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HIGHER ORDER OSCILLATION AND UNIFORM DISTRIBUTION

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ABSTRACT. It is known that the Möbius function in number theory is higher order oscillating. In this paper we show that there is another kind of higher order oscillating sequences in the form $(e^{2\pi i\alpha\beta^n g(\beta)})_{n\in\mathbb{N}}$, for a non-decreasing twice differentiable function g with a mild condition. This follows the result we prove in this paper that for a fixed non-zero real number α and almost all real numbers $\beta > 1$ (alternatively, for a fixed real number $\beta > 1$ and almost all real numbers α) and for all real polynomials $Q(x)$, sequences $\big(\alpha\beta^n g(\beta) + Q(n)\big)_{n\in\mathbb{N}}$ are uniformly distributed modulo 1.

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1. Introduction

We denote by $\mathbb{N}$ the set of positive integers. Suppose $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$, is a sequence of complex numbers. In [2] (see also [4]), an oscillating sequence is defined for the purpose of the study of Sarnak's conjecture (see [8, 9]) which is stated as

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the Möbius function is linearly disjoint from all zero entropy flows. Let us recall the definition of an oscillating sequence.

Definition 1 (Oscillation). The sequence $\mathbf{c}=(c_n)_{n\in\mathbb{N}}$ is said to be oscillating if

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}c_n e^{2\pi int}=0,\ \ \forall\, 0\leq t<1. \tag{1}$$

with the technical condition

$$\sum_{n=1}^{N}|c_n|^{\lambda}=O(N)\quad \text{for some}\quad \lambda>1. \tag{2}$$

Recall that the Möbius function $\mu(n)$ is, by definition, $\mu(n)=1$ if $n=1$; $\mu(n)=(-1)^r$ if $n=p_1\cdots p_r$ for r distinct prime numbers p_i; $\mu(n)=0$ if $p^2|n$ for some prime number p. The Möbius sequence $\mathbf{u}=(\mu(n))_{n\in\mathbb{N}}$ is the one generated by the Möbius function. Due to Davenport's theorem [1], the Möbius sequence is oscillating.

We proved in [2] that any oscillating sequence is linearly disjoint from all minimally mean attractable (MMA) and minimally mean-L-stable (MMLS) flows. In the same paper, we further proved that flows defined by all p-adic polynomials of integral coefficients, all p-adic rational maps with good reduction, all automorphisms of the 2-torus with zero topological entropy, all diagonalizable affine maps of the 2-torus with zero topological entropy, all orientation-preserving circle homeomorphisms are MMA and MMLS. Furthermore, in [4], we proved that flows defined by all continuous interval maps with zero topological entropy are MMA and MMLS. Therefore, we confirmed Sarnak's conjecture for these flows which form a large class of zero topological entropy flows. However, it is also shown in [2, Example 7], only the oscillation property is not enough for the study of Sarnak's conjecture. We need a higher order oscillation condition in the study of Sarnak's conjecture. We gave a definition of a higher order oscillating sequence in [5].

Definition 2 (Higher order oscillation). We call the sequence $\mathbf{c}=(c_n)_{n\in\mathbb{N}}$ a higher order oscillating sequence of order $m\geq 2$ if

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}c_n e^{2\pi iP(n)}=0 \tag{3}$$

for every real polynomial P of degree $\leq m$ with also the technical condition (2).

We prove in [5] that any higher order oscillating sequence of order d is linearly disjoint from all affine distal flows on the d-torus for all $d\geq 2$. One consequence of this result is that any higher order oscillating sequence of order 2 is linearly disjoint from all affine flows on the 2-torus with zero topological entropy. Thanks to Hua's result [3], the Möbius sequence $\mathbf{u}$ is an example of a higher order oscillating sequence of order m for all $m \geq 2$ (see also [7, Lemma 2.1] and [10]). Combining this with our main result in [5], we reconfirmed in [5] that Sarnak's conjecture for all affine flows on the 2-torus with zero topological entropy and for all affine distal flows on the d-torus for all $d > 2$ by using a much simple method. Then we have the following interesting question.

Question 1. Is there another kind of higher order oscillation sequences as defined in Definition 2 except for the one generated by an arithmetic function like the Möbius function?

We study this question in this paper.

2. Statement of the main result

For a real number x, let $[x]$ denote the integer part of x, that is, the greatest integer $\leq x$; let

$$\{x\} = x - [x]$$

be the fractional part of x, or the residue of x modulo 1.

Definition 3 (Uniform distribution)**.** We say a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of real numbers is uniformly distributed modulo 1 (abbreviated u. d. mod 1) if for any $0 \leq a < b \leq 1$, we have

$$\lim_{N\to\infty} \frac{\#(\{n \in [1, N] \mid \{x_n\} \in [a, b]\})}{N} = b - a.$$

We state three results in the u. d. mod 1 theory.

Theorem A (The Weyl criterion)**.** *The sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is u. d.* mod 1 *if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} = 0 \quad \textit{for all integers } h \neq 0.$$

Theorem B (Koksma's theorem)**.** *Let $\big(y_n(x)\big)_{n\in\mathbb{N}}$ be a sequence of real valued C^1 functions defined on an interval $[a, b]$. Suppose $y'_m(x) - y'_n(x)$ is monotone on $[a, b]$ for any two integers $m \neq n$ and suppose*

$$\inf_{m\neq n} \min_{x\in[a,b]} \big|y'_m(x) - y'_n(x)\big| > 0.$$

Then for almost all $x \in [a, b]$, the sequence $\mathbf{y} = \big(y_n(x)\big)_{n\in\mathbb{N}}$ is u. d. mod 1.

Theorem C (Van der Corput's theorem). *Let* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *be a sequence of real numbers. If for every positive integer* h *the sequence*

$$\mathbf{d_h x} = (x_{n+h} - x_n)_{n\in\mathbb{N}} \quad \textit{is u. d.} \quad \bmod 1,$$

then $\mathbf{x}$ *itself is u. d.* mod 1.

The reader who is interested in these three theorems can find proofs in [6, Theorem 2.1, Theorem 3.1, and Theorem 4.3].

For the convenience of notation, we understand that an empty sum is 0 and an empty product is one, i.e.,

$$\sum_{j=1}^{k}(\cdots) = 0 \quad \text{and} \qquad \prod_{j=1}^{k}(\cdots) = 1 \quad \text{when} \quad k = 0.$$

In this notation, we have

$$y_n(x) = x^n = x^n \prod_{j=1}^{k}(\cdots) \quad \text{and} \quad y_n(x)\sum_{j=1}^{k}(\cdots) = 0 \quad \text{when} \quad k = 0.$$

Take a non empty interval I in the real line $\mathbb{R}$, which can be closed, open or semi-open. Let $\mathcal{C}_+^k(I)$ be the space of all positive real valued k-times continuously differentiable functions on an interval I, whose i-th derivative is non-negative for $i \le k$. Then it is closed under addition and multiplication, that is, if

$$f, g \in \mathcal{C}_+^k(I),$$

then

$$f + g, \quad fg \in \mathcal{C}_+^k(I).$$

In what follows, we often use this *closure property* of $\mathcal{C}_+^k(I)$. Let $\mathbb{R}[x]$ denote the space of all real polynomials. The main result we prove in this paper is

Theorem 1 (Main theorem). *Let us take a function* $g \in \mathcal{C}_+^2((1,\infty))$*. Then, for a fixed real number* $\alpha \neq 0$ *and almost all real numbers* $\beta > 1$ *(alternatively, for a fixed real number* $\beta > 1$ *and almost all real numbers* α*) and for all real polynomials* $Q \in \mathbb{R}[x]$*, sequences*

$$\Big(\alpha\beta^n g(\beta) + Q(n)\Big)_{n\in\mathbb{N}} \quad \textit{are u. d.} \bmod 1. \tag{4}$$

As a countable union of exceptional null sets is null, we clearly have

Corollary 1. *Given a countable family $\{g_i \mid i \in \mathbb{N}\}$ in $\mathcal{C}^2_+((1,\infty))$. Then, for a fixed real number $\alpha \neq 0$ and almost all real numbers $\beta > 1$ (alternatively, for a fixed real number $\beta > 1$ and almost all real numbers α) and for all real polynomials $Q \in \mathbb{R}[x]$, sequences*

$$\Big(\alpha\beta^n g_i(\beta) + Q(n)\Big)_{n\in\mathbb{N}}, \qquad i \in \mathbb{N}, \quad \textit{are u. d.} \bmod 1.$$

Since $x^n - 1 \in \mathcal{C}^2_+((1,\infty))$ for all $n \geq 1$, we have that for any $g \in \mathcal{C}^2_+((1,\infty))$ and any integer $l \geq 0$ and $h_j \in \mathbb{N}$, $1 \leq j \leq l$,

$$g(x)\prod_{j=1}^{l}\big(x^{h_j} - 1\big) \in \mathcal{C}^2_+((1,\infty)).$$

Applying this countable family to Corollary 1, we obtain:

Theorem 2 (Equivalent statement). *Given any $g \in \mathcal{C}^2_+((1,\infty))$. Then, for a fixed real number $\alpha \neq 0$ and almost all real numbers $\beta > 1$ (alternatively, for a fixed real number $\beta > 1$ and almost all real numbers α), for all integers $l \geq 0$ and all l-tuple $(h_1, \ldots, h_l) \in \mathbb{N}^l$, and for all real polynomials $Q \in \mathbb{R}[x]$, sequences*

$$\left(\alpha\beta^n g(\beta)\prod_{j=1}^{l}\big(\beta^{h_j} - 1\big) + Q(n)\right)_{n\in\mathbb{N}} \qquad \textit{are u. d.} \bmod 1. \tag{5}$$

On the other hand, when $l = 0$, the product is empty, Theorem 2 is reduced to Theorem 1. So our Theorem 1 and Theorem 2 are equivalent. Our proof is based on the formulation of Theorem 2, which is already an interesting point in this paper. We will give the proof in § 3.

Theorem 1 combined with Theorem A answers the question (Question 1) affirmatively.

Corollary 2 (Main corollary). *Given any $g \in \mathcal{C}^2_+((1,\infty))$. Then, for a fixed real number $\alpha \neq 0$ and almost all real numbers $\beta > 1$ (alternatively, for a fixed real numbers $\beta > 1$ and almost all real numbers α), sequences*

$$\mathbf{c} = \left(e^{2\pi i\alpha\beta^n g(\beta)}\right) \tag{6}$$

are higher order oscillating sequences of order m for all $m \geq 2$.

Remark 1. In particular, taking a constant function $g \equiv 1$, we have

$$\mathbf{c} = \left(e^{2\pi i\alpha\beta^n}\right)_{n\in\mathbb{N}}.$$

3. Proof of the main theorem

We start the proof for the case that $\alpha \neq 0$ is fixed and figure out the exceptional set for $\beta > 1$. We may assume that $\alpha > 0$ and fixed it from now on. Take $g \in \mathcal{C}^2_+((1,\infty))$. For $n \in \mathbb{N}$ and an integer $l \geq 0$ and for a l-tuple $(h_1, \dots, h_l) \in \mathbb{N}^l$, define a function

$$y_n(x) = \alpha g(x) x^n \prod_{j=1}^{l} \bigl(x^{h_j} - 1\bigr).$$

Remember that when $l = 0$, $y_n(x) = \alpha x^n g(x)$.

For $n > m$, we have

$$\begin{aligned} y_n'(x) - y_m'(x) = \ & \alpha g(x)\Biggl(\bigl(nx^{n-1} - mx^{m-1}\bigr)\prod_{j=1}^{k}\bigl(x^{h_j} - 1\bigr) + \\ & \qquad \bigl(x^n - x^m\bigr)\sum_{j=1}^{k} h_j x^{h_j - 1}\prod_{i \neq j}\bigl(x^{h_i} - 1\bigr)\Biggr) + \\ & \qquad\qquad \alpha g'(x)\bigl(x^n - x^m\bigr)\prod_{j=1}^{l}\bigl(x^{h_j} - 1\bigr). \end{aligned}$$

Since $g', g'' \geq 0$ and since

$$nx^{n-1} - mx^{m-1} = x^{m-1}\bigl(nx^{n-m} - m\bigr)$$

and

$$nx^{n-m} - m \geq n - m \geq 1 \quad \text{for } \ x \geq 1,$$

we see that every term in the last expression are in $\mathcal{C}^1_+([1,\eta])$ for any $\eta > 1$. By the closure property of $\mathcal{C}^1_+([1,\eta])$, we have that

$$y_n'(x) - y_m'(x) \in \mathcal{C}^1_+([1,\eta]) \quad \forall n > m \in \mathbb{N}. \tag{7}$$

In particular, this imply that $y_n' - y_m'$ is increasing for $n > m \in \mathbb{N}$. Furthermore, if $x > a > 1$, then $x^h - 1 \geq a - 1$ for any $h \in \mathbb{N}$, we see that there is a constant $L > 0$ such that

$$|y_n'(x) - y_m'(x)| \geq L \qquad \forall n > m \in \mathbb{N}, \quad \forall\, a \leq x \leq \eta. \tag{8}$$

Inequalities (7) and (8) say that the sequence of real valued C^1 functions

$$\bigl(y_n(x)\bigr)_{n\in\mathbb{N}}$$

satisfies all hypothesizes of Theorem B.

Theorem B implies that for almost all x in

$$[(2^k+1)/2^k, (2^{k-1}+1)/2^{k-1}] \quad \text{or} \quad [k, k+1] \quad \text{for } k \geq 2,$$

the sequence

$$\big(\alpha y_n(x)\big)_{n\in\mathbb{N}} \quad \text{is u. d. mod 1.}$$

Further, this implies that for almost all

$$x \in (1,\infty) = \cup_{k=2}^{\infty}\left[\frac{2^k+1}{2^k}, \frac{2^{k-1}+1}{2^{k-1}}\right] \cup \cup_{k=2}^{\infty}[k, k+1]$$

the sequence $\big(\alpha y_n(x)\big)_{n\in\mathbb{N}}$ is u. d. mod 1.

Let

$$A_{(h_1,h_2,\ldots,h_l)} = \left\{\beta > 1 \;\middle|\; \left(\alpha\beta^n g(\beta)\prod_{j=1}^{l}\big(\beta^{h_j}-1\big)\right)_{n\in\mathbb{N}} \text{is not u. d. mod 1}\right\}.$$

Then it has one dimensional Lebesgue measure zero. By the above convention, we include the case $l = 0$ as well.

Since the set

$$U = \bigcup_{l=0}^{\infty}\{(h_1,\ldots,h_l) \mid h_j \in \mathbb{N}\}$$

is countable, the one dimensional Lebesgue measure of

$$\bigcup_{(h_1,\ldots,h_l)\in U} A_{(h_1,\ldots,h_l)}$$

is zero, too.

For the fix a real number $\alpha \neq 0$ in the theorem, take a real number

$$\beta \in (1,\infty) \Big\backslash \bigcup_{(h_1,\ldots,h_l)\in U} A_{(h_1,\ldots,h_l)}.$$

This says that the sequence

$$\left(\alpha\beta^n g(\beta)\prod_{j=1}^{l}(\beta^{h_j}-1)\right)_{n\in\mathbb{N}} \quad \text{is u. d. mod 1}$$

for all integers $l \geq 0$ and all l-tuple $(h_1,\ldots,h_l) \in \mathbb{N}^l$.

Define statements $P(k)$ for $k = 0, 1, \dots$ as follows.

$P(k)$: For any integer $l \geq 0$,

$$(h_1, \dots, h_l) \in \mathbb{N}^l \qquad \textit{and} \qquad t_i \in \mathbb{R} \ (i = 0, 1, 2, \dots, k),$$

the sequence

$$\left(\alpha \beta^n g(\beta) \prod_{j=1}^{l} \left(\beta^{h_j} - 1 \right) + \sum_{i=0}^{k} t_i n^i \right)_{n \in \mathbb{N}} \qquad \textit{is u. d} \bmod 1.$$

We claim that $P(k)$ holds for every integer $k \geq 0$. We prove the claim by induction.

By our choice of α and β, we know that $P(0)$ holds.

Assume $P(k-1)$ holds for $k \geq 1$. Let

$$x_n = \alpha \beta^n g(\beta) \prod_{j=1}^{\ell} \left(\beta^{h_j} - 1 \right) + \sum_{i=0}^{k} t_i n^i.$$

Then

$$x_{n+h} = \alpha \beta^{n+h} g(\beta) \prod_{j=1}^{\ell} \left(\beta^{h_j} - 1 \right) + \sum_{i=0}^{k} t_i (n+h)^i.$$

Consider the difference appeared in Theorem C:

$$\begin{aligned} x_{n+h} - x_n &= \alpha \beta^n (\beta^h - 1) g(\beta) \prod_{j=1}^{\ell} \left(\beta^{h_j} - 1 \right) + \sum_{i=0}^{k-1} T_i n^i \\ &= \alpha \beta^n g(\beta) \prod_{j=1}^{\ell+1} \left(\beta^{h_j} - 1 \right) + \sum_{i=0}^{k-1} T_i n^i \end{aligned}$$

$$\text{with} \quad h_{\ell+1} = h \qquad \text{and} \qquad T_i = -t_i + \sum_{j=i}^{k} t_j \binom{j}{i} h^{j-i}.$$

Since $P(k-1)$ is valid, the resulting sequence is u. d. $\bmod 1$ for all $h \in \mathbb{N}$. Now Theorem C implies that $P(k)$ holds too. We proved the claim. Therefore we completed the proof of Theorem 2.

The proof for the case that $\beta > 1$ is fixed and to obtain the exceptional set for α, is similar and easier. Let g be any positive function on $(1, \infty)$. For $n \in \mathbb{N}$ and an integer $l \geq 0$ and for a l-tuple $(h_1, \dots, h_l) \in \mathbb{N}^l$, define

$$y_n(x) = x g(\beta) \beta^n \prod_{j=1}^{l} \left(\beta^{h_j} - 1 \right). \tag{9}$$

Then

$$y_n'(x) - y_m'(x) = g(\beta)\big(\beta^n - \beta^m\big)\prod_{j=1}^{l}\big(\beta^{h_j} - 1\big)$$

is a positive constant for $n > m$ and satisfies the condition of Theorem B under the similar dissection of the interval $(1, \infty)$. The rest of the proof is the same.

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