

PALINDROMIC CLOSURES AND THUE-MORSE SUBSTITUTION FOR MARKOFF NUMBERS

CHRISTOPHE REUTENAUER — LAURENT VUILLON

Dedicated to the memory of Professor Pierre Liardet

ABSTRACT. We state a new formula to compute the Markoff numbers using iterated palindromic closure and the Thue-Morse substitution. The main theorem shows that for each Markoff number m , there exists a word $v \in \{a, b\}^*$ such that $m - 2$ is equal to the length of the iterated palindromic closure of the iterated antipalindromic closure of the word av . This construction gives a new recursive construction of the Markoff numbers by the lengths of the words involved in the palindromic closure. This construction interpolates between the Fibonacci numbers and the Pell numbers.

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1. Introduction

Markoff numbers are fascinating integers; the reader may use the recent book by Martin Aigner [A] for studying them. These numbers are related to number theory, hyperbolic geometry, continued fractions and Christoffel words [A, M1, M2, F, Re1, Re2]. Many great mathematicians have worked on these numbers and the famous uniqueness conjecture by Frobenius is still unsolved [B, M1, M2, F, C1, C2]. Markoff numbers are positive integers that appear in the solution of the Diophantine equation

$$x^2 + y^2 + z^2 = 3xyz.$$

The first Markoff numbers are 1, 2, 5, 13, 29, 34, 89, 169, 194, 233, 433, 610, 985, 1325, 1597, 2897, 4181, 5741, 6466, 7561, 9077, 10946, 14701, 28657, 33461,

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37666, 43261, 51641; they are listed in the Sloane Encyclopedia of Integer Sequences (sequence number A002559). One shows that if a *Markoff triple* (x, z, y) , that is, a triple satisfying the previous Diophantine equation, has maximum z , then the triple gives birth to two others, which are $(x, 3xy - z, z)$ and $(z, 3zy - x, y)$ (see [A] Section 3.1). One can construct a binary tree using these computations, where each node is a Markoff triple (see [A]). The Frobenius conjecture asserts that each Markoff number is the maximum of a unique Markoff triple ([A, Re2]). In the work of Markoff [M1, M2], one finds implicitly combinatorics on words and construction of balanced sequences [CF, BS, V, BdLR] on the alphabet $\{1, 2\}$. The Markoff numbers are also linked with approximation theory and continued fractions [BRS, B].

In this article, we find a new relation between Markoff numbers and combinatorics on words. The main theorem shows that for each Markoff number m there exists a word $v \in \{a, b\}^*$ such that $m - 2$ is equal to the length of the iterated palindromic closure of the iterated antipalindromic closure of the word av .

The *iterated palindromic closure* (due to Aldo de Luca) is used in combinatorics on words in order to generate standard Sturmian words and central words [dL, J, BS]. One defines first the *palindromic closure* $w^{(+)}$ of a word w : it is the shortest palindrome having w as prefix (it exists and is unique). The iterated palindromic closure $\text{Pal}(u)$ is then defined recursively by $\text{Pal}(1) = 1$ (the empty word), and $\text{Pal}(va) = (\text{Pal}(v)a)^{(+)}$ for any word v and any letter a .

The *iterated antipalindromic closure* appears in the literature in order to construct antipalindromes and to generalize the iterated palindromic closure [dLDL, BPTV]. In fact, when the alphabet is binary, the iterated antipalindromic closure of a word u is obtained by applying the Thue-Morse substitution to the iterated palindromic closure of u [dLDL].

As an application of the main theorem, we give a new computation of Markoff numbers by a recursive construction on the lengths of the words involved in the iterated palindromic closure. The lengths of these words allow us to state a recursive formula using a *directive sequence* $d = d_1d_2 \dots d_j$ with d_i on the alphabet $\{a, b\}$. One interesting property is to recover the usual Fibonacci recursive construction if $d_j \neq d_{j-1} \neq d_{j-2}$ and the usual Pell recursive construction if $d_j \neq d_{j-1} = d_{j-2}$ [C3, BRS].

Note that in the articles [F, P] we find two other decompositions of the Markoff numbers as sums of positive integers: using properties of continued fractions in the work of Frobenius and properties of snake graphs in the work of Poppel et al. (see also [A]).

2. Iterated palindromic closures

In the sequel we work with the usual notations in combinatorics on words [BS]. Let \mathcal{A} be a finite alphabet.

The *reversal* of a word $x = x_1x_2\dots x_n$ with $x_i \in \mathcal{A}$ is the word $\tilde{x} = x_nx_{n-1}\dots x_1$.

A word p is a *palindrome* if it is equal to its reversal (that is $p = \tilde{p}$).

The length of a word $u = u_1u_2\dots u_m$, where $u_i \in \mathcal{A}$, is equal to m and is denoted $|u|$.

The *concatenation* of two words $u = u_1u_2\dots u_m$ and $v = v_1v_2\dots v_n$ is the word of the length $m + n$ given by $u \cdot v = u_1u_2\dots u_mv_1v_2\dots v_n$.

In this article we use the *palindromic closure*, introduced by Aldo de Luca [dL] (more precisely, it is the *right palindromic closure*): the palindromic closure of a word x is the shortest palindrome having x as a prefix; it exists and is unique; it is denoted by $x^{(+)}$. For example, if

$$x = ab, \quad \text{then } x^{(+)} = aba.$$

It is known that

$$x^{(+)} = x'y\tilde{x}',$$

where $x = x'y$ with y the longest palindrome suffix of x . We consider the iterated palindromic closure (also introduced in [dL]), denoted by $\text{Pal}(d)$: it is a mapping from the free monoid on \mathcal{A} into itself, defined recursively by

$$\text{Pal}(d_1d_2\dots d_n) = (\text{Pal}(d_1d_2\dots d_{n-1})d_n)^{(+)}, \quad d_i \in \mathcal{A},$$

with the initial condition $\text{Pal}(1) = 1$, where 1 denotes the empty word. This mapping is injective and w is called the *directive word* of $\text{Pal}(w)$. For example, $\text{Pal}(aba) = abaaba$: indeed,

$$\text{Pal}(a) = a \quad \text{and} \quad \text{Pal}(ab) = (\text{Pal}(a)b)^{(+)} = (ab)^{(+)} = aba$$

and then

$$\text{Pal}(aba) = (\text{Pal}(ab)a)^{(+)} = (abaa)^{(+)} = abaaba.$$

We also use the *Thue-Morse substitution*, denoted by $\theta = (ab, ba)$: it is an endomorphism of the free monoid $\{a, b\}^*$ that maps the letter a to ab and the letter b to ba .

We have

$$\alpha(ab) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \mu(a), \quad \alpha(aabb) = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} = \mu(b).$$

Consider $\phi = (ab, aabb)$. Then $\mu = \alpha\phi$.

Note that for each word m , $b\phi(m) = \psi(m)b$, where $\psi = (ba, baab) = (ba, ab)G$ with $G = (a, ab)$. Using [BdLR, Corollary 3.2], we see that the length of the Christoffel word $a\text{Pal}(w)b$ is equal to $h + i + j + k$, where $\alpha(w) = \begin{pmatrix} h & i \\ j & k \end{pmatrix}$.

The word w is defined as follows: we have $\phi(u) = \phi(apb) = ab\phi(p)aabb$ and we define $w = b\phi(p)a$. Thus $\phi(u) = awabb$. Then we have

$$\begin{aligned} \mu(u) = \alpha\phi(u) = \alpha(awabb) &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h & i \\ j & k \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} * & h + i + j + k \\ * & * \end{pmatrix} \end{aligned}$$

and therefore $m = \mu(u)_{12} = h + i + j + k = |a\text{Pal}(w)b|$.

Furthermore,

$$w = b\phi(p)a = \psi(p)ba = ((ba, ab)G(p))((ba, ab)G(a)) = (ba, ab)G(pa).$$

Since $p = \text{Pal}(v)$, we obtain

$$G(pa) = G(\text{Pal}(v)a) = \text{Pal}(av),$$

by Justin's formula [Be, J]. Thus $w = (ba, ab) \circ \text{Pal}(av)$. The computation of m gives

$$\begin{aligned} m &= 2 + |\text{Pal}(w)| \\ &= 2 + |\text{Pal} \circ (ba, ab) \circ \text{Pal}(av)| \\ &= 2 + |E \circ \text{Pal} \circ (ba, ab) \circ \text{Pal}(av)| \\ &= 2 + |\text{Pal} \circ E \circ (ba, ab) \circ \text{Pal}(av)| \\ &= 2 + |\text{Pal} \circ (ab, ba) \circ \text{Pal}(av)| \\ &= 2 + |\text{Pal} \circ \theta \circ \text{Pal}(av)|. \end{aligned} \quad \square$$

A word z is an *antipalindrome* if it is equal to the exchange of its reversal (that is $z = E(\tilde{z})$). For example, $z = aababb$ is an antipalindrome because its reversal is $\tilde{z} = bbabaa$ and the exchange gives $E(\tilde{z}) = aababb$.

As for the palindromic case, we use the antipalindromic closure and the iterated antipalindromic closure which are defined in the work of de Luca and De Luca [dLDL]. The *antipalindromic closure* of a word x is the shortest antipalindrome having x as a prefix; it is denoted by x^\oplus . For example, if $x = ab$, then $x^\oplus = ab$ because ab is already an antipalindrome and if $x = aa$,

then $x^\oplus = abbb$. The iterated antipalindromic closure noted $\text{AntiPal}(d)$ is defined by the recursive formula $\text{AntiPal}(d_1 d_2 \cdots d_n) = (\text{AntiPal}(d_1 d_2 \cdots d_{n-1}) d_n)^\oplus$ and the initial condition $\text{AntiPal}(1) = 1$. For example, $\text{AntiPal}(aba) = abbaababbaab$; indeed, $\text{AntiPal}(a) = ab$, thus $\text{AntiPal}(ab) = (\text{AntiPal}(a)b)^\oplus = (abb)^\oplus = abbaab$ and then $\text{AntiPal}(aba) = (\text{AntiPal}(ab)a)^\oplus = (abbaaba)^\oplus = abbaababbaab$. We see that

$$\text{AntiPal}(aba) = ab \cdot ba \cdot ab \cdot ab \cdot ba \cdot ab = (ab, ba) \circ abaaba = (ab, ba) \circ \text{Pal}(aba).$$

This is a general fact, as shown in [dLdL] Theorem 7.6.

THEOREM 2 (de Luca, De Luca). *Let v be a word on the alphabet $\mathcal{A} = \{a, b\}$ and $\theta = (ab, ba)$ be the Thue-Morse substitution. Then*

$$\text{AntiPal}(v) = \theta \circ \text{Pal}(v).$$

COROLLARY 3. *For each word $v \in \{a, b\}^*$, the number $|\text{Pal} \circ \text{AntiPal}(av)| + 2$ is a Markoff number $\neq 1, 2$. The mapping defined in this way from $\{a, b\}^*$ into the set of Markoff numbers different from 1, 2 is surjective. Injectivity of this mapping is equivalent to the Frobenius conjecture.*

4. Computation of Markoff numbers

The previous corollary gives a new way to compute the Markoff numbers by using iterated antipalindromic closures and iterated palindromic closures. We now give a recursive formula for computing the Markoff numbers.

THEOREM 4. *Consider $d = \text{AntiPal}(av)$ with $v \in \{a, b\}^*$. We write $d = d_1 d_2 \dots d_{|d|}$ with $d_i \in \{a, b\}$. We let $L_0 = L_1 = 1$ and $L_2 = L_1 + L_0 = 2$. For $j \geq 3$ we define recursively the L_j :*

$$L_j = \begin{cases} L_{j-1} & \text{if } d_j = d_{j-1}, \\ L_{j-1} + L_{j-2} & \text{if } d_j \neq d_{j-1} \neq d_{j-2}, \\ L_{j-1} + L_{j-2} + L_{j-3} & \text{if } d_j \neq d_{j-1} = d_{j-2}. \end{cases}$$

Then the Markoff number m_v is given by

$$m_v = 1 + \sum_{j=0}^{|d|} L_j.$$

Consider the example $v = ab$.

We have $d = \text{AntiPal}(aab) = \theta(\text{Pal}(aab)) = \theta(aabaa) = ababbaabab$ and then $L_0 = 1$; $L_1 = 1$; $L_2 = L_1 + L_0 = 1 + 1 = 2$; $L_3 = L_2 + L_1 = 2 + 1 = 3$ (because $d_3 = a \neq d_2 = b \neq d_1 = a$); $L_4 = L_3 + L_2 = 3 + 2 = 5$

(because $d_4 = b \neq d_3 = a \neq d_2 = b$); $L_5 = L_4$ (because $d_5 = d_4 = b$).
 $L_6 = L_5 + L_4 + L_3 = 5 + 5 + 3 = 13$ (because $d_6 = a \neq d_5 = d_4 = b$);
 and so on. $L_7 = 13$; $L_8 = 31$; $L_9 = 44$; $L_{10} = 75$. Thus the sum of the L_j is 193
 and if we add 1, we find the Markoff number 194.

A more compact way of writing the L_i 's is to write d and above each letter
 the L_i :

$$\begin{array}{cccccccccc} d_v = & a & b & a & b & b & a & a & b & a & b \\ & 1 & 1 & 2 & 3 & 5 & 5 & 13 & 13 & 31 & 44 & 75 \end{array}$$

Proof. To prove the theorem, we use Justin's Formula [J, Be]

$$\text{Pal}(d'd'') = \psi_{d'}(d'') \cdot \text{Pal}(d')$$

with d' a word on $\{a, b\}^*$ and d'' a letter.

We recall that

$$\psi_{d'}(a) = \psi_{d_1} \left(\psi_{d_2} \left(\dots \left(\psi_{d_{|d'|}}(a) \right) \right) \right) \quad \text{with } \psi_a(a) = a, \psi_a(b) = ab$$

(ψ_a was previously denoted G) and $\psi_b(a) = ba, \psi_b(b) = b$.

In our construction, we use $d = \text{AntiPal}(av)$ with $v \in \{a, b\}^*$ and we have
 to study $\text{Pal}(d) = \text{Pal}(d_1 d_2 \dots d_{|d|-1} d_{|d|})$. By successive applications of Justin's
 Formula we find

$$\begin{aligned} \text{Pal}(d_1 d_2 \dots d_{|d|}) &= \psi_{d_1 d_2 \dots d_{|d|-1}}(d_{|d|}) \cdot \text{Pal}(d_1 d_2 \dots d_{|d|-1}), \\ \text{Pal}(d) &= \text{Pal}(d_1 d_2 \dots d_{|d|}) = \psi_{d_1 d_2 \dots d_{|d|-1}}(d_{|d|}) \dots \psi_{d_1 d_2}(d_3) \cdot \psi_{d_1}(d_2) \cdot d_1. \end{aligned}$$

We define

$$W_j = \psi_{d_1 d_2 \dots d_{j-1}}(d_j)$$

and

$$L'_j = |W_j| \quad \text{for } j = 1, \dots, |d|.$$

Thus we have $W_1 = d_1, W_2 = \psi_{d_1}(d_2), \dots, W_{|d|} = \psi_{d_1 d_2 \dots d_{|d|-1}}(d_{|d|})$ and the L'_j
 are the length of each W_j . We will prove that the L'_i 's satisfy the same recursive
 formula as the L_i 's; thus it will follow that $L_j = L'_j$. The recursive formula for
 the L'_j 's is constructed on the prefixes of d .

We investigate the base cases. Note that d begins by ab : indeed,

$$\theta(\text{Pal}(av)) = \theta(av') = abd''.$$

For the prefix of length one of d we find $W_1 = a$, in accordance with the base
 case $L'_1 = |W_1| = 1$. For a prefix of length two of d we have $W_2 = \psi_a(b) = ab$,
 and thus $L'_2 = |ab| = 2$.

Now we compute the recursive formula for prefixes of length at least three of d . We have six cases to consider, indeed, we use the directive word $d = (ab, ba) \circ \text{Pal}(av)$ and thus aaa and bbb are forbidden words in the directive word d . It is sufficient to use Justin's formula for the following prefixes of d :

$$d'aba, \quad d'bab, \quad d'baa, \quad d'aab, \quad d'baa \quad \text{and} \quad d'abb \quad \text{with} \quad d' \in \{a, b\}^*.$$

The first case of the recursive formula is given by the prefixes of d of the form $d'baa$. We write $d = d_1 d_2 \dots d_{j-1} d_j = d'baa$ for a given j and we are in the case $d_j = d_{j-1} = a$. Thus we have by the definition

$$W_j = \psi_{d'ba}(a) \quad \text{and} \quad W_{j-1} = \psi_{d'b}(a).$$

We have

$$W_j = \psi_{d'ba}(a) = \psi_{d'b}(\psi_a(a)) = \psi_{d'b}(a) = W_{j-1}, \quad \text{thus} \quad W_j = W_{j-1}.$$

We find $L'_j = L'_{j-1}$ for $d_j = d_{j-1} = a$. Similarly, by exchanging the roles of a and b that is by considering the prefixes of the form $d'abb$ we find $W_j = W_{j-1}$; thus $L'_j = L'_{j-1}$ for $d_j = d_{j-1} = b$.

The second case of the recursive formula is given by the prefixes of d of the form $d'aba$. We write $d_1 d_2 \dots d_{j-1} d_j = d'aba$ for a some j and we are in the case $d_j = a \neq d_{j-1} = b \neq d_{j-2} = a$. By the definition

$$W_j = \psi_{d'ab}(a) \quad \text{and} \quad W_{j-1} = \psi_{d'a}(b) \quad \text{and} \quad W_{j-2} = \psi_{d'}(a),$$

we have

$$\begin{aligned} W_j &= \psi_{d'ab}(a) = \psi_{d'a}(\psi_b(a)) = \psi_{d'a}(ba) \\ &= \psi_{d'a}(b) \cdot \psi_{d'a}(a) \\ &= \psi_{d'a}(b) \cdot \psi_{d'}(\psi_a(a)) \\ &= \psi_{d'a}(b) \cdot \psi_{d'}(a) \\ &= W_{j-1} \cdot W_{j-2}. \end{aligned}$$

Thus we have

$$W_j = W_{j-1} \cdot W_{j-2} \quad \text{and} \quad L'_j = L'_{j-1} + L'_{j-2}$$

for $d_j = a \neq d_{j-1} = b \neq d_{j-2} = a$. Similarly, for the prefixes of the form $d'bab$ by exchanging the roles of a and b we have

$$W_j = \psi_{d'ba}(b) = W_{j-1} W_{j-2} \quad \text{and thus} \quad L'_j = L'_{j-1} + L'_{j-2}$$

for $d_j = b \neq d_{j-1} = a \neq d_{j-2} = b$.

The third case is given by the prefixes of d of the form $d'''bba$. As bbb is forbidden in d , thus we write

$$d_1 d_2 \dots d_{j-1} d_j = d'abba \quad \text{and we have} \quad d_j = a \neq d_{j-1} = d_{j-2} = b.$$

By the definition

$$W_j = \psi_{d'abb}(a), \quad W_{j-1} = \psi_{d'ab}(b), \quad W_{j-2} = \psi_{d'a}(b) \quad \text{and} \quad W_{j-3} = \psi_{d'}(a)$$

for a given j we have

$$\begin{aligned} W_j &= \psi_{d'abb}(a) = \psi_{d'a}(\psi_b(\psi_b(a))) = \psi_{d'a}(\psi_b(ba)) = \psi_{d'a}(bba) \\ &= \psi_{d'a}(b) \cdot \psi_{d'a}(b) \cdot \psi_{d'a}(a) = \psi_{d'ab}(b) \cdot \psi_{d'a}(b) \cdot \psi_{d'a}(a) \\ &= \psi_{d'ab}(b) \cdot \psi_{d'a}(b) \cdot \psi_{d'}(a) \\ &= W_{j-1} \cdot W_{j-2} \cdot W_{j-3} \end{aligned}$$

Thus we have

$$W_j = \psi_{d'abb}(a) = W_{j-1} \cdot W_{j-2} \cdot W_{j-3}$$

and thus

$$L'_j = L'_{j-1} + L'_{j-2} + L'_{j-3} \quad \text{for } d_j = a \neq d_{j-1} = d_{j-2} = b,$$

And similarly, for the prefixes of the form $d'baab$ we find

$$W_j = \psi_{d'baa}(b) = W_{j-1} \cdot W_{j-2} \cdot W_{j-3}$$

and

$$L'_j = L'_{j-1} + L'_{j-2} + L'_{j-3} \quad \text{for } d_j = b \neq d_{j-1} = d_{j-2} = a.$$

Finally, we have to compute the Markoff numbers by using Corollary 3:

$$m = |\text{Pal}(d)| + 2 \quad \text{with } d = \text{AntiPal}(av).$$

Thus by Justin's Formula

$$\begin{aligned} m &= |\psi_{d_1 d_2 \dots d_{|d|-1}}(d_{|d|}) \cdots \psi_{d_1 d_2}(d_3) \cdot \psi_{d_1}(d_2) \cdot d_1| + 2 \\ &= |W_{|d|}| + |W_{|d|-1}| + \cdots + |W_2| + |W_1| + 2 \\ &= 2 + \sum_{j=1}^{|d|} L_j = 1 + \sum_{j=0}^{|d|} L_j. \end{aligned} \quad \square$$

Note that in the second case of the recursive formula, we have a Fibonacci recurrence

$$L_j = L_{j-1} + L_{j-2}.$$

In the third case of the recursive formula we have

$$L_j = L_{j-1} + L_{j-2} + L_{j-3} \quad \text{and} \quad d_{j-1} = d_{j-2}.$$

By application of the first case of the recursive formula we find

$$L_{j-1} = L_{j-2}$$

and then a Pell recurrence

$$L_j = 2L_{j-2} + L_{j-3}.$$

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Christophe Reutenauer
Université du Québec à Montréal
Département de Mathématiques
Montréal
CANADA
E-mail: christo@math.uqam.ca

Laurent Vuillon
Université de Savoie Mont Blanc
LAMA—UMR CNRS 5127
Chambéry
FRANCE
E-mail: Laurent.Vuillon@univ-smb.fr