

NOTES ON THE DISTRIBUTION OF ROOTS MODULO A PRIME OF A POLYNOMIAL

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ABSTRACT. Let $f(x)$ be a monic polynomial in $\mathbb{Z}[x]$ with roots $\alpha_1, \dots, \alpha_n$. We point out the importance of linear relations among $1, \alpha_1, \dots, \alpha_n$ over rationals with respect to the distribution of local roots of f modulo a prime. We formulate it as a conjectural uniform distribution in some sense, which elucidates data in previous papers.

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In this note, a polynomial means always a monic one over the ring \mathbb{Z} of integers and the letter p denotes a prime number, unless specified. Let

$$f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0 \quad (0)$$

be a polynomial of degree n . As in the previous papers, we put

$$\text{Spl}_X(f) := \{p \leq X \mid f(x) \text{ is fully splitting modulo } p\}$$

for a positive number X and $\text{Spl}(f) := \text{Spl}_\infty(f)$. In this note, we require the following conditions on the local roots $r_1, \dots, r_n (\in \mathbb{Z})$ of $f(x) \equiv 0 \pmod p$ for a prime $p \in \text{Spl}(f)$:

$$f(x) \equiv \prod_{i=1}^n (x - r_i) \pmod p, \quad (1)$$

$$0 \leq r_1 \leq r_2 \leq \dots \leq r_n < p. \quad (2)$$

We can determine local roots r_i uniquely with this global ordering. If f is irreducible and of $\deg(f) > 1$, and p is sufficiently large, then (2) is equivalent to $0 < r_1 < \dots < r_n < p$. Here, we consider two types of distribution of local roots r_i of f .

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Before stating them, let $\alpha_1, \dots, \alpha_n$ be roots of a polynomial f in (0) and suppose a linear relation

$$\sum_{i=1}^n m_i \alpha_i = m \quad (m_i, m \in \mathbb{Q}). \quad (3)$$

Let us give three typical examples of a linear relation (3) among roots:

The first example is

$$\sum_{i=1}^n \alpha_i = \text{tr}(f) \quad (:= -a_{n-1}).$$

We call a linear relation (3) trivial if $m_1 = \dots = m_n$, otherwise non-trivial. A trivial relation is reduced to the above. We know that for an irreducible polynomial f , there is only a trivial relation if the degree n is prime or the Galois group $\text{Gal}(\mathbb{Q}(\alpha_1, \dots, \alpha_n)/\mathbb{Q})$ is S_n or A_n ($n \geq 6$) as a permutation group of $\{\alpha_1, \dots, \alpha_n\}$ (Proposition 2).

The second is a reducible polynomial

$$f(x) = g(x)h(x) \quad \text{with} \quad 1 < \deg g < \deg f.$$

There is a non-trivial relation $\sum \beta_i = \text{tr}(g)$ for roots β_i of g , since a set of roots of g is a proper subset of roots of f .

The third is a decomposable polynomial, that is

$$f(x) = g(h(x)) \quad \text{with} \quad 1 < \deg h < \deg f.$$

For a root β of $g(x) = 0$, a set of solutions γ_i of $h(x) = \beta$ is a proper subset of roots of $f(x)$, and we have a non-trivial relation $\sum \gamma_i = \text{tr}(h - \beta) = \text{tr}(h) \in \mathbb{Z}$. Some other examples are given in [3] and in the text.

If the degree of f is less than 6, there is no non-trivial linear relation except the above two types, as shown below. In case of degree 6, other non-trivial linear relations appear.

The first subject is a kind of uniformity: Let f be a polynomial in (0) of degree n and put

$$\hat{\mathcal{D}}_n := \{(x_1, \dots, x_n) \in [0, 1]^n \mid 0 \leq x_1 \leq \dots \leq x_n < 1, \sum_{i=1}^n x_i \in \mathbb{Z}\} \quad (4)$$

and for a set $D \subset [0, 1]^n$ with $D = \overline{D^\circ}$

$$\text{Pr}_D(f, X) := \frac{\#\{p \in \text{Spl}_X(f) \mid (r_1/p, \dots, r_n/p) \in D\}}{\#\text{Spl}_X(f)},$$

where local roots r_i satisfy properties (1), (2).

We expect, under an assumption that a polynomial f has only a trivial linear relation (3) among roots.

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EXPECTATION 1.

$$\Pr_D(f) := \lim_{X \rightarrow \infty} \Pr_D(f, X) = \frac{\text{vol}(D \cap \hat{\mathfrak{D}}_n)}{\text{vol}(\hat{\mathfrak{D}}_n)}. \quad (5)$$

The set $\hat{\mathfrak{D}}_n$ is parametrized by x_1, \dots, x_{n-1} , since x_n equals $\lceil \sum_{i < n} x_i \rceil - \sum_{i < n} x_i$, and the volume of the projection \mathfrak{D}_n of $\hat{\mathfrak{D}}_n$ to a hyperplane \mathbb{R}^{n-1} defined by $x_n = 0$ is $1/n!$. Here, $\lceil x \rceil$ denotes an integer satisfying $x \leq \lceil x \rceil < x + 1$ for a real number x .

In case of $\text{tr}(f) = 0$, we may suppose that D is limited to a domain on $\hat{\mathfrak{D}}_n$ by virtue of $(r_1/p, \dots, r_n/p) \in \hat{\mathfrak{D}}_n$, and it is easy to see that the right-hand side of (5) is $\text{vol}(\text{pr}(D))/(1/n!)$, where pr is a projection $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_{n-1})$.

In general, a polynomial has non-trivial linear relations among roots, and suppose that a system of linear equations

$$\sum_{i=1}^n m_{j,i} \alpha_i = m_j \quad (j = 1, \dots, t) \quad (6)$$

is a basis of all linear relations (3) restricted to $m_{j,i}, m_j \in \mathbb{Z}$. If f is irreducible, then we see $(\sum_i m_{j,i}) \text{tr}(f) = nm_j$. We fix a numbering of roots α_i of f . For a prime $p \in \text{Spl}(f)$, there is a permutation $\sigma \in S_n$ (dependent on p) such that local roots r_i satisfy induced relations (cf. Proposition 1)

$$\sum_i m_{j,i} r_{\sigma(i)} \equiv m_j \pmod{p} \quad (1 \leq \forall j \leq t) \quad (7)$$

which implies $\sum_i m_{j,i} \cdot r_{\sigma(i)}/p - m_j/p \in \mathbb{Z}$. Let $\mathbf{x} = (x_1, \dots, x_n) \in [0, 1]^n$ be an accumulation point of $(r_1/p, \dots, r_n/p)$ with the same permutation σ above; then we see that \mathbf{x} is in the closure of

$$\mathfrak{D}(f, \sigma) := \left\{ (x_1, \dots, x_n) \in [0, 1]^n \mid \begin{array}{l} 0 \leq x_1 \leq \dots \leq x_n < 1, \\ \sum_i m_{j,i} x_{\sigma(i)} \in \mathbb{Z} \text{ for } 1 \leq \forall j \leq t \end{array} \right\}.$$

If \mathbf{x} is not in $\mathfrak{D}(f, \sigma)$, then x_n is equal to 1 and we neglect the case since we are concerned with the volume. We note that the set $\mathfrak{D}(f, \sigma)$ depends on a numbering of roots α_i and may be the same for distinct permutations.

If f has only a trivial linear relation, then $\mathfrak{D}(f, \sigma)$ is nothing but $\hat{\mathfrak{D}}_n$.

Put

$$\text{Spl}_X(f, \sigma) := \{p \in \text{Spl}_X(f) \mid \sum_i m_{j,i} r_{\sigma(i)} \equiv m_j \pmod{p} (1 \leq \forall j \leq t)\}.$$

If $\text{Spl}_\infty(f, \sigma_1) \cap \text{Spl}_\infty(f, \sigma_2)$ is an infinite set, then $\text{Spl}_\infty(f, \sigma_1)$ and $\text{Spl}_\infty(f, \sigma_2)$ are equal except a finite set. The following is a generalization of Expectation 1.

EXPECTATION 1'.

$$\begin{aligned} \Pr_D(f, \sigma) &:= \lim_{X \rightarrow \infty} \frac{\#\{p \in \text{Spl}_X(f, \sigma) \mid (r_1/p, \dots, r_n/p) \in D\}}{\#\text{Spl}_X(f, \sigma)} \\ &= \frac{\text{vol}(D \cap \mathfrak{D}(f, \sigma))}{\text{vol}(\mathfrak{D}(f, \sigma))} \end{aligned} \quad (8)$$

for a permutation σ with $\dim \mathfrak{D}(f, \sigma) = n - t$, and vol is the volume as a set of \dim being $n - t$. With respect to the density of a set $\text{Spl}(f, \sigma)$ of primes, our observation is

EXPECTATION 1''.

$$\lim_{X \rightarrow \infty} \frac{\#\text{Spl}_X(f, \sigma)}{\#\text{Spl}_X(f)} = c^{-1} \cdot \text{vol}(\mathfrak{D}(f, \sigma)),$$

where the constant c is independent of σ .

We give a remark on the numbering of roots : Since (6) and (7) are equivalent to $\sum_{i=1}^n m_{j, \sigma^{-1}(i)} \alpha_{\sigma^{-1}(i)} = m_j$ and $\sum_{i=1}^n m_{j, \sigma^{-1}(i)} r_i \equiv m_j \pmod{p}$, we may assume that σ in Expectation 1', 1'' is the identity for any numbering of roots with replacing “ c is independent of σ ” by “ c is independent of the numbering of roots of f ” in Expectation 1''.

We note that for a sufficiently large prime p , we see that $0 < r_1 + a_{n-1}/n, r_n + a_{n-1}/n < p$, and then $((r_1 + a_{n-1}/n)/p, \dots, (r_n + a_{n-1}/n)/p) \in \mathfrak{D}(f, \sigma)$ if f is irreducible and that a point $(r_1/p, \dots, r_n/p)$ is on $\mathfrak{D}(f, \sigma)$ if and only if $a_{n-1} = 0$. Is it possible to reduce the problem to the case of trace being 0, using $g(x) := n^n f((x - a_{n-1})/n) = x^n + 0 \cdot x^{n-1} + \dots$? The relation between local roots R_i of $g(x)$ and r_i of $f(x)$ is $R_i \equiv nr_{\sigma(i)} + a_{n-1} \pmod{p}$ for a permutation σ dependent on p .

The second subject is as follows.

For given integers $L (> 1)$, R_i with $0 \leq R_i < L$ and a prime $p \in \text{Spl}(f)$, we require a following congruence condition besides (1), (2) on the local roots r_1, \dots, r_n of $f(x) \equiv 0 \pmod{p}$:

$$r_i \equiv R_i \pmod{L} \quad (1 \leq \forall i \leq n). \quad (9)$$

We put

$$\Pr_X(f, L, \{R_i\}) := \frac{\#\{p \in \text{Spl}_X(f) \mid (1), (2), (9)\}}{\#\text{Spl}_X(f)} \quad (10)$$

and

$$\Pr(f, L, \{R_i\}) := \lim_{X \rightarrow \infty} \Pr_X(f, L, \{R_i\}). \quad (11)$$

Although the existence of the limit is not proved in this case either, there is no data to deny it. By putting

$$R_f := a_{n-1} + \sum_{i=1}^n R_i \quad \text{and} \quad d := (R_f, L),$$

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our second expectation is as follows:

For a polynomial f of degree ≥ 2 with only a trivial relation (3)

EXPECTATION 2.

$$\Pr(f, L, \{R_i\}) := \frac{1}{L^{n-1}} \sum_{K, q} \frac{E_n(K)}{[\mathbb{Q}(\zeta_L) : \mathbb{Q}(f) \cap \mathbb{Q}(\zeta_L/d)]}, \quad (12)$$

where K runs over a set of integers satisfying

$$1 \leq K \leq n-1, \quad (K, L) = d,$$

and $q \in (\mathbb{Z}/L\mathbb{Z})^\times$ satisfy the conditions

$$\begin{cases} R_f \equiv Kq \pmod{L} & (\Leftrightarrow R_f/d \equiv K/d \cdot q \pmod{L/d}), \\ [[q]] = [[1]] & \text{on } \mathbb{Q}(f) \cap \mathbb{Q}(\zeta_L/d). \end{cases}$$

Let us explain notations: $E_n(k)$ is the volume of the set $\{x \in [0, 1]^{n-1} \mid \lceil x_1 + \dots + x_{n-1} \rceil = k\}$. $E_n(k)$ is also defined as $E_n(k) := A(n-1, k)/(n-1)!$, using Eulerian numbers $A(n, k)$ ($1 \leq k \leq n$) defined recursively by

$$A(1, 1) = 1, \quad A(n, k) = (n-k+1)A(n-1, k-1) + kA(n-1, k).$$

ζ_L is a primitive L th root of unity, and $\mathbb{Q}(f)$ is a Galois extension of the rational number field \mathbb{Q} generated by all roots of f . For an abelian field F in $\mathbb{Q}(\zeta_c)$ and an integer a relatively prime to c , $[[a]]$ denotes an automorphism of F induced by $\zeta_c \rightarrow \zeta_c^a$.

Expectations 1, 2 are supported by numerical data by computer for irreducible and indecomposable polynomials of degree < 6 ([6],[7]), which are polynomials with only a trivial linear relation among roots. Expectation 2 fails for some polynomials of $\deg f = 6$ with non-trivial linear relations.

Let us refer to a relation with a one-dimensional distribution of r_i/p ($i = 1, \dots, n$): Let f be a polynomial of degree n with only a trivial linear relation among roots. Given number $a \in [0, 1]$, put $D_{i,a} := \{(x_1, \dots, x_n) \in [0, 1]^n \mid x_i \leq a\}$. Then we see

$$\begin{aligned} & \lim_{X \rightarrow \infty} \frac{\sum_{p \in \text{Spl}_X(f)} \#\{i \mid r_i/p \leq a, 1 \leq i \leq n\}}{n \cdot \#\text{Spl}_X(f)} \\ &= \lim_{X \rightarrow \infty} \frac{\sum_{p \in \text{Spl}_X(f)} \#\{i \mid (r_1/p, \dots, r_n/p) \in D_{i,a}\}}{n \cdot \#\text{Spl}_X(f)} \\ &= \sum_{i=1}^n \Pr_{D_{i,a}}(f)/n \\ &= \sum_{i=1}^n \frac{\text{vol}(D_{i,a} \cap \hat{\mathcal{D}}_n)}{n \cdot \text{vol}(\hat{\mathcal{D}}_n)} \quad (\text{by Expectation 1}) \end{aligned}$$

which is equal to a , as far as we check approximately by the Monte Carlo method (definitely for $n = 2, 3$), that is we have the equi-distribution of r_i/p .

Lastly, let us give a relation between Expectation 1 and a series of observations in the references. Let a polynomial f of degree n have only a trivial linear relation among roots, and put, for an integer k

$$D_k := \{(x_1, \dots, x_n) \in [0, 1]^n \mid \lceil x_1 + \dots + x_{n-1} \rceil = k\}.$$

Then, under Expectation 1, we have

$$\begin{aligned} & \lim_{X \rightarrow \infty} \frac{\#\{p \in \text{Spl}_X(f) \mid (r_1 + \dots + r_n - \text{tr}(f))/p = k\}}{\#\text{Spl}_X(f)} \\ &= \lim_{X \rightarrow \infty} \frac{\#\{p \in \text{Spl}_X(f) \mid \lceil r_1/p + \dots + r_{n-1}/p \rceil = k\}}{\#\text{Spl}_X(f)} \\ &= \lim_{X \rightarrow \infty} \frac{\#\{p \in \text{Spl}_X(f) \mid (r_1/p, \dots, r_n/p) \in D_k\}}{\#\text{Spl}_X(f)} \quad (= \text{Pr}_{D_k}(f)) \\ &= \frac{\text{vol}(\{(x_1, \dots, x_n) \in \hat{\mathcal{D}}_n \mid \lceil \sum_{i=1}^{n-1} x_i \rceil = k\})}{\text{vol}(\hat{\mathcal{D}}_n)} \quad (\text{by Expectation 1}) \\ &= \frac{\text{vol}(\{(x_1, \dots, x_n) \in [0, 1]^n \mid x_1 \leq \dots \leq x_n, \sum_{i=1}^n x_i = k\})}{\text{vol}(\hat{\mathcal{D}}_n)} \\ &= \frac{\text{vol}(\{(x_1, \dots, x_n) \in [0, 1]^n \mid \sum_{i=1}^n x_i = k\})}{n! \cdot \text{vol}(\{(x_1, \dots, x_n) \in [0, 1]^n \mid x_1 \leq \dots \leq x_n, \sum_{i=1}^n x_i \in \mathbb{Z}\})} \\ &= \text{vol} \left(\left\{ (x_1, \dots, x_{n-1}) \in [0, 1]^{n-1} \mid \left\lceil \sum_{i=1}^{n-1} x_i \right\rceil = k \right\} \right) \quad (\text{projected to } \mathbb{R}^{n-1}) \\ &= E_n(k), \end{aligned}$$

which elucidates most of numerical observations in previous papers. We note that the last vol is the usual volume on \mathbb{R}^{n-1} , but others are the one on hyperplanes defined by

$$\sum_{i=1}^n x_i \in \mathbb{Z} \quad \text{in } \mathbb{R}^n.$$

We discuss a linear relation among roots in the first section, and in the second section, we correct insufficient data given in [6] with respect to (12) and add new ones.

When we refer to an explicit value of a density, it is an approximation by computer, unless specified.

1. Linear relation among roots

Let $f(x)$ be a polynomial f in (0) of degree n with roots α_i ($i = 1, \dots, n$) and suppose a linear relation (3). We may suppose that $m = 0$ in (3) to discuss the non-triviality of a linear relation, if necessary. Because, if $\text{tr}(f) = -a_{n-1} = 0$ holds, then taking traces, we have $nm = (\sum_i m_i)\text{tr}(f) = 0$, hence $m = 0$. Otherwise, we have $\sum M_i \alpha_i = 0$ for $M_i := m_i + m/a_{n-1}$. The non-triviality of (3) is unchanging by this operation.

Just to make sure, let us see a relation between global relations of roots α_i and local relations of roots r'_i of $f(x) \equiv 0 \pmod p$.

PROPOSITION 1. *Let $f(x)$ be a polynomial of degree n with roots $\alpha_1, \dots, \alpha_n$ and suppose that it has no multiple roots, and let $g_j(x_1, \dots, x_n)$ ($j = 1, \dots, t$) be polynomials in $\mathbb{Z}[x_1, \dots, x_n]$.*

If there are global relations $g_j(\alpha_1, \dots, \alpha_n) = 0$ ($j = 1, \dots, t$), then there are roots r'_i of $f(x) \equiv 0 \pmod p$ satisfy $g_j(r'_1, \dots, r'_n) \equiv 0 \pmod p$ ($j = 1, \dots, t$) for $p \in \text{Spl}(f)$.

Conversely, if roots r'_i of $f(x) \equiv 0 \pmod p$ satisfy $g_j(r'_1, \dots, r'_n) \equiv 0 \pmod p$ ($j = 1, \dots, t$) for infinitely many primes $p \in \text{Spl}(f)$, then there is a permutation σ such that $g_j(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)}) = 0$ ($j = 1, \dots, t$).

Proof. Put $K = \mathbb{Q}(f)$ and take a prime $p \in \text{Spl}(f)$. K is a Galois extension and a prime p is fully splitting in K .

First, assume $g_j(\alpha_1, \dots, \alpha_n) = 0$ ($j = 1, \dots, t$): For a prime ideal \mathfrak{p} of K over p , take a rational integer r'_i satisfying $\alpha_i \equiv r'_i \pmod{\mathfrak{p}}$, which implies $g_j(r'_1, \dots, r'_n) \equiv 0 \pmod{\mathfrak{p}}$, hence $g_j(r'_1, \dots, r'_n) \equiv 0 \pmod p$. Let us see that r'_1, \dots, r'_n are roots of $f(x) \equiv 0 \pmod p$. We see $0 = f(\alpha_i) \equiv f(r'_i) \pmod{\mathfrak{p}}$, hence $f(r'_i) \equiv 0 \pmod p$. If p is sufficiently large, then we see that $\alpha_i \not\equiv \alpha_j \pmod{\mathfrak{p}}$ for $i \neq j$, hence $r'_i \not\equiv r'_j \pmod p$, that is, r'_1, \dots, r'_n are all distinct roots of $f(x) \equiv 0 \pmod p$.

Conversely, suppose that there are infinitely many primes $p \in \text{Spl}(f)$ such that $g_j(r'_1, \dots, r'_n) \equiv 0 \pmod p$ ($j = 1, \dots, t$) for roots r'_i of $f(x) \equiv 0 \pmod p$. For such a prime, we fix any prime ideal \mathfrak{p} over p ; then there is a permutation σ_p of $\{1, \dots, n\}$ such that $\alpha_{\sigma_p(i)} \equiv r'_i \pmod{\mathfrak{p}}$ as above. We take a permutation σ satisfying $\sigma = \sigma_p$ for infinitely many primes $p \in \text{Spl}(f)$. For such infinitely many primes p , we see $g_j(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)}) \equiv g_j(r'_1, \dots, r'_n) \equiv 0 \pmod{\mathfrak{p}}$, hence a global relation $g_j(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)}) = 0$ ($j = 1, \dots, t$). \square

We apply this to linear equations $g_j := \sum_{i=1}^n m_{j,i}x_i - m_j$ ($j = 1, \dots, t$) (cf. (6), (7)). The following is a sufficient condition to a polynomial being without non-trivial linear relation.

PROPOSITION 2. *Let $f(x)$ be an irreducible polynomial of degree n . If n is a prime number p , or the Galois group $\text{Gal}(\mathbb{Q}(f)/\mathbb{Q})$ is S_n or A_n ($n \geq 6$) as a permutation group of roots of f , then f has only a trivial linear relation among roots.*

Proof. First, suppose that the degree of a polynomial f is a prime p , and let $\alpha_1, \dots, \alpha_p$ be roots of f , and suppose a linear relation (3). Adding a trivial relation $\sum \alpha_i = \text{tr}(f)$ to (3) if necessary, we may assume that $\sum m_i \neq 0$. The Galois group $\text{Gal}(\mathbb{Q}(f)/\mathbb{Q})$ acts faithfully on the set of all roots and contains an element σ of order p , hence we may assume that $(\sigma(\alpha_1), \dots, \sigma(\alpha_p)) = (\alpha_2, \dots, \alpha_p, \alpha_1)$. Then from the assumption (3) follows

$$\begin{pmatrix} m_1 & m_2 & \dots & m_p \\ m_p & m_1 & \dots & m_{p-1} \\ \vdots & & & \\ m_2 & m_3 & \dots & m_1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_p \end{pmatrix} = \begin{pmatrix} m \\ m \\ \vdots \\ m \end{pmatrix}.$$

Since α_i 's are not rational, the determinant of the coefficient matrix of entries m_i vanishes, hence we have

$$\prod_{i=0}^{p-1} (m_1 + \zeta^i m_2 + \zeta^{2i} m_3 + \dots + \zeta^{(p-1)i} m_p) = 0$$

for a primitive p th root $\zeta := \zeta_p$ of unity, using a formula for cyclic determinant. By the assumption $\sum m_i \neq 0$, we have

$$m_1 + \zeta^i m_2 + \zeta^{2i} m_3 + \dots + \zeta^{(p-1)i} m_p = 0 \quad \text{for some } i \ (0 < i < p),$$

which implies $m_1 = \dots = m_p$, that is, (3) is trivial, since ζ^i is still a primitive p th root of unity.

Next, suppose that the Galois group $\text{Gal}(\mathbb{Q}(f)/\mathbb{Q})$ is the symmetric group S_n . For any $1 \leq i < j \leq n$, there is an automorphism σ which induces a transposition of α_i and α_j . Hence we have

$$m = \left(\sum_{k \neq i, j} m_k \alpha_k \right) + m_i \alpha_i + m_j \alpha_j = \left(\sum_{k \neq i, j} m_k \alpha_k \right) + m_i \alpha_j + m_j \alpha_i,$$

which implies

$$m_i(\alpha_i - \alpha_j) = m_j(\alpha_i - \alpha_j).$$

By $\alpha_i \neq \alpha_j$, we have $m_i = m_j$, thus (3) is trivial.

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Finally, suppose that $\text{Gal}(\mathbb{Q}(f)/\mathbb{Q})$ is the alternative group A_n and that (3) is non-trivial. Let us show that coefficients m_1, \dots, m_n are mutually distinct, first. Suppose that $m_1 = m_2$; acting an even permutation $\alpha_1 \rightarrow \alpha_2 \rightarrow \alpha_3 \rightarrow \alpha_1$ on (3), we have

$$\begin{aligned} m_1\alpha_1 + m_2\alpha_2 + m_3\alpha_3 &= m - \sum_{i>3} m_i\alpha_i, \\ m_3\alpha_1 + m_1\alpha_2 + m_2\alpha_3 &= m - \sum_{i>3} m_i\alpha_i, \end{aligned}$$

which imply $(m_1 - m_3)(\alpha_1 - \alpha_3) = 0$, hence $m_2 = m_1 = m_3$. Considering other α_i ($i > 3$) instead of α_3 , we get $m_1 = m_2 = \dots = m_n$, which contradicts the non-triviality of (3). Thus coefficients m_i are mutually distinct.

Next, considering even permutations:

$$\{\alpha_1 \leftrightarrow \alpha_2, \alpha_3 \leftrightarrow \alpha_4\}, \quad \{\alpha_1 \leftrightarrow \alpha_3, \alpha_2 \leftrightarrow \alpha_4\}, \quad \{\alpha_1 \leftrightarrow \alpha_4, \alpha_2 \leftrightarrow \alpha_3\},$$

we get

$$\begin{cases} m_1\alpha_1 + m_2\alpha_2 + m_3\alpha_3 + m_4\alpha_4 = m - \sum_{i>4} m_i\alpha_i, \\ m_2\alpha_1 + m_1\alpha_2 + m_4\alpha_3 + m_3\alpha_4 = m - \sum_{i>4} m_i\alpha_i, \\ m_3\alpha_1 + m_4\alpha_2 + m_1\alpha_3 + m_2\alpha_4 = m - \sum_{i>4} m_i\alpha_i, \\ m_4\alpha_1 + m_3\alpha_2 + m_2\alpha_3 + m_1\alpha_4 = m - \sum_{i>4} m_i\alpha_i, \end{cases}$$

which imply

$$\begin{aligned} (m_1 - m_2)(\alpha_1 - \alpha_2) + (m_3 - m_4)(\alpha_3 - \alpha_4) &= 0, \\ (m_3 - m_4)(\alpha_1 - \alpha_2) + (m_1 - m_2)(\alpha_3 - \alpha_4) &= 0, \end{aligned}$$

hence $(\alpha_1 - \alpha_2)^2 = (\alpha_3 - \alpha_4)^2$. Similarly, we have $(\alpha_1 - \alpha_2)^2 = (\alpha_3 - \alpha_5)^2$. Therefore we get

$$\begin{aligned} 0 &= (\alpha_3 - \alpha_4)^2 - (\alpha_3 - \alpha_5)^2 \\ &= (\alpha_3 - \alpha_4 + \alpha_3 - \alpha_5)(\alpha_3 - \alpha_4 - \alpha_3 + \alpha_5) \\ &= (2\alpha_3 - \alpha_4 - \alpha_5)(-\alpha_4 + \alpha_5), \end{aligned}$$

i.e., $2\alpha_3 = \alpha_4 + \alpha_5$, similarly, $2\alpha_3 = \alpha_4 + \alpha_6$. Thus we have a contradiction

$$\alpha_5 = \alpha_6. \quad \square$$

PROPOSITION 3. *Let $f = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ be an irreducible polynomial. If there is a non-trivial linear relation among roots of f , then f is decomposable, that is $f(x) = g(h(x))$ for quadratic polynomials $g(x), h(x)$.*

Proof. Let $\alpha_1, \dots, \alpha_4$ be the roots of f . Let $G := \text{Gal}(\mathbb{Q}(f)/\mathbb{Q})$ be the Galois group; then it operates faithfully on a set $\{\alpha_1, \dots, \alpha_4\}$ and there is a subgroup of order 4 in G . Noting that for permutations:

$$\begin{aligned} \sigma = (1, 2), \quad \mu = (1, 3) &\Rightarrow \sigma\mu \neq \mu\sigma, \\ \sigma = (1, 2)(3, 4), \quad \mu = (2, 3) &\Rightarrow \sigma\mu \neq \mu\sigma, \\ \sigma = (1, 2)(3, 4), \quad \mu = (1, 3)(2, 4) &\Rightarrow \sigma\mu = \mu\sigma, \end{aligned}$$

we see that:

- (i) there is a cyclic permutation σ in G so that

$$\sigma : (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \rightarrow (\alpha_2, \alpha_3, \alpha_4, \alpha_1),$$

- (ii) there are permutations σ_1, σ_2 in G so that

$$\sigma_1 : (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \rightarrow (\alpha_2, \alpha_1, \alpha_4, \alpha_3),$$

$$\sigma_2 : (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \rightarrow (\alpha_3, \alpha_4, \alpha_1, \alpha_2) \quad \text{or}$$

- (iii) there are permutations σ_1, σ_2 such that σ_1 (resp. σ_2) is a transposition of α_1 and α_2 (α_3 and α_4), respectively.

Suppose that (3) is non-trivial, that is $\exists m_i \neq \exists m_j$, and if $a_3 = 0$ happens, then considering $f(x-1)$ instead of $f(x)$, we may assume that $a_3 \neq 0$ and furthermore $\sum m_i \neq 0$, adding a trivial relation.

First, let us consider

Case (i). By linear equations $\sum m_i \sigma^j(\alpha_i) = m$ ($j = 0, 1, 2, 3$), we have

$$\begin{pmatrix} m_1 & m_2 & m_3 & m_4 \\ m_4 & m_1 & m_2 & m_3 \\ m_3 & m_4 & m_1 & m_2 \\ m_2 & m_3 & m_4 & m_1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_2 & \alpha_3 & \alpha_4 & \alpha_1 \\ \alpha_3 & \alpha_4 & \alpha_1 & \alpha_2 \\ \alpha_4 & \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix} = \begin{pmatrix} m \\ m \\ m \\ m \end{pmatrix}.$$

Since α_i 's are irrational, the determinant of coefficient matrix on m_i vanishes, i.e., $\prod_{i=0}^3 (m_1 + \zeta^i m_2 + \zeta^{2i} m_3 + \zeta^{3i} m_4) = 0$ for a primitive fourth root $\zeta := \zeta_4$ of unity. By the assumption $\sum m_i \neq 0$, we have

$$m_1 + \zeta^i m_2 + \zeta^{2i} m_3 + \zeta^{3i} m_4 = 0 \quad \text{for some } i = 1, 2, 3,$$

hence

(i.1) $m_1 - m_3 = m_2 - m_4 = 0$ in the case of $i = 1, 3$ or

(i.2) $m_1 - m_2 + m_3 - m_4 = 0$ in the case of $i = 2$.

Case of (i.1), i.e., $m_1 = m_3, m_2 = m_4$:

The difference of the first row and the second row gives

$$(m_1 - m_2)(\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4) = 0.$$

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If $m_1 = m_2$ holds, we have a contradiction $m_1 = \dots = m_4$. It implies $\alpha_1 + \alpha_3 = \alpha_2 + \alpha_4 = -a_3/2$, hence $f(x) = (x^2 + a_3x/2 + \alpha_1\alpha_3)(x^2 + a_3x/2 + \alpha_2\alpha_4)$ is a polynomial in $x^2 + a_3x/2$, that is f is decomposable.

Case of (i.2), hence $m_1 + m_3 = m_2 + m_4$:

It is easy to see that

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_2 & \alpha_3 & \alpha_4 & \alpha_1 \\ \alpha_3 & \alpha_4 & \alpha_1 & \alpha_2 \\ \alpha_4 & \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix} \begin{pmatrix} m_1 - m_2 \\ m_2 - m_3 \\ m_3 - m_4 \\ m_4 - m_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

By non-triviality $(m_1 - m_2, \dots, m_4 - m_1) \neq (0, \dots, 0)$, the cyclic determinant of coefficients matrix vanishes, i.e.,

$$a_3(\alpha_1 + \zeta\alpha_2 - \alpha_3 - \zeta\alpha_4)(\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4)(\alpha_1 - \zeta\alpha_2 - \alpha_3 + \zeta\alpha_4) = 0.$$

(i.2.1) Suppose $\alpha_1 + \zeta\alpha_2 - \alpha_3 - \zeta\alpha_4 = 0$, i.e., $\alpha_1 - \alpha_3 = -\zeta(\alpha_2 - \alpha_4)$. By equations $\sum m_i\alpha_i = m$ and (by acting σ^2 on it) $m_1\alpha_3 + m_2\alpha_4 + m_3\alpha_1 + m_4\alpha_2 = m$, we have

$$(m_1 - m_3)(\alpha_1 - \alpha_3) + (m_2 - m_4)(\alpha_2 - \alpha_4) = 0,$$

hence $((m_1 - m_3)(-\zeta) + m_2 - m_4)(\alpha_2 - \alpha_4) = 0$. Therefore we get $m_1 = m_3$, $m_2 = m_4$ and so a contradiction $m_1 = m_2 = m_3 = m_4$ by the assumption $m_1 + m_3 = m_2 + m_4$.

(i.2.2) Suppose that $\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 = 0$; it implies $\alpha_1 + \alpha_3 = \alpha_2 + \alpha_4$, which implies that f is decomposable as above.

(i.2.3) The case of $\alpha_1 - \zeta\alpha_2 - \alpha_3 + \zeta\alpha_4 = 0$ is similar to (i.2.1).

Thus we have shown that in the case of (i), f is decomposable.

Case (ii). The second case gives the following equations:

$$m_1\alpha_1 + m_2\alpha_2 + m_3\alpha_3 + m_4\alpha_4 = m, \tag{13}$$

$$m_2\alpha_1 + m_1\alpha_2 + m_4\alpha_3 + m_3\alpha_4 = m, \tag{14}$$

$$m_3\alpha_1 + m_4\alpha_2 + m_1\alpha_3 + m_2\alpha_4 = m, \tag{15}$$

$$m_4\alpha_1 + m_3\alpha_2 + m_2\alpha_3 + m_1\alpha_4 = m. \tag{16}$$

(13), (14) (resp. (15), (16)) give

$$(m_1 + m_2)(\alpha_1 + \alpha_2) + (m_3 + m_4)(\alpha_3 + \alpha_4) = 2m,$$

$$(m_3 + m_4)(\alpha_1 + \alpha_2) + (m_1 + m_2)(\alpha_3 + \alpha_4) = 2m,$$

hence if $m_1 + m_2 \neq m_3 + m_4$ holds, then $\alpha_1 + \alpha_2 = \alpha_3 + \alpha_4$ follows, i.e., f is decomposable. Hence we may suppose that $m_1 + m_2 = m_3 + m_4$.

Similarly, using (13), (16) (resp. (14), (15)), we may suppose $m_1 + m_4 = m_2 + m_3$, and $m_1 + m_3 = m_2 + m_4$, using (13), (15) (resp. (14), (16)). These give a contradiction $m_1 = m_2 = m_3 = m_4$.

Finally, let us consider:

Case (iii). Acting σ_1, σ_2 on $\sum m_i \alpha_i = m$, we have

$$\begin{aligned} m_1 \alpha_1 + m_2 \alpha_2 + m_3 \alpha_3 + m_4 \alpha_4 &= m, \\ m_1 \alpha_2 + m_2 \alpha_1 + m_3 \alpha_3 + m_4 \alpha_4 &= m, \\ m_1 \alpha_1 + m_2 \alpha_2 + m_3 \alpha_4 + m_4 \alpha_3 &= m, \end{aligned}$$

which implies

$$(m_1 - m_2)(\alpha_1 - \alpha_2) = (m_3 - m_4)(\alpha_3 - \alpha_4) = 0.$$

Since α_i 's are distinct, we have

$$m_1 = m_2, \quad m_3 = m_4.$$

By $m_1 \neq m_3$, the equations

$$(\alpha_1 + \alpha_2) + (\alpha_3 + \alpha_4) = -a_3, \quad m_1(\alpha_1 + \alpha_2) + m_3(\alpha_3 + \alpha_4) = m$$

imply

$$b_1 := \alpha_1 + \alpha_2 \in \mathbb{Q}, \quad b_2 := \alpha_3 + \alpha_4 \in \mathbb{Q}.$$

Therefore

$$f(x) = (x^2 - (\alpha_1 + \alpha_2)x + \alpha_1 \alpha_2)(x^2 - (\alpha_3 + \alpha_4)x + \alpha_3 \alpha_4)$$

is equal to

$$x^4 + a_3 x^3 + (\alpha_3 \alpha_4 + b_1 b_2 + \alpha_1 \alpha_2) x^2 - (b_1 \alpha_3 \alpha_4 + b_2 \alpha_1 \alpha_2) x + f(0),$$

hence we have

$$\alpha_1 \alpha_2 + \alpha_3 \alpha_4 = a_2 - b_1 b_2, \quad b_2 \alpha_1 \alpha_2 + b_1 \alpha_3 \alpha_4 = -a_1.$$

If $b_1 \neq b_2$ holds, then solving them, we have $\alpha_1 \alpha_2, \alpha_3 \alpha_4 \in \mathbb{Q}$, which implies that f is reducible. Thus we have $b_1 = b_2$ and then f is a polynomial in $x^2 - b_1 x$, that is decomposable.

The following is an easy corollary.

COROLLARY 1. *Let f be a polynomial of degree less than 6 and suppose that f has a non-trivial linear relation among roots. Then f is reducible or decomposable.*

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PROPOSITION 4. *Let $f = (x^2+ax)^2+b(x^2+ax)+c$ ($a, b, c \in \mathbb{Q}$) be an irreducible and decomposable polynomial, and put*

$$x^2 + bx + c = (x - \beta_1)(x - \beta_2), \quad x^2 + ax - \beta_i = (x - \alpha_{i,1})(x - \alpha_{i,2}).$$

Then equations $\alpha_{i,1} + \alpha_{i,2} = -a$ ($i = 1, 2$) are a basis of linear relations (3) among roots of f .

Proof. Let

$$m_{1,1}\alpha_{1,1} + m_{1,2}\alpha_{1,2} + m_{2,1}\alpha_{2,1} + m_{2,2}\alpha_{2,2} = m \quad (m_{i,j}, m \in \mathbb{Q})$$

be a linear relation. Using $\alpha_{i,1} + \alpha_{i,2} = -a$, we may suppose

$$m_{1,2}\alpha_{1,2} + m_{2,2}\alpha_{2,2} = m.$$

We have only to show $m_{2,2} = 0$, which implies $m_{1,2} = 0$, hence we complete the proof. Suppose that $m_{2,2} \neq 0$, and dividing $m_{2,2}$, we may assume

$$\alpha_{2,2} = m_1\alpha_{1,2} + m_2 \quad (m_1, m_2 \in \mathbb{Q}).$$

Hence $\alpha_{1,2}$ is a root of $g(x) = x^2 + ax - \beta_1$ and $h(x) = (m_1x + m_2)^2 + a(m_1x + m_2) - \beta_2 = (m_1x + m_2)^2 + a(m_1x + m_2) + b + \beta_1$, which are polynomials over a quadratic field $\mathbb{Q}(\beta_1)$. Since $g(x)$ is irreducible in $\mathbb{Q}(\beta_1)[x]$, we have $h(x) = m_1^2g(x)$, hence comparing constant terms $m_2^2 + am_2 + b + \beta_1 = -m_1^2\beta_1$. Thus we find a contradiction that β_1 is rational. \square

Let us give $\mathfrak{D}(f, \sigma)$ explicitly for a polynomial of degree 4. In case that f is irreducible and indecomposable, there is only a trivial relation, hence

$$\mathfrak{D}(f, \sigma) = \hat{\mathfrak{D}}_n.$$

In case that f is irreducible and decomposable, by using Proposition 4, we find, with $\dim \mathfrak{D}(f, \sigma) = 2$

$$\begin{aligned} \mathfrak{D}(f, \sigma) &= \left\{ (x_1, x_2, x_3, x_4) \left| \begin{array}{l} 0 \leq x_1 \leq x_2 \leq x_3 \leq x_4 \leq 1, \\ x_{\sigma(1)} + x_{\sigma(2)} \in \mathbb{Z}, x_{\sigma(3)} + x_{\sigma(4)} \in \mathbb{Z} \end{array} \right. \right\} \\ &= \{(x_1, x_2, 1 - x_2, 1 - x_1) \mid 0 \leq x_1 \leq x_2 \leq 1 - x_2 \leq 1 - x_1 < 1\}, \end{aligned}$$

which is parametrized by $0 \leq x_1 \leq x_2 \leq 1/2$. The dimension of a domain corresponding to $\{\sigma(1), \sigma(2)\} \neq \{1, 4\}, \{2, 3\}$ is less than 2. To confirm Expectation 1', i.e., (8), what we can do now is an approximate calculation by computer. The right hand of (8) is a ratio, hence we do not need to look for volumes themselves. By using a projection to (x_1, x_2) -plane, we can approximate the right hand of (8) by the Monte Carlo method.

In case that f is a product of two irreducible quadratic polynomials with distinct fundamental discriminants, relations are similar to the previous case and hence $\mathfrak{D}(f, \sigma)$ is the same.

In case that two irreducible quadratic factors have the same fundamental discriminant D , e.g., $f(x) = (x^2 - D)((x - 1)^2 - 4D)$, put

$$\alpha_1 = \sqrt{D}, \quad \alpha_2 = -\sqrt{D}, \quad \alpha_3 = 1 + 2\sqrt{D}, \quad \alpha_4 = 1 - 2\sqrt{D}.$$

A basis of linear relations (3) among roots are

$$\alpha_1 + \alpha_2 = 0, \quad 2\alpha_1 - \alpha_3 = -1, \quad 2\alpha_1 + \alpha_4 = 1,$$

hence hyperplanes necessary in $[0, 1]^4$ in question are

$$x_1 + x_2, \quad 2x_1 - x_3, \quad 2x_1 + x_4 \in \mathbb{Z}$$

and its permutations of indexes.

Thus we see that $\mathfrak{D}(f, \sigma)$ of dim 1 is one of

$$\begin{aligned} & \{(x, 2x, 1 - 2x, 1 - x) \mid 0 \leq x < 1/4\} \text{ for } \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}, \\ & \{(x, 1 - 2x, 2x, 1 - x) \mid 1/4 \leq x < 1/3\} \text{ for } \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}, \\ & \{(1 - 2x, x, 1 - x, 2x) \mid 1/3 \leq x < 1/2\} \text{ for } \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}. \end{aligned}$$

Let us see that Expectation 1'' is true in this case. It is easy to see that lengths are, in order $\sqrt{10}/4, \sqrt{10}/12, \sqrt{10}/6$. For densities of $\text{Spl}(f, \sigma)$, we invoke [1], [9], that is for an irreducible quadratic polynomial, $r_1/p, r_2/p$ are equi-distributed. Let r be a root of $r^2 \equiv D \pmod{p}$ with $0 < r < p/2$. Then other roots of $f(x) \equiv 0 \pmod{p}$ are $-r, 1 \pm 2r \pmod{p}$, and it is easy to see except for finitely many primes local roots r_1, \dots, r_4 are in order

$$(r_1, \dots, r_4) = \begin{cases} (r, 1 + 2r, p + 1 - 2r, p - r) & \text{if } r/p \in [0, 1/4), \\ (r, p + 1 - 2r, 1 + 2r, p - r) & \text{if } r/p \in [1/4, 1/3), \\ (p + 1 - 2r, r, p - r, 1 + 2r) & \text{if } r/p \in [1/3, 1/2). \end{cases}$$

The uniformity of r/p implies that densities are proportional to

$$1/4, 1/3 - 1/4 = 1/12, 1/2 - 1/3 = 1/6.$$

Hence the constant c in Expectation 1'' is independent of σ .

In case that a polynomial f is a product of irreducible quadratic polynomials with the same fundamental discriminant, Expectation 1', 1'' should be reduced to [1], [9].

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In case that $f(x) = \prod_{i=1}^a (x - \alpha_i) \cdot g(x)$, where α_i 's are all rational integer roots of f with $\alpha_1 \leq \dots \alpha_t < 0 \leq \alpha_{t+1} \leq \dots \alpha_a$. It is easy to see that

$$r_1 = \alpha_{t+1}, \dots, r_{a-t} = \alpha_a, r_{n+1-t} = p + \alpha_t, \dots, r_n = p + \alpha_1$$

for any prime $p \in \text{Spl}(f)$ except finitely many primes. Linear relations among roots are reduced to relations of g . Therefore the projection of $\mathfrak{D}(f, \sigma)$ to a hyperplane defined by

$$x_1 = \dots = x_a = 0 \quad \text{with} \quad \sigma(i) = i \quad (1 \leq i \leq a)$$

is $\mathfrak{D}(g, \sigma_{\{a+1, \dots, n\}})$ modulo a lower dimensional set, hence the problem is reduced to that of a polynomial $g(x)$ as expected.

Before we discuss the case of degree six, let us introduce a notion "type". For an irreducible polynomial f of degree 6, we define its type number 2, 3 temporarily as follows:

Denote a root of f by α . The type number of f is 2 if $\mathbb{Q}(\alpha)$ contains a quadratic subfield M_2 such that the trace of α to M_2 is rational.

The type number of f is 3 if $\mathbb{Q}(\alpha)$ contains a cubic subfield M_3 such that the discriminant D of the monic minimal quadratic polynomial $g_2(x)$ of α over M_3 is rational.

We note that the type number is independent of the choice of a root α of f , and type numbers of $f(x), f(x + a)$ ($a \in \mathbb{Q}$) are equal.

PROPOSITION 5. *Let $f = x^6 + a_5x^5 + \dots + a_0$ be an irreducible polynomial of degree 6 with roots $\alpha_1, \dots, \alpha_6$ and suppose that there is a non-trivial relation (3). Then we have:*

- (i) *The extension degree $[\mathbb{Q}(f) : \mathbb{Q}]$ is not divisible by 5.*
- (ii) *If $\mathbb{Q}(\alpha_1)$ is an abelian extension, then f is either of type 2 or 3, or decomposable.*
- (iii) *If $\mathbb{Q}(\alpha_1)$ is an S_3 Galois extension, then f is either of type 2 or 3, decomposable or there are a rational number c , two distinct roots α, α' of f , and a cubic subfield M_3 such that $\text{tr}_{K/M_3}(\alpha) + c \cdot \text{tr}_{K/M_3}(\alpha') \in \mathbb{Q}$ and α, α' are not conjugate over M_3 .*

Proof. Let (3) be a non-trivial relation.

(i) Suppose that the extension degree $[\mathbb{Q}(f) : \mathbb{Q}]$ is divisible by 5; then there is an automorphism σ of order 5 in $\text{Gal}(\mathbb{Q}(f)/\mathbb{Q})$, which acts faithfully on a set $\{\alpha_1, \dots, \alpha_6\}$, hence we may assume that

$$\sigma(\alpha_i) = \alpha_{i+1} \quad (i = 1, 2, 3, 4), \quad \sigma(\alpha_5) = \alpha_1 \quad \text{and} \quad \sigma(\alpha_6) = \alpha_6.$$

Adding a trivial relation, we may assume $\sum_{i=1}^5 m_i \neq 0$

$$\begin{pmatrix} m_1 & m_2 & m_3 & m_4 & m_5 \\ m_5 & m_1 & m_2 & m_3 & m_4 \\ m_4 & m_5 & m_1 & m_2 & m_3 \\ m_3 & m_4 & m_5 & m_1 & m_2 \\ m_2 & m_3 & m_4 & m_5 & m_1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{pmatrix} = (m - m_6 \alpha_6) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

If the determinant of the coefficient matrix does not vanish, then $\alpha_1, \dots, \alpha_5$ are in $\mathbb{Q}(\alpha_6)$, hence $\mathbb{Q}(\alpha_6) = \mathbb{Q}(\{\alpha_1, \dots, \alpha_6\}) = \mathbb{Q}(f)$ is a Galois extension of degree 6. This contradicts the assumption. Thus the determinant vanishes, hence there is a fifth root ζ of unity satisfying $\sum_{i=1}^5 m_i \zeta^{i-1} = 0$, i.e., $m_1 = \dots = m_5$. Thus (3) implies $m_1(\text{tr}(f) - \alpha_6) + m_6 \alpha_6 = m$, which implies $m_1 = m_6$, that is (3) is a trivial relation, contradicting the assumption. This completes the proof of (i). \square

(ii) Suppose that $\mathbb{Q}(\alpha_1)$ is an abelian extension, hence the Galois group is generated by an automorphism σ of order 6. We may assume that $\sigma(\alpha_i) = \alpha_{i+1}$, where $\alpha_j = \alpha_k$ if $j \equiv k \pmod{6}$. Otherwise, there is a fixed root α_i by σ . Let $\zeta = (1 + \sqrt{-3})/2$ be a primitive sixth root of unity, which satisfies $\zeta^2 - \zeta + 1 = 0$ and $\zeta^3 = -1$, and consider central idempotents

$$\chi_i = 6^{-1} \sum_{j \pmod{6}} \zeta^{ij} \sigma^j,$$

which satisfies

$$\sum_{i \pmod{6}} \chi_i = 1, \chi_i \chi_j = \delta_{i,j} \chi_j.$$

The equation (3) is equivalent to $\chi_i(m) = \chi_i(\sum_k m_k \alpha_k)$, hence

$$0 = \chi_i(m) = \left(\sum_{k \pmod{6}} \zeta^{-ik} m_k \right) \left(\sum_{l \pmod{6}} \zeta^{il} \alpha_l \right) \quad (i \not\equiv 0 \pmod{6}),$$

using $\chi_i(\alpha_k) = 6^{-1} \zeta^{-ki} \sum_{l \pmod{6}} \zeta^{li} \alpha_l$. Thus for

$$i \not\equiv 0 \pmod{6}, \quad \sum_{k \pmod{6}} \zeta^{ik} m_k = 0 \quad \text{or} \quad \sum_{l \pmod{6}} \zeta^{il} \alpha_l = 0 \quad \text{occurs.}$$

If $\sum_{l \pmod{6}} \zeta^{il} \alpha_l = 0$ holds for every $i = 1, \dots, 5$, then we have

$$\begin{aligned} 0 &= \sum_{i=1}^5 \left(\sum_{l \pmod{6}} \zeta^{il} \alpha_l \right) = 5\alpha_6 + \sum_{l=1}^5 \left(\sum_{i=1}^5 \zeta^{il} \right) \alpha_l \\ &= 5\alpha_6 - \sum_{l=1}^5 \alpha_l = 6\alpha_6 - \text{tr}(f), \end{aligned}$$

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which implies a contradiction $\alpha_6 \in \mathbb{Q}$. Hence, we have $\sum_{l \bmod 6} \zeta^{il} \alpha_l \neq 0$ for some $i \not\equiv 0 \pmod 6$, i.e.,

$$\zeta^{-i} \sum_{k=1}^6 \zeta^{ik} m_k = m_1 + \zeta^i m_2 + \zeta^{2i} m_3 + \cdots + \zeta^{5i} m_6 = 0. \quad (17)$$

By expressing the above as a linear form of ζ and 1, the equation (17) is

$$\begin{cases} m_1 - m_3 - m_4 + m_6 = m_2 + m_3 - m_5 - m_6 = 0 & (i = 1, 5), \\ m_1 - m_2 + m_4 - m_5 = m_2 - m_3 + m_5 - m_6 = 0 & (i = 2, 4), \\ m_1 + m_3 + m_5 = m_2 + m_4 + m_6 & (i = 3). \end{cases}$$

Suppose that (17) is true for both $i = 1$ and $i = 2$: Then we have

$$m_1 = m_3 = m_5 \quad \text{and} \quad m_2 = m_4 = m_6.$$

If (17) is true for $i = 3$ moreover, then (3) is a trivial relation, which is a contradiction. Thus (17) is false for $i = 3$, hence $\sum_{l \bmod 6} \zeta^{3l} \alpha_l = 0$ follows, that is $\alpha_1 + \alpha_3 + \alpha_5 = \alpha_2 + \alpha_4 + \alpha_6$. Putting $g = (x - \alpha_1)(x - \alpha_3)(x - \alpha_5)$ and $h = (x - \alpha_2)(x - \alpha_4)(x - \alpha_6)$, coefficients of polynomials g, h are in a quadratic subfield M_2 fixed by σ^2 and their second leading coefficient $\alpha_1 + \alpha_3 + \alpha_5 = \alpha_2 + \alpha_4 + \alpha_6 = \text{tr}(f)/2$ is rational, hence f is of type 2.

Suppose that (17) is true for $i = 1$, but false for $i = 2, 4$: Hence we have $\sum_{l \bmod 6} \zeta^{il} \alpha_l = 0$ for $i = 2, 4$, which implies two equations

$$\begin{aligned} (\alpha_1 - \alpha_2 + \alpha_4 - \alpha_5)\sqrt{-3} + (-\alpha_1 - \alpha_2 + 2\alpha_3 - \alpha_4 - \alpha_5 + 2\alpha_6) &= 0, \\ -(\alpha_1 - \alpha_2 + \alpha_4 - \alpha_5)\sqrt{-3} + (-\alpha_1 - \alpha_2 + 2\alpha_3 - \alpha_4 - \alpha_5 + 2\alpha_6) &= 0, \end{aligned}$$

hence $\alpha_1 + \alpha_4 = \alpha_2 + \alpha_5 = \alpha_3 + \alpha_6$. Thus f is a polynomial in $x^2 + (\alpha_1 + \alpha_4)x$, that is decomposable.

Finally, we assume that (17) is false for $i = 1$ hence for $i = 5$; similarly to the above, we have $\alpha_1 + \alpha_2 - \alpha_4 - \alpha_5 = 0$, i.e., $\alpha_1 - \alpha_4 = \alpha_5 - \alpha_2$. Hence we see that the discriminant $(\alpha_1 - \alpha_4)^2$ of a polynomial $(x - \alpha_1)(x - \alpha_4)$ fixed by σ^3 is fixed by σ , hence rational, that is f is of type 3.

(iii) Suppose that $K = \mathbb{Q}(\alpha_1)$ is an S_3 -extension: Then there are automorphisms σ, μ and a numbering $\beta_{i,j}$ ($i = 1, 2, j = 1, 2, 3$) of roots α_i of f such that $\sigma^3 = \eta^2 = 1, \eta\sigma\eta = \sigma^2$ and

$$\begin{aligned} \sigma(\beta_{i,1}) &= \beta_{i,2}, \sigma(\beta_{i,2}) = \beta_{i,3}, \sigma(\beta_{i,3}) = \beta_{i,1} \quad (i = 1, 2) \\ \eta(\beta_{1,1}) &= \beta_{2,1}, \eta(\beta_{1,2}) = \beta_{2,3}, \eta(\beta_{1,3}) = \beta_{2,2}, \end{aligned}$$

noting that σ, η have no fixed root. We divide a proof to several parts.

LEMMA 1.

- (1) If $\sum_j \beta_{1,j}$ is rational, then f is of type 2.
- (2) If one of $(\beta_{1,1} - \beta_{2,1})^2, (\beta_{1,2} - \beta_{2,3})^2, (\beta_{1,3} - \beta_{2,2})^2$ is fixed by σ , then f is of type 3.

Proof.

Suppose that $\sum_i \beta_{1,j} \in \mathbb{Q}$, and decompose f as $f = gh$ with $g = \prod(x - \beta_{1,j})$, $h = \prod(x - \beta_{2,j}) \in M_2[x]$, where M_2 is a quadratic subfield fixed by σ . Therefore f is of type 2. Since polynomials $(x - \beta_{1,1})(x - \beta_{2,1})$, $(x - \beta_{1,2})(x - \beta_{2,3})$ and $(x - \beta_{1,3})(x - \beta_{2,2})$ are fixed by η , their discriminants are also fixed by η . Therefore, if the discriminant is fixed by σ , it is a rational number, that is, f is of type 3. \square

LEMMA 2. *If there is a non-trivial relation*

$$\sum_j m_{1,j} \beta_{1,j} + \sum_j m_{2,j} \beta_{2,j} = m \quad (m_{i,j}, m \in \mathbb{Q}), \quad (18)$$

then f is of type 2, or there are rational numbers m_1, m_2, m_3 with $m_i \neq m_j$ for some i, j and $m_1 + m_2 + m_3 = 0$ such that

$$m_1(\beta_{1,1} + \beta_{2,1}) + m_2(\beta_{1,2} + \beta_{2,3}) + m_3(\beta_{1,3} + \beta_{2,2}) = 0. \quad (19)$$

Proof. We may suppose that $m = 0$ in (18) by the remark at the beginning of this section, and acting $id, \sigma, \sigma^2, \eta, \eta\sigma, \eta\sigma^2$ in order, we have

$$m_{1,1}\beta_{1,1} + m_{1,2}\beta_{1,2} + m_{1,3}\beta_{1,3} + m_{2,1}\beta_{2,1} + m_{2,2}\beta_{2,2} + m_{2,3}\beta_{2,3} = 0, \quad (20)$$

$$m_{1,3}\beta_{1,1} + m_{1,1}\beta_{1,2} + m_{1,2}\beta_{1,3} + m_{2,3}\beta_{2,1} + m_{2,1}\beta_{2,2} + m_{2,2}\beta_{2,3} = 0, \quad (21)$$

$$m_{1,2}\beta_{1,1} + m_{1,3}\beta_{1,2} + m_{1,1}\beta_{1,3} + m_{2,2}\beta_{2,1} + m_{2,3}\beta_{2,2} + m_{2,1}\beta_{2,3} = 0, \quad (22)$$

$$m_{2,1}\beta_{1,1} + m_{2,3}\beta_{1,2} + m_{2,2}\beta_{1,3} + m_{1,1}\beta_{2,1} + m_{1,3}\beta_{2,2} + m_{1,2}\beta_{2,3} = 0, \quad (23)$$

$$m_{2,3}\beta_{1,1} + m_{2,2}\beta_{1,2} + m_{2,1}\beta_{1,3} + m_{1,3}\beta_{2,1} + m_{1,2}\beta_{2,2} + m_{1,1}\beta_{2,3} = 0, \quad (24)$$

$$m_{2,2}\beta_{1,1} + m_{2,1}\beta_{1,2} + m_{2,3}\beta_{1,3} + m_{1,2}\beta_{2,1} + m_{1,1}\beta_{2,2} + m_{1,3}\beta_{2,3} = 0. \quad (25)$$

Equations (20)+(23), (21)+(24), (22)+(25) are

$$\begin{pmatrix} m_{1,1} + m_{2,1} & m_{1,2} + m_{2,3} & m_{1,3} + m_{2,2} \\ m_{1,3} + m_{2,3} & m_{1,1} + m_{2,2} & m_{1,2} + m_{2,1} \\ m_{1,2} + m_{2,2} & m_{1,3} + m_{2,1} & m_{1,1} + m_{2,3} \end{pmatrix} \begin{pmatrix} \beta_{1,1} + \beta_{2,1} \\ \beta_{1,2} + \beta_{2,3} \\ \beta_{1,3} + \beta_{2,2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (26)$$

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If entries in each row of the coefficient matrix in (26) are the same, that is

$$\begin{cases} m_{1,1} + m_{2,1} = m_{1,2} + m_{2,3} = m_{1,3} + m_{2,2} := a \text{ (say)}, \\ m_{1,3} + m_{2,3} = m_{1,1} + m_{2,2} = m_{1,2} + m_{2,1} := b, \\ m_{1,2} + m_{2,2} = m_{1,3} + m_{2,1} = m_{1,1} + m_{2,3} := c, \end{cases}$$

then we have

$$\begin{cases} m_{2,3} = a - m_{1,2} = b - m_{1,3} = c - m_{1,1}, \\ m_{2,2} = a - m_{1,3} = b - m_{1,1} = c - m_{1,2}, \\ m_{2,1} = a - m_{1,1} = b - m_{1,2} = c - m_{1,3}, \end{cases}$$

hence

$$m_{1,3} = m_{1,1} + b - c = m_{1,1} + a - b = m_{1,1} - a + c,$$

which imply

$$b - c = a - b = -a + c, \quad \text{hence} \quad a = b = c.$$

Therefore, we get $m_{1,1} = m_{1,2} = m_{1,3}$ and $m_{2,1} = m_{2,2} = m_{2,3}$. The non-triviality of (20) implies $m_{1,1} \neq m_{2,1}$, hence comparing it with trivial relation $\sum_i \beta_{1,i} + \sum_i \beta_{2,i} = \text{tr}(f)$, we see $\sum_i \beta_{1,i} \in \mathbb{Q}$. By lemma 1, f is of type 2.

If there are distinct entries in some row of the coefficient matrix in (26), then putting entries of the row by m_1, m_2, m_3 , we have

$$m_1(\beta_{1,1} + \beta_{2,1}) + m_2(\beta_{1,2} + \beta_{2,3}) + m_3(\beta_{1,3} + \beta_{2,2}) = 0.$$

If $\text{tr}(f) \neq 0$, then taking the trace, we have $(m_1 + m_2 + m_3)\text{tr}(f) = 0$, which implies $m_1 + m_2 + m_3 = 0$. If $\text{tr}(f) = 0$, then we have only to replace m_i by $m_i - (m_1 + m_2 + m_3)/3$.

Thus we have, by (19)

$$m_1(\beta_{1,1} + \beta_{2,1} - \beta_{1,3} - \beta_{2,2}) + m_2(\beta_{1,2} + \beta_{2,3} - \beta_{1,3} - \beta_{2,2}) = 0,$$

where $m_1 = m_2 = 0$ does not hold.

We divide the proof to several cases:

(I) Case of $m_1 = 0$ and $m_2 \neq 0$: We have $\beta_{1,2} + \beta_{2,3} - \beta_{1,3} - \beta_{2,2} = 0$, and $\beta_{1,3} + \beta_{2,1} - \beta_{1,1} - \beta_{2,3} = 0$, acting σ . Therefore we find that $\beta_{1,1} - \beta_{2,1} = \beta_{1,3} - \beta_{2,3} = \beta_{1,2} - \beta_{2,2}$, which is σ -invariant, hence by Lemma 1, f is of type 3.

(II) Case of $m_1 \neq 0$ and $m_2 = 0$: We have $\sigma(\beta_{1,1} + \beta_{2,1} - \beta_{1,3} - \beta_{2,2}) = \beta_{1,2} + \beta_{2,2} - \beta_{1,1} - \beta_{2,3} = 0$, and $\beta_{2,3} + \beta_{1,3} - \beta_{2,1} - \beta_{1,2} = 0$, acting η . Hence, we find that $\sigma(\beta_{1,2} - \beta_{2,3})^2 - (\beta_{1,2} - \beta_{2,3})^2 = (\beta_{1,3} - \beta_{2,1} + \beta_{1,2} - \beta_{2,3})(\beta_{1,3} - \beta_{2,1} - \beta_{1,2} + \beta_{2,3}) = 0$ and so by Lemma 1, f is of type 3.

(III) Case of $m_1 m_2 \neq 0$: Dividing by m_2 , we may suppose $m_2 = 1$, that is

$$m_1(\beta_{1,1} + \beta_{2,1} - \beta_{1,3} - \beta_{2,2}) + (\beta_{1,2} + \beta_{2,3} - \beta_{1,3} - \beta_{2,2}) = 0 \quad (27)$$

Suppose that $m_1 = 1$: Adding $3(\beta_{1,3} + \beta_{2,2})$ to the above, we have $\text{tr}(f) = 3(\beta_{1,3} + \beta_{2,2})$. Acting σ , we have $\beta_{1,3} + \beta_{2,2} = \beta_{1,1} + \beta_{2,3} = \beta_{1,2} + \beta_{2,1} = \text{tr}(f)/3$. Therefore, $f = (x - \beta_{1,3})(x - \beta_{2,2}) \cdot (x - \beta_{1,1})(x - \beta_{2,3}) \cdot (x - \beta_{1,2})(x - \beta_{2,1})$ is a polynomial in $x^2 - \text{tr}(f)/3 \cdot x$, that is decomposable.

Finally, we assume that $m_1 \neq 1$. Substituting $\beta_{1,2} + \beta_{2,3} = -(\beta_{1,1} + \beta_{2,1} + \beta_{1,3} + \beta_{2,2}) + \text{tr}(f)$ to (27), we get

$$(m_1 - 1)(\beta_{1,1} + \beta_{2,1}) - (m_1 + 2)(\beta_{1,3} + \beta_{2,2}) = -\text{tr}(f).$$

By denoting a cubic subfield fixed by η by M_3 , it means

$$(m_1 - 1)\text{tr}_{K/M_3}(\beta_{1,1}) - (m_1 + 2)\text{tr}_{K/M_3}(\beta_{1,3}) = -\text{tr}(f),$$

which completes the proof, putting

$$c = -(m_1 + 2)/(m_1 - 1), \quad \alpha = \beta_{1,1}, \quad \alpha' = \beta_{1,3}. \quad \square$$

In [3], there are examples of a polynomial of type 2, 3, but at that time the author did not recognize any S_3 -type polynomial satisfying the last condition in Proposition 5. In the next section, we give examples.

Let us give a basis of linear relations of an irreducible abelian polynomial of degree 6 without proof to describe a set $\mathfrak{D}(f, \sigma)$.

PROPOSITION 6. *Let f be an irreducible polynomial of degree 6 with a root α_1 and suppose that $\mathbb{Q}(\alpha_1)/\mathbb{Q}$ is an abelian extension of \mathbb{Q} and let σ be an automorphism satisfying $\sigma(\alpha_i) = \alpha_{i+1}$ ($\alpha_i = \alpha_j$ for $i \equiv j \pmod{6}$) for roots α_i of f . Then a basis of linear relations among roots are:*

(a) *In case that f is indecomposable and neither of type 2 nor of type 3.*

$$\sum_{i=1}^6 \alpha_i = \text{tr}(f).$$

(b) *in case that f is indecomposable and of type 2,*

$$\alpha_1 + \alpha_3 + \alpha_5 = \alpha_2 + \alpha_4 + \alpha_6 = \text{tr}(f)/2.$$

(c) *In case that f is indecomposable and of type 3,*

$$\sum \alpha_i = \text{tr}(f), \quad \alpha_1 + \alpha_2 - (\alpha_4 + \alpha_5) = 0, \quad \alpha_2 + \alpha_3 - (\alpha_5 + \alpha_6) = 0.$$

(d) *In case that $f(x) = g(h(x))$ is possible for a cubic polynomial g , but impossible for a quadratic polynomial g ,*

$$\alpha_1 + \alpha_4 = \alpha_2 + \alpha_5 = \alpha_3 + \alpha_6 = \text{tr}(f)/3.$$

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- (e) *In case that $f(x) = g(h(x))$ is possible for a quadratic polynomial g , but impossible for a cubic polynomial g ,*

$$\alpha_1 + \alpha_3 + \alpha_5 = \alpha_2 + \alpha_4 + \alpha_6 = \text{tr}(f)/2.$$

- (f) *In case that f is decomposable and $f(x) = g(h(x))$ is possible for both $\deg g = 2, 3$,*

$$\alpha_1 + \alpha_4 = \alpha_2 + \alpha_5 = \alpha_3 + \alpha_6 = \text{tr}(f)/3, \alpha_1 - \alpha_2 + \alpha_3 = \text{tr}(f)/6.$$

Except this abelian case, even the classification of non-trivial relations is incomplete.

2. Expectation 2 for a polynomial of degree 6

We have no data to deny Expectation 2 for an irreducible and indecomposable polynomial f in the case of degree ≤ 5 , but it fails in the case of degree 6. The two conditions irreducibility and indecomposability are equivalent to having no non-trivial linear relations among roots in the case of degree ≤ 5 as in the previous section. Data in [2] are less accurate in the case of degree 6, that is X in (11) was too small to guess the precise limit. We improve a method to guess the limit from approximate values.

Suppose that the limit in (11) exists and every $\Pr(f, L, \{R_i\})$ is rational: Then for the common denominator b , we see that

$$\sum_{\{R_i\}} \Pr(f, L, \{R_i\}) \cdot b = b, \gcd\left(\gcd_{\{R_i\}}(\Pr(f, L, \{R_i\}) \cdot b), b\right) = 1.$$

Supposing that b is less than 30000 and taking the above into account, let us consider integers d with $1 \leq d \leq 30000$, which satisfies

$$\sum_{\{R_i\}} r(\Pr_X(f, L, \{R_i\}) \cdot d) = d, \gcd\left(\gcd_{\{R_i\}}(r(\Pr_X(f, L, \{R_i\}) \cdot d)), d\right) = 1, \quad (28)$$

where $r(x)$ is an integer closest to x . Because, they must be satisfied if d is the common denominator of $\Pr(f, L, \{R_i\})$ and an approximation by $\Pr_X(f, L, \{R_i\})$ is sufficiently well. We consider the following four measures, abbreviating $\Pr_X(f, [3]L, \{R_i\})$ to \Pr_{X,R_i} :

$$er_1 := \max_{\{R_i\}} \left| \Pr_{X,R_i} \cdot d - r(\Pr_{X,R_i} \cdot d) \right|,$$

$$er_2 := \sum_{\{R_i\}} \left| \Pr_{X,R_i} \cdot d - r(\Pr_{X,R_i} \cdot d) \right|^2,$$

$$er_3 := \max_{\{R_i\}} \left| \Pr_{X,R_i} - r(\Pr_{X,R_i} \cdot d)/d \right|,$$

$$er_4 := \sum_{\{R_i\}} \left| \Pr_{X,R_i} - r(\Pr_{X,R_i} \cdot d)/d \right|^2.$$

If $\Pr_X(f, L, \{R_i\})$ approximates a rational number a/b well, they are close to 0 for $d = b$. So, we can find the conjectural denominator b of $\Pr(f, L, \{R_i\})$ by checking that there is a large number X satisfying that there is an integer d with $1 \leq d \leq 30000$ which gives the common minimum for four measures above. The first two d -adic measures er_1, er_2 seem to be more appropriate.

We put

$$\begin{aligned} f_1 &:= x^6 + 2x^5 + 4x^4 + x^3 + 2x^2 - 3x + 1, \\ f_2 &:= x^6 + 4x^5 + 16x^4 + 22x^3 + 39x^2 + 16x + 29, \\ f_3 &:= x^6 + 5x^5 + 11x^4 + 13x^3 + 23x^2 + 31x + 43, \\ f_4 &:= x^6 + 8x^5 + 43x^4 + 134x^3 + 372x^2 + 596x + 953. \end{aligned}$$

They are irreducible and indecomposable and define the same cyclotomic field $\mathbb{Q}(\zeta_7)$, and the type of f_1, f_2 is 2 and that of f_3, f_4 is 3.

Let $L = 2$. A 6-tuple (R_1, \dots, R_6) with $0 \leq R_i \leq L - 1$ corresponds to an integer r with $1 \leq r \leq L^6$ by

$$r = 1 + \sum_{i=1}^6 R_i L^{i-1}.$$

(I) The case that there is no non-trivial linear relation among roots:

For a polynomial $f = x^6 + 5x^5 + 1$ in [2], data were insufficient. Wrong values $7/512 = 0.0136\dots, 9/512 = 0.0175\dots$ on p.87 in [2] are close to the conjectural values $13/960 = 0.0135\dots, 17/960 = 0.0177\dots$ on p.84 respectively. The common denominator for four measures above is 960 for $X = 1.36 \cdot 10^{12}$ and the density matches with the conjecture (12). The density in (12) for $L = 2$ is given explicitly by

$$\Pr(f, 2, \{R_i\}) = \begin{cases} 13/960 & \text{if } \operatorname{tr}(f) + \sum R_i \equiv 0 \pmod{2}, \\ 17/960 & \text{if } \operatorname{tr}(f) + \sum R_i \equiv 1 \pmod{2}. \end{cases} \quad (29)$$

(II) Case that f is irreducible and indecomposable, and the type number is 2. For $f = f_1, f_2$, the common denominator of $\Pr_X(f, 2, \{R_i\})$ for four measures is 2304, which is attained for $X = 2 \cdot 10^{11}$. The following table of densities is arranged in the order of r above. For example, $(R_1, \dots, R_6) = (0, \dots, 0)$ corresponds to $r = 1$, and hence the density $\Pr(f_1, 2, (0, \dots, 0))$ is the first entry $36/2304$.

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$$\Pr(f_1, 2, \{R_i\}) = [36, 4, 15, 43, 43, 42, 23, 62, 29, 30, 35, 48, 36, 38, 49, 43, 57, 29, 36, 38, 37, 40, 29, 42, 49, 54, 43, 30, 23, 29, 36, 4, 68, 36, 43, 49, 42, 29, 18, 23, 30, 43, 32, 35, 34, 36, 43, 15, 29, 23, 34, 36, 24, 37, 42, 43, 10, 49, 30, 29, 29, 57, 68, 36] / 2304,$$

$$\Pr(f_2, 2, \{R_i\}) = [36, 68, 57, 29, 29, 30, 49, 10, 43, 42, 37, 24, 36, 34, 23, 29, 15, 43, 36, 34, 35, 32, 43, 30, 23, 18, 29, 42, 49, 43, 36, 68, 4, 36, 29, 23, 30, 43, 54, 49, 42, 29, 40, 37, 38, 36, 29, 57, 43, 49, 38, 36, 48, 35, 30, 29, 62, 23, 42, 43, 43, 15, 4, 36] / 2304.$$

Errors are

$$er_1(f_1) = 0.019615, \quad er_1(f_2) = 0.026945.$$

The density of another polynomial f of type 2 for $L = 2$ seems to be given by the above according to $\text{tr}(f) \pmod 4$. ($\text{tr}(f)$ is an even integer in this case.) Let $f(x)$ be of type 2; then a polynomial $f(x - 1)$ is also of type 2 and

$$\text{tr}(f(x - 1)) \equiv \text{tr}(f(x)) + 2 \pmod 4 \quad \text{is easy.}$$

If r_1, \dots, r_6 are local roots with $r_i \equiv R_i \pmod 2$, then $r'_1 := r_1 + 1, \dots, r'_6 := r_6 + 1$ are also local roots of $f(x - 1)$ with $r'_i \equiv R_i + 1 \pmod 2$. It is the reason why the densities $\Pr(f_1, 2, \{R_i\}), \Pr(f_2, 2, \{R_i\})$ are anti-symmetric. Some properties of $f = f_1, f_2$ are for $R = (R_1, \dots, R_6), R' = (R'_1, \dots, R'_6)$,

- (1) if $R_i + R'_i = 1$ for $1 \leq \forall i \leq 6$, then $\Pr(f, 2, \{R_i\}) + \Pr(f, 2, \{R'_i\}) = 72/2304 = 1/32$,
- (2) $\sum_{\sum R_i \equiv 0 \pmod 2} \Pr(f, 2, \{R_i\}) = \sum_{\sum R_i \equiv 1 \pmod 2} \Pr(f, 2, \{R_i\}) = 1/2$,
- (3) if $R_i = 1 - R'_{7-i}$ for $1 \leq \forall i \leq 6$, then $\Pr(f, 2, \{R_i\}) = \Pr(f, 2, \{R'_i\})$.

The third property is explained as follows: If we have $f(x) \equiv \prod(x - r_i) \pmod p$, then $f(-x) \equiv \prod(x - r'_i) \pmod p$ with $r'_i = p - r_{7-i}$ is easy to see. Hence the condition $r_i \equiv R_i \pmod 2$ implies $r'_i \equiv 1 - R_{7-i} \pmod 2$ for an odd prime p .

The author does not know how to give densities directly from $\{R_i\}$.

A basis of linear relations of $f = f_1, f_2$ with an appropriate numbering is

$$\alpha_1 + \alpha_3 + \alpha_5 = \alpha_2 + \alpha_4 + \alpha_6 = \text{tr}(f)/2.$$

(III) The case that f is irreducible and indecomposable, and the type number is 3. For $f = f_3, f_4$, the common denominator of $\Pr_X(f, 2, \{R_i\})$ for four measures is 15120, which is attained for $X = 2 \cdot 10^{11}$. The following table of densities is

arranged in the order of r as above.

$$\Pr(f_3, 2, \{R_i\}) = [525, 189, 63, 414, 63, 229, 176, 159, 63, 224, 288, 172, 544, 125, 204, 414, 63, 153, 394, 125, 288, 320, 300, 229, 176, 401, 300, 224, 204, 153, 189, 189, 189, 189, 153, 204, 224, 300, 401, 176, 229, 300, 320, 288, 125, 394, 153, 63, 414, 204, 125, 544, 172, 288, 224, 63, 159, 176, 229, 63, 414, 63, 189, 525] / 15120,$$

$$\Pr(f_4, 2, \{R_i\}) = [420, 288, 180, 279, 162, 157, 140, 336, 162, 110, 174, 229, 343, 176, 273, 279, 180, 84, 247, 176, 174, 485, 405, 157, 140, 602, 405, 110, 273, 84, 42, 288, 288, 42, 84, 273, 110, 405, 602, 140, 157, 405, 485, 174, 176, 247, 84, 180, 279, 273, 176, 343, 229, 174, 110, 162, 336, 140, 157, 162, 279, 180, 288, 420] / 15120.$$

Errors are

$$er_1(f_3) = 0.16450, er_1(f_4) = 0.16892.$$

We note that $\text{tr}(f_3)$ is odd and $\text{tr}(f_4)$ is even, and the density of another polynomial of type 3 for $L = 2$ seems to be given by the above according to $\text{tr}(f) \pmod 2$.

A basis of linear relations of $f = f_3, f_4$ is

$$\sum \alpha_i = \text{tr}(f), \alpha_1 + \alpha_2 = \alpha_4 + \alpha_5, \alpha_2 + \alpha_3 = \alpha_5 + \alpha_6.$$

(IV) The case that $K = \mathbb{Q}(\alpha_1)$ is an S_3 -extension and there are a rational number c , two distinct roots α, α' of f , and a cubic subfield M_3 such that $\text{tr}_{K/M_3}(\alpha) + c \cdot \text{tr}_{K/M_3}(\alpha') \in \mathbb{Q}$. Let us give two examples:

The first example is $f = x^6 + 3$. Putting $y := \sqrt[6]{-3}$ with $y^3 = \sqrt{-3}$, roots are

$$\begin{aligned} \beta_{1,1} &= y, & \beta_{1,2} &= (-1 + \sqrt{-3})/2 \cdot y, & \beta_{1,3} &= -(1 + \sqrt{-3})/2 \cdot y, \\ \beta_{2,1} &= -y, & \beta_{2,2} &= (1 - \sqrt{-3})/2 \cdot y, & \beta_{2,3} &= (1 + \sqrt{-3})/2 \cdot y. \end{aligned}$$

Automorphisms η, σ in the proof of Proposition 5 are given by $y \mapsto -y, y \mapsto (-1 + \sqrt{-3})/2 \cdot y$, respectively. A basis of linear relations is four equations

$$\beta_{1,2} = -\beta_{1,1} + \beta_{2,3}, \quad \beta_{1,3} = -\beta_{2,3}, \quad \beta_{2,1} = -\beta_{1,1}, \quad \beta_{2,2} = \beta_{1,1} - \beta_{2,3}.$$

The inclusion $\text{tr}_{K/M_3}(\alpha) + c \cdot \text{tr}_{K/M_3}(\alpha') \in \mathbb{Q}$ is obvious for $\alpha = y, c = 0$. Densities $\Pr(f, 2, \{R_i\})$ for $[R_1, \dots, R_6]$ are given by:

$$\begin{cases} 1/16 & \text{for } [1, 1, 1, 0, 0, 0], [1, 0, 0, 1, 1, 0], [0, 1, 0, 1, 0, 1], [0, 0, 1, 0, 1, 1], \\ 3/16 & \text{for } [1, 1, 0, 1, 0, 0], [1, 0, 1, 0, 1, 0], [0, 1, 1, 0, 0, 1], [0, 0, 0, 1, 1, 1], \\ 0 & \text{otherwise.} \end{cases}$$

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Since $-r$ is also a root for a local root r , a relation $r_i = p - r_{7-i}$ should hold for $i = 1, 2, 3$, hence $R_i + R_{7-i} \equiv 1 \pmod{2}$, i.e., $R_i + R_{7-i} = 1$. This elucidates the cases of density 0.

The second example is $f = x^6 + 100x^4 - 168x^3 + 5200x^2 + 16800x + 26256$, and let β be a root. Then we see that roots $\alpha_1, \dots, \alpha_6$ of f are in order $-1/2122960$ times -2122960β ,

$$\begin{aligned} & -1757\beta^5 + 2758\beta^4 - 189756\beta^3 + 699188\beta^2 - 11117792\beta - 9582496, \\ & -1463\beta^5 + 182\beta^4 - 158004\beta^3 + 159292\beta^2 - 8902208\beta - 22357664, \\ & 125\beta^5 + 710\beta^4 + 13500\beta^3 + 131500\beta^2 + 2458400\beta + 6844000, \\ & 1088\beta^5 - 2312\beta^4 + 117504\beta^3 - 553792\beta^2 + 7895888\beta + 1825664, \\ & 2007\beta^5 - 1338\beta^4 + 216756\beta^3 - 436188\beta^2 + 11788672\beta + 23270496. \end{aligned}$$

The polynomial f is irreducible, indecomposable and not of type 2, 3, and we see

$$\alpha_1 + \alpha_6 + 3(\alpha_2 + \alpha_5) = 0$$

and

$$(\alpha_1 + \alpha_6)^3 = 756 = 28 \cdot 3^3$$

and

$$(\alpha_1 + \alpha_5)^3 = -224 = -28 \cdot 2^3.$$

Hence $\alpha_1 + \alpha_6$ is a trace to a cubic subfield defined by $x^3 - 756 = 0$, hence

$$\text{tr}_{K/M_3}(\alpha) + c \cdot \text{tr}_{K/M_3}(\alpha') \in \mathbb{Q} \quad \text{for } \alpha = \alpha_1, \quad c = 3, \quad \alpha' = \alpha_2.$$

A basis of linear relations is three equations

$$\sum_i \alpha_i = 0, \quad \alpha_1 + \alpha_6 + 3(\alpha_2 + \alpha_5) = 0, \quad \alpha_1 + \alpha_5 - 2(\alpha_3 + \alpha_6) = 0.$$

And we see $(\alpha_3 + \alpha_6)^3 = -28$. The speed of convergence is slow,

$$er_1(f) > 0.3 \quad \text{even for } X = 4 \cdot 10^{13}.$$

The author checked the following: Let us consider following 16 polynomials

$$\begin{array}{ll} x^6 - 9x^4 - 4x^3 + 9x^2 + 3x - 1, & x^6 - 2x^3 + 9x^2 + 6x + 2, \\ x^6 - 7x^3 - 6x^2 - 9x - 3, & x^6 - 10x^4 - 4x^3 + 10x^2 - 1, \\ x^6 - 9x^4 - 8x^3 + 6x^2 + 6x + 1, & x^6 - 10x^4 - 7x^3 + 10x^2 - 1, \\ x^6 - 7x^4 + 8x^2 - 10x + 1, & x^6 - x^4 - 2x^3 + 7x^2 + x + 10, \\ x^6 - 8x^4 - 10x^3 - 3x^2 + 2x + 6, & x^6 - 5x^4 - 7x^3 - 3x^2 - x + 3, \\ x^6 - 10x^4 - 10x^3 - 10x^2 + 1, & x^6 - 9x^4 - 10x^3 - 2x^2 - 1, \end{array}$$

$$\begin{aligned}
&x^6 - 10x^4 - 10x^3 - 10x^2 - 7, & x^6 - 10x^4 - 8x^3 - 4x^2 + 8x - 2, \\
&x^6 - 10x^4 - 10x^3 + 5x^2 - 2x + 9, & x^6 - 10x^4 - 10x^3 - 10x^2 - 10x - 10
\end{aligned}$$

which exhaust all types of Galois closure, checked by pari/gp. Take a root α of one of them and a polynomial f whose root is

$$\sum_{i=1}^6 c_i \alpha^{i-1} \quad \text{with} \quad -1 \leq c_i \leq 1.$$

We consider irreducible and indecomposable ones of degree 6 only. We checked densities $\text{Pr}_X(f, 2, \{R_i\})$ approximate well densities of special polynomials f_1, f_2, f_3, f_4 or (29), where we say that for a rational number a/b and a real number x , x approximates well a/b if the nearest integer $r(bx)$ to bx is a , i.e., $r(bx) = a$.

For a polynomial $f = x^6 - 3x^5 + 6x^4 + 3x^3 - 9x^2 - 18x + 36$ of type 3 and $\text{tr}(f) \equiv 1 \pmod{2}$, which is given in [2, (5) on p.87], $\text{Pr}_X(f, 2, \{R_i\})$ ($X \leq 10^{11}$) approximates well densities given above and the densities on p.87 in [2] should be corrected by the above.

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