

# UPPER BOUNDS FOR DOUBLE EXPONENTIAL SUMS ALONG A SUBSEQUENCE

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**ABSTRACT.** We consider a class of double exponential sums studied in a paper of Sinai and Ulcigrai. They proved a linear bound for these sums along the sequence of denominators in the continued fraction expansion of  $\alpha$ , provided  $\alpha$  is badly-approximable. We provide a proof of a result, which includes a simple proof of their theorem, and which applies for all irrational  $\alpha$ .

*Communicated by Michael Drmota*

## 1. Introduction

### 1.1. Some notation

Let  $\alpha = [a_0; a_1, \dots]$  denote the continued fraction expansion of  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ . We write  $\|x\|$  for the distance from  $x$  to the nearest integer. The *convergents*  $p_n/q_n = [a_0; a_1, \dots, a_n]$ , where  $(p_n, q_n) = 1$ , give good approximations to  $\alpha$ . We call  $\{q_n\}_{n \in \mathbb{N}}$  the *sequence of denominators* of  $\alpha$ . We say that an irrational number  $\alpha$  is *badly-approximable* if there exists  $\varepsilon_\alpha > 0$  such that for all  $p, q \in \mathbb{Z}$ ,  $(p, q) = 1$ , we have

$$\left| \alpha - \frac{p}{q} \right| > \frac{\varepsilon_\alpha}{q^2}.$$

These correspond precisely with those numbers  $\alpha$  for which there exists  $N \in \mathbb{N}$  such that,  $a_n(\alpha) \leq N$  for all  $n \in \mathbb{N}$ . The set of all badly-approximable numbers is a set of Lebesgue measure zero.

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2010 Mathematics Subject Classification: 11J70, 11L03, 11L07.

Key words: continued fraction, badly-approximable  $\alpha$ , double-exponential sum, discrepancy, Koksma-Hlawka inequality, Ostrowski expansion.

The author's research is supported by an EPSRC Doctoral Training Grant.

When  $\alpha$  is badly approximable, we have the helpful bound that

$$\|q_n \alpha\| > \frac{\varepsilon_\alpha}{q_n}.$$

Since convergents give the best approximations for the distance to the nearest integer (see [7]), this means that for  $m \leq q_{n+1} - 1$  we have the bound

$$\|m\alpha\| > \frac{\varepsilon_\alpha}{q_n}.$$

We write  $f(n) = O(g(n))$  to mean that there exists a constant  $C$ , (which does not depend on  $n$ ), such that

$$f(n) \leq C \cdot g(n) \quad \text{for all } n \in \mathbb{N}.$$

Finally, we define the discrepancy of a sequence.

**DEFINITION 1.1.** Let  $(x_n)$  be a sequence of real numbers. For  $N \in \mathbb{N}$  the *discrepancy* of  $(x_n)$  modulo one,  $D_N(x_n)$ , is defined as:

$$D_N(\{x_m\}) := \sup_{I \subseteq \mathbb{R}/\mathbb{Z}} \left| \sum_{m=1}^N \chi_I(x_m) - N \cdot |I| \right|,$$

where  $I$  denotes an interval and  $\chi_I$  is the characteristic function of  $I$ .

## 1.2. Double exponential sums

In [9] Sinai and Ulcgrai studied double trigonometric sums of the form

$$T_M(\alpha) = \frac{1}{M} \sum_{m=0}^{M-1} \sum_{n=0}^{M-1} e(nm\alpha). \quad (1.2.1)$$

We want to determine when the absolute value of this sum is bounded uniformly (i.e., by a constant which depends only on  $\alpha$ ) over some subsequence

$$M \in \mathcal{A} \subseteq \mathbb{N}.$$

This will, obviously, depend on the Diophantine properties of  $\alpha$  and the subsequence  $\mathcal{A}$ .

We will see that the problem of bounding this sum depends importantly on controlling sums such as

$$\left| \frac{1}{M} \sum_{m=1}^M \frac{1}{\{\{m\alpha\}\}} \right|. \quad (1.2.2)$$

Here

$$\{\{x\}\} := \begin{cases} \{x\}, & x \in [0, \frac{1}{2}], \\ \{x\} - 1, & x \in (-\frac{1}{2}, 0), \end{cases}$$

is the *signed fractional part* of  $x \in \mathbb{R}$ .

In [9] the following is proved

**THEOREM 1.2** (Sinai, Ulcigrai). *Let  $\alpha$  be badly-approximable. Consider the following double trigonometric sum:*

$$T_M(\alpha) := \frac{1}{M} \sum_{m=0}^{M-1} \sum_{n=0}^{M-1} e(nm\alpha).$$

Then there exists a constant

$$C = C(\alpha) > 0$$

such that

$$|T_M| \leq C_\alpha \quad \text{for all } M \in \{q_n\}_{n \in \mathbb{N}}.$$

**REMARK 1.3.** The sum here is an example of a 2-dimensional finite theta sum. In [2] Cosentino and Flaminio prove bounds for far more general  $g$ -dimensional finite theta sums. A special case of one of their results implies that the above theorem is true for all  $M \in \mathbb{N}$ .

Our main theorem generalises Theorem 1.2.

**THEOREM 1.4.** *Let  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ . Then there exists a constant*

$$C = C(\alpha) > 0$$

such that

$$|T_{q_n}| \leq C_\alpha \cdot \max \left\{ \frac{\log(2 \cdot \max_{i \leq n} \{a_i\})}{a_{n+1}}, 1 \right\} \quad \text{for all } n \in \mathbb{N}. \quad (1.2.3)$$

**REMARK 1.5.** By examining signs it appears that the upper bound here is close to best possible. Equation (1.13) in [1] gives a lower bound for the largest terms in a sum that we will consider. While it is true that we use the triangle inequality earlier in our calculation, it does not make our estimate so much larger.

## 2. Proof of main result

### 2.1. Reducing $T_M$

Following the methods in [9], we split  $T_M$  into two separate sums.

By summing the terms for  $n = 0, \dots, M-1$ , we can rewrite (1.2.1) as

$$T_M = 1 + \frac{1}{M} \sum_{m=1}^{M-1} \frac{e(Mm\alpha) - 1}{e(m\alpha) - 1}.$$

Then we can write  $T_M = 1 + S'_M - S''_M$ , where

$$S'_M := \frac{1}{M} \sum_{m=1}^{M-1} \frac{e(Mm\alpha)}{e(m\alpha) - 1} \quad (2.1.1)$$

and

$$S''_M := \frac{1}{M} \sum_{m=1}^{M-1} \frac{1}{e(m\alpha) - 1}. \quad (2.1.2)$$

We will prove that there exist constants  $C', C'' \in \mathbb{R}$  such that

$$|S'_{q_n}| \leq C' \cdot \max \left\{ \frac{\log(2 \cdot \max_{i \leq n} \{a_i\})}{a_{n+1}}, 1 \right\} \quad (2.1.3)$$

and

$$|S''_{q_n}| \leq C'' \quad \text{for all } n \in \mathbb{N}.$$

These constants will depend only on  $\alpha$ .

### 2.2. The sum $S''_M$ (2.1.2)

Let us consider the ‘less intimidating’ sum first. We want to show that there exists a  $C'' \in \mathbb{R}$  such that  $|S''_{q_n}| \leq C''$  for all  $n \in \mathbb{N}$ .

Note that in [3], Hardy and Littlewood prove a similar theorem.

**THEOREM 2.1** (Hardy, Littlewood). *Let  $\alpha$  be badly-approximable. Then there exists  $C^* > 0$  such that  $|S''_M| \leq C^*$  for each  $M \in \mathbb{N}^+$ .*

We proceed by calculating real and imaginary parts.

$$\frac{1}{e(m\alpha) - 1} = -\frac{1}{2} - \frac{i}{2} \cot(\pi m\alpha).$$

The Taylor series expansion of  $\cot x$  is

$$\cot x = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n B_{2n}}{(2n)!} x^{2n-1} = \frac{1}{x} - \frac{x}{3} + \frac{x^3}{45} - \dots$$

with radius of convergence  $0 < |x| < \pi$ . Here  $B_n$  is the  $n$ th Bernoulli number.

Note that due to the symmetry of  $\cot x$ ,

$$\cot(\pi m\alpha) = \cot(\pi\{\{m\alpha\}\}).$$

So we can write

$$\cot(\pi m\alpha) = \frac{1}{\pi\{\{m\alpha\}\}} \left( 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 4^n B_{2n}}{(2n)!} (\pi\{\{m\alpha\}\})^{2n} \right).$$

Now the series on the right is negative and it takes values strictly between 0 (when  $\{\{m\alpha\}\}$  is close to 0) and  $-1$  (when  $\{\{m\alpha\}\}$  is close to  $\pm\frac{1}{2}$ ).

Hence, in order to prove that  $|S''_{q_n}|$  is bounded by a uniform constant for all  $n \in \mathbb{N}$ , we have to prove the following:

**LEMMA 2.2.** *Let  $\alpha \in \mathbb{R}$ . Then there exists  $C = C(\alpha) > 0$  such that,*

$$\left| \sum_{m=1}^{q_n-1} \frac{1}{q_n\{\{m\alpha\}\}} \right| \leq C \quad \text{for all } n \in \mathbb{N}. \quad (2.2.1)$$

We will consider two different proofs of Lemma 2.2. The first one is simpler, while the latter one will be applicable to estimating  $S'_M$  as well. The second proof is also malleable to proving Theorem 2.1.

### 2.3. Koksma-Hlawka Proof of Lemma 2.2

Recall the Koksma-Hlawka inequality.

**LEMMA 2.3.** *Let  $f$  be a real function with period 1 of bounded variation. Then for every sequence  $\{x_m\}$  and every integer  $N \geq 1$ ,*

$$\left| \frac{1}{N} \sum_{m=1}^N f(x_m) - \int_0^1 f(x) dx \right| \leq V(f) \frac{D_N(x_m)}{N},$$

where  $V(f)$  is the total variation of the function.

We wish to apply this inequality with

$$f(x) = \frac{1}{\{\{x\}\}}, \quad x_m = \{m\alpha\}, \quad \text{and} \quad N = q_n - 1.$$

Therefore we have to restrict the domain on which we define our function, in order to ensure that it is integrable.

We are able to use the following from [8]

$$\left| \alpha - \frac{p_{n-1}}{q_{n-1}} \right| > \frac{1}{2q_{n-1}q_n}. \quad (2.3.1)$$

So, for all  $m \leq N = q_n - 1$ , we have

$$||m\alpha|| > \frac{1}{2q_n}.$$

Hence we can restrict the domain of  $f$  to the interval  $[\frac{1}{2q_n}, 1 - \frac{1}{2q_n}]$ . Furthermore, since  $f$  is anti-symmetric about  $1/2$ , the integral of  $f$  over this interval is equal to 0. The total variation,  $V(f)$ , of  $f$  is

$$\sup_{\mathcal{P}} \sum_{i=1}^{n_p} \left| \frac{1}{\{\{x_{i+1}\}\}} - \frac{1}{\{\{x_i\}\}} \right|,$$

where  $\mathcal{P}$  is a partition of  $[\frac{1}{2q_n}, 1 - \frac{1}{2q_n}]$ . As  $f$  is monotone in this interval, the total variation is maximised when we take the trivial partition (that is the two endpoints). Therefore  $V(f) = 4q_n$ .

Finally, we move on to considering the Discrepancy.

Lemma 5.6 from [5] states that

$$D_N(m\alpha) \leq 3 \sum_{j=0}^r t_j,$$

where

$$N = \sum_{j=0}^r q_j t_j$$

is the *Ostrowski expansion* of  $N$ . This is defined in the next subsection (see Definition 2.5) but all we need to know here is that if  $N = q_n$ , then  $t_n = 1$  and  $t_i = 0$  for all  $i \neq n$ . So  $D_{q_n}(m\alpha) \leq 3$ .

Finally, we can apply all the estimates we have (with  $N = q_n - 1$  and  $f$  &  $\{x_m\}$  as above.)

$$\begin{aligned} \left| \sum_{m=1}^{q_n-1} f(x_m) - (q_n - 1) \int_0^1 f(x) dx \right| &\leq D_{q_n-1}(x_m)V(f) \\ &\leq (D_{q_n}(x_m) + 1)V(f) \\ &\leq 4 \cdot 4q_n = 16q_n \end{aligned}$$

Hence,

$$\left| \frac{1}{q_n} \sum_{m=1}^{q_n-1} \frac{1}{\{\{m\alpha\}\}} \right| \leq 16.$$

Here we used the obvious fact that  $D_M(x_m) \leq D_{M+1}(x_m) + 1$ .

## 2.4. The sum $S'_M$ (2.1.1)

We move on to considering the sum

$$S'_M := \frac{1}{M} \sum_{m=1}^{M-1} \frac{e(Mm\alpha)}{e(m\alpha) - 1}.$$

We will write this sum as a telescoping series and then take advantage of some cancellation to reduce our situation to considering the sum  $S''_M$  (2.1.2). Firstly,

$$\sum_{m=1}^{M-1} \frac{e(Mm\alpha)}{e(m\alpha) - 1} = \sum_{m=1}^{M-1} \left( e(Mm\alpha) - e(M(m+1)\alpha) \right) \sum_{k=1}^m \frac{1}{e(k\alpha) - 1} \quad (2.4.1)$$

$$+ e(M^2\alpha) \sum_{k=1}^{M-1} \frac{1}{e(k\alpha) - 1}. \quad (2.4.2)$$

We then consider the outer part of the sum on the right hand side of (2.4.1) (for  $M = q_n$ ),

$$\begin{aligned} e(mq_n\alpha) - e((m+1)q_n\alpha) &= e(mq_n\alpha) - e((m+1)q_n\alpha) \\ &= e(mq_n\alpha) - e(mq_n\alpha)e(q_n\alpha) \\ &= (1 - e(q_n\alpha))e(mq_n\alpha). \end{aligned}$$

In absolute value this is less than  $2\pi/q_{n+1}$ .

Now using the triangle inequality and Lemma 2.2, we see that (2.1.3) results from the following lemma.

**LEMMA 2.4.** *For all  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$  and for all  $m \leq q_n - 1$ ,*

$$\sum_{k=1}^m \frac{1}{\{\{k\alpha\}\}} = O\left(q_n \cdot \max_{i \leq n} \{1, \log a_i\}\right).$$

To prove this Lemma we will need to introduce some different techniques, which will also yield a new proof of Lemma 2.2.

## 2.5. The Ostrowski proof of Lemmas 2.2 and 2.4

Our alternative proof of Lemma 2.2 will involve decomposing the sum in (2.2.1) into segments where there is some obvious cancellation.

**DEFINITION 2.5.** Let  $\alpha$  be irrational. Then for every  $n \in \mathbb{N}$  there exists a unique integer  $M \geq 0$  and a unique sequence  $\{c_{k+1}\}_{k=0}^{\infty}$  of integers such that

$$q_M \leq m < q_{M+1} \quad \text{and} \quad m = \sum_{k=0}^{\infty} c_{k+1}q_k,$$

with

$$0 \leq c_1 < a_1, \quad 0 \leq c_{k+1} \leq a_{k+1} \quad \text{for} \quad k \geq 1,$$

$$c_k = 0 \quad \text{whenever} \quad c_{k+1} = a_{k+1} \quad \text{for some} \quad k \geq 1,$$

and

$$c_{k+1} = 0 \quad \text{for} \quad k > M.$$

This is known as the *Ostrowski expansion*.

We will consider segments of our sum which ‘spread out’ in the unit interval. We take our inspiration from a set of intervals discussed in [6].

**DEFINITION 2.6** (Special intervals). For fixed  $\alpha$  define  $\mathcal{A}(i)$  to be the collection of non-negative integers  $n$  with Ostrowski expansions of the form

$$n = \sum_{k=i}^{\infty} c_{k+1} q_k.$$

Then for each  $i \in \mathbb{N}$  and for each  $\gamma \in \mathbb{R}/\mathbb{Z}$  we define a subset  $\mathcal{J}(i, \gamma)$  (which turns out to be an interval, see [6]) of  $\mathbb{R}/\mathbb{Z}$  by

$$\mathcal{J}(i, \gamma) = \gamma + \overline{\{n\alpha : n \in \mathcal{A}(i)\}}.$$

These intervals have some very nice properties such as

$$\sup_{N \in \mathbb{N}} \sup_{\mathcal{J} \subseteq \mathbb{R}/\mathbb{Z}} \left| \sum_{n=1}^N \chi_{\mathcal{J}}(n\alpha) - N \cdot |\mathcal{J}| \right| \leq K,$$

where  $K$  is a universal constant and the inner supremum is taken over all special intervals  $\mathcal{J}$  for  $\alpha$ . We will use what these intervals tell us about the distribution of  $n\alpha$  on the unit interval to achieve cancellation in (2.2.1).

Let

$$m = \sum_{i=0}^{n-1} c_{i+1} q_i \leq q_n - 1, \quad 0 \leq c_{i+1} \leq a_{i+1}$$

and

$$n(i, c) := \sum_{j=0}^{i-1} c_{j+1} q_j + c q_i.$$

We will use this decomposition to sum up to  $m$ .

$$\sum_{k=1}^m \frac{1}{\{\{k\alpha\}\}} = \sum_{i=0}^{n-1} \sum_{c=0}^{c_{i+1}-1} \sum_{l=n(i,c)+1}^{n(i,c)+q_i} \frac{1}{\{\{l\alpha\}\}}.$$

Note that

$$n(i, c) + q_i = n(i, c+1) \quad \text{and} \quad n(i, c_{i+1} - 1) + q_i = n(i+1, 0).$$

Let us consider a situation, where we are studying

$$\sum_{l=n(i,c)+1}^{n(i,c)+q_i} \frac{1}{\{\{l\alpha\}\}}. \quad (2.5.1)$$

We wish to approximate  $\alpha$  by  $p_i/q_i$  and achieve (almost) complete cancellation in the main term that we get.

Obviously, problems can occur. Specifically, if  $l \cdot p_i \equiv 0(q_i)$ , then we do not want to divide by 0, so we want to isolate these terms and deal with them separately. Note that since  $(p_i, q_i) = 1$ , we have a complete set of residue classes modulo  $q_i$ , so in each sum (2.5.1) we will have exactly one term,  $l = (c+1)q_i$ , where this happens. Also, there exists  $r \leq q_i$  such that

$$\begin{aligned} n(i, 0) + r &= q_i, \\ n(i, 1) + r &= 2q_i, \\ &\vdots \\ n(i, c_{i+1} - 1) + r &= c_{i+1}q_i. \end{aligned}$$

So we can consider all of these terms separately.

Finally, we consider summing over a complete set of residue classes modulo  $q_i$ . We will first consider the simple case,  $(1 \leq k \leq q_{i-1})$ , which will give us a second proof of Lemma 2.2. We write

$$\alpha = \frac{p_i}{q_i} + \frac{\xi_i}{q_i q_{i+1}}, \quad \text{where } \frac{1}{2} < |\xi_i| < 1.$$

Now

$$\sum_{k=1}^{q_i-1} \frac{1}{\{\{k\alpha\}\}} = \sum_{k=1}^{q_i-1} \frac{1}{\{\{k \frac{p_i}{q_i} + \frac{k\xi_i}{q_i q_{i+1}}\}\}}. \quad (2.5.2)$$

Now we use the fact that

$$\left\{ \left\{ \frac{kp_i}{q_i} + \frac{k\xi_i}{q_i q_{i+1}} \right\} \right\} = \left\{ \left\{ \frac{kp_i}{q_i} \right\} \right\} + \left\{ \left\{ \frac{k\xi_i}{q_i q_{i+1}} \right\} \right\},$$

unless perhaps if  $kp_i \equiv \frac{q_i}{2}$  modulo  $q_i$  (when  $2|q_i$ ), or if  $kp_i \equiv \frac{q_i \pm 1}{2}$  (when  $2|q_i + 1$ ).

Now (2.5.2) equals<sup>1</sup>

$$\sum_{k=1}^{q_i-1} \frac{1}{\{\{k \frac{p_i}{q_i}\}\}} \left( \frac{1}{1 + \{\{k \frac{p_i}{q_i}\}\}^{-1} \frac{k\xi_i}{q_i q_{i+1}}} \right) + O(1).$$

<sup>1</sup>The one or two extra term/s mentioned just above have been removed from the sum and are accounted for by the  $O(1)$  term.

Furthermore,

$$\begin{aligned} \left( \frac{1}{1 + \left\{ \left\{ \frac{kp_i}{q_i} \right\} \right\}^{-1} \frac{k\xi_i}{q_i q_{i+1}}} \right) &= 1 - \left\{ \left\{ \frac{kp_i}{q_i} \right\} \right\}^{-1} \frac{k\xi_i}{q_i q_{i+1}} \\ &\quad + \left\{ \left\{ \frac{kp_i}{q_i} \right\} \right\}^{-2} \left( \frac{k\xi_i}{q_i q_{i+1}} \right)^2 - \dots \end{aligned}$$

There exists  $n_k$  such that  $1 \leq n_k \leq q_i - 1$  and  $n_k \equiv kp_i \pmod{q_i}$ . Now we define  $n'_k$  as follows

$$n'_k := \begin{cases} n_k, & n_k \leq \frac{q_i}{2}, \\ n_k - q_i, & n_k > \frac{q_i}{2}. \end{cases}$$

Then,

$$\left\{ \left\{ \frac{kp_i}{q_i} \right\} \right\}^{-1} \frac{k\xi_i}{q_i q_{i+1}} = \frac{k\xi_i}{n'_k q_{i+1}}.$$

We then know that for all  $k$ ,

$$\left\{ \left\{ \frac{kp_i}{q_i} \right\} \right\}^{-1} \frac{k\xi_i}{q_i q_{i+1}} + \left\{ \left\{ \frac{kp_i}{q_i} \right\} \right\}^{-2} \left( \frac{k\xi_i}{q_i q_{i+1}} \right)^2 - \dots = C_k \frac{k\xi_i}{n'_k q_{i+1}}.$$

We need  $|n'_k| \geq 2$  in order to have a uniform bound over  $k$  for the constant  $C_k$ . When this is the case

$$-2 < C_k < -\frac{1}{2}$$

(apart from the one or two exceptions mentioned previously.) So we have to isolate another two terms. We write  $k_1, k_{-1}$  for the numbers, where  $k_1 p_i \equiv 1 \pmod{q_i}$  and  $k_{-1} p_i \equiv -1 \pmod{q_i}$ , respectively. So (2.5.2) becomes

$$\begin{aligned} \sum_{n_k=2}^{q_i-2} \left( \frac{1}{\left\{ \left\{ \frac{n_k}{q_i} \right\} \right\}} + C_k \left( \frac{k\xi_i q_i}{(n'_k)^2 q_{i+1}} \right) \right) &+ \frac{1}{\left\{ \left\{ k_1 \alpha \right\} \right\}} + \frac{1}{\left\{ \left\{ k_{-1} \alpha \right\} \right\}} + O(1) \\ &= \sum_{n_k=2}^{q_i-2} C_k \left( \frac{k\xi_i q_i}{(n'_k)^2 q_{i+1}} \right) + O(q_i). \end{aligned}$$

(Here we used the basic approximation from Khinchin (2.3.1) to deal with the two extra terms.)

By the rearrangement inequality, (see [4], Theorem 368), this first sum is smaller (in modulus) than

$$4q_i \left( \frac{1}{2^2} + \frac{1}{3^2} + \cdots \right),$$

which in turn is bounded above by  $4q_i$ . So

$$\sum_{k=1}^{q_i-1} \frac{1}{\{\{k\alpha\}\}} = O(q_i),$$

as required. Now, we move on to a proof of Lemma 2.4. We wish to prove, (for all  $i$ ), that

$$\sum_{c=0}^{c_{i+1}-1} \sum_{l=n(i,c)+1}^{n(i,c)+q_i} \frac{1}{\{\{l\alpha\}\}} = O(q_{i+1} \log c_{i+1}).$$

Note that if we sum

$$\sum_{l=n(i,c)+1}^{n(i,c)+q_i} \frac{1}{\{\{l\alpha\}\}},$$

then a similar argument to the proof of Lemma 2.2 shows that this is equal to

$$O(q_i) + \frac{1}{\{\{k_{(1,c)}\alpha\}\}} + \frac{1}{\{\{k_{(-1,c)}\alpha\}\}} + \frac{1}{\{\{(c+1)q_i\alpha\}\}},$$

where

$$n(i, c) + 1 \leq k_{(\pm 1, c)} \leq n(i, c) + q_i, \quad \text{and} \quad k_{(\pm 1, c)} p_i \equiv \pm 1 \pmod{q_i}.$$

Clearly,

$$k_{(\pm 1, c)} = k_{(\pm 1, 0)} + cq_i.$$

Furthermore, as  $n(i, 0) < q_i$ ,

$$k_{(\pm 1, c_{i+1}-r)} < n(i, (c_{i+1} - r)) + q_i < q_{i+1} - (r - 1)q_i.$$

Now, we calculated earlier that

$$\frac{1}{\{\{k\alpha\}\}} = \frac{1}{\{\{k \frac{p_i}{q_i}\}\}} \left( \frac{1}{1 + \{\{ \frac{kp_i}{q_i} \}\}^{-1} \frac{k\xi_i}{q_i q_{i+1}}} \right).$$

Letting  $k = k_{(1,0)}$ ,

$$\begin{aligned} \frac{1}{\{\{k_{(1,0)}\alpha\}\}} &= q_i \left( \frac{1}{1 + \frac{k_{(1,0)}\xi_i}{q_{i+1}}} \right) \\ &= \frac{q_i q_{i+1}}{q_{i+1} + k_{(1,0)}\xi_i}. \end{aligned}$$

Hence,

$$\frac{1}{\{\{k_{(1,c)}\alpha\}\}} = \frac{q_i q_{i+1}}{q_{i+1} + (k_{(1,0)} + cq_i)\xi_i}$$

and also

$$\frac{1}{\{\{k_{(-1,c)}\alpha\}\}} = \frac{-q_i q_{i+1}}{q_{i+1} - (k_{(-1,0)} + cq_i)\xi_i}.$$

Now, without loss of generality, assume that  $\xi_i > 0$ . Then

$$\frac{1}{\{\{k_{(1,c)}\alpha\}\}} < q_i \quad \text{for all } c.$$

Hence,

$$\begin{aligned} \sum_{c=0}^{c_{i+1}-1} \sum_{l=n(i,c)+1}^{n(i,c)+q_i} \frac{1}{\{\{l\alpha\}\}} &= O(q_{i+1}) + \sum_{c=0}^{c_{i+1}-1} \frac{1}{\{\{(c+1)q_i\alpha\}\}} \\ &\quad + \sum_{c=0}^{c_{i+1}-1} \frac{-q_i q_{i+1}}{q_{i+1} - (k_{(-1,0)} + cq_i)\xi_i} \\ &= O(q_{i+1}) + O(q_{i+1} \log c_{i+1}) \\ &\quad + \sum_{c=0}^{c_{i+1}-1} \frac{-q_i q_{i+1}}{q_{i+1} - (k_{(-1,0)} + cq_i)\xi_i}. \end{aligned}$$

Finally, (using  $k_{(-1,0)} \leq q_i + q_{i-1}$  and  $\xi < 1$ ),

$$\begin{aligned} \left| \sum_{c=0}^{c_{i+1}-1} \frac{-q_i q_{i+1}}{q_{i+1} - (k_{(-1,0)} + cq_i)\xi_i} \right| &\leq \frac{q_{i+1}}{c_{i+1}-1} + \cdots + \frac{q_{i+1}}{2} + q_{i+1} + 2q_{i+1} \\ &= O(q_{i+1} \log c_{i+1}). \end{aligned}$$

As this is true for all  $i$ , the condition for Lemma 2.4 follows.

**REMARK 2.7.** Equation (1.13) in [1] tells us that the sum

$$\sum_{c=0}^{c_{i+1}-1} \frac{1}{\{\{(c+1)q_i\alpha\}\}}$$

can be no smaller than  $O(q_{i+1} \log c_{i+1})$ .

**REMARK 2.8.** In our final calculation we have ignored the cancellation between the positive and negative terms. However, when  $c_{i+1} \approx a_{i+1}/2$ , for example, we get very little cancellation and our main term is

$$O(q_{i+1} \log a_{i+1}).$$

**REMARK 2.9.** In fact since convergents  $p_i/q_i$  give lower bounds for  $\alpha$  when  $i$  is even this implies that  $\xi_i$  is positive when  $i$  is even (and negative when  $i$  is odd). Now from the well-known formula for continued fractions

$$p_{i-1}q_i - p_iq_{i-1} = (-1)^i,$$

we see that for the negative terms we separated,

$$k_{(-1,c)} = cq_i + q_{i-1}.$$

These correspond to the *semiconvergents*

$$\frac{cp_i + p_{i-1}}{cq_i + q_{i-1}}$$

in the continued fraction expansion of  $\alpha$ .

**ACKNOWLEDGMENT.** The author would like to thank Alan Haynes for his invaluable advice and tutelage, and Jens Marklof for pointing out the work of Cosentino and Flaminio.

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Received November 1, 2015  
Accepted October 10, 2016

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