^{(2)}(h)$ with

$$b_H^{(1)}(0) = \begin{cases} \frac{2 - m + \sqrt{d}}{2\varphi^{k_0}} & \text{if } k = 2k_0, \quad k_0 \in \mathbb{Z}, \\ \frac{1}{\varphi^{k_0}} & \text{if } k = 2k_0 + 1, \ k_0 \in \mathbb{Z}, \end{cases}$$

$$b_H^{(2)}(0) = \begin{cases} \frac{1}{\varphi^{k_0}} & \text{if } k = 2k_0, \quad k_0 \in \mathbb{Z}, \\ \frac{-m + \sqrt{d}}{2\varphi^{k_0}} & \text{if } k = 2k_0 + 1, \ k_0 \in \mathbb{Z}, \end{cases}$$

and for $i = 1, 2$,

$$|b_H^{(i)}(h)| \le \min\left(b_H^{(i)}(0), \frac{1}{|h|} \right) \quad \text{if } h \ne 0,$$

$$|c_H^{(i)}(h)| \le 2$$

such that

$$e\big(\gamma S_{\alpha,k}(n)\big) = \sum_{i=1}^{2}\left(\sum_{|h|\leq H} b_H^{(i)}(h)e\big(hp_k(n)\big)M_k^{(i)}(h,\gamma)\right)$$

$$+ O\left(\frac{1}{H}\sum_{|h|\leq H} c_H^{(1)}(h)e\big(hp_k(n)\big)\sum_{0\leq u<q_{k-1}} e\big(-hp_k(u)\big)\right)$$

$$+ O\left(\frac{1}{H}\sum_{|h|\leq H} c_H^{(2)}(h)e\big(hp_k(n)\big)\sum_{q_{k-1}\leq u<q_k} e\big(-hp_k(u)\big)\right),$$

where the expressions within the O-parentheses are non-negative real numbers.

P r o o f. Let $u \in [0, q_k)$ be an integer and let

$$\chi_u = \chi_{R_k(u)+\mathbb{Z}},$$

denote the indicator function of $R_k(u) + \mathbb{Z}$. Using Corollary 3.3, we get

$$e\big(\gamma S_{\alpha,k}(n)\big) = e\Big(\gamma S_\alpha\big(t_\alpha(n;k)\big)\Big)$$

$$= \sum_{0\leq u<q_k} e\big(\gamma S_\alpha(u)\big)\chi_u\big(p_k(n)\big)$$

$$= \sum_{0\leq u<q_{k-1}} e\big(\gamma S_\alpha(u)\big)\chi_u\big(p_k(n)\big)$$

$$+ \sum_{q_{k-1}\leq u<q_k} e\big(\gamma S_\alpha(u)\big)\chi_u\big(p_k(n)\big).$$

Using Vaaler's [22] trigonometric polynomial approximation to the function $\{x\} - 1/2$, one obtains (see [20, Eqn. (5.7)]),

$$\chi_{[a,b)+\mathbb{Z}}(x) = \sum_{|h|\leq H} a_H'(h)e\big(h(x-b)\big) + O\big(\kappa_H(x-b) + \kappa_H(x-a)\big),$$

where

$$a_H'(0) = b-a, \ |a_H'(h)| \leq \min\left(b-a, \frac{1}{|h|}\right) \text{ if } h \neq 0 \qquad (10)$$

and

$$\kappa_H(t) = \frac{1}{2(H+1)}\sum_{|h|\leq H}\left(1 - \frac{|h|}{H+1}\right)e(ht).$$

Further, $\kappa_H(t)$ is a non-negative real number for all real numbers t. Write

$$R_k(u) = [p_k(u) + c, p_k(u) + d),$$

where the values of c and d can be seen from Lemma 3.1.

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Then,

$$\sum_{0 \le u < q_{k-1}} e\big(\gamma S_\alpha(u)\big) \chi_u\big(p_k(n)\big)$$

$$= \sum_{|h| \le H} b_H^{(1)}(h) e\big(hp_k(n)\big) \sum_{0 \le u < q_{k-1}} e\big(\gamma S_\alpha(u) - hp_k(u)\big)$$

$$+ O\left(\sum_{\beta \in \{c,d\}} \sum_{0 \le u < q_{k-1}} \kappa_H\big(p_k(n) - p_k(u) - \beta\big) \right),$$

where

$$b_H^{(1)}(h) = a_H'(h) e(-hd).$$

Using (10), we get

$$b_H^{(1)}(0) = d - c$$

and

$$\left| b_H^{(1)}(h) \right| \le \min\left(b_H^{(1)}(0), \frac{1}{|h|} \right) \text{ if } h \ne 0.$$

Let $\beta \in \{c, d\}$. Since $\kappa_H(t)$ is a non-negative real number for all real numbers t, we have

$$\left| \sum_{0 \le u < q_{k-1}} \kappa_H(p_k(n) - p_k(u) - \beta) \right|$$

$$\le \frac{1}{H} \sum_{|h| \le H} \left(1 - \frac{|h|}{H+1} \right) \sum_{0 \le u < q_{k-1}} e(h(p_k(n) - p_k(u) - \beta)).$$

Thus, we obtain the term with $i = 1$ claimed in the lemma with

$$c_H^{(1)}(h) = \left(1 - \frac{|h|}{H+1} \right) \big(e(-hc) + e(-hd) \big),$$

whose absolute value is at most 2. Similarly, by considering the sum

$$\sum_{q_{k-1} \le u < q_k} e\big(\gamma S_\alpha(u)\big) \chi_u\big(p_k(n)\big),$$

we obtain the term with $i = 2$. $\qquad\square$

LEMMA 3.5. *Let $\gamma \in \mathbb{R}$ with $\|m\gamma\| \ne 0$. Then there exist $C, \eta > 0$ such that for all $\beta \in \mathbb{R}$ and $k \ge 2$, we have*

$$\left| \frac{1}{q_k} \sum_{0 \le u < q_k} e(\gamma S_\alpha(u) + \beta u) \right| \le C e^{-k\eta}.$$

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P r o o f. Set

$$\mu_k = \mu_k(\gamma, \beta) = \frac{1}{q_k} \sum_{0 \leq u < q_k} e\big(\gamma S_\alpha(u) + \beta u\big),$$

$$\tilde{M}_k = \max(|\mu_k|, |\mu_{k-1}|) \quad \text{and} \quad c_k = \gamma + \beta q_k.$$

By [11, Lemma 4 & p. 333], $(\tilde{M}_k)$ is a decreasing sequence and for $k \geq 1$,

$$\tilde{M}_{k+2} \leq \tilde{M}_{k-1} \left(1 - \min\left(\frac{1}{12}, \frac{3}{20} \left(1 + \frac{1}{a_k} - \frac{1}{a_{k+1}} \right) a_{k+1}^2 \|c_k\|^2 \right) \right)$$

$$\leq \tilde{M}_{k-1} \left(1 - \min\left(\frac{1}{12}, \frac{3}{20m} \|c_k\|^2 \right) \right) =: \tilde{M}_{k-1} \psi_k, \tag{11}$$

as $a_i \in \{1, m\}$ for $i \geq 1$. Observe that

$$\|a_{k+3}\gamma\| = \|a_{k+3}\gamma + \beta(a_{k+3}q_{k+2} + q_{k+1} - q_{k+3})\|$$

$$\leq \|c_{k+1}\| + a_{k+3}\|c_{k+2}\| + \|c_{k+3}\|.$$

Thus

$$\min(\|\gamma\|, \|m\gamma\|) \leq m \sum_{1 \leq i \leq 3} \|c_{k+i}\|,$$

implying that for every k, there exists $i \in \{1, 2, 3\}$ such that

$$\|c_{k+i}\| \geq \frac{\min(\|\gamma\|, \|m\gamma\|)}{3m}$$

and hence

$$\psi_{k+i} \leq 1 - \min\left(\frac{1}{12}, \frac{\min(\|\gamma\|, \|m\gamma\|)^2}{60m^3} \right)$$

$$= 1 - \frac{\min(\|\gamma\|, \|m\gamma\|)^2}{60m^3} =: \psi.$$

Since $\|m\gamma\| \neq 0$, $\psi < 1$. Fix k. Then there exists $k_i \in \{k-2, k-3, k-4\}$ such that $\psi_{k_i} \leq \psi$. Using (11) and the fact that $(\tilde{M}_j)$ is a decreasing sequence, we get

$$\tilde{M}_k \leq \tilde{M}_{k_i+2} \leq \tilde{M}_{k_i-1}\psi_{k_i} \leq \tilde{M}_{k-5}\psi.$$

Applying this repeatedly, we get

$$\tilde{M}_k \leq \tilde{M}_5 \psi^{k/5-2},$$

completing the proof of the lemma. □

4. Proof of Theorem 1.2

With all the ingredients in place, the theorem follows as in [20]. We include the details below. Set

$$R = \lfloor N^{a-\eta a/(2\ln\varphi)} \rfloor, \ H = \lfloor N^{4a+\eta a/(2\ln\varphi)} \rfloor,$$
$$t = \lfloor a\ln N/\ln q \rfloor, \qquad k = 1 + \lfloor 2a\ln N/\ln\varphi \rfloor,$$

where η is as in Lemma 3.5, φ is as in (4) and a is sufficiently small. Let

$$g_q(n) = e\big(\theta S_q(n)\big), \ g_{q,t}(n) = e\big(\theta S_{q,t}(n)\big),$$
$$g_\alpha(n) = e\big(\gamma S_\alpha(n)\big), \ g_{\alpha,k}(n) = e\big(\gamma S_{\alpha,k}(n)\big)$$

and

$$g(n) = g_q(n)g_\alpha(n).$$

By Lemma 2.3, we get

$$\left| \sum_{n<N} g(n) \right|^2 \ll \frac{N}{R} \sum_{|r|<R} \left(1 - \frac{|r|}{R}\right) \left| \sum_{0\leq n, n+r<N} g(n+r)\overline{g(n)} \right|.$$

Removing the condition $n + r < N$ from the second sum gives an error which is

$$\ll \frac{N}{R} \sum_{|r|<R} \left(1 - \frac{|r|}{R}\right) \left| \sum_{N\leq n+r<N+R} g(n+r)\overline{g(n)} \right|$$
$$\leq \frac{N}{R} \sum_{|r|<R} \left(1 - \frac{|r|}{R}\right) \sum_{N\leq n+r<N+R} 1$$
$$\leq \frac{N}{R} \sum_{|r|<R} \left(1 - \frac{|r|}{R}\right) R = \frac{N}{R} \sum_{|r|<R} (R - |r|) \ll NR.$$

Thus

$$\left| \sum_{n<N} g(n) \right|^2 \ll \frac{N}{R} \sum_{|r|<R} \left(1 - \frac{|r|}{R}\right) \left| \sum_{0\leq n<N} g(n+r)\overline{g(n)} \right| + O(NR). \qquad (12)$$

We denote by N' the largest multiple of q^t not exceeding N. Restricting the second sum in (12) to N' gives an error which is

$$\ll \frac{N}{R} \sum_{|r|<R} \left(1 - \frac{|r|}{R}\right) q^t = \frac{Nq^t}{R^2} \sum_{|r|<R} (R - |r|) \ll Nq^t.$$

Hence

$$\left|\sum_{n<N} g(n)\right|^2 \ll \frac{N}{R} \sum_{|r|<R} \left(1 - \frac{|r|}{R}\right) \left|\sum_{0\leq n<N'} g(n+r)\overline{g(n)}\right|$$

$$+ O(NR + Nq^t). \tag{13}$$

From the definitions of the functions g, g_q, g_α, we get

$$g(n+r)\overline{g(n)} = g_q(n+r)g_\alpha(n+r)\overline{g_q(n)}\ \overline{g_\alpha(n)}$$

$$= e\big(\theta S_q(n+r)\big)e\big(\gamma S_\alpha(n+r)\big)e\big(-\theta S_q(n)\big)e\big(-\gamma S_\alpha(n)\big)$$

$$= e\Big(\theta\big(S_q(n+r) - S_q(n)\big)\Big)e\Big(\gamma\big(S_\alpha(n+r) - S_\alpha(n)\big)\Big).$$

Therefore for a fixed r, we have

$$\left|\sum_{0\leq n<N'} g(n+r)\overline{g(n)}\right|$$

$$= \left|\sum_{0\leq n<N'} e\Big(\theta\big(S_q(n+r) - S_q(n)\big)\Big)e\Big(\gamma\big(S_\alpha(n+r) - S_\alpha(n)\big)\Big)\right|$$

$$\leq \left|\sum_{0\leq n<N'} e\Big(\theta\big(S_{q,t}(n+r) - S_{q,t}(n)\big)\Big)e\Big(\gamma\big(S_{\alpha,k}(n+r) - S_{\alpha,k}(n)\big)\Big)\right|$$

$$+ \sum_{0\leq n<N'}{}' 1$$

$$= \left|\sum_{0\leq n<N'} g_{q,t}(n+r)\overline{g_{q,t}(n)}g_{\alpha,k}(n+r)\overline{g_{\alpha,k}(n)}\right| + \sum_{0\leq n<N'}{}' 1,$$

where the second sum $\sum'$ runs over those integers n for which either

$$e\Big(\theta\big(S_q(n+r) - S_q(n)\big)\Big) \neq e\Big(\theta\big(S_{q,t}(n+r) - S_{q,t}(n)\big)\Big)$$

or

$$e\Big(\gamma\big(S_\alpha(n+r) - S_\alpha(n)\big)\Big) \neq e\Big(\gamma\big(S_{\alpha,k}(n+r) - S_{\alpha,k}(n)\big)\Big).$$

Substituting in (13), we get

$$\left| \sum_{n<N} g(n) \right|^2 \ll \frac{N}{R} \sum_{|r|<R} \left(1 - \frac{|r|}{R}\right) \left| \sum_{0 \le n < N'} g_{q,t}(n+r)\overline{g_{q,t}(n)} g_{\alpha,k}(n+r)\overline{g_{\alpha,k}(n)} \right|$$

$$+ \frac{N}{R} \sum_{|r|<R} \left(1 - \frac{|r|}{R}\right) \left({\sum_{0 \le n < N'}}' 1 \right) + O(NR + Nq^t). \qquad (14)$$

Now by Lemma 2.4,

$${\sum_{0 \le n < N'}}' 1 \le \left(\frac{Nr}{q^t} + r\right) + \frac{Nr}{q_{k-1}}.$$

Therefore,

$$\frac{N}{R} \sum_{|r|<R} \left(1 - \frac{|r|}{R}\right) \left({\sum_{0 \le n < N'}}' 1 \right) \le \frac{N}{R} \sum_{|r|<R} {\sum_{0 \le n < N'}}' 1$$

$$\le \frac{N}{R} \sum_{|r|<R} \left(\frac{Nr}{q^t} + r + \frac{Nr}{q_{k-1}}\right) = \frac{N}{R} \left(\frac{N}{q^t} + 1 + \frac{N}{q_{k-1}}\right) \sum_{|r|<R} r$$

$$\ll \frac{N}{R} \left(\frac{N}{q^t} + 1 + \frac{N}{q_{k-1}}\right) R^2 = \frac{N^2 R}{q^t} + NR + \frac{N^2 R}{q_{k-1}}.$$

Substituting in (14), we obtain

$$\left| \sum_{n<N} g(n) \right|^2 \ll \frac{N}{R} \sum_{|r|<R} \left(1 - \frac{|r|}{R}\right) \left| \sum_{0 \le n < N'} g_{q,t}(n+r)\overline{g_{q,t}(n)} g_{\alpha,k}(n+r)\overline{g_{\alpha,k}(n)} \right|$$

$$+ O\left(NR + Nq^t + \frac{N^2 R}{q^t} + \frac{N^2 R}{q_{k-1}}\right).$$

From (5), it follows that $\varphi^{(k-1)/2} \ll q_{k-1}$. Using this and the choice of R, t, k, we find that

$$NR + Nq^t + \frac{N^2 R}{q^t} + \frac{N^2 R}{q_{k-1}}$$

$$\ll N^{1+a-\eta a/(2 \ln \varphi)} + Nq^{a \ln_q N} + \frac{N^2 N^{a-\eta a/(2 \ln \varphi)}}{q^{a \ln_q N}} + \frac{N^2 N^{a-\eta a/(2 \ln \varphi)}}{\varphi^{a \ln_\varphi N}}$$

$$\ll N^{1+a} + N^{2-\eta a/(2 \ln \varphi)}.$$

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Thus

$$\left|\sum_{n<N} g(n)\right|^2 \ll \frac{N}{R}\sum_{|r|<R}\left(1-\frac{|r|}{R}\right)\left|\sum_{0\leq n<N'} g_{q,t}(n+r)\overline{g_{q,t}(n)}g_{\alpha,k}(n+r)\overline{g_{\alpha,k}(n)}\right|$$
$$+ O\left(N^{1+a}+N^{2-\eta a/(2\ln\varphi)}\right). \quad (15)$$

We use the expressions for $g_{q,t}(n+r), g_{q,t}(n), g_{\alpha,k}(n+r), g_{\alpha,k}(n)$ from (7), (8) and Lemma 3.4. In the product $g_{\alpha,k}(n+r)g_{\alpha,k}(n)$, there are sixteen summands of the following three kinds: four products of the main terms in the expressions for $g_{\alpha,k}(n+r)$ and $g_{\alpha,k}(n)$, eight products of the main terms and error terms, four products of the error terms. We now consider these three cases separately.

Case I. (*Summands with both factors as main terms.*)
Let $h = h_1 + h_2$ and $\ell = \ell_1 + \ell_2$. We need to estimate

$$\frac{N}{R^2}\sum_{\substack{\ell_1,\ell_2<q^t \\ |h_1|,|h_2|\leq H}} G_t(\ell_1)\overline{G_t(-\ell_2)}b_H^{(i)}(h_1)\overline{b_H^{(j)}(-h_2)}M_k^{(i)}(h_1,\gamma)\overline{M_k^{(j)}(-h_2,\gamma)}$$
$$\times \sum_{|r|<R}(R-|r|)e\left(r\left(\frac{\ell_1}{q^t}+h_1(-1)^k\varphi\right)\right)\sum_{n<N'}e\left(n\left(\frac{\ell}{q^t}+(-1)^kh\varphi\right)\right). \quad (16)$$

We first consider the subcase when $h = 0$. If $\ell \not\equiv 0 \pmod{q^t}$, then

$$\sum_{n<N'}e\left(n\ell q^{-t}\right) = 0$$

as $q^t|N'$. Hence we assume that $\ell \equiv 0 \pmod{q^t}$. Therefore, $G_t(-\ell_2) = G_t(\ell_1)$, implying that
$$G_t(\ell_1)\overline{G_t(-\ell_2)} = |G_t(\ell_1)|^2. \quad (17)$$

We will estimate the sum over n trivially. By Lemma 2.1(ii) and (i),

$$\sum_{|r|<R}(R-|r|)e\left(r\left(\frac{\ell_1}{q^t}+h_1(-1)^k\varphi\right)\right) = \left|\sum_{r<R}e\left(r\left(\frac{\ell_1}{q^t}+h_1(-1)^k\varphi\right)\right)\right|^2$$

$$\leq \left(\min\left(R,\frac{1}{2}\left\|\ell_1 q^{-t}+h_1(-1)^k\varphi\right\|^{-1}\right)\right)^2$$

$$\leq \min\left(R^2,\left\|\frac{\ell_1}{q^t}+h_1(-1)^k\varphi\right\|^{-2}\right). \quad (18)$$

Next,
$$|M_k^{(j)}(-h_2,\gamma)| \leq q_k \ll q_{k-1} \quad (19)$$
as
$$q_k \leq (m+1)q_{k-1}.$$

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By Lemma 3.4,
$$\left|b_H^{(j)}(0)\right| \le \frac{2}{\varphi^{k_0}} \le \frac{2}{q_k},$$
since $q_k \le \varphi^{k_0}$ by (5). Again, by Lemma 3.4 and the fact that $h = h_1 + h_2 = 0$, we get
$$\left|b_H^{(j)}(-h_2)\right| = \left|b_H^{(j)}(h_1)\right| \ll \min\left(\frac{1}{q_k}, \frac{1}{|h_1|}\right) \text{ if } h_1 \ne 0. \tag{20}$$

Similarly,
$$\left|b_H^{(i)}(h_1)\right| \ll \frac{1}{q_k}, \tag{21}$$
implying that
$$\left|b_H^{(i)}(h_1) M_k^{(i)}(h_1, \gamma)\right| \ll \frac{1}{q_k}\left|\sum_u e\big(\gamma S_\alpha(u) - h p_k(u)\big)\right|,$$
where the sum either runs over $u < q_{k-1}$ or $q_{k-1} \le u < q_k$. Thus by the triangle inequality
$$\left|b_H^{(i)}(h_1) M_k^{(i)}(h_1, \gamma)\right|$$
$$\le \frac{1}{q_k}\left|\sum_{u < q_{k-1}} e\big(\gamma S_\alpha(u) - h p_k(u)\big)\right| + \frac{1}{q_k}\left|\sum_{u < q_k} e\big(\gamma S_\alpha(u) - h p_k(u)\big)\right|$$
$$\le C e^{-(k-1)\eta} + C e^{-k\eta} \ll e^{-(k-1)\eta}, \tag{22}$$
where the last step follows from Lemma 3.5 and the inequality $q_{k-1} \le q_k$. Combining the estimates (17)-(20) and (22), we find that the expression in (16) is
$$\ll \frac{N^2 q_{k-1}}{e^{\eta(k-1)} R^2} \sum_{\ell_1 < q^t} |G_t(\ell_1)|^2 \sum_{|h_1| \le H} \min\left(\frac{1}{q_k}, \frac{1}{|h_1|}\right) \min\left(R^2, \left\|\frac{\ell_1}{q^t} + h_1(-1)^k \varphi\right\|^{-2}\right).$$

Using (9), the above expression is
$$\ll \frac{N^2 q_{k-1}}{e^{\eta(k-1)} R^2} \sup_{\ell_1 \in \mathbb{Z}} \sum_{|h| \le H} \min\left(\frac{1}{q_k}, \frac{1}{|h|}\right) \min\left(R^2, \|\ell_1 q^{-t} + |h|\varphi\|^{-2}\right).$$

Write
$$|h| = s q_k + \tilde{h}, \qquad 0 \le \tilde{h} < q_k.$$

Then
$$|h|^{-1} \le (s q_k)^{-1} \quad \text{and} \quad \min\big(1/(s q_k), 1/q_k\big) = 1/(s q_k).$$

Hence the preceeding expression is
$$\ll \frac{N^2 q_{k-1}}{e^{\eta(k-1)} R^2} \sum_{s \le H/q_k} \frac{1}{s q_k} \times \sup_{\ell_1 \in \mathbb{Z}} \sum_{\tilde{h} < q_k} \min\left(R^2, \|\ell_1 q^{-t} + (s q_k + \tilde{h})\varphi\|^{-2}\right).$$

By Lemma 2.2 and the inequality $\sum_{s \leq x} s^{-1} \ll \ln x$, we obtain the bound

$$\ll \frac{N^2}{e^{\eta(k-1)} R^2} \ln H (Rq_k + R^2 \ln q_k) \ll \frac{N^2}{e^{\eta(k-1)}} \ln H (q_{k-1} R^{-1} + \ln q_{k-1}).$$

By the choice of H, R, k and the fact that $q_{k-1} \leq \varphi^{(k-1)/2}$ (from (5)), the contribution in this subcase is

$$\ll \left(4a + \frac{\eta a}{2 \ln \varphi} \right) \frac{N^2 \ln N}{e^{\eta(2a \ln N / \ln \varphi)}} \left(\frac{\varphi^{a \ln_\varphi N}}{N^{a - \eta a/(2 \ln \varphi)}} + \ln \varphi^{a \ln_\varphi N} \right)$$

$$= \left(4a + \frac{\eta a}{2 \ln \varphi} \right) \ln N \left(N^{2 - 3\eta a/(2 \ln \varphi)} + aN^{2 - 2\eta a/\ln \varphi} \ln N \right).$$

Next, we consider the case when $h \neq 0$. Since φ is badly approximable, there is a constant c_1 such that for all integers ℓ', h' with $h' \neq 0$,

$$\left| \varphi + \frac{\ell'}{h' q^t} \right| > \frac{c_1}{(h' q^t)^2}.$$

Thus

$$\|h\varphi + \ell q^{-t}\| = \min_{j \in \mathbb{Z}} |h\varphi - j + \ell q^{-t}| = \min_{j \in \mathbb{Z}} |h| \left| \varphi - \frac{jq^t - \ell}{hq^t} \right|$$

$$> |h| \frac{c_1}{(hq_t)^2} = \frac{c_1}{|h| q^{2t}}. \tag{23}$$

We estimate G_t and the sum over r trivially. Next, as in (21), $|b_H^{(i)}(h_1)| \ll q_k^{-1}$. Further, since $|M_k^{(i)}(h_1, \gamma)| \leq q_k$, we get that $|b_H^{(i)}(h_1) M_k^{(i)}(h_1, \gamma)| \ll 1$. Similarly,

$$|b_H^{(j)}(-h_2) M_k^{(j)}(-h_2, \gamma)| \ll 1.$$

Thus we obtain that the expression in (16) is

$$\ll Nq^{2t} \sum_{\substack{|h_1|, |h_2| \leq H \\ h_1 + h_2 \neq 0}} \sup_{\ell \in \mathbb{Z}} \left| \sum_{n < N'} e\big(n(\ell q^{-t} + (-1)^k h\varphi)\big) \right|.$$

Let $h \in \mathbb{Z}$ with $1 \leq |h| \leq 2H$. The number of pairs (h_1, h_2) with $(-1)^k (h_1 + h_2) = h$ is $2H + 1 - |h|$. Hence the above expression equals

$$Nq^{2t} \sum_{1 \leq |h| \leq 2H} (2H + 1 - |h|) \sup_{\ell \in \mathbb{Z}} \left| \sum_{n < N'} e\big(n(\ell q^{-t} + h\varphi)\big) \right|$$

$$\ll Nq^{2t} \sum_{1 \leq |h| \leq 2H} (2H + 1 - |h|) \sup_{\ell \in \mathbb{Z}} \left(\min\left(N, \frac{1}{2\|\ell q^{-t} + h\varphi\|} \right) \right)$$

$$\ll NHq^{2t} \sum_{1 \leq |h| \leq 2H} |h| q^{2t} \ll NH^3 q^{4t},$$

by Lemma 2.1 and (23).

Further, by the choice of H, t, the contribution in this subcase is

$$\ll N^{1+12a+3\eta a/(2\ln\varphi)}q^{4a\ln_q N} = N^{1+16a+3\eta a/(2\ln\varphi)}.$$

Case II. (*Summands with exactly one factor as main term.*)
Suppose that the main term comes from the expression for $g_{\alpha,t}(n+r)$.
(The other case is similar.) Then, we need to estimate

$$\frac{N}{R}\sum_{|r|<R}\left(1-\frac{|r|}{R}\right)\left|\sum_{n<N'}\sum_{\substack{\ell_1,\ell_2<q^t\\|h_1|\leq H}} e\big(\ell_1(n+r)q^{-t}+\ell_2 nq^{-t}+h_1p_k(n+r)\big)\right.$$

$$G_t(\ell_1,\theta)\overline{G_t(-\ell_2,\theta)}b_H^{(i)}(h_1)M_k^{(i)}(h_1,\gamma)\times$$

$$\left. O\left(\frac{1}{H}\sum_{|h_2|\leq H}c_H^{(j)}(h_2)e\big(h_2p_k(n)\big)\sum_u e\big(-h_2p_k(u)\big)\right)\right|.$$

Recall that the expression in the error term is a non-negative real number. We use the inequality $|M_k^{(i)}(h_1,\gamma)|\leq q_k$, and estimate G_t and the sum over r trivially. Using Lemma 3.4, we obtain the following upper bound.

$$\frac{Nq^{2t}q_k}{H}\sum_{|h_1|\leq H}\min\left(b_H^{(i)}(0),\frac{1}{|h_1|}\right)$$

$$\times\sum_{|h_2|\leq H}\left|\sum_u e(-h_2p_k(u))\right|\left|\sum_{n<N'}e(h_2p_k(n))\right|$$

$$\ll\frac{Nq^{2t}q_k\ln H}{H}\sum_{|h_2|\leq H}\left|\sum_u e(-h_2p_k(u))\right|\left|\sum_{n<N'}e(h_2p_k(n))\right|,\qquad(24)$$

where the last step follows from the inequality $\sum_{s\leq x}s^{-1}\ll\ln x$. By Lemma 2.1, the above expression is

$$\ll\frac{Nq^{2t}q_k\ln H}{H}\sum_{|h_2|\leq H}\min\left(q_k,\frac{1}{||h_2\varphi||}\right)\min\left(N,\frac{1}{||h_2\varphi||}\right)$$

$$\ll\frac{Nq^{2t}q_k\ln H}{H}\sum_{|h_2|\leq H}\min\left(q_kN,\frac{1}{||h_2\varphi||^2}\right)$$

$$\ll N^{3/2}q^{2t}q_k^{3/2}\ln H+N^2q^{2t}q_k^2\frac{(\ln H)^2}{H},$$

where the last inequality follows from Lemma 2.2. By the choice of H, t, k and the fact that $q_k \leq \varphi^{k/2}$, we find that the contribution in this case is

$$\ll \left(4a + \frac{\eta a}{2 \ln \varphi} \right) N^{3/2} q^{2a \ln_q N} \varphi^{(3a \ln_\varphi N)/2} \ln N$$

$$+ \left(4a + \frac{\eta a}{2 \ln \varphi} \right)^2 N^2 q^{2a \ln_q N} \varphi^{2a \ln_\varphi N} \frac{(\ln N)^2}{N^{4a + \eta a/(2 \ln \varphi)}}$$

$$\leq \left(4a + \frac{\eta a}{2 \ln \varphi} \right) N^{(3+7a)/2} \ln N + \left(4a + \frac{\eta a}{2 \ln \varphi} \right)^2 N^{2 - \eta a/(2 \ln \varphi)} (\ln N)^2 .$$

Case III. (*Summands with both factors as error terms.*)
We need to estimate

$$\frac{N}{R} \sum_{|r| < R} \left(1 - \frac{|r|}{R} \right) \left| \sum_{n < N'} \sum_{\ell_1, \ell_2 < q^t} e\big(\ell_1 (n+r) q^{-t} + \ell_2 n q^{-t}\big) \right.$$

$$G_t(\ell_1, \theta) \overline{G_t(-\ell_2, \theta)}$$

$$\times O\left(\frac{1}{H} \sum_{|h_1| \leq H} c_H^{(i)}(h_1) e\big(h_1 p_k(n+r)\big) \sum_u e\big(-h_1 p_k(u)\big) \right)$$

$$\times O\left(\frac{1}{H} \sum_{|h_2| \leq H} c_H^{(j)}(h_2) e\big(h_2 p_k(n)\big) \sum_u e\big(-h_2 p_k(u)\big) \right) \Bigg| .$$

We estimate one of the error terms trivially by q_k. Further, we estimate G_t and the sum over r trivially to get

$$\ll \frac{N q^{2t} q_k}{H} \sum_{|h_2| \leq H} \left| \sum_u e\big(-h_2 p_k(u)\big) \right| \left| \sum_{n < N'} e\big(h_2 p_k(n)\big) \right| .$$

This is the product of the expression in (24) with $(\ln H)^{-1}$. Therefore, proceeding as in Case II, we obtain the bound

$$\ll N^{3/2} q^{2t} q_k^{3/2} + N^2 q^{2t} q_k^2 \frac{\ln H}{H}$$

$$\ll N^{(3+7a)/2} + \left(4a + \frac{\eta a}{2 \ln \varphi} \right) N^{2 - \eta a/(2 \ln \varphi)} \ln N .$$

Combining the three cases and (15), we get

$$\left| \sum_{n < N} g(n) \right|^2 \ll N^{2 - 2\delta} \quad \text{for some } \delta > 0 .$$

$\square$

P r o o f o f C o r o l l a r y 1.3.

We first recall an estimate of Gel'fond [13]: If k_1, m_1, q are positive integers with

$$m_1 \geq 2, \quad k_1 < m_1 \quad \text{and} \quad \gcd(m_1, q-1) = 1,$$

then there exists $\delta_1 > 0$ such that for every $a \in \mathbb{R}$, we have

$$\sum_{n<N} e\left(an + \frac{k_1}{m_1} S_q(n)\right) = O(N^{1-\delta_1}). \tag{25}$$

Now, since

$$\frac{1}{b} \sum_{0 \leq \ell < b} e\left(\frac{a}{b}\ell\right) = 1 \text{ or } 0$$

according to whether b divides a or not, we have

$$|\{0 \leq n < N : S_q(n) \equiv r_1 \ (\mathrm{mod}\ m_1), \quad S_\alpha(n) \equiv r_2 \ (\mathrm{mod}\ m_2)\}|$$

$$= \sum_{n<N} \frac{1}{m_1} \sum_{0 \leq k_1 < m_1} e\left(k_1 \frac{S_q(n) - r_1}{m_1}\right) \frac{1}{m_2} \sum_{0 \leq k_2 < m_2} e\left(k_2 \frac{S_\alpha(n) - r_2}{m_2}\right)$$

$$= \frac{1}{m_1 m_2} \sum_{\substack{0 \leq k_1 < m_1 \\ 0 \leq k_2 < m_2}} e\left(-\frac{k_1 r_1}{m_1} - \frac{k_2 r_2}{m_2}\right) \sum_{n<N} e\left(\frac{k_1}{m_1} S_q(n) + \frac{k_2}{m_2} S_\alpha(n)\right)$$

$$= \frac{N}{m_1 m_2} + O\left(\frac{1}{m_1 m_2} \sum_{1 \leq k_1 < m_1} \left|\sum_{n<N} e\left(\frac{k_1}{m_1} S_q(n)\right)\right|\right.$$

$$\left. + \frac{1}{m_1 m_2} \sum_{\substack{0 \leq k_1 < m_1 \\ 1 \leq k_2 < m_2}} \left|\sum_{n<N} e\left(\frac{k_1}{m_1} S_q(n) + \frac{k_2}{m_2} S_\alpha(n)\right)\right|\right)$$

$$= \frac{N}{m_1 m_2} + O(N^{1-\delta'}),$$

where the last equality follows from (25) and Theorem 1.2. $\qquad\square$

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Divyum Sharma
Department of Pure Mathematics,
University of Waterloo,
Ontario, Canada.

Current Address:
Department of Mathematics
Birla Institute of Technology and Science, Pilani
Vidya Vihar, Rajasthan INDIA

E-mail: divyum.sharma@pilani.bits-pilani.ac.in