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DISTRIBUION OF LEADING DIGITS OF NUMBERS II

Yukio Ohkubo[1] — Oto Strauch[2]

[1]Dept. of Businessv Administration, The International University of Kagoshima, Kagoshima, JAPAN

[2]Institue of Mathematics, Slovak Academy of Sciences, Bratislava, SLOVAKIA

ABSTRACT. In this paper, we study the sequence $(f(p_n))_{n\ge1}$, where p_n is the nth prime number and f is a function of a class of slowly increasing functions including $f(x)=\log_b x^r$ and $f(x)=\log_b(x\log x)^r$, where $b\ge 2$ is an integer and $r>0$ is a real number. We give upper bounds of the discrepancy $D^*_{N_i}(f(p_n),g)$ for a distribution function g and a sub-sequence $(N_i)_{i\ge1}$ of the natural numbers. Especially for $f(x)=\log_b x^r$, we obtain the effective results for an upper bound of $D^*_{N_i}(f(p_n),g)$.

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1. Introduction

In [8], we gave some results about the first digit problem in base $b\ge 2$ for the sequence $(n^r)_{n\ge1}$ and for the sequence $(p_n^r)_{n\ge1}$, where p_n is the nth prime number and $r>0$ is a real number. In this paper, we generalize and sharpen them.

Throughout this paper, for $x\in\mathbb{R}$, let

$$\lfloor x\rfloor=\max\{n\in\mathbb{Z}:n\le x\},\quad \lceil x\rceil=\min\{n\in\mathbb{Z}:x\le n\},\quad \{x\}=x-\lfloor x\rfloor,$$

let c_E denote the characteristic function of the set $E\subset\mathbb{R}$, and let $\pi(x)$ be the number of primes not exceeding x.

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Let $(y_n)_{n\geq 1}$ be a sequence of real numbers. If there exists a strictly increasing sequence of positive integers $(N_i)_{i\geq 1}$ such that

$$\lim_{i\to\infty}\frac{1}{N_i}\sum_{n=1}^{N_i} c_{[0,x)}(\{y_n\}) = g(x) \quad \text{for every} \quad x\in[0,1],$$

then g is called a distribution functions (d.f.) mod 1 of $(y_n)_{n\geq 1}$. Let $G(y_n \bmod 1)$ be the set of all d.f.'s of $(y_n)_{n\geq 1}$ (see [12]). If $G(y_n \bmod 1) = \{g(x) = x\}$, then the sequence $(y_n)_{n\geq 1}$ is said to be uniformly distributed mod1 (abbreviating u.d.mod1).

Suppose that g is a non-decreasing function on $[0,1]$ with $g(0) = 0$, $g(1) = 1$. Then the extremal discrepancy $D_N(y_n, g)$ and the star discrepancy $D_N^*(y_n, g)$ of $(y_n)_{n\geq 1}$ with respect to g are defined by

$$D_N(y_n, g) = \sup_{0\leq\alpha<\beta\leq 1}\left|\frac{1}{N}\sum_{n=1}^{N} c_{[\alpha,\beta)}(\{y_n\}) - (g(\beta) - g(\alpha))\right|$$

and

$$D_N^*(y_n, g) = \sup_{x\in[0,1]}\left|\frac{1}{N}\sum_{n=1}^{N} c_{[0,x)}(\{y_n\}) - g(x)\right|,$$

respectively (see [12, 1.10.1]). If $g(x) = x$ for $0\leq x\leq 1$, then g is omitted in $D_N^*(y_n, g)$ or $D_N(y_n, g)$.

Let $b\geq 2$ be an integer considered as a base for the development of a real number $x > 0$ and $M_b(x)$ be the mantissa of x defined by $x = M_b(x)\times b^{n(x)}$ such that $1\leq M_b(x) < b$ holds, where $n(x)$ is a uniquely determined integer. Let $D = d_1 d_2\cdots d_s$ be a positive integer expressed in the base b, that is

$$D = d_1 b^{s-1} + d_2 b^{s-2} + \cdots + d_{s-1} b + d_s,$$

where $d_1\neq 0$ and at the same time $D = d_1 d_2\cdots d_s$ is considered as an s-consecutive block of digits in the base b. Note that for x of the type $x = 0.00\cdots 0\, d_1 d_2\cdots d_s\cdots$, $d_1 > 0$, we have $M_b(x) = d_1.d_2\cdots d_s\cdots$ and the first zero digits are omitted. Thus a positive real number x has the first s-digits, starting a non-zero digit, that equal to $D = d_1 d_2\cdots d_s$ if and only if

$$\frac{D}{b^{s-1}} = d_1.d_2\cdots d_s \leq M_b(x) < d_1.d_2\cdots(d_s+1) = \frac{D+1}{b^{s-1}}. \tag{1}$$

Let (x_n) be a sequence of positive number, and let $F(D, s, N, x_n)$ be the number of integers $1\leq n\leq N$ such that the leading block of s digits (beginning with $\neq 0$) of x_n equals D. The sequence $(x_n)_{n\geq 1}$ is said to satisfy *Benford's law* (abbreviated by B. L.) in base b, if for every $s = 1, 2, \ldots$ and every s-digits integer $D = d_1 d_2\cdots d_s$

$$\lim_{N\to\infty} \frac{F(D,s,N,x_n)}{N} = \log_b\left(\frac{D+1}{b^{s-1}}\right) - \log_b\left(\frac{D}{b^{s-1}}\right)$$

(see [3]). Since $\log_b M_b(x) = \{\log_b x\}$ by (1), the leading block of s digits (beginning with $\neq 0$) of x_n equals D if and only if

$$\log_b\left(\frac{D}{b^{s-1}}\right) \leq \{\log_b x_n\} < \log_b\left(\frac{D+1}{b^{s-1}}\right).$$

Hence, the sequence $(x_n)_{n\geq 1}$ satisfies Benford's law if and only if $(\log_b x_n)_{n\geq 1}$ is uniformly distributed mod1 (see [3], [9]).

If $g \in G(\log_b x_n \bmod 1)$ and

$$\lim_{i\to\infty} \frac{1}{N_i} \sum_{n=1}^{N_i} c_{[0,x)}(\{\log_b x_n\}) = g(x)$$

for $x \in [0,1]$, then for every s-digits integer $D = d_1 d_2 \cdots d_s$, $s = 1, 2, \ldots,$

$$\lim_{i\to\infty} \frac{F(D,s,N_i,x_n)}{N_i} = g\left(\log_b\left(\frac{D+1}{b^{s-1}}\right)\right) - g\left(\log_b\left(\frac{D}{b^{s-1}}\right)\right)$$

(see [9], [8]).

For $r > 0$, the sequence $(n^r)_{n\geq 1}$ and $(p_n^r)_{n\geq 1}$ do not satisfy Benford's law, that is, the sequence $(\log_b n^r)_{n\geq 1}$ and $(\log_b p_n^r)_{n\geq 1}$ are not u.d. mod1. Eliahou, Massé, and Schneider [4] showed that for a positive integer r, the discrepancy of the sequence $(y_n) = (\log_{10} n^r)_{n\geq 1}$, $(\log_{10}(n \log n)^r)_{n\geq 1}$, or $(\log_{10} p_n^r)_{n\geq 1}$ satisfies $D_{\phi(r)}(y_n) = O(r^{-1})$ with $\phi(r) = \lfloor e^r \rfloor$. Thus, when $r \to \infty$, these sequences satisfy Benford's law in a sense.

It is also known that

$$G(\log_b n^r \bmod 1) = G(\log_b p_n^r \bmod 1) = \{g_w(x); 0 \leq w \leq 1\},$$

where

$$g_w(x) = \frac{1}{b^{\frac{w}{r}}} \frac{b^{\frac{x}{r}} - 1}{b^{\frac{1}{r}} - 1} + \frac{\min(b^{\frac{x}{r}}, b^{\frac{w}{r}}) - 1}{b^{\frac{w}{r}}} \quad (0 \leq x \leq 1),$$

and if $\lim_{i\to\infty}\{\log_b p_{N_i}^r\} = w$, then for $x \in [0,1]$

$$\lim_{i\to\infty} \frac{1}{N_i} \sum_{n=1}^{N_i} c_{[0,x)}(\{\log_b p_n^r\}) = g_w(x)$$

(see [13], [6], [12, 2.12.1], and [7]), and so

$$\lim_{i\to\infty} \frac{F(D,s,N_i,p_n^r)}{N_i} = g_w\left(\log_b\left(\frac{D+1}{b^{s-1}}\right)\right) - g_w\left(\log_b\left(\frac{D}{b^{s-1}}\right)\right). \tag{2}$$

As examples of N_i and w, in [8] we gave the following:

- if $0 < w < 1$ and $N_i = \pi(b^{\frac{i+w}{r}})$, then $\lim_{i\to\infty}\{\log_b p_{N_i}^r\} = w$,
- if $N_i = \pi(b^{\frac{i}{r}})$, then $\lim_{i\to\infty}\{\log_b p_{N_i}^r\} = 1$.

In [8], for rate of convergence of (2) we also proved that

$$\left|\frac{F(D,s,N_i,p_n^r)}{N_i} - \frac{(\frac{D+1}{b^{s-1}})^{\frac{1}{r}} - (\frac{D}{b^{s-1}})^{\frac{1}{r}}}{b^{\frac{1}{r}}-1}\right| \le C_{b,r}\left(\frac{1}{\log p_{N_i}}\right), \tag{3}$$

for $N_i = \pi(b^{\frac{i}{r}})$, $i = 1, 2, \dots$ and a constant $C_{b,r} > 0$ depending on b and r.

In this paper, to refine (3), we consider a generalized sequence $(f(p_n))_{n\ge1}$, where f is a function of a class of slowly increasing functions including

$$f(x) = \log_b x^r \quad \text{and} \quad f(x) = \log_b(x\log x)^r.$$

Since

$$\left|\frac{F(D,s,N,x_n)}{N} - \left(g\left(\log_b\frac{D+1}{b^{s-1}}\right) - g\left(\log_b\frac{D}{b^{s-1}}\right)\right)\right| \le 2D_N^*(\log_b x_n, g),$$

an upper bound of $D_{N_i}^*(\log_b x_n, g)$ is the rate of convergence of $\frac{F(D,s,N_i,x_n)}{N_i}$. Especially, we obtain an effective result for an upper bound of $D_N^*(\log_b p_n^r, g)$.

2. Preliminaries

To characterize the class of slowly increasing functions, we need some definitions and results concerning regular variation.

Definition 1 ([2], p.18)**.** A function f is said to be regularly varying of index ρ if it is real-valued, positive and measurable on $[t_0,\infty)$ for some $t_0 > 0$, and if

$$\lim_{t\to\infty}\frac{f(\lambda t)}{f(t)} = \lambda^\rho \quad (\forall\lambda > 0)$$

for some real number ρ. Then we write $f \in R_\rho$. If $f \in R_0$, then f is said to be slowly varying.

Lemma 1 ([2], Theorem 1.3.1)**.** *The function ℓ is slowly varying if and only if it may be written in the form*

$$\ell(x) = c(x)\exp\left\{\int_a^x \varepsilon(u)\mathrm{d}u/u\right\} \quad (x \ge a)$$

for some $a > 0$, where $c(x)$ is measurable and $c(x) \to c \in (0,\infty)$, $\varepsilon(x) \to 0$ as $x \to \infty$.

Lemma 2. *If a function f is positive, strictly increasing, differentiable on $[t_0, \infty)$ for some $t_0 > 0$, and satisfies*

$$\lim_{t\to\infty} \frac{tf'(t)}{f(t)} = \rho \neq 0,$$

then $f \in R_\rho$ and $f^{-1} \in R_{1/\rho}$.

Proof. Let $\ell(t) = t^{-\rho} f(t)$ for $t \geq t_0$. Then we have

$$\varepsilon(t) := \frac{t\ell'(t)}{\ell(t)} = -\rho + \frac{tf'(t)}{f(t)} \to 0 \quad (t \to \infty). \tag{4}$$

Since

$$\int_{t_0}^{t} \frac{\varepsilon(u)}{u} \mathrm{d}u = \int_{t_0}^{t} \frac{\ell'(u)}{\ell(u)} \mathrm{d}u = \int_{t_0}^{t} \big(\log \ell(u)\big)' \mathrm{d}u = \log \ell(t) - \log \ell(t_0),$$

we have

$$\ell(t) = \ell(t_0) \exp\left(\int_{t_0}^{t} \frac{\varepsilon(u)}{u} \mathrm{d}u \right).$$

From Lemma 1, it follows that $\ell(t)$ is slowly varying. Hence $f(t) = t^\rho \ell(t)$ is regularly varying of index ρ, i.e., $f \in R_\rho$.

Let $h(t) = \log f(e^t)$. Then

$$h'(t) = \frac{f'(e^t)e^t}{f(e^t)} = \frac{\{\rho e^{(\rho-1)t}\ell(e^t) + e^{\rho t}\ell'(e^t)\}e^t}{e^{\rho t}\ell(e^t)} = \rho + \frac{e^t \ell'(e^t)}{\ell(e^t)} = \rho + \varepsilon(e^t).$$

Therefore by (4) we have

$$h'(t) \to \rho \quad (t \to \infty).$$

Hence

$$(h^{-1})'(t) = \frac{1}{h'(h^{-1}(t))} \to \frac{1}{\rho} \quad (t \to \infty). \tag{5}$$

Since $h^{-1}(t) = \log f^{-1}(e^t)$, we have

$$f^{-1}(t) = \exp\big(h^{-1}(\log t)\big).$$

Hence

$$\begin{aligned} \frac{f^{-1}(\lambda t)}{f^{-1}(t)} &= \frac{\exp(h^{-1}(\log \lambda + \log t))}{\exp(h^{-1}(\log t))} \\ &= \exp\{h^{-1}(\log \lambda + \log t) - h^{-1}(\log t)\} \\ &= \exp\{(\log \lambda)(h^{-1})'(\xi_t)\}, \end{aligned}$$

where $\log t \leq \xi_t \leq \log t + \log \lambda$. By this and (5) we have

$$\frac{f^{-1}(\lambda t)}{f^{-1}(t)} \to \exp\left(\frac{1}{\rho} \log \lambda\right) = \lambda^{1/\rho} \quad (t \to \infty).$$

Thus $f^{-1} \in R_{1/\rho}$. □

Lemma 3 (Uniform Convergence Theorem: [2], [11])**.** *If $f \in R_\rho$, then for every $[a,b]$ with $0 < a < b < \infty$*

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^\rho \quad \textit{uniformly in } \lambda \in [a,b].$$

Lemma 4. *If a real-valued function f is strictly increasing and differentiable on $[t_0,\infty)$ for some $t_0 > 0$, and satisfies $\lim_{t\to\infty} tf'(t) = \rho > 0$, then for $-\infty < a < b < \infty$*

$$\sup_{x\in[a,b]} \left| \frac{f^{-1}(t+x)}{f^{-1}(t)} - e^{x/\rho} \right| \to 0 \quad (t\to\infty).$$

P r o o f. Let $g(t) = \exp(f(t))$. From the assumption,

$$\frac{tg'(t)}{g(t)} = tf'(t) \to \rho \quad (t\to\infty).$$

By Lemma 2 we have $g^{-1} \in R_{1/\rho}$. Since $g^{-1}(y) = f^{-1}(\log y)$, Lemma 3 implies for $[a',b']$ with $0 < a' < b' < \infty$

$$\lim_{y\to\infty} \frac{f^{-1}(\log(\lambda y))}{f^{-1}(\log y)} = \lambda^{1/\rho} \qquad \text{uniformly in } \lambda \in [a',b'],$$

and so

$$\lim_{t\to\infty} \frac{f^{-1}(t+x)}{f^{-1}(t)} = e^{x/\rho} \quad (t\to\infty)$$

uniformly for $x \in [a,b]$ $(-\infty < a < b < \infty)$, that is,

$$\sup_{x\in[a,b]} \left| \frac{f^{-1}(t+x)}{f^{-1}(t)} - e^{x/\rho} \right| \to 0 \quad (t\to\infty).$$

□

Lemma 5. *If a real-valued function f is strictly increasing, differentiable on $[t_0,\infty)$, and satisfies $\lim_{t\to\infty} tf'(t) = \rho > 0$, then for $-\infty < a < b < \infty$*

$$\sup_{x\in[a,b]} \left| \frac{B(t+x)}{B(t)} - e^{x/\rho} \right| \to 0 \quad (t\to\infty),$$

where $B(t) = \frac{f^{-1}(t)}{\log f^{-1}(t) - 1}$.

P r o o f. Since $\lim_{t\to\infty}\frac{f(t)}{\log t}=\lim_{t\to\infty}tf'(t)=\rho$, we have $\lim_{t\to\infty}f(t)=\infty$, and so $\lim_{t\to\infty}f^{-1}(t)=\infty$. By Lemma 4 we have

$$\left|\frac{\log f^{-1}(t)-1}{\log f^{-1}(t+x)-1}-1\right|=\frac{\log\frac{f^{-1}(t+x)}{f^{-1}(t)}}{\log f^{-1}(t+x)-1}\le\frac{\log\frac{f^{-1}(t+1)}{f^{-1}(t)}}{\log f^{-1}(t)-1}\to 0\quad (t\to\infty)$$

for $t\ge 0$ and $0\le x\le 1$.
Therefore,

$$\sup_{x\in[a,b]}\left|\frac{B(t+x)}{B(t)}-e^{x/\rho}\right|=\sup_{0\le x\le 1}\left|\frac{f^{-1}(t+x)}{f^{-1}(t)}\cdot\frac{\log f^{-1}(t)-1}{\log f^{-1}(t+x)-1}-e^{x/\rho}\right|$$

$$\le\sup_{0\le x\le 1}\left(\frac{f^{-1}(t+x)}{f^{-1}(t)}\left|\frac{\log f^{-1}(t)-1}{\log f^{-1}(t+x)-1}-1\right|\right)+\sup_{0\le x\le 1}\left|\frac{f^{-1}(t+x)}{f^{-1}(t)}-e^{x/\rho}\right|$$

$$\le\frac{f^{-1}(t+1)}{f^{-1}(t)}\sup_{0\le x\le 1}\left|\frac{\log f^{-1}(t)-1}{\log f^{-1}(t+x)-1}-1\right|+\sup_{0\le x\le 1}\left|\frac{f^{-1}(t+x)}{f^{-1}(t)}-e^{x/\rho}\right|$$

$$\to 0\quad (t\to\infty).$$ □

3. Estimate of discrepancy of $(f(p_n))$

In [8], we obtained an upper bound of $D_N^*\big(\log_b(p_n^r),g_w\big)$ for $w\in[0,1]$. In slightly different fashion from [8], we have the following estimate of $D_N^*\big(f(p_n),g_w\big)$ for $w\in[0,1]$, where f is a strictly increasing continuous function on $[X,\infty)$ for some $X>0$.

Theorem 1. *Suppose that f is a strictly increasing continuous function on $[p_{m+1},\infty)$ for some integer $m\ge 0$, k_0 is an integer satisfying $f^{-1}(k_0)>\max(p_{m+1},e)$, and ψ is an increasing positive function on $[0,1]$ with $\psi(0)=1$. Let*

$$B(x)=\frac{f^{-1}(x)}{\log f^{-1}(x)-1},\quad K_N=\lfloor f(p_N)\rfloor,\quad \textit{and}\quad w_N=\{f(p_N)\}.$$

Then for $0\le w\le 1$ the sequence $\big(f(p_{m+n})\big)_{n\ge1}$, $n=1,2,\dots$ satisfies

$$D_N^*\big(f(p_{m+n}),g_w\big)\le\frac{2}{N}\sum_{k=k_0}^{K_N-1}B(k)\sup_{x\in[0,1]}\left|\frac{B(k+x)}{B(k)}-\psi(x)\right|$$

$$+\big(\psi(1)+1\big)\frac{B(K_N)}{N}\sup_{x\in[0,1]}\left|\frac{B(K_N+x)}{B(K_N)}-\psi(x)\right|$$

$$+\left(\psi(1)\frac{B(K_N)}{N}+1\right)|\psi(w)-\psi(w_N)|+\frac{B(k_0)}{N}$$

$$+ O\left(\frac{1}{N}\sum_{k=k_0+1}^{K_N+1}\frac{f^{-1}(k)}{(\log f^{-1}(k))^3}\right) + O\left(\frac{\psi(1)B(K_N)}{N(\log f^{-1}(K_N))^2}\right)$$

$$+ O\left(\frac{f^{-1}(k_0)+m}{N}\right) + O\left(\frac{\psi(1)\log f^{-1}(K_N)}{N(\log f^{-1}(K_N)-1)}\right),$$

where

$$g_w(x) = \frac{1}{\psi(w)}\frac{\psi(x)-1}{\psi(1)-1} + \frac{\min(\psi(x),\psi(w))-1}{\psi(w)} \quad (0 \le x \le 1).$$

P r o o f. Set

$$F_N(x) = \frac{\#\{1 \le n \le N; \{f(p_{m+n})\} \in [0,x)\}}{N} \quad \text{for} \quad x \in [0,1].$$

Then, for $0 \le x \le 1$

$$\begin{aligned} F_N(x) =& \frac{\#\{1 \le n \le N-m; \{f(p_{m+n})\} \in [0,x)\}}{N} \\ &+ \frac{\#\{N-m+1 \le n \le N; \{f(p_{m+n})\} \in [0,x)\}}{N} \\ =& \frac{1}{N}\sum_{k=k_0}^{K_N-1}\big(A(k+x)-A(k)\big) \\ &+ \frac{\min(A(K_N+x),A(K_N+w_N))-A(K_N)}{N} + \frac{O(f^{-1}(k_0)+m)}{N}, \quad (6) \end{aligned}$$

where

$$A(y) = \#\{1 \le n \le N : f(p_{m+n}) < y\} \quad \text{for } y \ge f(p_{m+1})$$

and

$$O\big(f^{-1}(k_0)+m\big) \le f^{-1}(k_0)+m.$$

Then, for $y \ge f(p_{m+1})$

$$A(y) = \pi\big(f^{-1}(y)\big) - m + O(1).$$

Since

$$\pi(x) = \int_2^x \frac{1}{\log t}\mathrm{d}t + O\left(\frac{x}{(\log x)^3}\right)$$

(see [5]) and that

$$\int_2^x \frac{1}{\log t}\mathrm{d}t = \frac{x}{\log x - 1} + O\left(\frac{x}{(\log x)^3}\right),$$

we have

$$\pi(x) = \frac{x}{\log x - 1} + O\left(\frac{x}{(\log x)^3}\right).$$

Therefore, for $y \ge f(p_{m+1})$

$$A(y) = B(y) + O\left(\frac{f^{-1}(y)}{(\log f^{-1}(y))^3}\right), \quad (7)$$

and so,

$$A(k+x) - A(k) = B(k+x) - B(k) + O\left(\frac{f^{-1}(k+1)}{(\log f^{-1}(k+1))^3}\right) \tag{8}$$

for $k \geq k_0$ and $0 \leq x \leq 1$.

Since $B(x)$ and $f^{-1}(x)/\big(\log f^{-1}(x)\big)^3$ are non-decreasing, by (7) and (8) we have

$$\min\big(A(K_N+x), A(K_N+w_N)\big) = \\ \min\big(B(K_N+x), B(K_N+w_N)\big) + O\left(\frac{f^{-1}(K_N+1)}{(\log f^{-1}(K_N+1))^3}\right).$$

Hence, by (6) and (8) we have

$$F_N(x) = \frac{1}{N}\sum_{k=k_0}^{K_N-1}\big(B(k+x)-B(k)\big) + \frac{\min(B(K_N+x), B(K_N+w_N)) - B(K_N)}{N} + \\ O\left(\frac{1}{N}\sum_{k=k_0+1}^{K_N+1}\frac{f^{-1}(k)}{\log f^{-1}(k))^3}\right) + \frac{O(f^{-1}(k_0)+m)}{N}. \tag{9}$$

We can express the first term in the right-hand side of (9) as

$$\frac{\sum_{k=k_0}^{K_N-1}(B(k+x)-B(k))}{\sum_{k=k_0}^{K_N-1}(B(k+1)-B(k))} \cdot \frac{B(K_N)-B(k_0)}{N},$$

and the second term as

$$\left(\min\left(\frac{B(K_N+x)}{B(K_N)}, \frac{B(K_N+w_N)}{B(K_N)}\right) - 1\right)\frac{B(K_N)}{N}.$$

Hence we have for $x \in [0,1]$

$$F_N(x) - g_w(x) \tag{10}$$

$$= \frac{\sum_{k=k_0}^{K_N-1}(B(k+x)-B(k))}{\sum_{k=k_0}^{K_N-1}(B(k+1)-B(k))} \cdot \frac{B(K_N)-B(k_0)}{N} - \frac{\psi(x)-1}{\psi(1)-1} \cdot \frac{1}{\psi(w)}$$

$$+ \left(\min\left(\frac{B(K_N+x)}{B(K_N)}, \frac{B(K_N+w_N)}{B(K_N)}\right) - 1\right)\frac{B(K_N)}{N} - \frac{\min(\psi(x),\psi(w)) - 1}{\psi(w)}$$

$$+ O\left(\frac{1}{N}\sum_{k=k_0+1}^{K_N+1}\frac{f^{-1}(k)}{\log f^{-1}(k))^3}\right) + \frac{O(f^{-1}(k_0)+m)}{N}$$

$$= \left(\frac{\sum_{k=k_0}^{K_N-1}(B(k+x)-B(k))}{\sum_{k=k_0}^{K_N-1}(B(k+1)-B(k))} - \frac{\psi(x)-1}{\psi(1)-1}\right)\frac{B(K_N)-B(k_0)}{N} \tag{11}$$

$$+\left(\frac{B(K_N)-B(k_0)}{N}-\frac{1}{\psi(w)}\right)\frac{\psi(x)-1}{\psi(1)-1} \tag{12}$$

$$+\left(\min\left(\frac{B(K_N+x)}{B(K_N)},\frac{B(K_N+w_N)}{B(K_N)}\right)-\min\big(\psi(x),\psi(w)\big)\right)\frac{B(K_N)}{N} \tag{13}$$

$$+\left(\psi(u)-\frac{N}{B(K_N)}\right)\left(\frac{\min(\psi(x),\psi(w))-1}{\frac{N}{B(K_N)}\psi(w)}\right) \tag{14}$$

$$+O\left(\frac{1}{N}\sum_{k=k_0+1}^{K_N+1}\frac{f^{-1}(k)}{\log f^{-1}(k))^3}\right)+\frac{O(f^{-1}(k_0)+m)}{N}.$$

In order to estimate (11), we set

$$S_j=\sum_{k=k_0}^{j-1}\big(B(k+x)-B(k)\big),\quad T_j=\sum_{k=k_0}^{j-1}\big(B(k+1)-B(k)\big)$$

for $j\ge k_0+1$, $S_{k_0}=T_{k_0}=0$, and $U=\frac{\psi(x)-1}{\psi(1)-1}$. Then we have, successively,

$$\frac{\sum_{k=k_0}^{K_N-1}(B(k+x)-B(k))}{\sum_{k=k_0}^{K_N-1}(B(k+1)-B(k))}-\frac{\psi(x)-1}{\psi(1)-1}=\frac{S_{K_N}}{T_{K_N}}-U$$

$$=\frac{1}{T_{K_N}}\sum_{k=k_0}^{K_N-1}(S_{k+1}-S_k)-\frac{U}{T_{K_N}}\sum_{k=k_0}^{K_N-1}(T_{k+1}-T_k)$$

$$=\frac{1}{T_{K_N}}\sum_{k=k_0}^{K_N-1}(T_{k+1}-T_k)\left(\frac{S_{k+1}-S_k}{T_{k+1}-T_k}-U\right)$$

$$=\frac{1}{B(K)-B(k_0)}\times\sum_{k=k_0}^{K_N-1}\big(B(k+1)-B(k)\big)\left(\frac{B(k+x)-B(k)}{B(k+1)-B(k)}-\frac{\psi(x)-1}{\psi(1)-1}\right).$$

Then, we obtain

$$|(11)|\le\frac{1}{N}\sum_{k=k_0}^{K_N-1}\big(B(k+1)-B(k)\big)\left|\frac{B(k+x)-B(k)}{B(k+1)-B(k)}-\frac{\psi(x)-1}{\psi(1)-1}\right|$$

$$=\frac{1}{N}\sum_{k=k_0}^{K_N-1}\big(B(k+1)-B(k)\big)\left|\frac{\Psi_k(x)-1}{\Psi_k(1)-1}-\frac{\psi(x)-1}{\psi(1)-1}\right|$$

$$\le\frac{1}{N}\sum_{k=k_0}^{K_N-1}\big(B(k+1)-B(k)\big)$$

$$\times\left|\frac{(\Psi_k(x)-\psi(x))(\psi(1)-1)-(\Psi_k(1)-\psi(1))(\psi(1)-1)}{(\Psi_k(1)-1)(\psi(1)-1)}\right|$$

$$\leq \frac{1}{N}\sum_{k=k_0}^{K_N-1}\frac{B(k+1)-B(k)}{\Psi_k(1)-1}\big(|\Psi_k(x)-\psi(x)|+|\Psi_k(1)-\psi(1)|\big)$$
$$\leq \frac{2}{N}\sum_{k=k_0}^{K_N-1}B(k)\sup_{x\in[0,1]}|\Psi_k(x)-\psi(x)|, \tag{15}$$

where $\Psi_k(x)=\frac{B(k+x)}{B(k)}$.

Every second factor in (12), (13) and (14) is bonded. In fact, for (12)

$$0\leq \frac{\psi(x)-1}{\psi(1)-1}\leq 1 \quad \text{for } x\in[0,1]. \tag{16}$$

Since

$$N{=}A(K_N{+}w_N){+}m{+}1{=}B(K_N{+}w_N){+}O\left(\frac{f^{-1}(K_N{+}w_N)}{(\log f^{-1}(K_N{+}w_N))^3}\right){+}m{+}1, \tag{17}$$

we have

$$\frac{N}{B(K_N)}=\Psi_{K_N}(w_N)+R(N), \tag{18}$$

where

$$R(N)=O\left(\frac{1}{(\log f^{-1}(K_N))^2}\right)+O\left(\frac{(m+1)\log f^{-1}(K_N)}{f^{-1}(K_N)}\right)\to 0 \quad (N\to\infty). \tag{19}$$

By (16), we have

$$\begin{aligned}
|(12)| &\leq \left|\frac{B(K_N)-B(k_0)}{N}-\frac{1}{\psi(w)}\right|\\
&\leq \left|\frac{1}{\frac{N}{B(K_N)}}-\frac{1}{\psi(w_N)}\right|+\left|\frac{1}{\psi(w_N)}-\frac{1}{\psi(w)}\right|+\frac{B(k_0)}{N}\\
&= \left|\frac{1}{\Psi_{K_N}(w_N)+R(N)}-\frac{1}{\psi(w_N)}\right|+\frac{|\psi(w)-\psi(w_N)|}{\psi(w_N)\psi(w)}+\frac{B(k_0)}{N}\\
&\leq \frac{|\psi(w_N)-\Psi_{K_N}(w_N)|+R(N)}{\psi(w_N)(\Psi_{K_N}(w_N)+R(N))}+|\psi(w)-\psi(w_N)|+\frac{B(k_0)}{N}\\
&\leq \frac{B(K_N)}{N}\big(|\psi(w_N)-\Psi_{K_N}(w_N)|+R(N)\big)+|\psi(w)-\psi(w_N)|+\frac{B(k_0)}{N}\\
&\leq \frac{B(K_N)}{N}\sup_{x\in[0,1]}|\psi(x)-\Psi_{K_N}(x)|+|\psi(w)-\psi(w_N)|+\frac{B(k_0)}{N}\\
&\quad +O\left(\frac{f^{-1}(K_N)}{N(\log f^{-1}(K_N))^2(\log f^{-1}(K_N)-1)}\right)\\
&\quad +O\left(\frac{\log f^{-1}(K_N)}{N(\log f^{-1}(K_N)-1)}\right).
\end{aligned} \tag{20}$$

We can rewrite (13) to the form

$$|(13)| \le \frac{B(K_N)}{N}\left(\left|\min\left(\frac{B(K_N+x)}{B(K_N)}, \frac{B(K_N+w_N)}{B(K_N)}\right) - \min\bigl(\psi(x),\psi(w_N)\bigr)\right| + \left|\min\bigl(\psi(x),\psi(w_N)\bigr) - \min\bigl(\psi(x),\psi(w)\bigr)\right|\right).$$

If $x < w_N$, then $B(K_N+x) < B(K_N+w_N)$, then $\psi(x) < \psi(w_N)$ and then

$$\left|\min\left(\frac{B(K_N+x)}{B(K_N)}, \frac{B(K_N+w_N)}{B(K_N)}\right) - \min\bigl(\psi(x),\psi(w_N)\bigr)\right| = |\Psi_{K_N}(x) - \psi(x)| \le \sup_{x\in[0,1]} |\Psi_{K_N}(x) - \psi(x)|.$$

If $x < w_N$ and $x < w$, then $\min\bigl(\psi(x),\psi(w_N)\bigr) - \min\bigl(\psi(x),\psi(w)\bigr) = 0$. If $w < x < w_N$, then $|\min\bigl(\psi(x),\psi(w_N)\bigr) - \min\bigl(\psi(x),\psi(w)\bigr)| = |\psi(x) - \psi(w)| \le |\psi(w_N) - \psi(w)|$. Similarly, in all other cases. Thus we have

$$|(13)| \le \frac{B(K_N)}{N}\left(\sup_{x\in[0,1]} |\Psi_{K_N}(x) - \psi(x)| + |\psi(w_N) - \psi(w)|\right). \tag{21}$$

By (19) we obtain

$$\begin{aligned}
|(14)| &\le \frac{B(K_N)}{N}\bigl(\psi(1)-1\bigr)\left(|\psi(w) - \psi(w_N)| + \left|\psi(w_N) - \frac{N}{B(K_N)}\right|\right)\\
&= \frac{B(K_N)}{N}\bigl(\psi(1)-1\bigr)\Bigl(|\psi(w) - \psi(w_N)| + |\psi(w_N) - \Psi_{K_N}(w_N)| + R(N)\Bigr)\\
&\le \frac{B(K_N)}{N}\bigl(\psi(1)-1\bigr)\Biggl(|\psi(w) - \psi(w_N)| + \sup_{x\in[0,1]} |\Psi_{K_N}(x) - \psi(x)| +\\
&\qquad O\left(\frac{1}{(\log f^{-1}(K_N))^2}\right) + O\left(\frac{\log f^{-1}(K_N)}{f^{-1}(K_N)}\right)\Biggr)\\
&\le \bigl(\psi(1)-1\bigr)\Bigl(|\psi(w) - \psi(w_N)| + \sup_{x\in[0,1]} |\Psi_{K_N}(x) - \psi(x)|\Bigr)\frac{B(K_N)}{N}\\
&\quad + O\left(\frac{(\psi(1)-1)f^{-1}(K_N)}{N(\log f^{-1}(K_N))^2(\log f^{-1}(K_N)-1)}\right)\\
&\quad + O\left(\frac{(\psi(1)-1)\log f^{-1}(K_N)}{N(\log f^{-1}(K_N)-1)}\right).
\end{aligned} \tag{22}$$

Combining (10), (15), (20), (21), and (22), we have

$$
\begin{aligned}
&|F_N(x) - g_w(x)| \\
&\le \frac{2}{N}\sum_{k=k_0}^{K_N-1} B(k) \sup_{x\in[0,1]} |\Psi_k(x) - \psi(x)| + \frac{B(K_N)}{N} \sup_{x\in[0,1]} |\Psi_{K_N}(x) - \psi(x)| \\
&\quad + |\psi(w) - \psi(w_N)| + \frac{B(k_0)}{N} + O\left(\frac{f^{-1}(K_N)}{N(\log f^{-1}(K_N))^2(\log f^{-1}(K_N)-1)}\right) \\
&\quad + O\left(\frac{\log f^{-1}(K_N)}{N(\log f^{-1}(K_N)-1)}\right) \\
&\quad + \frac{B(K_N)}{N}\left(\sup_{x\in[0,1]} |\Psi_{K_N}(x) - \psi(x)| + |\psi(w_N) - \psi(w)|\right) \\
&\quad + \big(\psi(1)-1\big)\frac{B(K_N)}{N}\left(|\psi(w) - \psi(w_N)| + \sup_{x\in[0,1]} |\Psi_{K_N}(x) - \psi(x)|\right) \\
&\quad + O\left(\frac{(\psi(1)-1)f^{-1}(K_N)}{N(\log f^{-1}(K_N))^2(\log f^{-1}(K_N)-1)}\right) + O\left(\frac{(\psi(1)-1)\log f^{-1}(K_N)}{N(\log f^{-1}(K_N)-1)}\right) \\
&\quad + O\left(\frac{1}{N}\sum_{k=k_0+1}^{K_N+1} \frac{f^{-1}(k)}{(\log f^{-1}(k))^3}\right) + O\left(\frac{f^{-1}(k_0)+m}{N}\right) \\
&\le \frac{2}{N}\sum_{k=k_0}^{K_N-1} B(k) \sup_{x\in[0,1]} |\Psi_k(x) - \psi(x)| + \big(\psi(1)+1\big)\frac{B(K_N)}{N} \sup_{x\in[0,1]} |\Psi_{K_N}(x) - \psi(x)| \\
&\quad + \left(\psi(1)\frac{B(K_N)}{N} + 1\right)|\psi(w) - \psi(w_N)| + \frac{B(k_0)}{N} \\
&\quad + O\left(\frac{\psi(1)B(K_N)}{N(\log f^{-1}(K_N))^2}\right) + O\left(\frac{1}{N}\sum_{k=k_0+1}^{K_N+1} \frac{f^{-1}(k)}{(\log f^{-1}(k))^3}\right) \\
&\quad + O\left(\frac{f^{-1}(k_0)+m}{N}\right) + O\left(\frac{\psi(1)\log f^{-1}(K_N)}{N(\log f^{-1}(K_N)-1)}\right).
\end{aligned}
$$

This proves the theorem. □

From Theorem 1, we deduce the following corollaries.

Corollary 1. *Suppose f is a strictly increasing differentiable function on $[p_{m+1}, \infty)$ for some integer $m \geq 0$, and satisfies*

$$\lim_{t\to\infty} tf'(t) = \rho > 0. \tag{23}$$

Let $0 < w \leq 1$ and $N_i = \pi\big(f^{-1}(i+w)\big)$. Then

$$\lim_{i\to\infty} \{f(p_{N_i})\} = w$$

and

$$\lim_{i\to\infty} D^*_{N_i}\big(f(p_{m+n}), g_w\big) = 0,$$

where

$$g_w(x) = \frac{1}{e^{\frac{w}{\rho}}} \frac{e^{\frac{x}{\rho}} - 1}{e^{\frac{1}{\rho}} - 1} + \frac{\min(e^{\frac{x}{\rho}}, e^{\frac{w}{\rho}}) - 1}{e^{\frac{w}{\rho}}} \quad (0 \leq x \leq 1).$$

Proof. Let $0 < w \leq 1$, $K_{N_i} = \lfloor f(p_{N_i})\rfloor$, $w_{N_i} = \{f(p_{N_i})\}$. By the prime number theorem, we have

$$\begin{aligned}
&\pi\big(f^{-1}(i+w)\big) - \pi\big(f^{-1}(i)\big)\\
&= \frac{f^{-1}(i+w)}{\log f^{-1}(i+w)} - \frac{f^{-1}(i)}{\log f^{-1}(i)} + O\left(\frac{f^{-1}(i+w)}{\log^2 f^{-1}(i+w)}\right)\\
&= \frac{f^{-1}(i+w)}{\log f^{-1}(i+w)} \left\{1 - \frac{f^{-1}(i)}{f^{-1}(i+w)} \frac{\log f^{-1}(i+w)}{\log f^{-1}(i)} + O\left(\frac{1}{\log f^{-1}(i+w)}\right)\right\}.
\end{aligned} \tag{24}$$

By the assumption, we obtain

$$\lim_{x\to\infty} f^{-1}(x) = \infty, \quad \text{and so} \quad \lim_{i\to\infty} \frac{f^{-1}(i+w)}{\log f^{-1}(i+w)} = \infty.$$

By Lemma 4, we have $\lim_{i\to\infty} \frac{f^{-1}(i+w)}{f^{-1}(i)} = e^{\frac{w}{\rho}} > 1$. Hence

$$\lim_{i\to\infty} \frac{\log f^{-1}(i+w)}{\log f^{-1}(i)} = \lim_{i\to\infty} \left(\frac{1}{\log f^{-1}(i)} \log \frac{f^{-1}(i+w)}{f^{-1}(i)}\right) + 1 = 1. \tag{25}$$

Therefore, (24) and (25) yield

$$\lim_{i\to\infty} \pi\big(f^{-1}(i+w)\big) - \pi\big(f^{-1}(i)\big) = \infty.$$

Hence, there exists i_0 such that $\pi\big(f^{-1}(i+w)\big) - \pi\big(f^{-1}(i)\big) \geq 1$ for $i \geq i_0$. Thus, for $i \geq i_0$

$$f^{-1}(i) \leq p_{N_i} < f^{-1}(i+w) \leq p_{N_i+1}, \quad \text{so} \quad i \leq f(p_{N_i}) < i + w \leq f(p_{N_i+1}),$$

and so

$$K_{N_i} = \lfloor f(p_{N_i})\rfloor = i. \tag{26}$$

Successively, we obtain

$$\begin{aligned} w_{N_i} =&\{f(p_{N_i})\} = f(p_{N_i}) - i < w, \\ w >& w_{N_i} = w_{N_i} - w + w = f(p_{N_i}) - (i+w) + w \geq f(p_{N_i}) - f(p_{N_i+1}) + w, \\ 0 <& w - w_{N_i} \leq f(p_{N_i+1}) - f(p_{N_i}). \end{aligned} \tag{27}$$

It is shown that

$$p_{n+1} - p_n = O\left(p_n^{\theta}\right) \tag{28}$$

for some $0 < \theta < 1$, namely, $p_{n+1} - p_n = O(p_n^{0.525})$ due to R. C. Baker, G. Harman and J. Pintz [1]. Hence, by the assumption (23)

$$f(p_{N_i+1}) - f(p_{N_i}) = (p_{N_i+1} - p_{N_i})f'(\xi) = O(p_{N_i}^{\theta-1}), \quad p_{N_i} \leq \xi \leq p_{Ni+1}.$$

Therefore, by (27) we have

$$w_{N_i} \to w \quad (i \to \infty). \tag{29}$$

Applying Theorem 1 with $\psi(x) = e^{\frac{x}{\rho}}$, we have

$$\begin{aligned} &D^*_{N_i}\big(f(p_{m+n}), g_w\big) \\ &\leq \frac{2}{N_i}\sum_{k=k_0}^{i-1} B(k) \sup_{x\in[0,1]}\left|\frac{B(k+x)}{B(k)} - e^{\frac{x}{\rho}}\right| + \left(e^{\frac{1}{\rho}}+1\right)\frac{B(i)}{N_i}\sup_{x\in[0,1]}\left|\frac{B(i+x)}{B(i)} - e^{\frac{x}{\rho}}\right| \\ &+ \left(e^{\frac{1}{\rho}}\frac{B(i)}{N_i}+1\right)\left|e^{\frac{w}{\rho}} - e^{\frac{w_{N_i}}{\rho}}\right| + \frac{B(k_0)}{N_i} + O\left(\frac{1}{N_i}\sum_{k=k_0+1}^{i+1}\frac{f^{-1}(k)}{(\log f^{-1}(k))^3}\right) \\ &+ O\left(\frac{e^{\frac{1}{\rho}}B(i)}{N_i(\log f^{-1}(i))^2}\right) + \frac{O(f^{-1}(k_0+m))}{N_i} + O\left(\frac{e^{\frac{1}{\rho}}\log f^{-1}(i)}{N_i(\log f^{-1}(i)-1)}\right). \end{aligned} \tag{30}$$

We note that

$$\frac{1}{N_i}\sum_{k=k_0}^{i-1} B(k)\left|\frac{B(k+x)}{B(k)} - e^{\frac{x}{\rho}}\right| = \frac{1}{N_i}\sum_{k=k_0}^{i-1}\big(B(k+1)-B(k)\big) \times \frac{\sup_{x\in[0,1]}\left|\frac{B(k+x)}{B(k)} - e^{\frac{x}{\rho}}\right|}{\frac{B(k+1)}{B(k)} - 1}. \tag{31}$$

By Lemma 5 we have

$$\lim_{k\to\infty}\sup_{x\in[0,1]}\left|\frac{B(k+x)}{B(k)} - e^{\frac{x}{\rho}}\right| = 0.$$

Combining this with $\lim_{k\to\infty}\frac{B(k+1)}{B(k)} = \psi(1) = e^{\frac{1}{\rho}} > 1$, we get

$$\lim_{k\to\infty}\frac{\sup_{x\in[0,1]}\left|\frac{B(k+x)}{B(k)} - e^{\frac{x}{\rho}}\right|}{\frac{B(k+1)}{B(k)} - 1} = 0. \tag{32}$$

By Lemma 5 and (29) we have

$$\left|\frac{B(K_{N_i}+w_{N_i})}{B(K_{N_i})}-e^{\frac{w}{\rho}}\right| \le \left|\frac{B(K_{N_i}+w_{N_i})}{B(K_{N_i})}-e^{\frac{w_{N_i}}{\rho}}\right| + \left|e^{\frac{w_{N_i}}{\rho}}-e^{\frac{w}{\rho}}\right|$$
$$\le \sup_{0\le x\le 1}\left|\frac{B(K_{N_i}+x)}{B(K_{N_i})}-e^{\frac{x}{\rho}}\right| + \left|e^{\frac{w_{N_i}}{\rho}}-e^{\frac{w}{\rho}}\right| \to 0 \qquad (i\to\infty),$$

so we have

$$\lim_{i\to\infty}\frac{B(K_{N_i}+w_{N_i})}{B(K_{N_i})}=e^{\frac{w}{\rho}}. \tag{33}$$

Therefore, by (18) and (19) we have

$$\lim_{i\to\infty}\frac{B(i)}{N_i}=\frac{1}{e^{\frac{w}{\rho}}}. \tag{34}$$

By (31), (32) and (34), we have

$$\lim_{i\to\infty}\frac{1}{N_i}\sum_{k=0}^{i-1}B(k)\sup_{x\in[0,1]}|\Psi_k(x)-e^{\frac{x}{\rho}}|=0.$$

Lastly, we show that $\frac{1}{N_i}\sum_{k=k_0+1}^{i+1}\frac{f^{-1}(k)}{(\log f^{-1}(k))^3}\to 0$. By Cauchy-Stolz lemma, it suffices to prove that

$$\lim_{i\to\infty}\frac{(\log f^{-1}(i+1))^3\,(N_i-N_{i-1})}{f^{-1}(i+1)} \;=\; \infty. \tag{35}$$

By (17) we have

$$N_i-N_{i-1}=B(i+w_{N_i})-B(i-1+w_{N_{i-1}})+O\left(\frac{f^{-1}(i+w_{N_i})}{(\log f^{-1}(i+w_{N_i}))^3}\right). \tag{36}$$

Since

$$\begin{aligned}
&f^{-1}(i+w_{N_i})\big(\log f^{-1}(i-1+w_{N_{i-1}})-1\big)\\
&\quad - f^{-1}(i-1+w_{N_{i-1}})\big(\log f^{-1}(i+w_{N_i})-1\big)\\
&= f^{-1}(i+w_{N_i})\log\frac{f^{-1}(i-1+w_{N_{i-1}})}{f^{-1}(i+w_{N_i})}\\
&\quad + \left\{f^{-1}(i+w_{N_i})-f^{-1}(i-1+w_{N_{i-1}})\right\}\left\{\log f^{-1}(i+w_{N_i})-1\right\}\\
&\ge f^{-1}(i)\log\frac{f^{-1}(i-1+w_{N_{i-1}})}{f^{-1}(i+w_{N_i})},
\end{aligned}$$

we obtain

$$B(i+w_{N_i})-B(i-1+w_{N_{i-1}})\ge\frac{f^{-1}(i)\log\frac{f^{-1}(i-1+w_{N_{i-1}})}{f^{-1}(i+w_{N_i})}}{(\log f^{-1}(i)-1)(\log f^{-1}(i+1)-1)}.$$

Therefore,

$$\begin{aligned}
&\frac{(\log f^{-1}(i))^3(B(i+w_{N_i})-B(i-1+w_{N_{i-1}}))}{f^{-1}(i)}\\
&\geq \frac{(\log f^{-1}(i))^3 \log \frac{f^{-1}(i-1+w_{N_{i-1}})}{f^{-1}(i+w_{N_i})}}{(\log f^{-1}(i)-1)(\log f^{-1}(i+1)-1)}\\
&\geq \left(\log f^{-1}(i)\right)\frac{1}{1-\frac{1}{\log f^{-1}(i)}}\frac{1}{\frac{\log f^{-1}(i+1)}{\log f^{-1}(i)}-\frac{1}{\log f^{-1}(i)}}\log\frac{f^{-1}(i-1+w_{N_{i-1}})}{f^{-1}(i+w_{N_i})}.
\end{aligned} \tag{37}$$

By Lemma 4 and (29), we have

$$\lim_{i\to\infty}\frac{f^{-1}(i+w_{N_i})}{f^{-1}(i)}=\psi(w), \tag{38}$$

so

$$\frac{f^{-1}(i-1+w_{N_{i-1}})}{f^{-1}(i+w_{N_i})}=\frac{\frac{f^{-1}(i-1+w_{N_{i-1}})}{f^{-1}(i-1)}}{\frac{f^{-1}(i+w_{N_i})}{f^{-1}(i)}\cdot\frac{f^{-1}(i)}{f^{-1}(i-1)}}\to\frac{\psi(w)}{\psi(w)\psi(1)}=\frac{1}{\psi(1)} \quad (i\to\infty). \tag{39}$$

From (25), (37), and (39), it follows that

$$\lim_{i\to\infty}\frac{(\log f^{-1}(i))^3(B(i+w_{N_i})-B(i-1+w_{N_{i-1}}))}{f^{-1}(i)}=\infty. \tag{40}$$

By (25) and (38) we have

$$\lim_{i\to\infty}\frac{\log f^{-1}(i)}{\log f^{-1}(i+w_{N_i})}=1. \tag{41}$$

Hence, by (36), (40), and (41) we have

$$\begin{aligned}
\frac{(\log f^{-1}(i))^3(N_i-N_{i-1})}{f^{-1}(i)} &= \\
&\frac{(\log f^{-1}(i))^3(B(i+w_{N_i})-B(i-1+w_{N_{i-1}}))}{f^{-1}(i)}+\\
&O\left(\frac{f^{-1}(i+w_{N_i})}{f^{-1}(i)}\cdot\left(\frac{\log f^{-1}(i)}{\log f^{-1}(i+w_{N_i})}\right)^3\right)\to\infty\\
&\text{as}\quad i\to\infty.
\end{aligned} \tag{42}$$

From (42),

$$\lim_{i\to\infty} f^{-1}(i)/f^{-1}(i+1) = 1/e^{1/\rho}, \quad \text{and} \quad \lim_{i\to\infty} \log f^{-1}(i+1)/\log f^{-1}(i) = 1,$$

we arrive at (35). Consequently, the right hand side of (30) tends to 0 as $i \to \infty$. Thus $\lim_{i\to\infty} D^*_{N_i}(f(p_{m+n}) \bmod 1, g_w) = 0$. $\square$

Corollary 2. *Let $r > 0$, $0 < w \le 1$, let $N_i = \pi(b^{\frac{i+w}{r}})$, $i = 1, 2, \dots$, and let*

$$g_w(x) = \frac{1}{b^{\frac{w}{r}}} \frac{b^{\frac{x}{r}} - 1}{b^{\frac{1}{r}} - 1} + \frac{\min(b^{\frac{x}{r}}, b^{\frac{w}{r}}) - 1}{b^{\frac{w}{r}}} \quad (0 \le x \le 1).$$

Then we have

$$\begin{aligned} D^*_{N_i}(\log_b p_n^r, g_w) \le{}& b^{\frac{1}{r}}(\log b)\left(2 + (b^{\frac{1}{r}} + 1)\frac{\log b}{r}\right)\frac{\log p_{N_i}}{\left(\log p_{N_i} - (1 + \frac{1}{r})\log b\right)^2} \\ &+ C_1\left(\frac{b^{\frac{2}{r}} r}{\log b} + b^{\frac{1}{r}}\log b\right)\frac{\log p_{N_i}}{\left(\log p_{N_i} - (1 + \frac{1}{r})\log b\right)^3} \\ &+ C_2\left(b^2 + \frac{b^2}{2\log b - 1}\right) b^{\frac{2}{r}} r \frac{\log p_{N_i}}{p_{N_i}^{1-\theta}} \\ &+ C_3 b^{\frac{2}{r}} r \frac{(\log p_{N_i})^2 + \log p_{N_i}\log\log p_{N_i}}{p_{N_i}(r\log p_{N_i} - r - \log b)}, \end{aligned} \tag{43}$$

where $N_i \ge 6$ for $i = 1, 2, \dots$ and C_j is a constant for $j = 1, 2, 3$.

P r o o f. Applying Theorem 1 and Corollary 1 with

- $f(t) = \log_b t^r$ for $t \ge 1$,
- $m = 0$,
- $k_0 = \lceil 2r \rceil$,
- $f^{-1}(t) = b^{\frac{t}{r}}$ for $t \ge 0$,
- $\rho = \lim_{t\to\infty} t f'(t) = \frac{r}{\log b}$,
- $\psi(x) = b^{\frac{x}{r}}$,
- $B(x) = \frac{r b^{\frac{x}{r}}}{x \log b - r}$,
- $N_i = \pi(b^{\frac{i+w}{r}})$,
- $K_{N_i} = \lfloor r \log_b p_{N_i} \rfloor = i$ by (26), $w_{N_i} = \{r \log_b p_{N_i}\}$,

we have

$$D^*_{N_i}(\log_b p_n^r, g_w) \le \frac{2r}{N_i} \sum_{2r \le k \le i-1} \frac{b^{\frac{k}{r}}}{k - r} \sup_{x\in[0,1]} \left| b^{\frac{x}{r}} \left(\frac{k - r}{k + x - r} - 1 \right) \right|$$

$$+\frac{(b^{\frac{1}{r}}+1)rb^{\frac{i}{r}}}{(i-r)N_i}\sup_{x\in[0,1]}\left|b^{\frac{x}{r}}\left(\frac{i-r}{i+x-r}-1\right)\right|+\left(\frac{rb^{\frac{i+1}{r}}}{N_i}+1\right)\left|b^{\frac{w}{r}}-b^{\frac{w_{N_i}}{r}}\right|$$
$$+\frac{b^{2+\frac{1}{r}}}{(2\log b-1)N_i}+O\left(\frac{r^3}{(\log b)^3N_i}\sum_{2r+1\le k\le i+1}\frac{b^{\frac{k}{r}}}{k^3}\right)+O\left(\frac{r^3b^{\frac{i+1}{r}}}{(\log b)^2i^2(i-r)N_i}\right)$$
$$+O\left(\frac{b^{2+\frac{1}{r}}}{N_i}\right)+O\left(\frac{b^{\frac{1}{r}}(\log b)i}{N_i(i\log b-r)}\right). \tag{44}$$

Since

$$\sup_{x\in[0,1]}\left|b^{\frac{x}{r}}\left(\frac{k-r}{k+x-r}-1\right)\right|<\frac{b^{\frac{1}{r}}}{k-r} \tag{45}$$

and that $\frac{b^{\frac{x}{r}}}{(x-r)^2}$ is increasing for $x\ge 2r$, we have

$$\sum_{2r\le k\le i-1}\frac{b^{\frac{k}{r}}}{k-r}\sup_{x\in[0,1]}\left|b^{\frac{x}{r}}\left(\frac{k-r}{k+x-r}-1\right)\right|\le b^{\frac{1}{r}}\int_{2r}^{i}\frac{b^{\frac{x}{r}}}{(x-r)^2}\mathrm{d}x.$$

Using the integration by parts, we have

$$\begin{aligned}\int_{2r}^{i}\frac{b^{\frac{x}{r}}}{(x-r)^2}\mathrm{d}x&\le\frac{rb^{\frac{i}{r}}}{(\log b)(i-r)^2}+2\frac{r}{\log b}\int_{2r}^{i}\frac{b^{\frac{x}{r}}}{(x-r)^3}\mathrm{d}x\\&\le\frac{rb^{\frac{i}{r}}}{(\log b)(i-r)^2}\\&\quad+2\frac{r}{\log b}\left\{\frac{rb^{\frac{i}{r}}}{(\log b)(i-r)^3}+3\frac{r}{\log b}\int_{2r}^{i}\frac{b^{\frac{x}{r}}}{(x-r)^4}\mathrm{d}x\right\}\\&\le\frac{rb^{\frac{i}{r}}}{(\log b)(i-r)^2}+\frac{8r^2b^{\frac{i}{r}}}{(\log b)^2(i-r)^3},\end{aligned}$$

because $\frac{e^{\frac{x}{r}}}{(x-r)^4}$ is increasing for $x\ge 2r$. Hence

$$\begin{aligned}&\sum_{2r\le k<i}\frac{b^{\frac{k}{r}}}{k-r}\sup_{x\in[0,1]}\left|b^{\frac{x}{r}}\left(\frac{k-r}{k+x-r}-1\right)\right|\\&\qquad\le\frac{rb^{\frac{i+1}{r}}}{(\log b)^2(i-r)^2}\left(\log b+\frac{8r}{i-r}\right).\end{aligned} \tag{46}$$

Similarly, we have

$$\sum_{2r+1\le k\le i+1} \frac{b^{\frac{k}{r}}}{k^3} \le \frac{4rb^{\frac{i+2}{r}}}{(\log b)(i+2)^3}. \tag{47}$$

By (27), we have for $w \in [0,1)$,

$$\begin{aligned} \left| b^{\frac{w}{r}} - b^{\frac{w_{N_i}}{r}} \right| &\le \frac{b^{\frac{1}{r}} \log b}{r} \big(f(p_{N_i+1}) - f(p_{N_i}) \big) \\ &\le b^{\frac{1}{r}} \left(\frac{p_{N_i+1} - p_{N_i}}{p_{N_i}} \right). \end{aligned} \tag{48}$$

By (28) and (48), we obtain

$$\left| b^{\frac{u}{r}} - b^{\frac{w_{N_i}}{r}} \right| = O\left(\frac{b^{\frac{1}{r}}}{p_{N_i}^{1-\theta}} \right) = O\left(\frac{b^{\frac{1}{r}}}{N_i^{1-\theta} (\log N_i)^{1-\theta}} \right), \tag{49}$$

where $\theta = 0.525$.
By (44), (45), (46), (47), and (49), we obtain

$$\begin{aligned} D^*_{N_i}(\log_b p_n^r, g_w) \le{}& \left(\frac{2}{\log b} b^{\frac{1}{r}} r^2 + (b^{\frac{2}{r}} + b^{\frac{1}{r}}) r \right) \frac{b^{\frac{i}{r}}}{(i-r)^2 N_i} \\ &+ \frac{16 b^{\frac{1}{r}} r^3}{(\log b)^2} \frac{b^{\frac{i}{r}}}{(i-r)^3 N_i} \\ &+ \text{const } b^{\frac{2}{r}} r \frac{b^{\frac{i}{r}}}{N_i^{2-\theta} (\log N_i)^{1-\theta}} + \text{const } b^{\frac{1}{r}} \frac{1}{N_i^{1-\theta} (\log N_i)^{1-\theta}} \\ &+ \frac{b^{2+\frac{1}{r}}}{2\log b - 1} \frac{1}{N_i} \\ &+ \text{const } \frac{b^{\frac{2}{r}} r^4}{(\log b)^4} \frac{b^{\frac{i}{r}}}{i^3 N_i} + \text{const } \frac{b^{\frac{1}{r}} r^3}{(\log b)^2} \frac{b^{\frac{i}{r}}}{i^2 (i-r) N_i} \\ &+ \text{const } b^{2+\frac{1}{r}} \frac{1}{N_i} + \text{const } (\log b) b^{\frac{1}{r}} \frac{i}{N_i (i \log b - r)}, \end{aligned} \tag{50}$$

where each "const" means a positive constant. Since

$$r \log_b p_{N_i} = i + w_{N_i} \le i + 1,$$

$$i \ge \frac{r}{\log b} \log p_{N_i} - 1. \tag{51}$$

Since

$$i < i + w_{N_i} = \frac{r}{\log b} \log p_{N_i} \tag{52}$$

and that

$$p_n < n(\log n + \log\log n) \qquad \text{for } n \geq 6$$

by J. B. Rosser and L. Schoenfeld [10], we have

$$i < \frac{r}{\log b}\big(\log N_i + \log(\log N_i + \log\log N_i)\big), \tag{53}$$

and so

$$b^{\frac{i}{r}} < N_i(\log N_i + \log\log N_i) \tag{54}$$

for $N_i \geq 6$. By (51), (53), (54), and

$$n \log n < p_n \qquad \text{for } n \geq 1,$$

by J. B. Rosser and L. Schoenfeld [10], we have

$$\frac{b^{\frac{i}{r}}}{(i-r)^2 N_i} < \frac{\log(N_i \log N_i)}{(\frac{r}{\log b}\log p_{N_i} - r - 1)^2} < \frac{\log p_{N_i}}{(\frac{r}{\log b}\log p_{N_i} - r - 1)^2},$$

$$\frac{b^{\frac{i}{r}}}{(i-r)^3 N_i} < \frac{\log(N_i \log N_i)}{(\frac{r}{\log b}\log p_{N_i} - r - 1)^3} < \frac{\log p_{N_i}}{(\frac{r}{\log b}\log p_{N_i} - r - 1)^3},$$

$$\frac{b^{\frac{i}{r}}}{N_i^{2-\theta}(\log N_i)^{1-\theta}} < \frac{\log(N_i \log N_i)}{N_i^{1-\theta}(\log N_i)^{1-\theta}},$$

$$\frac{b^{\frac{i}{r}}}{i^3 N_i} < \frac{\log(N_i \log N_i)}{(\frac{r}{\log b}\log p_{N_i} - 1)^3} < \frac{\log p_{N_i}}{(\frac{r}{\log b}\log p_{N_i} - 1)^3},$$

$$\begin{aligned}\frac{b^{\frac{i}{r}}}{i^2(i-r) N_i} &< \frac{\log(N_i \log N_i)}{(\frac{r}{\log b}\log p_{N_i} - 1)^2(\frac{r}{\log b}\log p_{N_i} - r - 1)} \\ &< \frac{\log p_{N_i}}{(\frac{r}{\log b}\log p_{N_i} - 1)^2(\frac{r}{\log b}\log p_{N_i} - r - 1)},\end{aligned}$$

$$\begin{aligned}\frac{i}{N_i(i-r)} &< \frac{r}{\log b}\frac{\log(N_i \log(N_i \log N_i))}{N_i(\frac{r}{\log b}\log p_{N_i} - r - 1)} \\ &< \frac{r}{\log b}\frac{\log p_{N_i}}{N_i(\frac{r}{\log b}\log p_{N_i} - r - 1)},\end{aligned}$$

$$\begin{aligned}\frac{1}{N_i} &< \frac{1}{\pi(b^{\frac{i}{r}})} \sim \frac{\log b}{r}\frac{i}{b^{\frac{i}{r}}} < \frac{\log b}{r}\frac{\frac{r}{\log b}\log p_{N_i}}{b^{(\log_b p_{N_i} - \frac{1}{r})}} \\ &= b^{\frac{1}{r}}\frac{\log p_{N_i}}{p_{N_i}}.\end{aligned}$$

Overall, by (50), we have

$$
\begin{aligned}
D^*_{N_i}(\log_b p_n^r, g_w) \le{}& \left(\frac{2b^{\frac{1}{r}}r^2}{\log b} + \left(b^{\frac{2}{r}} + b^{\frac{1}{r}}\right)r\right)\frac{\log p_{N_i}}{(\frac{r}{\log b}\log p_{N_i} - r - 1)^2}\\
&+ \frac{16b^{\frac{1}{r}}r^3}{(\log b)^2}\frac{\log p_{N_i}}{(\frac{r}{\log b}\log p_{N_i} - r - 1)^3} + \text{const } b^{\frac{2}{r}}r\frac{\log p_{N_i}}{p_{N_i}^{1-\theta}}\\
&+ \text{const } b^{\frac{1}{r}}\frac{1}{p_{N_i}^{1-\theta}} + \text{const }\frac{b^{2+\frac{2}{r}}}{2\log b - 1}\frac{\log p_{N_i}}{p_{N_i}}\\
&+ \text{const }\frac{b^{\frac{2}{r}}r^4}{(\log b)^4}\frac{\log p_{N_i}}{(\frac{r}{\log b}\log p_{N_i} - 1)^3}\\
&+ \text{const }\frac{b^{\frac{1}{r}}r^3}{(\log b)^2}\frac{\log p_{N_i}}{(\frac{r}{\log b}\log p_{N_i} - 1)^2(\frac{r}{\log b}\log p_{N_i} - r - 1)}\\
&+ \text{const } b^{2+\frac{2}{r}}\frac{\log p_{N_i}}{p_{N_i}}\\
&+ \text{const } b^{\frac{2}{r}}r\frac{(\log p_{N_i})^2 + \log p_{N_i}\log\log p_{N_i}}{p_{N_i}(r\log p_{N_i} - r - \log b)}\\
\le{}& b^{\frac{1}{r}}(\log b)\left(2 + (b^{\frac{1}{r}} + 1)\frac{\log b}{r}\right)\frac{\log p_{N_i}}{\left(\log p_{N_i} - (1 + \frac{1}{r})\log b\right)^2}\\
&+ \text{const }\left(\frac{b^{\frac{2}{r}}r}{\log b} + b^{\frac{1}{r}}\log b\right)\frac{\log p_{N_i}}{\left(\log p_{N_i} - (1 + \frac{1}{r})\log b\right)^3}\\
&+ \text{const }\left(b^2 + \frac{b^2}{2\log b - 1}\right)b^{\frac{2}{r}}r\frac{\log p_{N_i}}{p_{N_i}^{1-\theta}}\\
&+ \text{const } b^{\frac{2}{r}}r\frac{(\log p_{N_i})^2 + \log p_{N_i}\log\log p_{N_i}}{p_{N_i}(r\log p_{N_i} - r - \log b)}
\end{aligned}
$$

for $N_i \ge 6$. □

From Corollary 2 it follows immediately:

Corollary 3. *Let $r > 0$, $0 < w \le 1$, let $N_i = \pi(b^{\frac{i+w}{r}})$, $i = 1, 2, \dots,$ and let*

$$g_w(x) = \frac{1}{b^{\frac{w}{r}}}\frac{b^{\frac{x}{r}} - 1}{b^{\frac{1}{r}} - 1} + \frac{\min(b^{\frac{x}{r}}, b^{\frac{w}{r}}) - 1}{b^{\frac{w}{r}}} \quad (0 \le x \le 1).$$

Then we have

$$\lim_{i\to\infty}(\log p_{N_i})D^*_{N_i}(\log_b p_n^r, g_w) \le b^{\frac{1}{r}}(\log b)\left(2 + (b^{\frac{1}{r}} + 1)\frac{\log b}{r}\right).$$

Corollary 4. *Let r be a positive integer. Then*

$$\lim_{r\to\infty} rD^*_{\pi(b^r)}(\log_b p_n^r, g_1) \le 2,$$

where

$$g_1(x) = \frac{b^{\frac{x}{r}} - 1}{b^{\frac{1}{r}} - 1} \quad (0 \le x \le 1).$$

P r o o f. Let $r \ge 3$. Put $w = 1$ and $i = r^2 - 1$ in Corollary 3. Then $N_i = \pi(b^r)$. By (52), we have

$$\log p_{N_i} > \left(r - \frac{1}{r}\right) \log b,$$

and so

$$\frac{r}{\log p_{N_i}} \le \frac{r}{(r - \frac{1}{r}) \log b} \to \frac{1}{\log b}$$

as $r \to \infty$. From this and Corollary 3, the desired inequality follows. □

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Yukio Ohkubo
Department of Business Administration
The International University of Kagoshima
8-34-1 Sakanoue, Kagoshima-shi 891-0197
JAPAN
E-mail: ohkubo@eco.iuk.ac.jp

Oto Strauch
Mathematical Institute
Slovak Academy of Sciences
Štefánikova 49
SK-814 73 Bratislava
SLOVAKIA
E-mail: strauch@mat.savba.sk