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Pierre Liardet (1943 – 2014†)*

Let us remind Professor Pierre Liardet during his trip around High Tatra in 2009.

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Dedicated to the memory of Professor Pierre Liardet

CONTENTS

Articles

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distribution
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Unif. Distrib. Theory **11** (2016), no.1, i–xi

PIERRE LIARDET (1943–2014)

Cor Kraaikamp

Pierre Liardet was a unique person, as a mathematician, as a friend and also as a collaborator. In papers like this often the life and work of the deceased is explained in some detail; see for example the nice and well-written obituary on Pierre's life and work by Guy Barat, Peter Grabner and Peter Hellekalek ([BGH]), or Pierre's own obituary of Gérard Rauzy ([L55, L56]). Here I hardly will do this and take a more personal approach and will speak about this complex and unique man as I remember him.

Figure 1. Pierre's picture on his own webpage.

When I met Pierre for the first time in 1986 he was already an accomplished if not famous mathematician. I had just started my Ph.D.-work with Henk Jager, who was visiting Pierre during a sabbatical year in Marseille. What immediately struck me, was the openness and hospitality of Pierre and his wife Josy. The expression "our house is your house" definitely applied to both of them, and this did not change over the years. What also struck me, was the scope of Pierre's knowledge; Pierre became famous as a Ph.D.-student of Gérard Rauzy for solving a conjecture by Serge Lang in algebraic number theory (see [L5]), and Lang even attended his Ph.D. defence. But Pierre was also a medical doctor, who worked at night at SAMU[1] in Marseille. Pierre studied medicine after he finished his Ph.D. in mathematics, and even got a Ph.D.-degree in medicine.

Henk Jager visited Pierre because he wanted to extend his knowledge on ergodic theory, a field in which Pierre had become a leading specialist after switching from algebraic number theory. A few year before this visit, Henk–together with Wieb Bosma and Freek Wiedijk–had solved a conjecture of Hendrik Lenstra (and–as turned out later–Wolfgang Döblin) on the almost sure distribution of the so-called "approximation coefficients". The proof was based on the natural extension of the regular continued fraction expansion, as studied by Shunji Ito, Hitoshi Nakada and Shigeru Tanaka, and in 1986-87 Pierre and Henk were able to adapt this approach in a very original way. In a way this paper ([L17]) testifies of many of Pierre's favourite subjects: uniform distribution, natural extensions and skew products, and continued fractions.

Pierre invited me to visit Marseille in May and June 1987. At that time he and Gérard Rauzy were busy organizing a large conference on ergodic theory and number theory[2] at the CIRM in Luminy. As a beginner I had no idea of all the people invited; a large group of Japanese mathematicians (Shunji Ito, Yuji Ito, Hitoshi Nakada, Makoto Mori, Teturo Kamae), from Austria (Peter Hellekalek, Gerhard Larcher), Holland (Michel Dekking, Mike Keane), from the US (Doug Lind), and of course from all over France! Due to the atmosphere of openness and friendliness before, during and after this conference and all the subsequent conferences which Pierre organized over the years I made friendships and collaborations which have lasted until today. These conferences were not just about mathematics; for example, after the 1987 conference there was a large party where Hitoshi and I, together with Brigitte Mossé played Bach, and the next day quite a number of us participated in a demonstration in Marseille against the FN. Often during these conferences there were large groups of people visiting Pierre and Josy in Saint Cannat, where we had wonderful dinners in their garden.

[1] *Service d'Aide Médicale Urgente.*

[2] *Arithmetics and Coding Systems*, Marseille-Luminy, June 1987.

Shortly after the '87 conference, Pierre and Josy's daughter became my girlfriend, and due to this I saw Pierre even more often; in the years which followed I visited Marseille (and in particular their home in Saint-Cannat) regularly, and often for several months on end (the longest period was in 1989–90, when I was appointed in Luminy for six months and shared Gérard Rauzy's office). Very quickly Pierre and I started to work on a project, and in doing so I got to know Pierre's personality very well. In a (at that time) recent paper on continued fractions I had found a 'gap' in the proof of the result. Pierre and I quickly decided that the result should hold (and actually could be generalized), but that the approach of the author would not lead to a proof. So we tried several other ways to get the result. For several months Pierre came up with various ideas, which I 'took down' with counter-examples; the last time this happened was during Christmas breakfast, were I mentioned a counter-example to an approach we had been working on for weeks. At first Pierre seemed hurt and a bit down (it was a stupid thing of me to do this at Christmas breakfast), but then he blurted out a vague outline of a possible other approach. After almost two days of hard work I could explain to him after dinner that he was completely correct! We wrote a paper and submitted this to a British journal. After a few months we received a letter from the journal containing an almost unintelligible handwritten referee-report of about two lines and a photocopy of the original paper, with the request to first read the original paper (in our paper we did not mention the gap in the proof). Pierre was livid, and immediately sent the journal an email in which he expressed the opinion that 'they' were complete idiots. It happened so fast that the email was sent before I could react. After this email I thought there was no point in explaining to this particular journal why we had written our paper. So we sent it to another journal, where the paper was rather promptly accepted. However, one day we were having tea in Josy's garden with some of the editors of this journal, and we were told that there is a page-fee; on the spot Pierre withdrew the paper! For me this was a rather unpleasant development; I was about to finish my Ph.D.-thesis, and therefore needed accepted papers to get at least a post-doc position. Pierre wanted to re-write the paper, but this time it was my turn to make a decision, and we sent the paper to the same journal were the original paper had appeared.

Pierre was a passionate and intense man; not only about doing math, but also in his love for his wife, children and grandchildren, or his dedication to his friends. This of course does not imply that it was always easy to be with Pierre. Like many other Frenchmen at that time Pierre had a very personal way of driving a car (meaning: very fast). During one of the conferences in Luminy Yuji Ito, Doug Lind, Michel Dekking and I had to sign some paperwork, so Pierre drove us down from Luminy to St. Charles in his Golf. On the way to St. Charles, Yuji was quietly sitting next to Pierre, while I was sitting in the middle on the backseat. Before we left St. Charles to go back to Luminy, Michel thought that it was perhaps better if Doug would sit in the middle; in this way it would be easier for both of them to have a conversation. After two minutes Doug puts his face in his hands, and shouts "I can't stand this." I look at Michel, who shrugs his shoulders (both me and Michel had been in the car of many French mathematician (Gérard Rauzy, Christian Maudiet, François Blanchard), so we were quite used to this). At that moment Pierre turns around, and asks me: "Ça va?" To which I answer "Oui Pierre, ça va très bien!" and at the same time I see we're doing about 100 km an hour in the inner city of Marseille. In fact this intensity extended to everything Pierre did, whether it was mathematics, repairing his car, walking in the mountains (which he loved, and did very often after he was told he was critically ill), or trying to safe their cat which has been poisoned by some of the neighbours.

Figure 2. Pierre's picture on some webpage on which he briefly describes his own life.

Apart from intensity there is always originality and beauty in what he did. This is obvious to anyone familiar with his famous papers, but is certainly also true for the lesser known ones, such as his papers with Pierre Stambul ([L31, L36]). Pierre curiosity never was really satisfied, and over the years he 'dived' into many new subjects. Apart from working in ergodic theory, dynamical systems and number theory, Pierre also became interested in cryptology, which he liked a lot and which offered him the opportunity to work with his oldest son Pierre-Yvan.

Figure 3. Pierre and Josy during my Ph.D.-defence in Amsterdam, April 23, 1990.

Christine and I separated in 1990, and in the years that followed my life literarily changed directions; I became a post-doc in College Park and Seattle, and met Karma in New Haven during a conference, where Mike Keane offered me a job in Delft. As a consequence I saw much less of Pierre. For me, working with Pierre was like going to a party; he was always full of ideas, and always willing to listen and if necessary to change his opinions and views. It therefore saddened me, that I saw Pierre much less frequently than before; Karma and I visited Marseille for a month at the end of the 1990's, and Pierre (together with Josy, and once with Christine and her daughter Lucy in 2002) visited Delft a number of times. Unfortunately, together with Karma I wrote only one other paper with Pierre, and it grieves me that it will never be possible to work with, or even talk to and see this beautiful man, who fought bravely against death, even when he knew that this was a fight which could not be won.

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AN IMPROVED BOUND FOR THE STAR DISCREPANCY OF SEQUENCES IN THE UNIT INTERVAL

Gerhard Larcher — Florian Puchhammer

Dedicated to the memory of Professor Pierre Liardet

ABSTRACT. It is known that there is a constant $c > 0$ such that for every sequence $x_1, x_2, \dots$ in $[0,1)$ we have for the star discrepancy D_N^* of the first N elements of the sequence that $ND_N^* \geq c \cdot \log N$ holds for infinitely many N. Let c^* be the supremum of all such c with this property. We show $c^* > 0.065664679\dots$, thereby slightly improving the estimates known until now.

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1. Introduction and statement of the result

Let $x_1, x_2, \dots$ be a point sequence in $[0,1)$. By D_N^* we denote the star discrepancy of the first N elements of the sequence, i.e.,

$$D_N^* = \sup_{x \in [0,1]} \left| \frac{\mathcal{A}_N(x)}{N} - x \right|,$$

where $\mathcal{A}_N(x) := \#\{1 \leq n \leq N | x_n < x\}$. The sequence $x_1, x_2, \dots$ is uniformly distributed in $[0,1)$ if and only if $\lim_{N\to\infty} D_N^* = 0$.

In 1972 W. M. Schmidt [7] showed that there is a positive constant c such that for all sequences $x_1, x_2, \dots$ in $[0,1)$ we have

$$D_N^* > c \cdot \frac{\log N}{N} \tag{1}$$

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for infinitely many N. The order $\frac{\log N}{N}$ is best possible. There are many known sequences for which $D_N^* \leq c' \cdot \frac{\log N}{N}$ holds for all N with an absolute constant c'. For all necessary details on discrepancy and low-discrepancy sequences see the monographs [2] or [5].

So it makes sense to define the *one-dimensional star discrepancy constant* c^* to be the supremum over all c such that (1) holds for all sequences $x_1, x_2, \ldots$ in $[0, 1)$ for infinitely many N. Or, in other words,

$$c^* := \inf_{\omega} \limsup_{N\to\infty} \frac{N D_N^*(\omega)}{\log N},$$

where the infimum is taken over all sequences $\omega = x_1, x_2, \ldots$ in $[0, 1)$, and where $D_N^*(\omega)$ denotes the star discrepancy of the first N elements of ω.

The currently best known estimates for c^* are

$$0.0646363 \ldots \leq c^* \leq 0.222 \ldots$$

The upper bound was given by Ostromoukhov [6] (thereby slightly improving earlier results of Faure (see, for example, [1])). The lower bound was given by Larcher [3].

It is the aim of this paper to improve the above lower bound for c^*. That is, what we prove

Theorem 1. *For the one-dimensional star discrepancy constant we have*

$$c^* \geq 0.065664679 \ldots$$

The idea of the proof follows a method introduced by Liardet [4] which was also used by Tijdeman and Wagner in [8] and by Larcher in [3].

2. Main ideas and proof of Theorem 1

We will heavily make use of the idea, the notation, and most of the results used and obtained in [3]. In this paper we extend the analysis carried out in the aforementioned paper. In this section we therefore repeat the most important notation and facts from [3] and explain how we extend the method to prove Theorem 1.

We consider a finite point set $\mathcal{P} = \{x_1, x_2, \ldots, x_N\}$ in $[0, 1)$ with $N = [a^t]$ for some real a, $3 \leq a \leq 3.7$, and some $t \in \mathbb{N}$. Further, we divide the index-set $A = \{1, 2, \ldots, N\}$ into index-subsets A_0, A_1, A_2, where $A_0 = \{1, 2, \ldots, [a^{t-1}]\}$, $A_2 = \{[a^t] - [a^{t-1}] + 1, \ldots, [a^t]\}$, and $A_1 = A \setminus (A_0 \cup A_2)$.

For simplicity, let us first of all assume that a^t and a^{t-1} are integers (of course this can only happen if $a = 3$). For $x \in [0, 1)$ we consider the discrepancy function $D_n(x) := \#\{i \leq n | x_i < x\} - nx = \mathcal{A}_n(x) - nx$ and we define the function $f(x) := \max_{n \in A_2} D_n(x) - \max_{n \in A_0} D_n(x)$.

In [3] it was shown that the function f has the following properties:

(i) $f(0) = f(1) = 0$.

(ii) f is piecewise linear, piecewise monotonically decreasing, and $|f|$ is bounded by a^t.

(iii) f is left-continuous and each discontinuity constitutes a positive jump.

(iv) The slope of f is always between $-a^t$ and $s_0 := -a^{t-1}(a-2)$.

(v) If f is continuous on $[x, y]$ then the slope of $f(x)$ and $f(y)$ can differ at most by a^{t-1}.

(vi) f has discontinuities with a jump of height at least 1 in all points x_i with $i \in A_1$.

Further it was shown in [3, Lemma 2.11] that for given a and t there exists a function $f^*_{\text{strong}} : \ [0, 1] \to \mathbb{R}$ satisfying (i)–(vi) for some $x_1, \ldots, x_N$ (we say f^*_{strong} is *strongly admissible*) such that

$$\int_0^1 \left| f^*_{\text{strong}}(x) \right| \, \mathrm{d}x = \min_{g \text{ strongly admissible}} \int_0^1 |g(x)| \, \mathrm{d}x,$$

and (in [3, Lemma 2.14]) that for every $\varepsilon > 0$ and (now arbitrary) $a \in [3, 4]$ and t with $t \geq t(\varepsilon)$

$$\int_0^1 |f^*_{\text{strong}}(x)| \, \mathrm{d}x \geq \frac{(a-2)(8a+3)}{8(1-2a)^2} - \varepsilon.$$

Finally, we finished the proof of the Theorem in [3] in the following way:

It was shown that (see Section 3 in [3])

$$\begin{aligned}
\int_0^1 \left(\max_{n \in A} D_n(x) - \min_{n \in A} D_n(x) \right) \mathrm{d}x &\geq t \int_0^1 \left| f^*_{\text{strong}}(x) \right| \, \mathrm{d}x \\
&\geq t \left(\frac{(a-2)(8a+3)}{8(1-2a)^2} - \varepsilon \right) \\
&\geq \frac{\log N}{\log a} \cdot \left(\frac{(a-2)(8a+3)}{8(1-2a)^2} - \varepsilon \right) \\
&\geq 2 \log N \cdot 0.0646363 \ldots
\end{aligned}$$

if we choose $a = 3.71866\ldots$ and N large enough. Hence there exist $x \in [0, 1]$ and $n \leq N$ with

$$D_n(x) \geq 0.0646363 \ldots \cdot \log N$$

and Theorem 1.1 from [3] follows.

To improve the above result from [3] in the present paper we proceed as follows: We show that f has to satisfy an even more restrictive property (vi′) instead of property (vi) and we call a function g satisfying (i)–(v) and (vi′) *strictly admissible*. Moreover, we show that there exists a strictly admissible function $f^*_{\text{strict}}\colon\ [0,1] \to \mathbb{R}$ with

$$\int_0^1 |f^*_{\text{strict}}(x)|\ \mathrm{d}x = \min_{g \text{ strictly admissible}} \int_0^1 |g(x)|\ \mathrm{d}x$$

and

$$\int_0^1 |f^*_{\text{strict}}(x)|\ \mathrm{d}x \geq \frac{(a-2)\left(12a+9+(a-2)(4a-3)\log\left(1+\frac{1}{a-2}\right)\right)}{a\left(a-\frac{1}{2}\right)^2\left(3+(a-2)\log\left(1+\frac{1}{a-2}\right)\right)} - \varepsilon$$

for all $a \in (3, 3.7]$ and $t \geq t(\varepsilon)$.

Note that, in the following, we will work with a^t and a^{t-1} as if they were integers and we will obtain the above result without "$-\varepsilon$" and for all $t \geq t_0$ in this case. For working with $[a^{t-1}]$ and $[a^t]$ instead of a^{t-1} and a^t we then easily obtain the stated result.

In the very same way as in [3] and as described above we then obtain $D_n(x) \geq 0.065664679\ldots \cdot \log N$ for some $x \in [0,1]$ and $n \geq N$ by choosing $a = 3.62079\ldots$ Consequently, Theorem 1 follows.

So it remains to prove the two main auxiliary results, namely, that a stronger property (vi′) for f as well as the lower bound for $\int_0^1 |f^*_{\text{strict}}(x)|\,\mathrm{d}x$ as stated above hold. This is carried out in the next section. For the proofs of these two results we will have to use some facts already obtained in [3], again.

3. Proof of the auxiliary results

Lemma 1. *Let $j \in A_2$, i.e., $j = a^t - a^{t-1} + k$ for some integer k, $1 \leq k < a^{t-1}$, and assume that $f(x) = \max_{n \in A_2} D_n(x) - \min_{n \in A_0} D_n(x)$ has a discontinuity in x_j. Let further l_j, $r_j \in A$ such that $\mathcal{P} \cap (x_{l_j}, x_{r_j}) = \{x_j\}$. If there exists an $\overline{x} \in (x_j, x_{r_j})$ such that, in $\overline{x}$ f has slope $s(\overline{x}) > s_0 - k$, then $f(\underline{x}) \geq f(\overline{x}) - s_0(\overline{x} - \underline{x})$ for all $\underline{x} \in [x_{l_j}, x_j)$. Here, again, $s_0 = a^{t-1}(a-2)$ as defined in property (iv) above.*

Remark. The meaning of Lemma 1 is illustrated in Figure 1. Using the same notation $f(\underline{x})$ lies above the line with slope s_0 reaching back from the point $(\overline{x}, f(\overline{x}))$ (dashed) in case the slope of f (solid) becomes flatter than $s_0 - k$.

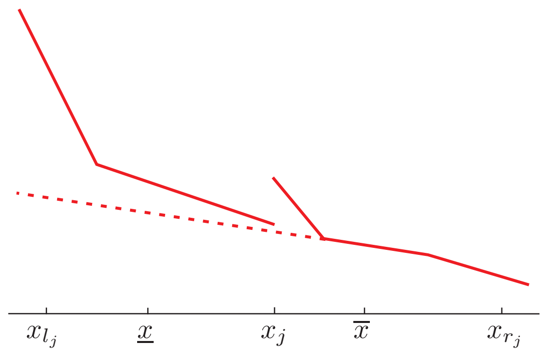

FIGURE 1.

P r o o f o f L e m m a 1. Let $\underline{x}$, $\overline{x}$ be like above with $s(\overline{x}) > s_0 - k$. We set

$$\overline{n}_i = n_i(\overline{x}) \quad \text{and} \quad \underline{n}_i = n_i(\underline{x})$$

such that

$$D_{\overline{n}_i}(\overline{x}) = \max_{n\in A_i} D_n(\overline{x}) \quad \text{and} \quad D_{\underline{n}_i}(\underline{x}) = \max_{n\in A_i} D_n(\underline{x}).$$

So

$$f(\overline{x}) = D_{\overline{n}_2}(\overline{x}) - D_{\overline{n}_0}(\overline{x}) \quad \text{and} \quad f(\underline{x}) = D_{\underline{n}_2}(\underline{x}) - D_{\underline{n}_0}(\underline{x}).$$

First we show that $\overline{n}_2 < j$. Indeed, we have

$$a^{t-1} - \overline{n}_2 \geq \overline{n}_0 - \overline{n}_2 = s(\overline{x}) > s_0 - k = -a^{t-1}(a-2) - k.$$

Thus, $\overline{n}_2 < a^t - a^{t-1} + k = j$.

Since $\mathcal{A}_n$ does not change its value in x_j for $n < j$, $D_{\overline{n}_2}$ does not have a jump in x_j. Consequently, $D_{\overline{n}_2}(\overline{x}) = D_{\overline{n}_2}(\underline{x}) - \overline{n}_2(\overline{x} - \underline{x})$. This observation yields

$$D_{\underline{n}_2}(\underline{x}) - D_{\overline{n}_2}(\overline{x}) \geq D_{\overline{n}_2}(\underline{x}) - D_{\overline{n}_2}(\overline{x}) = \overline{n}_2(\overline{x} - \underline{x}).$$

By the same argument we additionally obtain

$$D_{\underline{n}_0}(\underline{x}) - D_{\overline{n}_0}(\overline{x}) \leq D_{\underline{n}_0}(\underline{x}) - D_{\underline{n}_0}(\overline{x}) = \underline{n}_0(\overline{x} - \underline{x}).$$

Alltogether

$$\begin{aligned} f(\underline{x}) - f(\overline{x}) &= \big(D_{\underline{n}_2}(\underline{x}) - D_{\overline{n}_2}(\overline{x})\big) - \big(D_{\underline{n}_0}(\underline{x}) - D_{\overline{n}_0}(\overline{x})\big) \\ &\geq (\overline{n}_2 - \underline{n}_0)(\overline{x} - \underline{x}) \geq -s_0(\overline{x} - \underline{x}) \end{aligned}$$

and the result follows. □

In addition to the new property of f obtained in Lemma 1 one can easily convince oneself that f is continuous at x_1. This result is not very efficient yet but nice for calculation purposes.We will use this fact in the following concept of strict admissibility.

Definition 1. A function $g\colon [0,1] \to \mathbb{R}$ is called strictly admissible if it satisfies conditions (i)–(v) and the following additional condition (vi′).
There exists a set $\Gamma = \{\xi_1, \xi_2, \dots, \xi_{a^t-1}\} \subset [0,1)$ such that:

a) If g has a jump in ξ then $\xi \in \Gamma$.
b) There exists a set $\Gamma_1 \subset \Gamma$, $|\Gamma_1| = a^{t-1}(a-2)$, such that f has a jump of height at least one in each $\xi \in \Gamma_1$.
c) There exist $a^{t-1} - 1$ further points $\{\xi_{k_1}, \xi_{k_2}, \dots, \xi_{k_{a^{t-1}-1}}\} =: \Gamma_2$ with the following property: For each $1 \leq n < a^{t-1}$ let $\xi_{l_n}, \xi_{r_n} \in \Gamma \cup \{0,1\}$ such that $\Gamma \cap (\xi_{l_n}, \xi_{r_n}) = \{\xi_{k_n}\}$. Now, if there is an $\overline{x} \in (\xi_{l_k}, \xi_{r_k})$ with
$$s(\overline{x}) > s_0 - n \tag{2}$$
then
$$g(\underline{x}) \geq g(\overline{x}) - s_0(\overline{x} - \underline{x}) \tag{3}$$
for all $\underline{x} \in [\xi_{l_n}, \xi_{k_n})$. Here, $s(x)$ denotes the slope of g in x.

From the paper [3] and from Lemma 1 it follows that f is strictly admissible. The space of strictly admissible functions, again, is obviously closed with respect to pointwise convergence. Hence, there exists f^*_{strict} strictly admissible with
$$\int_0^1 |f(x)| \,\mathrm{d}x \geq \min_{g \text{ strictly admissible}} \int_0^1 |g(x)| \,\mathrm{d}x = \int_0^1 |f^*_{\text{strict}}(x)| \,\mathrm{d}x.$$
We intend to estimate $\int_0^1 |f^*_{\text{strict}}(x)| \,\mathrm{d}x$ from below. To this end we have to derive some properties of f^*_{strict}.

Lemma 2. *Let f^*_{strict} have a discontinuity in γ. Then there exist two zeros α, β of f^*_{strict} with $\alpha < \gamma < \beta$ such that γ is the only discontinuity in the interval (α, β).*

Proof. First of all, if γ is the only point at which f^*_{strict} has a jump, the claim is fulfilled with $\alpha = 0$ and $\beta = 1$. Hence it suffices to show the following statement: Let f^*_{strict} have two successive discontinuities in, say, a_1 and a_2, $0 < a_1 < a_2 < 1$. Then f^*_{strict} has a zero in the interval (a_1, a_2).

For contradiction we assume $f^*_{\text{strict}} > 0$ on (a_1, a_2) (the case $f^*_{\text{strict}} < 0$ can be treated quite similarly). In what follows, we will construct a strictly admissible function $\tilde{f}$ such that

$$\int_0^1 |\tilde{f}(x)|\,\mathrm{d}x < \int_0^1 |f^*_{\text{strict}}(x)|\,\mathrm{d}x,$$

which clearly contradicts the definition of f^*_{strict}.

Naturally, we need to take special care in constructing $\tilde{f}$ if either $a_1 \in \Gamma_2$ or $a_2 \in \Gamma_2$ which was defined in Definition 1. Moreover, if we manage to preserve the height of any existing jump in any other case then (vi′.b) is automatically fulfilled for $\tilde{f}$.

First of all, we notice that f^*_{strict} cannot have a bend at, say, $y \in (a_1, a_2)$ such that the slope before the bend is greater than afterwards. We say f^*_{strict} has a *bend* in y if f^*_{strict} is continuous in y and if it changes its slope in y. Indeed, let $\delta > 0$ such that the slope of f^*_{strict} is constant on $[y-\delta, y)$ as well as on $(y, y+\delta]$. Then, as can be seen in Figure 2, we may interchange those pieces such that the resulting function $\tilde{f}$ (solid) remains continuous in $[y-\delta, y+\delta]$. Its absolute integral, however, is smaller than that of f^*_{strict} (dashed). Thus, we need only

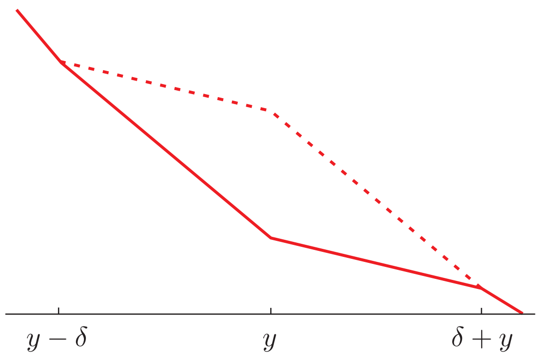

FIGURE 2.

consider bends where f^*_{strict} becomes flatter.

Let now $a_2 \notin \Gamma_2$. We choose $\delta_1 > 0$ such that the slope of f^*_{strict} is a constant s_1 on $(a_2, a_2 + \delta_1)$. Furthermore, we set

$$s = \min\{s^*(x) :\ x \in (a_1, a_2 + \delta_1)\},$$

where s^* denotes the slope of f^*_{strict} and where we define $s^*(a_2)$ as its left limit. Now, let $0 < \delta \le \min\{-2f^*_{\text{strict}}(a_2)/(s_1 + s), \delta_1\}$. With this choice of δ we have

$$f^*_{\text{strict}}(a_2) + s\delta > -f^*_{\text{strict}}(a_2 + \delta).$$

In this case we may thus construct $\tilde{f}$ by moving the discontinuity to $\tilde{a}_2 = a_2 + \delta$. The missing part of $\tilde{f}$ on the left of $\tilde{a}_2$ of length δ is then chosen such that $\tilde{f}$

is continuous in a_2 and such that it has constant slope s. This construction is visualized in Figure 3 (again f^*_{strict} is represented by the dashed and $\tilde{f}$ by the solid line). This choice for the slope guarantees that the height of the jump is preserved and, additionally, property (vi′.c) from Definition 1, too, cannot be violated by this construction if $a_1 \in \Gamma_2$.

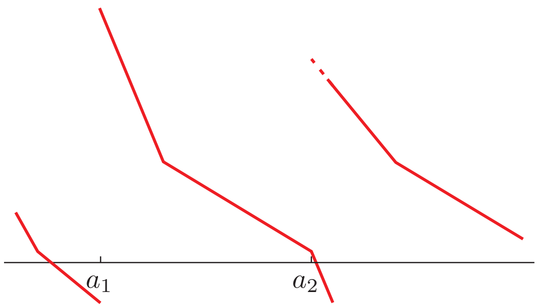

Figure 3.

Certainly, the same construction also works if $a_2 = \xi_{k_n} \in \Gamma_2$ for a suitable k_n with $s^* \leq -a^{t-1}(a-2) - n$ between a_2 and the next discontinuity of f^*_{strict}.

However, if there is some point $x > a_2$ before the next jump of f^*_{strict} with $s^*(x) > -a^{t-1}(a-2) - n$ we have to proceed differently. In this case, we keep the discontinuity at a_2 and take the smallest such x, call it $\overline{x}$. We define

$$\tilde{f}(x) := \begin{cases} s_0(\overline{x} - x) + f^*_{\text{strict}}(\overline{x}) & \text{if } x \in [\overline{x} - \delta, \overline{x}), \\ s^*(\overline{x})(\overline{x} - \delta - x) + \tilde{f}(\overline{x} - \delta) & \text{if } x \in [a_2, \overline{x} - \delta), \\ f^*_{\text{strict}}(x) & \text{else,} \end{cases}$$

where $\delta > 0$ is such that we still have a positive jump in a_2. Recall that a discontinuity always constitutes a positive jump, hence this is possible. Figure 4 shows $\tilde{f}$ (solid) as well as f^*_{strict} (dashed) in this case. Notice that, again,

$$\int_0^1 |\tilde{f}(x)|\,\mathrm{d}x < \int_0^1 |f^*_{\text{strict}}(x)|\,\mathrm{d}x$$

and that (vi′.c) from Definition 1 is not violated for a_2. Additionally, the condition on δ guarantees that (vi′.c) is not violated for a_1 if $a_1 \in \Gamma_2$ either. Moreover, we need not take care of the height of the jump in a_2, since Γ_1 and Γ_2 are disjoint. The dotted line represents the line with slope s_0 reaching back from $\{\overline{x}, f^*_{\text{strict}}(\overline{x})\}$ which occurs in Definition 1. $\square$

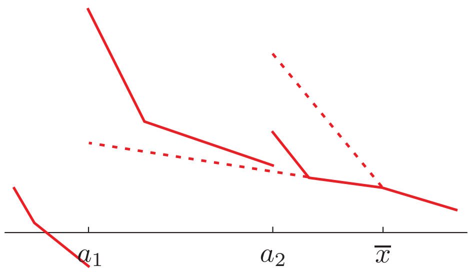

FIGURE 4.

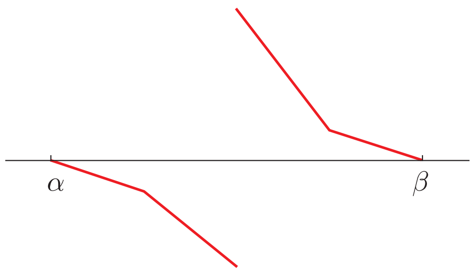

FIGURE 5.

Thus, f^*_{strict} consists of parts Q, each of which is defined on an interval $[\alpha, \beta]$ with $f^*_{\text{strict}}(\alpha) = f^*_{\text{strict}}(\beta) = 0$ and such that there is exactly one discontinuity in (α, β), see Figure 5.

In the following we determine the number of such Q's for f^*_{strict}.

LEMMA 3. *The function f^*_{strict} has exactly $a^t - 1$ discontinuities.*

P r o o f. Assume that the total number of discontinuities of f^*_{strict} is less than $a^t - 1$. Then, in the following, we will define a strictly admissible function $\tilde{f}$ from f^*_{strict} whose absolute integral is smaller than that of f^*_{strict}. Let Γ^* be the set Γ from property (vi$'$) for the function f^*_{strict}.

By assumption there is a $\xi^* \in \Gamma^*$ such that f^*_{strict} is continuous in ξ^*. The definition of Γ^*_1 (i.e., the set Γ_1 for f^*_{strict}) guarantees $\xi^* \notin \Gamma^*_1$. Assume that $\xi^* \in \Gamma^*_2$ (the case $\xi^* \in \Gamma^*_0 := \Gamma^* \setminus (\Gamma^*_1 \cup \Gamma^*_2)$ can be treated quite analogously).

Now choose $\gamma \in \Gamma^*$ such that f^*_{strict} has a jump in γ. We show that $\gamma \in \Gamma^*_1$ and that f^*_{strict} has a jump of height 1 in γ (case d) below). Indeed, à priori we are in one of the following four cases:

a) $\gamma \in \Gamma^*_0$,

b) $\gamma \in \Gamma^*_2$,

c) $\gamma \in \Gamma^*_1$ with a jump of height greater than 1, or

d) $\gamma \in \Gamma^*_1$ with a jump of height exactly equal to 1 in γ.

Assume that $\gamma \in \Gamma^*_2$ (case b). By Lemma 2 γ is isolated by two successive zeros of f^*_{strict}. Hence (3) from property (vi$'$) cannot hold, and therefore (2) from the same property does not hold either. Consequently, (see Fig. 6) we can take a point $\tilde{\xi}$ on the left of γ and insert a short piece of minimal slope on $[\tilde{\xi}, \gamma)$ without interferring with property (vi′.c). Again, the dashed line represents f^*_{strict} and the solid one the resulting new function $\tilde{f}$. The new set $\tilde{\Gamma}$ is the set Γ^* with ξ^* replaced by $\tilde{\xi}$.

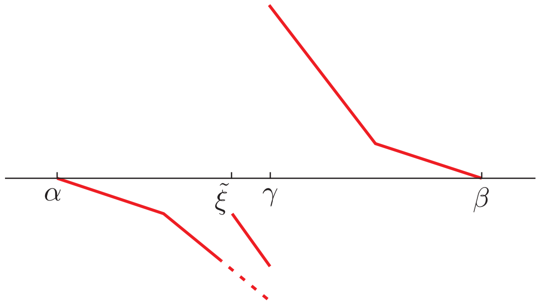

Figure 6.

This construction also works for case a) in the same way, and, with some special care, i.e., the jump of $\tilde{f}$ in γ maintains a height of at least one, for the case c) too.

Consequently, f^*_{strict} can only have the $a^{t-1}(a-2)$ jumps at the positions given by Γ^*_1. All these jumps have height exactly equal to one and there are absolutely no further jumps. Obviously, f^*_{strict} cannot have slope $-a^t$ everywhere, since then

$$0 > a^{t-1}(a-2) - a^t = f^*_{\text{strict}}(1),$$

a contradiction to property (i). Thus, there exists an interval $[\delta_1, \delta_2]$ such that $f^*_{\text{strict}} > 0$ (or $f^*_{\text{strict}} < 0$) on $[\delta_1, \delta_2]$ and its slope is greater than $-a^t$. We choose $\delta' \in (\delta_1, \delta_2)$ sufficiently close to δ_1 (or to δ_2) and define

$$\tilde{f}(x) = \begin{cases} f^*_{\text{strict}}(\delta_1) - a^t(x - \delta_1) & \text{if } x \in (\delta_1, \delta'], \\ f^*_{\text{strict}}(x) & \text{else,} \end{cases}$$

or

$$\tilde{f}(x) = \begin{cases} f^*_{\text{strict}}(\delta_2) - a^t(x - \delta_2) & \text{if } x \in (\delta', \delta_2], \\ f^*_{\text{strict}}(x) & \text{else,} \end{cases}$$

respectively. See Figures 7 and 8.

□

From the above results we obtain that f^*_{strict} has to be of the following form: It divides $[0, 1)$ into $a^t - 1$ parts $[\alpha, \beta)$ with $f^*_{\text{strict}}(\alpha) = f^*_{\text{strict}}(\beta) = 0$, and f^*_{strict} has exactly one discontinuity $\gamma \in (\alpha, \beta)$. We say that $[\alpha, \beta)$ is of type Q_i if $\gamma \in \Gamma^*_i$ for $i = 0, 1, 2$.

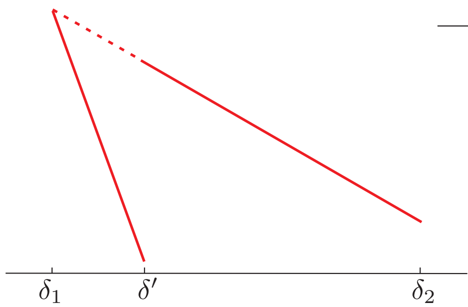

FIGURE 7. Case $f^*_{\text{strict}} > 0$ on $[\delta_1, \delta_2]$

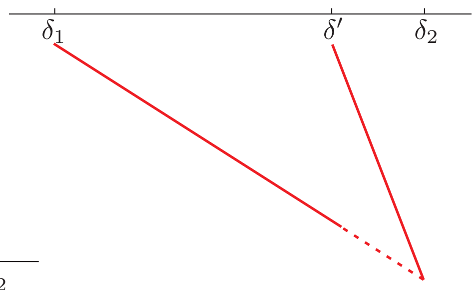

FIGURE 8. Case $f^*_{\text{strict}} < 0$ on $[\delta_1, \delta_2]$

From [3], the equation (2), we know that, for an interval of type the Q_0 (this corresponds to the type Q'' in the abovementioned paper), we have

$$\int_\alpha^\beta |f^*_{\text{strict}}(x)| \, \mathrm{d}x \geq \chi^2 \, \frac{a^{t-1}(a-2)}{4}, \qquad \chi = \beta - \alpha,$$

and from [3, Lemma 2.12] and the considerations following the proof of this lemma we know that for an interval of type Q_1 (this corresponds to the type Q' in the abovementioned paper) we have

$$\int_\alpha^\beta |f^*_{\text{strict}}(x)| \, \mathrm{d}x \geq \frac{\chi\left(4 - a^{t-1}\chi\right)}{16}, \qquad \chi = \beta - \alpha.$$

Moreover, we know from [3, Lemma 2.10] that for f^*_{strict} all a^{t-1} intervals Q_0 have the same length and all $a^t - 2a^{t-1}$ intervals Q_1 have the same length.

Lemma 4. *For $1 \leq n \leq a^{t-1} - 1$ let $Q_2^{(n)}$ be given by the interval $[\alpha, \beta)$. Then we have*

$$\int_{Q_2^{(n)}} |f^*_{strict}(x)| \, \mathrm{d}x \geq (\beta - \alpha)^2 \frac{|s_0|(n + |s_0|)}{2(n + 2|s_0|)}.$$

P r o o f. This follows from the remark preceeding Lemma 4 and simple calculations. □

To finish the proof of our theorem we finally show:

Lemma 5. *For all $3 \leq a \leq 3.7$ we have*

$$\int_0^1 |f^*_{strict}(x)| \, \mathrm{d}x \geq \frac{(a-2)\left(12a + 9 + (a-2)(4a-3)\log\left(1 + \frac{1}{a-2}\right)\right)}{16\left(a - \frac{1}{2}\right)^2 \left(3 + (a-2)\log\left(1 + \frac{1}{a-2}\right)\right)}.$$

Proof. Due to Lemma 4 and the remarks preceeding it we have to minimize the right hand-side of

$$\begin{aligned}\int_0^1 |f^*_{\text{strict}}(x)|\,\mathrm{d}x &\geq a^{t-1}\cdot\chi_0^2\,\frac{a^{t-1}(a-2)}{4}+a^{t-1}(a-2)\cdot\frac{\chi_1\left(4-a^{t-1}\chi_1\right)}{16}\\ &\quad+\sum_{n=1}^{a^{t-1}-1}\left(\chi_2^{(n)}\right)^2\frac{|s_0|(n+|s_0|)}{2(n+2|s_0|)}\\ &=: a^{t-1}\cdot\chi_0^2\tilde{A}_0+a^{t-1}(a-2)\cdot\frac{\chi_1\left(4-a^{t-1}\chi_1\right)}{16}\\ &\quad+\sum_{n=1}^{a^{t-1}-1}\left(\chi_2^{(n)}\right)^2\tilde{A}_n\end{aligned}$$

with respect to $\chi_0,\chi_1,\chi_2^{(n)}\geq 0$ (these quantities denote the lengths of the intervals $Q_0,Q_1,Q_2^{(n)}$) under the constraint

$$a^{t-1}\chi_0+a^{t-1}(a-2)\chi_1+\sum_{n=1}^{a^{t-1}-1}\chi_2^{(n)}=1.$$

The Lagrangian approach immediately implies

$$\tilde{A}_0\chi_0=\tilde{A}_n\chi_2^{(n)}\quad\text{for all }\ 1\leq n<a^{t-1}.$$

The constraint therefore yields

$$\chi_0=\frac{1-a^{t-1}(a-2)\chi_1}{a^{t-1}+\sum_{n=1}^{a^{t-1}-1}\frac{\tilde{A}_0}{\tilde{A}_n}}.$$

Moreover, the denominator in the above equation simplifies to

$$\begin{aligned}a^{t-1}+\sum_{n=1}^{a^{t-1}-1}\frac{\tilde{A}_0}{\tilde{A}_n}&=a^{t-1}+\sum_{n=1}^{a^{t-1}-1}\left(1-\frac{n}{2(|s_0|+n)}\right)\\ &=2a^{t-1}-1-\frac{1}{2}\sum_{n=|s_0|+1}^{a^{t-1}-1+|s_0|}\left(1-\frac{|s_0|}{n}\right)\\ &=\frac{1}{2}\left(3a^{t-1}-1+|s_0|\sum_{n=|s_0|+1}^{a^{t-1}-1+|s_0|}\frac{1}{n}\right).\end{aligned}$$

The latter sum can be bounded by $\log(1+1/(a-2))$ from above. We summarize our intermediate findings and obtain

$$\int_0^1 |f^*_{\text{strict}}(x)|\,\mathrm{d}x \geq \frac{(a-2)\left(1-a^{t-1}(a-2)\chi_1\right)^2}{2\left(3+(a-2)\log\left(1+\frac{1}{a-2}\right)\right)} + a^{t-1}(a-2)\frac{\chi_1(4-a^{t-1}\chi_1)}{16} =: p(\chi_1).$$

Now, our goal is to minimize the function p. We immediately see that p is a polynomial of degree two and its leading coefficient is positive for all $3 < a \leq 3.7$. Thus, it attains its minimum at its only critical point

$$\chi_{\text{crit}} = a^{1-t}\,\frac{2\left(4a-11-(a-2)\log\left(1+\frac{1}{a-2}\right)\right)}{29+8a(a-4)-(a-2)\log\left(1+\frac{1}{a-2}\right)}.$$

On the other hand, from the proof of Lemma 2.13 in [3] we know that we have the following bounds for χ_1

$$\chi_{\min} := \frac{a^{1-t}}{a-\frac{1}{2}} \leq \chi_1 \leq \frac{a^{1-t}}{a-\frac{3}{2}}.$$

We will show that $\chi_{\text{crit}} \leq \chi_{\min}$. Indeed, it can easily be verified that the denominator of χ_{crit} is positive. Thus, $\chi_{\text{crit}} > \chi_{\min}$ if and only if

$$0 > 3a-9-(a-1)(a-2)\log\left(1+\frac{1}{a-2}\right) =: q(a).$$

We observe that $q(3.7) < 0$ and, additionally, that $q'(a) > 0$ for all $a \in (3, 3.7]$. Hence

$$\chi_1 = \frac{a^{1-t}}{a-\frac{1}{2}}$$

and by inserting this value into the function p the result follows. □

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UNIFORM DISTRIBUTION OF THE SEQUENCE OF BALANCING NUMBERS MODULO m

Prasanta Kumar Ray — Bijan Kumar Patel

Dedicated to the memory of Professor Pierre Liardet

ABSTRACT. The balancing numbers and the balancers were introduced by Behera et al. in the year 1999, which were obtained from a simple diophantine equation. The goal of this paper is to investigate the moduli for which all the residues appear with equal frequency with a single period in the sequence of balancing numbers. Also, it is claimed that, the balancing numbers are uniformly distributed modulo 2, and this holds for all other powers of 2 as well. Further, it is shown that the balancing numbers are not uniformly distributed over odd primes.

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1. Introduction

The concept of balancing numbers were first introduced by Behera and Panda [1] when they considered the integer solutions of the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r)$$

for some positive integers n and r, calling r the balancer corresponding to the balancing number n. The nth balancing number is denoted by B_n which satisfy the recurrence relation

$$B_{n+1} = 6B_n - B_{n-1}, \text{ for n} \geq 1,$$

with initial values $B_0 = 0$, $B_1 = 1$. The closed form of balancing numbers which is also called as Binet's formula for balancing number is given by $B_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}$, where $\lambda_1 = 3 + \sqrt{8}$ and $\lambda_2 = 3 - \sqrt{8}$ are the roots of the equation $\lambda^2 - 6\lambda + 1 = 0$.

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In the year 1960, Wall studied the periodicity of Fibonacci numbers modulo arbitrary natural numbers [8]. He explicitly determined the period length of $a_{n+2} = a_{n+1} + a_n \pmod m$ in terms a_0, a_1 and m. By Niederreiter [3], the Fibonacci sequence

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n$$

is uniformly distributed modulo m if and only if $m = 5^k$, $k = 1, 2, \ldots$ Recently, Panda et al. established the periodicity of balancing numbers modulo primes and studied the periods of sequence of balancing numbers modulo balancing, The Pell and the associated Pell numbers [6]. In addition, the authors also shown that the period of this sequence coincides with the modulus of congruence if there exists the modulus in any power of 2. In [4], it is given that for the sequence of rational integers $\{a_n\}$, define $A(j, m; N)$ as the number of terms among $a_1, a_2, a_3, \ldots, a_N$ that satisfy the congruence $a_n \equiv j \ (mod\ m)$. Then $\{a_n\}$ is uniformly distributed mod m in case

$$\lim_{N\to\infty} \frac{A(j, m; N)}{N} = \frac{1}{m}, \quad \text{for each} \quad j = 0, 1, \ldots, m-1. \tag{1}$$

In this article, we survey how frequently each residue is excepted to appear. It is proved that, the balancing numbers are uniformly distributed modulo even prime 2, and are not uniformly distributed over odd primes. Further, it is also shown that the balancing numbers attains uniform values from each residue class modulo 2^k for any natural number k.

2. Distribution of sequence of balancing numbers modulo primes

Panda [5] and Ray [7] established some interesting results of divisibility and the greatest common divisor of balancing numbers. We summarize few of them.

Lemma 2.1. *If n and k are natural numbers, then B_k divides B_{nk}.*

Lemma 2.2. *If m and n are natural numbers, then $(B_m, B_n) = B_{(m,n)}$.*

Lemma 2.3. *If m and n are any natural numbers which $m > 1$, then $B_m | B_n$ if and only if $m|n$.*

The following result can be easily verified.

Lemma 2.4. *Every positive integer m divides some balancing number whose index does not exceed m^2.*

The following result is proved in [6] .

Lemma 2.5. *If p is a prime of the form $8x \pm 1$, then $B_{p-1} \equiv 0 \pmod{p}$, $B_p \equiv 0 \pmod{p}$; further if the prime p is of the form $8x \pm 3$, then $B_p \equiv -1 \pmod{p}$, $B_{p+1} \equiv 0 \pmod{p}$.*

The following result shows that the sequence of balancing number is uniformly distributed modulo even prime.

Theorem 2.6. *The sequence of balancing numbers is uniformly distributed modulo 2.*

Proof. If all the balancing numbers are reduced modulo 2, then the sequence of the least residues is obtained where the initial string is

$$\text{“ } 0, 1, 0, 1, 0, 1, \ldots \text{”},$$

which is periodic with the period length 2. Using (1), it would seem that

$$\lim_{n\to\infty} \frac{1}{N} A(j,\, 2;\, N) = \frac{1}{2} \quad \text{for } \ j = 0, 1,$$

which means that the sequence of balancing numbers is uniformly distributed modulo 2. □

For any odd prime, say for $p = 5$, it is observed that the sequence of balancing numbers is not uniformly distributed modulo 5. Because, if all balancing numbers are reduced modulo 5, then the sequence $0, 1, 1, 0, 4, 4, 0, 1, 1, 0, 4, 4, 0, \ldots$ of the least residues is obtained which is periodic with the period length 6. But it would seem that among the first six elements of the sequence, it does not occur at the most in one element from each residue class modulo 5. We generalize this result as follows:

Theorem 2.7. *The sequence of balancing numbers is not uniformly distributed modulo p for any odd prime p.*

Proof. Let p be an odd prime. Let $\lambda > 0$ be the minimal such that $B_\lambda \equiv 0 \pmod{p}$. Then $B_{k\lambda} \equiv 0 \pmod{p}$ for all $k = 1, 2, \ldots$ If $B_\gamma \equiv 0 \pmod{p}$ and $k\lambda < \gamma < (k+1)\lambda$, then for n.s.d.$(\lambda, \gamma) = max\{n \in \mathbb{N} : n|\lambda \wedge n|\gamma\} = \beta$, we have $B_\beta \equiv 0 \pmod{p}$ and $0 < \beta < \lambda$, which is a contradiction. Thus all $B_n \equiv 0 \pmod{p}$ are $B_{k\lambda} \equiv 0 \pmod{p}$ for $k = 1, 2, \ldots$ It follows that,

$$A(0,\, p;\, N) = \frac{N}{\lambda} + O(1),$$

which ends the proof. □

For each natural number m, $\pi(m)$ denotes the periodic length of the sequence of balancing numbers modulo m. In [6], Panda et al. shown some divisibility properties concerning the periodicity of balancing numbers. Using those results, we have the following observations.

Observation 2.8. We know that for any natural number $m > 1$, $\pi(m) = m$ if and only if $m = 2^k$ for some natural number k. Therefore, if the sequence of balancing numbers modulo m is uniform, then evidently $m|\pi(m)$.

Observation 2.9. Let p be a prime. If $p = 8x \pm 1$, then the sequence of balancing numbers modulo p cannot be uniform for such p, because $\pi(p)|p-1$ but $p \nmid p-1$ and so, $p \nmid \pi(p)$.

Observation 2.10. If $p = 8x \pm 3$, then the sequence of balancing numbers modulo p cannot be uniform. As $\pi(p)|p+1$ and $p \nmid p+1$, therefore $p \nmid \pi(p)$.

Observation 2.11. If the sequence of balancing numbers modulo m is uniform and n divides m, then the sequence of balancing numbers modulo n is also uniform.

Observation 2.12. If $\pi(n)$ divides $\pi(m)$ is b, then with in $\pi(n)$ terms each residue appears b divides x times. Then, the sequence of balancing numbers modulo n is uniform.

The following result is useful while proving the subsequent result.

Lemma 2.13. *Let B_n be the nth balancing number, then*

$$B_n = 3^{n-1} \sum_{j=0}^{\infty} \left(\frac{\sqrt{8}}{3}\right)^{2j} \binom{n}{2j+1}.$$

P r o o f. Using Binet's formula for balancing numbers and expanding by binomial theorem, we get

$$\begin{aligned} B_n &= \frac{\lambda_1^n - \lambda_2^n}{2\sqrt{8}} \\ &= \frac{3^n}{2\sqrt{8}} \left[\left(1 + \frac{\sqrt{8}}{3}\right)^n - \left(1 - \frac{\sqrt{8}}{3}\right)^n \right] \\ &= \frac{3^n}{2\sqrt{8}} \left[\left(1 + \binom{n}{1} \left(\frac{\sqrt{8}}{3}\right) + \binom{n}{2} \left(\frac{\sqrt{8}}{3}\right)^2 + \cdots \right) \right. \\ &\qquad \left. - \left(1 - \binom{n}{1} \left(\frac{\sqrt{8}}{3}\right) + \binom{n}{2} \left(\frac{\sqrt{8}}{3}\right)^2 - \cdots \right) \right] \end{aligned}$$

$$= 3^{n-1}\left[\binom{n}{1} + \binom{n}{3}\left(\frac{\sqrt{8}}{3}\right)^2 + \cdots\right]$$
$$= 3^{n-1}\sum_{j=0}^{\infty}\left(\frac{\sqrt{8}}{3}\right)^{2j}\binom{n}{2j+1},$$

which completes the proof. □

Theorem 2.14. *The sequence of balancing numbers is uniformly distributed modulo 2^k for all integers $k \geq 1$.*

Proof. In [6], Panda et al. shown that for any natural number $n > 1$, $\pi(n) = n$ if and only if $n = 2^k$ for some natural number k. It will be adequate to show that among the first 2^k elements of the sequence, we found at most one element from each residue class $\bmod 2^k$. The basis step is clear by Theorem 2.6. Assume that the sequence of balancing numbers is uniform modulo 2^{r-1}, where r is any positive integer less than or equal to k. Now we proceed by induction on r.

Let $0 \leq m, n < 2^r$, and without loss of generality, say $m \leq n$. We will prove that, in fact, $m = n$. By virtue of Lemma 2.13, the expression $B_n \equiv B_m \pmod{2^r}$ becomes

$$\sum_{j=0}^{r-1} 2^{3j}\cdot 3^{-2j}\binom{n}{2j+1} \equiv 3^{-(n-m)}\sum_{j=0}^{\infty} 2^{3j}\cdot 3^{-2j}\binom{m}{2j+1} \pmod{2^r}. \quad (2)$$

As $3^{-(2^{r-1})} \equiv 1 \pmod{2^r}$ by the Euler-Fermat theorem, the equation (2) reduces to the following:

$$\sum_{j=0}^{2^{r-1}} 2^{3j}\cdot 3^{-2j}\left[\binom{n}{2j+1} - \binom{m}{2j+1}\right] \equiv 0 \pmod{2^r}. \quad (3)$$

Using the Cauchy combinatorial formula

$$\binom{a+b}{c} = \sum_{d=0}^{a}\binom{a}{d}\binom{b}{c-d},$$

we obtain

$$\binom{n}{2j+1} = \binom{m}{2j+1} + \sum_{i=1}^{2j+1}\binom{n-m}{i}\binom{m}{2j+1-i}.$$

Substituting this result into (3) gives

$$(n-m)\sum_{j=0}^{2^{r-1}}\left[\sum_{i=1}^{2j+1} 2^{3j}3^{-2j}\binom{n-m}{i}\frac{1}{(n-m)}\binom{m}{2j+1-i}\right]\equiv 0 \pmod{2^r}. \tag{4}$$

The term $j=0$ and $i=1$ in (4) is equal to 1 and every other terms for $j \geq 1$ and $1 \leq i \leq 2j+1$ is even. This follows from the fact that the maximal α for which $2^\alpha | i!$ is $\alpha < i$. Since $i \leq 2j+1$, then $\alpha \leq 2j$ and

$$\frac{8^j}{2^\alpha} \geq \frac{2^{3j}}{2^{2j}} = 2^j.$$

Thus

$$2^{3j}3^{-2j}\frac{(n-m-1)(n-m-2)\dots(n-m-i+1)}{i!}\binom{m}{2j+1-i} \quad \text{is even.}$$

Now, rewrite the formula (4) as $(n-m)(1+A)$.

To prove $B_n \equiv B_m \pmod{p^r}$ implies $n \equiv m \pmod{p^r}$, there can be used the following implication

$$2^r|(n-m)(1+A) \quad \text{and} \quad 2|A \Rightarrow 2^r|(n-m).$$

We know that $0 \leq m,\ n < 2^r$, and so we must conclude that $n = m$. Therefore any residue modulo 2^r appears once, which completes the induction. □

Theorem 2.15. *The sequence of balancing numbers is not uniformly distributed modulo m for any composite integer $m \neq 2^k$ for every value of $k \in \mathbb{N}$.*

Proof. Suppose the sequence of balancing numbers is uniformly distributed modulo m for any composite integer $m \neq 2^k$. In [4] Niven proved that a sequence is uniformly distributed modulo any positive integer n, then it is uniformly distributed modulo every divisor of n. Since, every composite number can be written as the product of two or more (not necessarily distinct) primes, we have that the sequence of balancing numbers is uniformly distributed modulo p, where p is some prime factors of $m \neq 2^k$. This contradicts Theorem 2.7. □

Remark 2.16. Note that, our proof is similar but different as the proof in Niederreiter [3]. Also Nathanson [2] studied the distribution of a sequence $x_{n+2} = ax_{n+1} + bx_n$ and proved that if p is an odd prime, then the sequence x_n is uniformly distributed modulo p if and only if $p|(a^2+4b)$, $p \nmid a$ and $p \nmid (2x_2 - ax_1)$. The sequence x_n is uniformly distributed modulo 2 if and only if $2|a$, $2 \nmid b$ and $2 \nmid (x_2 - x_1)$. The balancing numbers satisfy such conditions.

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DISTRIBUTION OF LEADING DIGITS OF NUMBERS

Yukio Ohkubo—Oto Strauch

Dedicated to the memory of Professor Pierre Liardet

ABSTRACT. Applying the theory of distribution functions of sequences we find the relative densities of the first digits also for sequences x_n not satisfying Benford's law. Especially for sequence $x_n = n^r$, $n = 1, 2, \dots$ and $x_n = p_n^r$, $n = 1, 2, \dots$, where p_n is the increasing sequence of all primes and $r > 0$ is an arbitrary real. We also add rate of convergence to such densities.

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1. Introduction

In this paper we give a complete solution of the first digit problem in base $b \geq 2$ for the sequence n^r, $n = 1, 2, \dots$, and for the sequence p_n^r, $n = 1, 2, \dots$, where p_n is the nth prime and $r > 0$ is an arbitrary real. They do not satisfy Benford's law. For example for n^r we use the following main steps:

- Denote $F_N(x) = \frac{1}{N}\#\{n \leq N; \log_b(n^r) \bmod 1 \in [0, x)\}$.
- Then $F_N\Big(\log_b\left(\frac{D+1}{b^{s-1}}\right)\Big) - F_N\Big(\log_b\left(\frac{D}{b^{s-1}}\right)\Big)$ = the density of the number of $n \leq N$ for which the leading block of s digits of n^r is equal to D, $D = d_1 d_2 \cdots d_s$.
- Let N_i be an increasing sequence of positive integer such that $\log_b N_i^r \bmod 1 \to w$ for some $w \in [0, 1]$.

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- Then $$\lim_{i\to\infty}\left(F_{N_i}\left(\log_b\left(\frac{D+1}{b^{s-1}}\right)\right)-F_{N_i}\left(\log_b\left(\frac{D}{b^{s-1}}\right)\right)\right)$$ $$=g_w\left(\log_b\left(\frac{D+1}{b^{s-1}}\right)\right)-g_w\left(\log_b\left(\frac{D}{b^{s-1}}\right)\right),$$

- where $g_w(x)=\frac{1}{b^{\frac{w}{r}}}\cdot\frac{b^{\frac{x}{r}}-1}{b^{\frac{1}{r}}-1}+\frac{\min(b^{\frac{x}{r}},b^{\frac{w}{r}})-1}{b^{\frac{w}{r}}}$.

In Theorem 5 we give a rate of such limit. The same limit holds also for the sequence p_n^r, $n=1,2,\dots$, but the rate of convergence described in Theorem 6 is more complicated.

In the following we present some clarifications.

1.1. Benford's law

For a real number x, let $[x]$ be the integral part of x, and let $x \bmod 1=\{x\}=x-[x]$. Let $b\geq 2$ be an integer considered as a base for the development of a real number $x>0$ and $M_b(x)$ be the mantissa of x defined by $x=M_b(x)\times b^{n(x)}$ such that $1\leq M_b(x)<b$ holds, where $n(x)$ is a uniquely determined integer. Let $D=d_1d_2\cdots d_s$ be a positive integer expressed in the base b, that is

$$D=d_1\times b^{s-1}+d_2\times b^{s-2}+\cdots+d_{s-1}\times b+d_s\,,$$

where $d_1\neq 0$ and at the same time $D=d_1d_2\cdots d_s$ is considered as an $s-$consecutive block of digits in the base b. Note that for x of the type

$$x=0.00\cdots 0d_1d_2\cdots d_s\cdots\,,$$

where $d_1>0$, we have $M_b(x)=d_1.d_2\cdots d_s\cdots$ and the first zero digits is omitted. Thus arbitrary $x>0$ has the first s-digits, starting a non-zero digit, equal to $d_1d_2\cdots d_s$ if and only if [1]

$$d_1.d_2\cdots d_s\leq M_b(x)<d_1.d_2\cdots(d_s+1). \tag{1}$$

Since $\log_b M_b(x)=\log_b x \bmod 1$ the inequality (1) is equivalent to

$$\log_b\left(\frac{D}{b^{s-1}}\right)\leq\log_b x \bmod 1<\log_b\left(\frac{D+1}{b^{s-1}}\right). \tag{2}$$

Underline once again

$$\frac{D}{b^{s-1}}=d_1.d_2\cdots d_s,\quad \frac{D+1}{b^{s-1}}=d_1.d_2\cdots(d_s+1).$$

[1] If $d_1=d_2=\cdots=d_s=b-1$, then we have $d_1.d_2\cdots(d_s+1)=b$.

Definition 1 (P. Diaconis (1977)). A sequence x_n, $n = 1, 2, \dots$, of positive real numbers satisfies *Benford's law* (abbreviated to B.L.) [2] in base b, if for every $s = 1, 2, \dots$ and every s-digits integer $D = d_1 d_2 \cdots d_s$ we have the density

$$\lim_{N\to\infty} \frac{\#\{n \le N;\ \text{leading block of } s \text{ digits (beginning with} \ne 0) \text{ of } x_n = D\}}{N}$$
$$= \log_b\left(\frac{D+1}{b^{s-1}}\right) - \log_b\left(\frac{D}{b^{s-1}}\right). \tag{3}$$

From (2) and from definition (3) it follows immediately:

Theorem 1. *A sequence x_n, $n = 1, 2, \dots$, of positive real numbers satisfies B.L. in base b if and only if the sequence* $\log_b x_n \bmod 1$ *is uniformly distributed in* $[0, 1)$ *(for definition see page 26).*

1.2. Historical comments

The first digit problem: An infinite sequence $x_n \ge 1$ of real numbers satisfies *Benford's law*, if the frequency (the asymptotic density) of occurrences of a given first digit a, when x_n is expressed in the decimal form is given by $\log_{10}\left(1 + \frac{1}{a}\right)$ for every $a = 1, 2, \dots, 9$ (0 as a possible first digit is not admitted). Since x_n has the first digit a if and only if $\log_{10} x_n \bmod 1 \in [\log_{10} a, \log_{10}(a+1))$, Benford's law for x_n follows from the u.d. of $\log_{10} x_n \bmod 1$. F. Benford (1938) compared the empirical frequency of occurrences of a with $\log_{10}\big((a+1)/a\big)$ in twenty different domains such as the areas of 335 rivers; the size of 3259 U.S. populations; the street address of first 342 persons listed in American Men of Sciences, etc. which led him to the conclusion that "the logarithmic law applies particularly to those outlaw numbers that are without known relationships ..." For the asymptotic density of the second-place digit b he found $\sum_{a=1}^{9} \log_{10}\left(1 + \frac{1}{10a+b}\right)$.

F. Benford rediscovered Newcomb's observation from (1881).

Many authors think that if the sequence x_n does not satisfy B.L., then the relative density of indices n for which the b-expansion of x_n start with leading digits $D = d_1 d_2 \cdots d_s$

$$\frac{1}{N}\#\left\{n \le N;\ \log_b\left(\frac{D}{b^{s-1}}\right) \le \{\log_b x_n\} < \log_b\left(\frac{D+1}{b^{s-1}}\right)\right\}$$

does not follow any distribution in the sense of natural density, see S. Eliahou, B. Massé and D. Schneider [5]. These authors as an alternate result shown that the sequence $\log_{10} n^r \bmod 1$, $n = 1, 2, \dots, [e^r]$ and the sequence $\log_{10} p_n^r \bmod 1$, $n = 1, 2, \dots, [e^r]$, where p_n are all prime numbers, have the discrepancy $O(r^{-1})$.

[2]precisely generalized or strong

Thus, for $r \to \infty$, these sequences tends to uniform distribution and thus n^r and p_n^r tends to B.L.

1.3. General scheme of solution of the First digit problem

Definition 2. Let y_n, $n = 1, 2, \dots$, be a sequence of real numbers and define the step distribution function of y_n mod 1

$$F_N(x) = \frac{\#\{n \le N; y_n \bmod 1 \in [0, x)\}}{N} \tag{4}$$

for $x \in [0, 1]$. The limit $g(x)$ of a subsequence $F_{N_k}(x)$ of $F_N(x)$

$$\lim_{k\to\infty} F_{N_k}(x) = g(x) \tag{5}$$

for every $x \in [0, 1]$, [3] is called distribution function (abbreviating d.f.) of y_n, where $N_1 < N_2 < \cdots$ is related sequence of indices. Let $G(y_n \bmod 1)$ be the set of all possible limits (5). If $G(y_n \bmod 1) = \{g(x)\}$, $g(x) = x$, then the sequence y_n, $n = 1, 2, \dots$ is called uniformly distributed mod 1 (abbreviating u.d. mod1.)

From Definition 2 and (2) immediately follows the basic theorem.

Theorem 2. *Let $g(x) \in G(\log_b x_n \bmod 1)$ and $\lim_{i\to\infty} F_{N_i}(x) = g(x)$. Then for $D = d_1 d_2 \cdots d_s$*

$$\lim_{N_i\to\infty} \frac{\#\{n \le N_i;\ \textit{first } s \textit{ digits (starting a non-zero digit) of } x_n = D\}}{N_i}$$
$$= g\left(\log_b\left(\frac{D+1}{b^{s-1}}\right)\right) - g\left(\log_b\left(\frac{D}{b^{s-1}}\right)\right). \tag{6}$$

This is the general scheme of solution of the First digit problem for the sequence x_n, $n = 1, 2, \dots$, for which $\log_b x_n$ mod 1 need not be u.d. sequence. This approach is also presented in [2].

Proof. Using $F_N(x) = \frac{1}{N}\#\{n \le N; x_n \bmod 1 \in [0, x)\}$ we have

$$F_N\left(\log_b\left(\frac{D+1}{b^{s-1}}\right)\right) - F_N\left(\log_b\left(\frac{D}{b^{s-1}}\right)\right)$$
$$= \frac{1}{N}\#\{n \le N;\ \text{first } s \text{ digits (starting a non-zero digit) of } x_n = D\}.$$

□

[3] Similar theory of d.f.s can be found by using weak limits, i.e. the limit (5) for every continuity point x of $g(x)$. But in the following we use only continuous $g(x)$.

2. Distribution functions of sequences involving logarithm

D.f.'s of $\log_b x_n \bmod 1$ which we need in (6) can be computed by following theorems:

Theorem 3 ([9]). *Let the real-valued function $f(x)$ be strictly increasing for $x \geq 1$ and let $f^{-1}(x)$ be its inverse function and*

$$F_N(x) = \frac{1}{N}\#\{n \leq N; f(n) \bmod 1 \in [0,x)\}.$$

Assume that

(i) $\lim_{x\to\infty} f'(x) = 0$,

(ii) $\lim_{k\to\infty} f^{-1}(k+1) - f^{-1}(k) = \infty$,

(iii) $\lim_{k\to\infty} \frac{f^{-1}(k+w(k))}{f^{-1}(k)} = \psi(w)$ *for every sequence* $w(k) \in [0,1]$ *for which* $\lim_{k\to\infty} w(k) = w$, *where this limit defines the function* $\psi : [0,1] \to [1, \psi(1)]$,

(iv) $\psi(1) > 1$.

Then for the sequence $f(n) \bmod 1$, $n = 1, 2, \dots$, we have

$$G(f(n) \bmod 1) = \left\{ g_w(x) = \frac{1}{\psi(w)} \frac{\psi(x)-1}{\psi(1)-1} + \frac{\min(\psi(x),\psi(w))-1}{\psi(w)};\ w \in [0,1] \right\}. \tag{7}$$

Now, if $f(N_i) \bmod 1$ is a subsequence $f(n) \bmod 1$ such that $f(N_i) \bmod 1 \to w$, then $F_{N_i}(x) \to g_w(x)$ for every $x \in [0,1]$.

Similar theorem is valid also for $f(p_n)$, where p_n are primes.

Theorem 4 ([7]). *Let $F_N(x) = \frac{1}{N}\#\{n \leq N; f(p_n) \bmod 1 \in [0,x)\}$ for $x \in [0,1]$, where p_n is the increasing sequence of all primes. Assume* (i)-(iv) *from Theorem 3. Then the sequence $f(p_n) \bmod 1$, $n = 1, 2, \dots$, has*

$$G\big(f(p_n) \bmod 1\big) = G\big(f(n) \bmod 1\big).$$

Now, if $f(p_{N_i}) \bmod 1$ is a subsequence $f(p_n) \bmod 1$ such that $f(p_{N_i}) \bmod 1 \to w$, then $F_{N_i}(x) \to g_w(x)$ for every $x \in [0,1]$.

2.1. Natural numbers

Applying Theorem 3 to the sequence $f(n) = \log_b n^r$, $n = 1, 2, \dots$ we have

$$f^{-1}(x) = b^{\frac{x}{r}}, \lim_{k\to\infty} \frac{f^{-1}(k+w)}{f^{-1}(k)} = \frac{b^{\frac{k+w}{r}}}{b^{\frac{k}{r}}} = b^{\frac{w}{r}} = \psi(w),$$

$$G(\log_b n^r \bmod 1) = \left\{ g_w(x) = \frac{1}{b^{\frac{w}{r}}} \frac{b^{\frac{x}{r}} - 1}{b^{\frac{1}{r}} - 1} + \frac{\min(b^{\frac{x}{r}}, b^{\frac{w}{r}}) - 1}{b^{\frac{w}{r}}};\quad w \in [0,1] \right\}.$$

If

$$\lim_{i\to\infty}\{f(N_i)\} = \lim_{i\to\infty}\{\log_b(N_i^r)\} = w,$$

then we have

$$\lim_{i\to\infty}\frac{\#\{n\le N_i;\ \text{first } s \text{ digits of } n^r \text{ are } d_1d_2\dots d_s\}}{N_i}$$
$$= g_w\big(\log_b d_1.d_2\dots(d_s+1)\big) - g_w\big(\log_b d_1.d_2\dots d_s\big). \tag{8}$$

Some examples of w :

1. Assume that $N_i = b^i$ and r is a positive integer, then

$$\lim_{i\to\infty}\{\log_b(b^{ir})\} = 0 = w$$

and thus for (8) we have

$$=\frac{b^{(\log_b d_1.d_2\dots(d_s+1))/r}-1}{b^{1/r}-1} - \frac{b^{(\log_b d_1.d_2\dots d_s)/r}-1}{b^{1/r}-1}$$
$$=\frac{(d_1.d_2\dots(d_s+1))^{(1/r)}-(d_1.d_2\dots d_s))^{(1/r)}}{b^{1/r}-1}$$
$$=\frac{1}{b^{s-1}}\frac{1}{b-1} \text{ if } r=1.$$

2. Put $N_i = [b^{\frac{i+w}{r}}]$, where i is a positive integer. The $\lim_{i\to\infty}\{\log_b(N_i^r)\} = w$.

Proof. We have $N_i = [b^{\frac{i+w}{r}}] = b^{\frac{i+w'}{r}}$, where $w'\le w$ and $\{\log_b(N_i^r)\} = w'$. Further $|b^{\frac{i+w'}{r}} - b^{\frac{i+w}{r}}| = |w'-w| b^{\frac{i+x}{r}}(\log b)\frac{1}{r}$. Since $|b^{\frac{i+w'}{r}} - b^{\frac{i+w}{r}}| < 1$, then $|w'-w|\to 0$. Thus $\lim_{i\to\infty}\{\log_b(N_i^r)\} = w$. □

2.2. Primes

Applying Theorem 4 for the sequence

$$f(p_n) = \log_b p_n^r, \quad n = 1,2,\dots,$$

where p_n is the nth prime and $r>0$, we have

$$G(\log_b p_n^r \bmod 1) = \Big\{g_w(x) = \frac{1}{b^{\frac{w}{r}}}\frac{b^{\frac{x}{r}}-1}{b^{\frac{1}{r}}-1} + \frac{\min(b^{\frac{x}{r}}, b^{\frac{w}{r}})-1}{b^{\frac{w}{r}}};\ w\in[0,1]\Big\}.$$

If $\{f(p_{N_i})\} = \{\log_b(p_{N_i}^r)\}\to w$, then

$$\lim_{i\to\infty}\frac{\#\{n\le N_i;\ \text{first } s \text{ digits of } p_n^r = d_1d_2\dots d_s\}}{N_i}$$
$$= g_w\big(\log_b d_1.d_2\dots(d_s+1)\big) - g_w\big(\log_b d_1.d_2\dots d_s\big).$$

Some examples of w and N_i:

1. If $0 < w < 1$ and $N_i = \pi(b^{\frac{i+w}{r}})$, then $\lim_{i\to\infty}\{\log_b p^r_{N_i}\} = w$.

If $N_i = \pi(b^{\frac{i}{r}}) + 1$, then $\lim_{i\to\infty}\{\log_b p^r_{N_i}\} = 0$.

If $N_i = \pi(b^{\frac{i}{r}})$, then $\lim_{i\to\infty}\{\log_b p^r_{N_i}\} = 1$.

P r o o f. Let $0 < w < 1$ and $N_i = \pi(b^{\frac{i+w}{r}})$. We have

$$p_{N_i} \le b^{\frac{w+i}{r}} < p_{N_i+1},$$

and so

$$\log_b p^r_{N_i} \le w + i < \log_b p^r_{N_i+1}.$$

Now we have

$$\begin{aligned}\log_b p^r_{N_i+1} - \log_b p^r_{N_i} &= r\log_b \frac{p_{N_i+1}}{p_{N_i}}\\ &= r\log_b\Big(1 + \frac{p_{N_i+1} - p_{N_i}}{p_{N_i}}\Big)\\ &= r\log_b\left(1 + \frac{1}{p^{1-\theta}_{N_i}}\right) \to 0 \quad (i\to\infty),\end{aligned} \tag{9}$$

where the last equality is obtained by using the result

$$p_{n+1} - p_n = O(p_n^\theta) \quad \text{with} \quad \theta = 0.525 \quad \text{(see [1])}.$$

Therefore there exists i_0 such that for $i \ge i_0$, $[\log_b p^r_{N_i}] = i$ and

$$\begin{aligned}0 \le w - \{\log_b p^r_{N_i}\} &= i + w - (i + \{\log_b p^r_{N_i}\}) = i + w - \log_b p^r_{N_i}\\ &< \log_b p^r_{N_i+1} - \log_b p^r_{N_i} \to 0 \quad (i\to\infty).\end{aligned}$$

Hence, $\lim_{i\to\infty}\{\log p^r_{N_i}\} = w$. Similarly the other two cases are proved. □

2.3. Summary

From the above it follows that the sequences

$$\log_b n^r \bmod 1, \quad n = 1, 2, \dots, \quad \text{and} \quad \log_b p_n^r \bmod 1, \quad n = 1, 2, \dots,$$

have the same distribution functions $g_w(x) = \frac{1}{b^{\frac{w}{r}}}\frac{b^{\frac{x}{r}}-1}{b^{\frac{1}{r}}-1} + \frac{\min(b^{\frac{x}{r}}, b^{\frac{w}{r}})-1}{b^{\frac{w}{r}}}$; $w \in [0, 1]$.

Since $\lim_{r\to\infty}\frac{b^{\frac{x}{r}}-1}{b^{\frac{1}{r}}-1} = x$, then $\lim_{r\to\infty} g_w(x) = x$. Thus, as $r \to \infty$ the sequences n^r and p_n^r tends to B.L. This is qualitative proof of results in [5].

3. Rate of convergence

3.1. The first digit problem for the sequence $n^r, n = 1, 2, \dots$

Theorem 5. *Let N, b be positive integers, $b > 1$, $r > 0$, $w_0 \in [0,1]$. Denote*

$$F_N(x) = \frac{\#\{n \le N; \log_b(n^r) \bmod 1 \in [0,x)\}}{N},$$
$$g_{w_0}(x) = \frac{1}{b^{\frac{w_0}{r}}} \cdot \frac{b^{\frac{x}{r}} - 1}{b^{\frac{1}{r}} - 1} + \frac{\min(b^{\frac{x}{r}}, b^{\frac{w_0}{r}}) - 1}{b^{\frac{w_0}{r}}},$$
$$w = \{r \log_b N\}.$$

Then for every $x \in [0,1]$ we have

$$|F_N(x) - g_{w_0}(x)| \le \frac{|w - w_0|}{r} \cdot \log b \cdot b^{\frac{1}{r}} \cdot (b^{\frac{1}{r}} + 1) + \frac{3}{N} + \frac{r \log_b N}{N}.$$

Corollary 1. *Let $r > 0$, $w = \{r \log_b N\}$. Then for $D = d_1 d_2 \cdots d_s$ the ration $\frac{1}{N}\#\{1 \le n \le N;$ the leading block of s digits of n^r is equal to $D\}$ it can be approximated by*

$$g_w\left(\log_b\left(\frac{D+1}{b^{s-1}}\right)\right) - g_w\left(\log_b\left(\frac{D}{b^{s-1}}\right)\right)$$

with the error term

$$2\left(\frac{3}{N} + \frac{r \log_b N}{N}\right) = O_{b,r}\left(\frac{\log N}{N}\right).$$

P r o o f o f C o r r o l a r y 1. We have

$$\frac{1}{N}\#\{1 \le n \le N; \text{the leading block of } s \text{ digits of } n^r \text{ is equal to } D\}$$
$$= F_N\left(\log_b\left(\frac{D+1}{b^{s-1}}\right)\right) - F_N\left(\log_b\left(\frac{D}{b^{s-1}}\right)\right).$$

Applying Theorem 5 with $w_0 = w$, we have $|F_N(x) - g_w(x)| \le \frac{3}{N} + \frac{r \log_b N}{N}$. □

Example 1. Let $r > 0$ and i be a positive integer. By Corollary 1 for $D = d_1 d_2 \cdots d_s$ the fraction

$$\frac{1}{b^i}\#\{1 \le n \le b^i; \text{the leading block of } s \text{ digits of } n^r \text{ is equal to } D\}$$

it can be approximated by

$$\frac{\left(\frac{D+1}{b^{s-1}}\right)^{\frac{1}{r}} - \left(\frac{D}{b^{s-1}}\right)^{\frac{1}{r}}}{b^{\frac{1}{r}} - 1} \quad \text{with the error term} \quad \frac{2ri+6}{b^i}.$$

Now we prove Theorem 5.

P r o o f. Firstly we repeat a proof of (7). For a positive integer N define

- $K_N = [f(N)]$, abbreviating $K_N = K$,
- $w_N = \{f(N)\}$, abbreviating $w_N = w$,
- $A_N([x,y)) = \#\{n \le N; f(n) \in [x,y)\}$,
- $F_N(x) = \frac{\#\{n\le N; f(n) \bmod 1 \in [0,x)\}}{N}$.

Clearly $f^{-1}(K+w) = N$ and for every $x \in [0,1]$ and $F_N(x)$ in (4) we have

$$F_N(x) = \frac{\sum_{k=0}^{K-1} A_N([k,k+x))}{N} + \\ + \frac{A_N([K,K+x) \cap [K,K+w))}{N} + \frac{O(A_N([1,f^{-1}(0)))}{N}$$

From monotonicity of $f(x)$ it follows $A_N([x,y)) = f^{-1}(y) - f^{-1}(x) + \theta$, where $|\theta| \le 1$. Thus

$$F_N(x) = \frac{\sum_{k=0}^{K-1}(f^{-1}(k+x) - f^{-1}(k))}{N} \\ + \frac{\min(f^{-1}(K+x), f^{-1}(K+w)) - f^{-1}(K)}{N} + \frac{O(K)}{N} + \frac{O(f^{-1}(0))}{N}, \quad (10)$$

where

$$O(K) \le K+1 \text{ and } O(f^{-1}(0)) \le f^{-1}(0).$$

The assumption (ii) implies $1/(f^{-1}(k+1) - f^{-1}(k)) \to 0$ which together with Cauchy-Stolz (other name is Stolz-Cesàro, see [10, p. 4–7]) lemma implies that

$$K/f^{-1}(K) \to 0 \quad \text{and thus} \quad K/N \to 0 \quad \text{as} \quad N \to \infty.$$

Furthermore we can express the first term of $F_N(x)$ in (10) as

$$\frac{\sum_{k=0}^{K-1}(f^{-1}(k+x) - f^{-1}(k))}{\sum_{k=0}^{K-1}(f^{-1}(k+1) - f^{-1}(k))} \cdot \frac{f^{-1}(K) - f^{-1}(0)}{f^{-1}(K+w)} \quad (11)$$

and the second term as

$$\frac{\min\left(\frac{f^{-1}(K+x)}{f^{-1}(K)}, \frac{f^{-1}(K+w)}{f^{-1}(K)}\right) - 1}{\frac{f^{-1}(K+w)}{f^{-1}(K)}}. \quad (12)$$

Using the assumption (ii) and (iii) the Cauchy-Stolz lemma implies

$$\lim_{K\to\infty} \frac{\sum_{k=0}^{K-1}(f^{-1}(k+x) - f^{-1}(k))}{\sum_{k=0}^{K-1}(f^{-1}(k+1) - f^{-1}(k))} = \lim_{k\to\infty} \frac{f^{-1}(k+x) - f^{-1}(k)}{f^{-1}(k+1) - f^{-1}(k)} = \frac{\psi(x)-1}{\psi(1)-1}. \quad (13)$$

Now, for increasing subsequence N_i of indices N, denote

$$K_i = [f(N_i)] \quad \text{and} \quad w_i = \{f(N_i)\}.$$

If $w_i \to w_0$, then by (iii)

$$f^{-1}(K_i)/f^{-1}(K_i+w_i) \to 1/\psi(w_0), \quad \text{and} \quad f^{-1}(K_i+x)/f^{-1}(K_i) \to \psi(x).$$

Thus (11), (12), (13) imply (7)

$$F_{N_i}(x) \to g_{w_0}(x) = \frac{1}{\psi(w_0)} \cdot \frac{\psi(x)-1}{\psi(1)-1} + \frac{\min(\psi(x),\psi(w_0))-1}{\psi(w_0)} \quad \text{for all} \quad x \in [0,1).$$

In the following we prove a quantitative form of (7). Put

- $K = [f(N)]$,
- $w = \{f(N)\}$,
- $N = f^{-1}(K+w)$. Then

$$\begin{aligned}
&F_N(x) - g_{w_0}(x)\\
&= \left(\frac{\sum_{k=0}^{K-1}(f^{-1}(k+x) - f^{-1}(k))}{\sum_{k=0}^{K-1}(f^{-1}(k+1) - f^{-1}(k))} \cdot \frac{f^{-1}(K) - f^{-1}(0)}{f^{-1}(K+w)} - \frac{\psi(x)-1}{\psi(1)-1} \cdot \frac{1}{\psi(w_0)} \right)\\
&\quad + \left(\frac{\min\left(\frac{f^{-1}(K+x)}{f^{-1}(K)}, \frac{f^{-1}(K+w)}{f^{-1}(K)}\right) - 1}{\frac{f^{-1}(K+w)}{f^{-1}(K)}} - \frac{\min(\psi(x),\psi(w_0)) - 1}{\psi(w_0)} \right)\\
&\quad + \left(\frac{O(K)}{N} + \frac{O(f^{-1}(0))}{N} \right)\\
&= (\mathrm{I}) \left(\frac{\sum_{k=0}^{K-1}(f^{-1}(k+x) - f^{-1}(k))}{\sum_{k=0}^{K-1}(f^{-1}(k+1) - f^{-1}(k))} - \frac{\psi(x)-1}{\psi(1)-1} \right) \left(\frac{f^{-1}(K) - f^{-1}(0)}{f^{-1}(K+w)} \right) && (14)\\
&\quad + (\mathrm{II}) \left(\frac{f^{-1}(K) - f^{-1}(0)}{f^{-1}(K+w)} - \frac{1}{\psi(w_0)} \right) \left(\frac{\psi(x)-1}{\psi(1)-1} \right) && (15)\\
&\quad + (\mathrm{III}) \left(\min\left(\frac{f^{-1}(K+x)}{f^{-1}(K)}, \frac{f^{-1}(K+w)}{f^{-1}(K)} \right) - \min\bigl(\psi(x),\psi(w_0)\bigr) \right) \left(\frac{1}{\frac{f^{-1}(K+w)}{f^{-1}(K_i)}} \right) && (16)\\
&\quad + (\mathrm{IV}) \left(\psi(w_0) - \frac{f^{-1}(K+w)}{f^{-1}(K)} \right) \left(\frac{\min(\psi(x),\psi(w_0)) - 1}{\frac{f^{-1}(K+w)}{f^{-1}(K)} \cdot \psi(w_0)} \right) && (17)\\
&\quad + (\mathrm{V}) \left(\frac{O(K)}{N} + \frac{O(f^{-1}(0))}{N} \right). && (18)
\end{aligned}$$

Every second term in (I)=(14), (II)=(15), (III)=(16), (IV)=(17) is bounded precisely

$$\left(\frac{f^{-1}(K)-f^{-1}(0)}{f^{-1}(K+w)}\right)\le 1,\quad \left(\frac{\psi(x)-1}{\psi(1)-1}\right)\le 1,\quad \left(\frac{1}{\frac{f^{-1}(K+w)}{f^{-1}(K)}}\right)\le 1,$$

$$\left(\frac{\min(\psi(x),\psi(w_0))-1}{\frac{f^{-1}(K+w)}{f^{-1}(K)}\cdot\psi(w_0)}\right)\le\psi(1)-1.$$

Now put $f(x)=\log_b(x^r)$. This function satisfies Theorem 3 and

$$f^{-1}(x)=b^{\frac{x}{r}},$$

$$\lim_{k\to\infty}\frac{f^{-1}(k+x)}{f^{-1}(k)}=\frac{b^{\frac{k+x}{r}}}{b^{\frac{k}{r}}}=b^{\frac{x}{r}}=\psi(x),$$

$$g_{w_0}(x)=\frac{1}{b^{\frac{w_0}{r}}}\cdot\frac{b^{\frac{x}{r}}-1}{b^{\frac{1}{r}}-1}+\frac{\min(b^{\frac{x}{r}},b^{\frac{w_0}{r}})-1}{b^{\frac{w_0}{r}}},$$

$$f^{-1}(0)=b^{\frac{0}{r}}=1,\quad \psi(1)=b^{\frac{1}{r}},$$

$$K=[r\log_b N],\quad \omega=\{r\log_b N\},\quad N=b^{\frac{K+w}{r}}.$$

Then for the first terms in (I)=(14), (II)=(15), (III)=(16), (IV)=(17) we have:

$$\text{(I)}\left|\frac{\sum_{k=0}^{K-1}(f^{-1}(k+x)-f^{-1}(k))}{\sum_{k=0}^{K-1}(f^{-1}(k+1)-f^{-1}(k))}-\frac{\psi(x)-1}{\psi(1)-1}\right|$$

$$=\left|\frac{\sum_{k=0}^{K-1}(b^{\frac{k+x}{r}}-b^{\frac{k}{r}})}{\sum_{k=0}^{K-1}(b^{\frac{k+1}{r}}-b^{\frac{k}{r}})}-\frac{b^{\frac{x}{r}}-1}{b^{\frac{1}{r}}-1}\right|=0,$$

$$\text{(II)}\left|\frac{f^{-1}(K)-f^{-1}(0)}{f^{-1}(K+w)}-\frac{1}{\psi(w_0)}\right|=\left|\frac{b^{\frac{K}{r}}-1}{b^{\frac{K+w}{r}}}-\frac{1}{b^{\frac{w_0}{r}}}\right|\le\left|\frac{1}{b^{\frac{w}{r}}}-\frac{1}{b^{\frac{w_0}{r}}}\right|+\frac{1}{N},$$

$$\text{(III)}\left|\min\left(\frac{f^{-1}(K+x)}{f^{-1}(K)},\frac{f^{-1}(K+w)}{f^{-1}(K)}\right)-\min\bigl(\psi(x),\psi(w_0)\bigr)\right|$$

$$=\left|\min\bigl(\psi(x),\psi(w)\bigr)-\min\bigl(\psi(x),\psi(w_0)\bigr)\right|$$

$$\le|\psi(w)-\psi(w_0)|=|b^{\frac{w}{r}}-b^{\frac{w_0}{r}}|(\text{ for a proof see in appendix}),\tag{19}$$

$$\text{(IV)}\left|\psi(w_0)-\frac{f^{-1}(K+w)}{f^{-1}(K)}\right|\bigl(\psi(1)-1\bigr)=|b^{\frac{w}{r}}-b^{\frac{w_0}{r}}|(b^{\frac{1}{r}}-1),$$

$$\text{(V)}\left(\frac{O(K)}{N}+\frac{O(f^{-1}(0))}{N}\right)\le\frac{K+1+f^{-1}(0)}{N}\le\frac{r\log_b N+2}{N}.$$

In the end of proof we use

$$|b^{\frac{w}{r}} - b^{\frac{w_0}{r}}| \leq \frac{|w - w_0|}{r} \cdot \log b \cdot b^{\frac{1}{r}}. \qquad \square$$

3.2. First digit problem for p_n^r, $n = 1, 2, \dots$, where p_n is the nth prime

Theorem 6. *Let N be an arbitrary integer, $r > 0$, $w_0 \in [0,1]$, Denote*

$$F_N(x) = \frac{\#\{n \leq N; \log_b({p_n}^r) \bmod 1 \in [0,x)\}}{N},$$

$$g_{w_0}(x) = \frac{1}{b^{\frac{w_0}{r}}} \cdot \frac{b^{\frac{x}{r}} - 1}{b^{\frac{1}{r}} - 1} + \frac{\min(b^{\frac{x}{r}}, b^{\frac{w_0}{r}}) - 1}{b^{\frac{w_0}{r}}},$$

$$K = [r \log_b p_N], \quad w = \{r \log_b p_N\}.$$

Let k_1 be an integer such that

$$\max\left(\frac{\log 59}{\log b} r, \frac{5rb^{\frac{1}{r}} + \frac{r}{2} + b^{\frac{1}{r}} \log b}{(b^{\frac{1}{r}} - 1)\log b}\right) < k_1 < K.$$

Then for every $x \in [0,1]$ we have

$$\begin{aligned}|F_N(x) - g_{w_0}(x)| \leq &\frac{b^{\frac{1}{r}}(2\log b + 11r)}{(\log b)(b^{\frac{1}{r}} - 1)k_1 - (5r + \log b)b^{\frac{1}{r}} - \frac{r}{2}} \\ &+ \frac{K}{k_1} \frac{1}{b^{\frac{K-k_1}{r}}} \frac{\log b + \frac{3}{2k_1}}{\log b + \frac{2}{K}} + 3(b^{\frac{1}{r}})^2 \frac{\log b + \frac{11}{2}r}{K \log b + \frac{r}{4}} \\ &+ 3(b^{\frac{1}{r}})^2 \frac{|w - w_0|}{r} \log b + \frac{r \log_b p_N + 1}{N}.\end{aligned}$$

Corollary 2. *Let $r > 0$, i be a positive integer, and let $N = \pi(b^{\frac{i}{r}})$. Then for $D = d_1 d_2 \cdots d_s$ the fraction*

$\frac{1}{N}\#\{1 \leq n \leq N;$ *the leading block of s digits of p_n^r is equal to* $D\}$

it can be approximated by

$$\frac{\left(\frac{D+1}{b^{s-1}}\right)^{\frac{1}{r}} - \left(\frac{D}{b^{s-1}}\right)^{\frac{1}{r}}}{b^{\frac{1}{r}} - 1}$$

with the error term

$$\left|F_N\left(\frac{D+1}{b^{s-1}}\right) - g_1\left(\frac{D+1}{b^{s-1}}\right)\right| + \left|F_N\left(\frac{D}{b^{s-1}}\right) - g_1\left(\frac{D}{b^{s-1}}\right)\right|$$
$$= O_{b,r}\left(\frac{1}{\log p_N}\right)$$

given by Theorem 6.

P r o o f o f C o r o l l a r y 2. For $N = \pi(b^{\frac{i}{r}})$ we have $r \log_b p_N \le i < r \log_b p_{N+1}$. Since $r \log_b p_{N+1} - r \log_b p_N \to 0$ by (9), we obtain $[r \log_b p_N] = i - 1$ for sufficiently large i. Therefore

$$\begin{aligned}|w-1| &= |\log_b p_N^r - (i-1) - 1| = i - \log_b p_N^r \le r(\log_b p_{N+1} - \log_b p_N)\\ &\le \frac{r}{\log b} \cdot \frac{1}{p_N}(p_{N+1} - p_N) \le \frac{r}{\log b} \cdot \frac{C}{p_N^{1-\theta}},\end{aligned}$$

where C is a positive constant, $\theta = 0.525$ (see [1]). We apply Theorem 6 with $N = \pi(b^{\frac{i}{r}})$, $w_0 = 1$, $K = [r \log_b p_N]$, $k_1 = [\frac{r}{2} \log_b p_N]$, $w = \{r \log_b p_N\}$. $\square$

P r o o f o f T h e o r e m 6. Firstly we repeat the proof of Theorem 4. Let $0 \le x \le 1$. We have

$$\begin{aligned}&\{f(p_n)\} < x \iff 0 \le f(p_n) - k < x \iff\\ &k \le f(p_n) < k + x \iff f^{-1}(k) \le p_n < f^{-1}(k+x).\end{aligned}$$

Let $\pi(t) = \#\{p \le t; p - \text{prime}\}$. Using $\pi(t) = \#\{p < t; p - \text{prime}\} + O(1)$, then we have

$$\begin{aligned}F_N(x) =&\frac{\sum_{k=0}^{K-1}(\pi(f^{-1}(k+x)) - \pi(f^{-1}(k)))}{N} +\\ &+ \frac{\min(\pi(f^{-1}(K+x)), \pi(f^{-1}(K+w))) - \pi(f^{-1}(K))}{N}\\ &+ \frac{O(K)}{N} + \frac{O(\pi(f^{-1}(0)))}{N},\end{aligned} \tag{20}$$

where

$$O(K) \le K + 1, \qquad O\Big(\pi\big(f^{-1}(0)\big)\Big) \le \pi\big(f^{-1}(0)\big).$$

Let $K = [f(p_N)]$, $w = \{f(p_N)\}$. Then $p_N = f^{-1}(K+w)$. Then we have

$$N = \pi\big(f^{-1}(K+w)\big).$$

We can express the first term of $F_N(x)$ in (20) as

$$\frac{\sum_{k=0}^{K-1}(\pi(f^{-1}(k+x)) - \pi(f^{-1}(k)))}{\sum_{k=0}^{K-1}(\pi(f^{-1}(k+1)) - \pi(f^{-1}(k)))} \cdot \frac{\pi(f^{-1}(K)) - \pi(f^{-1}(0))}{\pi(f^{-1}(K+w))}$$

and the second term as

$$\frac{\min\left(\frac{\pi(f^{-1}(K+x))}{\pi(f^{-1}(K))}, \frac{\pi(f^{-1}(K+w))}{\pi(f^{-1}(K))}\right) - 1}{\frac{\pi(f^{-1}(K+w))}{\pi(f^{-1}(K))}}. \tag{21}$$

The Cauchy-Stolz lemma implies

$$\lim_{K\to\infty} \frac{\sum_{k=0}^{K-1}(\pi(f^{-1}(k+x))-\pi(f^{-1}(k)))}{\sum_{k=0}^{K-1}(\pi(f^{-1}(k+1))-\pi(f^{-1}(k)))} = \lim_{k\to\infty} \frac{\pi(f^{-1}(k+x))-\pi(f^{-1}(k))}{\pi(f^{-1}(k+1))-\pi(f^{-1}(k))}. \tag{22}$$

The Prime number theorem in the form

$$\pi(t) = \frac{t}{\log t} + O\left(\frac{t}{\log^2 t}\right)$$

gives

$$\frac{\pi(f^{-1}(k+x))}{\pi(f^{-1}(k)))} = \frac{f^{-1}(k+x)}{f^{-1}(k)} \cdot \frac{\log f^{-1}(k)}{\log f^{-1}(k+x)} \cdot \frac{1+O\left(\frac{1}{\log f^{-1}(k+x)}\right)}{1+O\left(\frac{1}{\log f^{-1}(k)}\right)}.$$

We need

$$\lim_{k\to\infty} \frac{\log f^{-1}(k)}{\log f^{-1}(k+x)} = 1. \tag{23}$$

The limit (23) is proved in [7] by using the following steps:

$$\begin{aligned} \frac{\log f^{-1}(k+x)}{\log f^{-1}(k)} &= \frac{1}{\log f^{-1}(k)}\big(\log f^{-1}(k+x) - \log f^{-1}(k)\big) + 1 \\ &= \frac{1}{\log f^{-1}(k)} \log\left(\frac{f^{-1}(k+x)}{f^{-1}(k)}\right) + 1 \to 1 \end{aligned}$$

since by assumption $\frac{f^{-1}(k+x)}{f^{-1}(k)} \to \psi(x) \geq 1$. Thus

$$\lim_{k\to\infty} \frac{\pi(f^{-1}(k+x))}{\pi(f^{-1}(k))} = \lim_{k\to\infty} \frac{f^{-1}(k+x)}{f^{-1}(k)} = \psi(x) \tag{24}$$

and (22) has the form $\frac{\psi(x)-1}{\psi(1)-1}$. Applying (24) on (21) we have

$$\lim_{K\to\infty} \frac{\min\left(\frac{\pi(f^{-1}(K+x))}{\pi(f^{-1}(K))}, \frac{\pi(f^{-1}(K+w))}{\pi(f^{-1}(K))}\right) - 1}{\frac{\pi(f^{-1}(K+w))}{\pi(f^{-1}(K))}} = \frac{\min(\psi(x),\psi(w)) - 1}{\psi(w)}.$$

Here for given N,

p_N is Nth prime, $K = [f(p_N)]$, $w = \{f(p_N)\}$ and $w_0 \in [0,1]$ is arbitrary.

Similarly as for the sequence $f(n) \bmod 1$ we have (18), (14), (15), (16), (17) for the sequence $f(p_n) \bmod 1$ and by applying $\pi(x)$ we have the following

$$F_N(x) - g_{w_0}(x)$$

$$= \left(\frac{\sum_{k=0}^{K-1}(\pi(f^{-1}(k+x)) - \pi(f^{-1}(k)))}{\sum_{k=0}^{K-1}(\pi(f^{-1}(k+1)) - \pi(f^{-1}(k)))} \cdot \frac{\pi(f^{-1}(K)) - \pi(f^{-1}(0))}{\pi(f^{-1}(K+w))} - \frac{\psi(x)-1}{\psi(1)-1} \cdot \frac{1}{\psi(w_0)} \right)$$

$$+ \left(\frac{\min\left(\frac{\pi(f^{-1}(K+x))}{\pi(f^{-1}(K))}, \frac{\pi(f^{-1}(K+w))}{\pi(f^{-1}(K))}\right) - 1}{\frac{\pi(f^{-1}(K+w))}{\pi(f^{-1}(K))}} - \frac{\min(\psi(x), \psi(w_0)) - 1}{\psi(w_0)} \right)$$

$$+ \left(\frac{O(K)}{N} + \frac{O(\pi(f^{-1}(0)))}{N} \right)$$

$$= (\mathrm{I}) \left(\frac{\sum_{\mathrm{k}=0}^{\mathrm{K}-1}(\pi(\mathrm{f}^{-1}(\mathrm{k}+\mathrm{x})) - \pi(\mathrm{f}^{-1}(\mathrm{k})))}{\sum_{\mathrm{k}=0}^{\mathrm{K}-1}(\pi(\mathrm{f}^{-1}(\mathrm{k}+1)) - \pi(\mathrm{f}^{-1}(\mathrm{k})))} - \frac{\psi(\mathrm{x})-1}{\psi(1)-1} \right) \left(\frac{\pi(\mathrm{f}^{-1}(\mathrm{K})) - \pi(\mathrm{f}^{-1}(0))}{\pi(\mathrm{f}^{-1}(\mathrm{K}+\mathrm{w}))} \right) \tag{25}$$

$$+ (\mathrm{II}) \left(\frac{\pi(\mathrm{f}^{-1}(\mathrm{K})) - \pi(\mathrm{f}^{-1}(0))}{\pi(\mathrm{f}^{-1}(\mathrm{K}+\mathrm{w}))} - \frac{1}{\psi(\mathrm{w}_0)} \right) \left(\frac{\psi(\mathrm{x}) - 1}{\psi(1) - 1} \right) \tag{26}$$

$$+ (\mathrm{III}) \left(\min\left(\frac{\pi(\mathrm{f}^{-1}(\mathrm{K}+\mathrm{x}))}{\pi(\mathrm{f}^{-1}(\mathrm{K}))}, \frac{\pi(\mathrm{f}^{-1}(\mathrm{K}+\mathrm{w}))}{\pi(\mathrm{f}^{-1}(\mathrm{K}))} \right) - \min\big(\psi(\mathrm{x}), \psi(\mathrm{w}_0)\big) \right) \left(\frac{1}{\frac{\pi(\mathrm{f}^{-1}(\mathrm{K}+\mathrm{w}))}{\pi(\mathrm{f}^{-1}(\mathrm{K}))}} \right) \tag{27}$$

$$+ (\mathrm{IV}) \left(\psi(\mathrm{w}_0) - \frac{\pi(\mathrm{f}^{-1}(\mathrm{K}+\mathrm{w}))}{\pi(\mathrm{f}^{-1}(\mathrm{K}))} \right) \left(\frac{\min(\psi(\mathrm{x}), \psi(\mathrm{w}_0)) - 1}{\frac{\pi(\mathrm{f}^{-1}(\mathrm{K}+\mathrm{w}))}{\pi(\mathrm{f}^{-1}(\mathrm{K}))} \cdot \psi(\mathrm{w}_0)} \right) \tag{28}$$

$$+ (\mathrm{V}) \left(\frac{\mathrm{O}(\mathrm{K})}{\mathrm{N}} + \frac{\mathrm{O}(\pi(\mathrm{f}^{-1}(0)))}{\mathrm{N}} \right). \tag{29}$$

Every second term in (I)-(V)=(25), (26), (27) and (28) is bounded, precisely

$$\left(\frac{\pi(f^{-1}(K)) - \pi(f^{-1}(0))}{\pi(f^{-1}(K+w))} \right) \leq 1,$$

$$\left(\frac{\psi(x) - 1}{\psi(1) - 1} \right) \leq 1, \qquad \left(\frac{1}{\frac{\pi(f^{-1}(K+w))}{\pi(f^{-1}(K))}} \right) \leq 1,$$

$$\left(\frac{\min(\psi(x), \psi(w_0)) - 1}{\frac{\pi(f^{-1}(K+w))}{\pi(f^{-1}(K))} \cdot \psi(w_0)} \right) \leq \psi(1) - 1.$$

Thus for the first terms (again writing as (I)-(V)) we have

$$|F_N(x) - g_{w_0}(x)|$$

$$\leq \text{(I)} \left| \frac{\sum_{\mathrm{k}=0}^{\mathrm{K}-1} (\pi(\mathrm{f}^{-1}(\mathrm{k}+\mathrm{x})) - \pi(\mathrm{f}^{-1}(\mathrm{k})))}{\sum_{\mathrm{k}=0}^{\mathrm{K}-1} (\pi(\mathrm{f}^{-1}(\mathrm{k}+1)) - \pi(\mathrm{f}^{-1}(\mathrm{k})))} - \frac{\psi(\mathrm{x}) - 1}{\psi(1) - 1} \right| \tag{30}$$

$$+ \text{(II)} \left| \frac{\pi(\mathrm{f}^{-1}(\mathrm{K})) - \pi(\mathrm{f}^{-1}(0))}{\pi(\mathrm{f}^{-1}(\mathrm{K}+\mathrm{w}))} - \frac{1}{\psi(\mathrm{w}_0)} \right| \tag{31}$$

$$+ \text{(III)} \left| \min\left(\frac{\pi(\mathrm{f}^{-1}(\mathrm{K}+\mathrm{x}))}{\pi(\mathrm{f}^{-1}(\mathrm{K}))}, \frac{\pi(\mathrm{f}^{-1}(\mathrm{K}+\mathrm{w}))}{\pi(\mathrm{f}^{-1}(\mathrm{K}))} \right) - \min(\psi(\mathrm{x}), \psi(\mathrm{w}_0)) \right| \tag{32}$$

$$+ \text{(IV)} \left| \psi(\mathrm{w}_0) - \frac{\pi(\mathrm{f}^{-1}(\mathrm{K}+\mathrm{w}))}{\pi(\mathrm{f}^{-1}(\mathrm{K}))} \right| (\psi(1) - 1) \tag{33}$$

$$+ \text{(V)} \left(\frac{\mathrm{O}(\mathrm{K})}{\mathrm{N}} + \frac{\mathrm{O}(\mathrm{f}^{-1}(0))}{\mathrm{N}} \right). \tag{34}$$

Now in the following we use Prime number theorem in the form

$$\pi(x) = \frac{x}{\log x} \theta(x), \quad \theta(x) \to 1, \text{ if } x \to \infty. \tag{35}$$

Using (see [8, Theorem 1])

$$\frac{t}{\log t} \left(1 + \frac{1}{2 \log t}\right) < \pi(t) < \frac{t}{\log t} \left(1 + \frac{3}{2 \log t}\right)$$

for $t \geq 59$, we have

$$1 + \frac{1}{2 \log t} < \theta(t) < 1 + \frac{3}{2 \log t}. \tag{36}$$

In our case

$$f(x) = \log_b x^r, \quad f^{-1}(x) = b^{\frac{x}{r}}, \quad \psi(x) = b^{\frac{x}{r}},$$

$$g_{w_0}(x) = \frac{1}{b^{\frac{w_0}{r}}} \cdot \frac{b^{\frac{x}{r}} - 1}{b^{\frac{1}{r}} - 1} + \frac{\min(b^{\frac{x}{r}}, b^{\frac{w_0}{r}}) - 1}{b^{\frac{w_0}{r}}}$$

and by (35)

$$\frac{\pi(f^{-1}(k+x))}{\pi(f^{-1}(k))} = b^{\frac{x}{r}} (\frac{k}{k+x}) \frac{\theta(b^{\frac{k+x}{r}})}{\theta(b^{\frac{k}{r}})}.$$

For an upper bound of (I)=(30) we use the inequality proved in Appendix

$$
\begin{aligned}
&(\mathrm{I})\left|\frac{\sum_{\mathrm{k=0}}^{\mathrm{K-1}}(\pi(\mathrm{f}^{-1}(\mathrm{k+x}))-\pi(\mathrm{f}^{-1}(\mathrm{k})))}{\sum_{\mathrm{k=0}}^{\mathrm{K-1}}(\pi(\mathrm{f}^{-1}(\mathrm{k+1}))-\pi(\mathrm{f}^{-1}(\mathrm{k})))}-\frac{\psi(\mathrm{x})-1}{\psi(1)-1}\right| \\
&\qquad\qquad \leq \Psi_{k_1}+\left|\frac{\pi(f^{-1}(k_1))-\pi(f^{-1}(0))}{\pi(f^{-1}(K))-\pi(f^{-1}(0))}\right|, \qquad (37)
\end{aligned}
$$

where $K \geq k_1$, $\Psi_{k_1}=\sup_{k\geq k_1}\Phi_k$ and

$$
\Phi_k=\left|\frac{\pi(f^{-1}(k+x))\,-\,\pi(f^{-1}(k))}{\pi(f^{-1}(k+1))\,-\,\pi(f^{-1}(k))}\,-\,\frac{\psi(x)-1}{\psi(1)-1}\right|. \qquad (38)
$$

Applying (35) to (38) we have

$$
\Phi_k=\left|\frac{b^{\frac{x}{r}}\left(\left(\frac{k}{k+x}\right)\frac{\theta(b^{\frac{k+x}{r}})}{\theta(b^{\frac{k}{r}})}-1\right)+b^{\frac{x}{r}}-1}{b^{\frac{1}{r}}\left(\left(\frac{k}{k+1}\right)\frac{\theta(b^{\frac{k+1}{r}})}{\theta(b^{\frac{k}{r}})}-1\right)+b^{\frac{1}{r}}-1}-\frac{b^{\frac{x}{r}}-1}{b^{\frac{1}{r}}-1}\right|. \qquad (39)
$$

For (II)=(31) we have

$$
\begin{aligned}
(\mathrm{II})\left|\frac{\pi(\mathrm{b}^{\frac{\mathrm{K}}{\mathrm{r}}})-\pi(1)}{\pi(\mathrm{b}^{\frac{\mathrm{K+w}}{\mathrm{r}}})}-\frac{1}{\mathrm{b}^{\frac{\mathrm{w_0}}{\mathrm{r}}}}\right| &= \left|\frac{1}{b^{\frac{w}{r}}}\Big(\frac{K+w}{K}\Big)\frac{\theta(b^{\frac{K}{r}})}{\theta(b^{\frac{K+w}{r}})}-\frac{1}{b^{\frac{w_0}{r}}}\right| \\
&\leq \frac{1}{b^{\frac{w}{r}}}\left|\Big(\frac{K+w}{K}\Big)\frac{\theta(b^{\frac{K}{r}})}{\theta(b^{\frac{K+w}{r}})}-1\right|+\left|\frac{1}{b^{\frac{w}{r}}}-\frac{1}{b^{\frac{w_0}{r}}}\right|. \qquad (40)
\end{aligned}
$$

For(III)=(32) we use

$$
\begin{aligned}
(\mathrm{III}) \leq &\left|\min\left(\frac{\pi(f^{-1}(K+x))}{\pi(f^{-1}(K))},\frac{\pi(f^{-1}(K+w))}{\pi(f^{-1}(K))}\right)-\min(\psi(x),\psi(w))\right| \\
&+|\min(\psi(x),\psi(w))-\min(\psi(x),\psi(w_0))| \qquad (41)\\
\leq &\max\left(\left|\frac{\pi(f^{-1}(K+x))}{\pi(f^{-1}(K))}-\psi(x)\right|,\quad \left|\frac{\pi(f^{-1}(K+w))}{\pi(f^{-1}(K))}-\psi(w)\right|\right) \\
&+|\psi(w)-\psi(w_0)| \\
= &\max\left(b^{\frac{x}{r}}\left|\left(\frac{K}{K+x}\right)\frac{\theta(b^{\frac{K+x}{r}})}{\theta(b^{\frac{K}{r}})}-1\right|,\quad b^{\frac{w}{r}}\left|\left(\frac{K}{K+w}\right)\frac{\theta(b^{\frac{K+w}{r}})}{\theta(b^{\frac{K}{r}})}-1\right|\right) \\
&+|b^{\frac{w}{r}}-b^{\frac{w_0}{r}}|. \qquad (42)
\end{aligned}
$$

For (IV)=(33) we have

$$\text{(IV)}\left|\mathrm{b}^{\frac{\mathrm{w}_0}{\mathrm{r}}}-\frac{\pi(\mathrm{b}^{\frac{\mathrm{K+w}}{\mathrm{r}}})}{\pi(\mathrm{b}^{\frac{\mathrm{K}}{\mathrm{r}}})}\right|(\mathrm{b}^{\frac{1}{\mathrm{r}}}-1)$$
$$\leq\left(\left|b^{\frac{w_0}{r}}-b^{\frac{w}{r}}\right|+b^{\frac{w}{r}}\left|\left(\frac{K}{K+w}\right)\frac{\theta(b^{\frac{K+w}{r}})}{\theta(b^{\frac{K}{r}})}-1\right|\right)(b^{\frac{1}{r}}-1). \quad (43)$$

For (V)=(34) we have

$$\text{(V)}\left(\frac{\mathrm{O(K)}}{\mathrm{N}}+\frac{\mathrm{O}(\pi(\mathrm{f}^{-1}(0)))}{\mathrm{N}}\right)$$
$$\leq\frac{K+1+\pi(1)}{N}=\frac{K+1}{N}=\frac{[f(p_N)]+1}{N}\leq\frac{r\log_b p_N+1}{N}.$$

In all (I)=(39), (II)=(40), (III)=(42) and (IV)=(43) we have common factors

$$\left|\left(\frac{K}{K+w}\right)\frac{\theta(b^{\frac{K+w}{r}})}{\theta(b^{\frac{K}{r}})}-1\right| \quad \text{or} \quad \left|\left(\frac{K+w}{K}\right)\frac{\theta(b^{\frac{K}{r}})}{\theta(b^{\frac{K+w}{r}})}-1\right|.$$

Assuming $b^{\frac{k}{r}}\geq 59$, i.e., $k\geq r\log_b 59$ then (36) implies

$$\frac{\log b+\frac{r}{2(k+x)}}{\log b+\frac{3r}{2k}}<\frac{\theta(b^{\frac{k+x}{r}})}{\theta(b^{\frac{k}{r}})}<\frac{\log b+\frac{3r}{2(k+x)}}{\log b+\frac{r}{2k}}$$

and we have

$$\frac{-x\log b-\frac{r}{2}\cdot\frac{2k^2+6kx+3x^2}{k(k+x)}}{(k+x)\log b+\frac{3r}{2}\cdot\frac{k+x}{k}}<\left(\frac{k}{k+x}\right)\frac{\theta(b^{\frac{k+x}{r}})}{\theta(b^{\frac{k}{r}})}-1$$
$$<\frac{-x\log b+\frac{r}{2}\cdot\frac{2k^2-2kx-x^2}{k(k+x)}}{(k+x)\log b+\frac{k+x}{k}\cdot\frac{r}{2}}$$

which gives

$$\left|\left(\frac{k}{k+x}\right)\frac{\theta(b^{\frac{k+x}{r}})}{\theta(b^{\frac{k}{r}})}-1\right|\leq\frac{\log b+\frac{11}{2}r}{k\log b+\frac{r}{2}} \quad (44)$$

or

$$\left|\left(\frac{k+x}{k}\right)\frac{\theta(b^{\frac{k}{r}})}{\theta(b^{\frac{k+x}{r}})}-1\right|\leq\frac{\log b+\frac{11}{2}r}{k\log b+\frac{r}{4}}. \quad (45)$$

Applying (44) and (45) to (I)=(39), for $k \geq r \log_b 59$ we find

$$\Phi_k \leq \frac{2b^{\frac{1}{r}\frac{\log b+\frac{11}{2}r}{k\log b+\frac{r}{2}}}}{b^{\frac{1}{r}} - 1 - b^{\frac{1}{r}\frac{\log b+\frac{11}{2}r}{k\log b+\frac{r}{2}}}}, \quad \text{provided that} \quad k > \frac{5rb^{\frac{1}{r}} + \frac{r}{2} + b^{\frac{1}{r}}\log b}{(b^{\frac{1}{r}} - 1)\log b}. \tag{46}$$

In this case (46) is decreasing with respect to k and in (37) it can be using (46) for $k = k_1$. Then for approximate (I) we need approximate second term in (37):

$$\left|\frac{\pi(f^{-1}(k_1)) - \pi(f^{-1}(0))}{\pi(f^{-1}(K)) - \pi(f^{-1}(0))}\right| = \frac{K}{k_1}\frac{1}{b^{\frac{K-k_1}{r}}}\frac{\theta(b^{\frac{k_1}{r}})}{\theta(b^{\frac{K}{r}})} < \frac{K}{k_1}\frac{1}{b^{\frac{K-k_1}{r}}}\frac{\log b + \frac{3}{2k_1}r}{\log b + \frac{1}{2K}r}.$$

Also applying (44) and (45) to (I)=(37), (II)=(40), (III)=(42) and (IV)=(43), we find

$$\begin{aligned}
&|F_N(x) - g_{w_0}(x)| \\
&\leq \text{(I)}\ \frac{2\mathrm{b}^{\frac{1}{\mathrm{r}}\frac{\log \mathrm{b}+\frac{11}{2}\mathrm{r}}{\mathrm{k}_1\log \mathrm{b}+\frac{1}{2}\mathrm{r}}}}{\mathrm{b}^{\frac{1}{\mathrm{r}}} - 1 - \mathrm{b}^{\frac{1}{\mathrm{r}}\frac{\log \mathrm{b}+\frac{11}{2}\mathrm{r}}{\mathrm{k}_1\log \mathrm{b}+\frac{1}{2}\mathrm{r}}}} + \frac{\mathrm{K}}{\mathrm{k}_1}\frac{1}{\mathrm{b}^{\frac{\mathrm{K}-\mathrm{k}_1}{\mathrm{r}}}}\cdot\frac{\log \mathrm{b} + \frac{3}{2\mathrm{k}_1}\mathrm{r}}{\log \mathrm{b} + \frac{1}{2\mathrm{K}}\mathrm{r}} \\
&\quad + \text{(II)}\ \frac{1}{\mathrm{b}^{\frac{\mathrm{w}}{\mathrm{r}}}}\cdot\frac{\log \mathrm{b} + \frac{11}{2}\mathrm{r}}{\mathrm{K}\log \mathrm{b} + \frac{\mathrm{r}}{4}} + \left|\frac{1}{\mathrm{b}^{\frac{\mathrm{w}_0}{\mathrm{r}}}} - \frac{1}{\mathrm{b}^{\frac{\mathrm{w}}{\mathrm{r}}}}\right| \\
&\quad + \text{(III)}\ \mathrm{b}^{\frac{1}{\mathrm{r}}}\cdot\frac{\log \mathrm{b} + \frac{11}{2}\mathrm{r}}{\mathrm{K}\log \mathrm{b} + \frac{\mathrm{r}}{2}} + |\mathrm{b}^{\frac{\mathrm{w}_0}{\mathrm{r}}} - \mathrm{b}^{\frac{\mathrm{w}}{\mathrm{r}}}| \\
&\quad + \text{(IV)}\ \mathrm{b}^{\frac{\mathrm{w}}{\mathrm{r}}}(\mathrm{b}^{\frac{1}{\mathrm{r}}} - 1)\frac{\log \mathrm{b} + \frac{11}{2}\mathrm{r}}{\mathrm{K}\log \mathrm{b} + \frac{\mathrm{r}}{2}} + (\mathrm{b}^{\frac{1}{\mathrm{r}}} - 1)\cdot|\mathrm{b}^{\frac{\mathrm{w}_0}{\mathrm{r}}} - \mathrm{b}^{\frac{\mathrm{w}}{\mathrm{r}}}| \\
&\quad + \text{(V)}\ \frac{\mathrm{r}\log_{\mathrm{b}}\mathrm{p}_{\mathrm{N}} + 1}{\mathrm{N}}.
\end{aligned}$$

It can be simplified as

$$\text{(II)} + \text{(III)} + \text{(IV)} \leq 3(b^{\frac{1}{r}})^2\frac{\log b + \frac{11}{2}r}{K\log b + \frac{r}{4}} + 3(b^{\frac{1}{r}})^2\frac{|w - w_0|}{r}\log b$$

and

$$\text{the first term of (I)} \leq \frac{b^{\frac{1}{r}}(2\log b + 11r)}{(b^{\frac{1}{r}} - 1)(\log b)k_1 - 5rb^{\frac{1}{r}} - b^{\frac{1}{r}}\log b - \frac{r}{2}}.$$

□

4. Appendix

4.1. Proof of (37).

Denote $\Psi_{k_1} = \sup_{k\geq k_1} \Phi_k$ and

$$\Phi_k = \left| \frac{\pi(f^{-1}(k+x)) - \pi(f^{-1}(k))}{\pi(f^{-1}(k+1)) - \pi(f^{-1}(k))} - \frac{\psi(x)-1}{\psi(1)-1} \right|.$$

For $k \geq k_1$ and $0 \leq x \leq 1$ we have

$$-\Psi_{k_1} \leq \frac{\pi(f^{-1}(k+x)) - \pi(f^{-1}(k))}{\pi(f^{-1}(k+1)) - \pi(f^{-1}(k))} - \frac{\psi(x)-1}{\psi(1)-1} \leq \Psi_{k_1}.$$

Then

$$\pi\big(f^{-1}(k+x)\big) - \pi(f^{-1}(k)) \leq \left(\Psi_{k_1} + \frac{\psi(x)-1}{\psi(1)-1}\right)\Big(\pi\big(f^{-1}(k+1)\big) - \pi\big(f^{-1}(k)\big)\Big),$$

$$\Big(\pi\big(f^{-1}(k+1)\big) - \pi\big(f^{-1}(k)\big)\Big)\left(-\Psi_{k_1} + \frac{\psi(x)-1}{\psi(1)-1}\right) \leq \pi\big(f^{-1}(k+x)\big) - \pi\big(f^{-1}(k)\big).$$

Thus we have

$$-\Psi_{k_1} + \frac{\psi(x)-1}{\psi(1)-1} \leq \frac{\sum_{k=k_1}^{K-1}(\pi(f^{-1}(k+x)) - \pi(f^{-1}(k)))}{\sum_{k=k_1}^{K-1}(\pi(f^{-1}(k+1)) - \pi(f^{-1}(k)))} \leq \Psi_{k_1} + \frac{\psi(x)-1}{\psi(1)-1}. \quad (47)$$

Abbreviated

$$A = \sum_{k=k_1}^{K-1}\Big(\pi\big(f^{-1}(k+x)\big) - \pi\big(f^{-1}(k)\big)\Big);$$

$$B = \sum_{k=k_1}^{K-1}\Big(\pi\big(f^{-1}(k+1)\big) - \pi\big(f^{-1}(k)\big)\Big);$$

$$C = \sum_{k=0}^{k_1-1}\Big(\pi\big(f^{-1}(k+x)\big) - \pi\big(f^{-1}(k)\big)\Big);$$

$$D = \sum_{k=0}^{k_1-1}\Big(\pi\big(f^{-1}(k+1)\big) - \pi\big(f^{-1}(k)\big)\Big);$$

Then

$$\frac{A+C}{B+D} - \frac{A}{B} = \frac{D\big(\frac{C}{D} - \frac{A}{B}\big)}{B+D}, \quad \text{where} \quad 0 \leq \frac{C}{D} \leq 1, \;\; 0 \leq \frac{A}{B} \leq 1,$$

and

$$D = \pi(f^{-1}(k_1)) - \pi(f^{-1}(0)), \;\; B + D = \pi(f^{-1}(K)) - \pi(f^{-1}(0)).$$

From it

$$\left|\frac{A+C}{B+D}-\frac{A}{B}\right| \leq \frac{\pi(f^{-1}(k_1))-\pi(f^{-1}(0))}{\pi(f^{-1}(K))-\pi(f^{-1}(0))},$$

using

$$\left|\frac{A+C}{B+D}-\frac{\psi(x)-1}{\psi(1)-1}\right| \leq \left|\frac{A+C}{B+D}-\frac{A}{B}\right| + \left|\frac{A}{B}-\frac{\psi(x)-1}{\psi(1)-1}\right|,$$

and (47) imply (39).

4.2. Proof of the inequality

$$A := |\min(\psi(x),\psi(w)) - \min(\psi(x),\psi(w_0))| \leq |\psi(w)-\psi(w_0)|$$

in (19) and (41).

We compute all cases:

$$\begin{aligned}
&x < w < w_0; \quad A = \psi(x)-\psi(x) = 0,\\
&x < w_0 < w; \quad A = \psi(x)-\psi(x) = 0,\\
&w_0 < x < w; \quad A = |\psi(x)-\psi(w_0)| < |\psi(w_0)-\psi(w)|,\\
&w_0 < w < x; \quad A = |\psi(w)-\psi(w_0)|,\\
&w < w_0 < x; \quad A = |\psi(w)-\psi(w_0)|,\\
&w < x < w_0; \quad A = |\psi(w)-\psi(x)| < |\psi(w_0)-\psi(w)|.
\end{aligned}$$

4.3. Discrepancy D_N of the sequence

$$\{\log_b 1^r\}, \quad \{\log_b 2^r\}, \quad \{\log_b 3^r\}, \dots, \{\log_b N^r\}.$$

Put

$$F_N(x) = \frac{1}{N}\#\{n \leq N; \{\log_b n^r\} \in [0,x)\}.$$

Then discrepancy is defined as

$$D_N = \sup_{x\in[0,1]} |F_N(x)-x|.$$

Clearly, we have

$$D_N \leq \sup_{x\in[0,1]} |F_N(x)-g_{w_0}(x)| + \sup_{x\in[0,1]} |g_{w_0}(x)-x|. \tag{48}$$

For the first part of (48) we use Theorem 5. Now we put $w_0 = 0$ and for the second part we need found upper bound of

$$x - g_0(x) = x - \frac{b^{\frac{x}{r}} - 1}{b^{\frac{1}{r}} - 1}.$$

By Lagrange's theorem

$$b^{\frac{x}{r}} - 1 = (x-0)b^{\frac{x_1}{r}}\frac{\log b}{r}, \quad x_1 \in (0,x),$$
$$b^{\frac{1}{r}} - 1 = (1-0)b^{\frac{x_2}{r}}\frac{\log b}{r}, \quad x_2 \in (0,1).$$

Thus

$$x - g_0(x) = x\big(1 - b^{\frac{x_1-x_2}{r}}\big) = x\big(0 - (x_1 - x_2)\big)b^{\frac{x_3}{r}}\frac{\log b}{r}, \quad x_3 \in (x_1, x_2).$$

The upper bound is

$$|x - g_0(x)| < b^{\frac{1}{r}}\frac{\log b}{r}$$

and applying Theorem 5 we have

$$D_N \le \frac{|w-0|}{r} \cdot \log b \cdot b^{\frac{1}{r}} \cdot (b^{\frac{1}{r}} + 1) + \frac{3}{N} + \frac{r\log_b N}{N} + b^{\frac{1}{r}}\frac{\log b}{r},$$

where $w = \{r \log_b N\}$.

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ON THE PSEUDORANDOMNESS OF THE LIOUVILLE FUNCTION OF POLYNOMIALS OVER A FINITE FIELD

László Mérai — Arne Winterhof

Dedicated to the memory of Professor Pierre Liardet

ABSTRACT. We study several pseudorandom properties of the Liouville function and the Möbius function of polynomials over a finite field. More precisely, we obtain bounds on their balancedness as well as their well-distribution measure, correlation measure, and linear complexity profile.

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1. Introduction

In analogy to the Liouville λ-function and the Möbius μ-function for integers, Carlitz [2] introduced the mappings λ and μ for polynomials over the finite field $\mathbb{F}_q$ by

$$\lambda(F) = (-1)^{\omega(F)}, \quad F \in \mathbb{F}_q[X],$$

where $\omega(F)$ denotes the number of irreducible factors of F (counted in multiplicities), and

$$\mu(F) = \begin{cases} \lambda(F) & \text{if } F \text{ is squarefree}, \\ 0 & \text{otherwise}, \end{cases} \qquad F \in \mathbb{F}_q[X].$$

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Carlitz [2] proved

$$\sum_{\deg F=d} \lambda(F) = (-1)^d q^{\lfloor (d+1)/2 \rfloor} \tag{1}$$

and

$$\sum_{\deg F=d} \mu(F) = \begin{cases} 0 & \text{if } d \geq 2, \\ -q & \text{if } d = 1, \end{cases} \tag{2}$$

where the sums are over all monic polynomials $F \in \mathbb{F}_q[X]$ of degree d.

For $\ell \geq 2$, $d \geq 2$, distinct polynomials $D_1, \ldots, D_\ell \in \mathbb{F}_q[X]$ of degree smaller than d, q odd, and $(\epsilon_1, \ldots, \epsilon_\ell) \in \{0,1\}^\ell \setminus (0, \ldots, 0)$, Carmon and Rudnick [3, Theorem 1.1] have recently proved that

$$\sum_{\deg F=d} \mu(F+D_1)^{\epsilon_1} \cdots \mu(F+D_\ell)^{\epsilon_\ell} = O\big(\ell d q^{d-1/2}\big), \quad d \geq 2. \tag{3}$$

(For $d = 1$ the sum trivially equals $(-1)^{\sum_{j=1}^{\ell} \epsilon_j} q$.) Since the number of monic squarefree polynomials over $\mathbb{F}_q$ of degree $d \geq 2$ is $q^d - q^{d-1}$ (see for example [12, Proposition 2.3]) the same result holds for λ instead of μ as well.

(1), (2), and (3) are results on the *global* pseudorandomness of polynomials of degree d over $\mathbb{F}_q$. More precisely, (1), (2), and (3) are essentially results on two measures of pseudorandomness, the balancedness and the correlation measure of order ℓ, respectively, for *all* monic polynomials of degree d. In this article we focus on the *local* pseudorandomness, that is, we deal only with the first $N < p^d$ monic polynomials of degree d (in the lexicographic order). The main motivation for doing this is to derive binary sequences and to analyze several measures of pseudorandomness for binary sequences: the balancedness, the well-distribution measure, the correlation measure of order ℓ, and the linear complexity profile. In particular, to obtain a lower bound on the linear complexity profile we need a local analog of (3). Although our results can be extended to any finite field of odd characteristic we focus on prime fields to avoid a more complicated notation. More precisely, let $p > 2$ be a prime and denote by $\mathbb{F}_p$ the finite field of p elements which we identify with the set of integers $\{0, 1, \ldots, p-1\}$ equipped with the usual arithmetic modulo p. We order the monic polynomials over $\mathbb{F}_p$ of degree $d \geq 2$ in the following way. For $0 \leq n < p^d$ put

$$F_n(X) = X^d + n_{d-1}X^{d-1} + \cdots + n_1 X + n_0$$

if

$$n = n_0 + n_1 p + \cdots + n_{d-1} p^{d-1}, \quad 0 \leq n_0, n_1, \ldots, n_{d-1} < p.$$

We study finite binary sequences $S_{p^d} = (s_0, \ldots, s_{p^d-1}) \in \{-1,+1\}^{p^d}$ with the property

$$s_n = \lambda(F_n) = \mu(F_n), \quad F_n \text{ squarefree}. \tag{4}$$

Our results will be independent of the choice of $s_n \in \{-1,+1\}$ for non-squarefree F_n.

First we prove the following local analog of (1) and (2) on the balancedness of the sequence S_{p^d}.

Theorem 1. *For $d \ge 2$, $1 \le N < p^d$, and s_n satisfying* (4) *for all $n = 0, 1, \dots, N-1$ such that F_n is squarefree, we have*

$$\sum_{n=0}^{N-1} s_n = O\Big(d\big(Np^{-1/2} + p^{1/2}\log p\big)\Big) \quad \text{if } d \text{ is even}$$

and

$$O\Big(d\big(Np^{-1/2} + p^{3/2}\log p\big)\Big) \quad \text{if } d \text{ is odd.}$$

Next, we study several pseudorandom properties of S_{p^d}. For a survey on pseudorandom sequences and their desirable properties we refer to [17].

For a given binary sequence

$$E_N = (e_0, \dots, e_{N-1}) \in \{-1,+1\}^N.$$

Mauduit and Sárközy [9] defined the *well-distribution measure* of E_N by

$$W(E_N) = \max_{a,b,t} \left| \sum_{j=0}^{t-1} e_{a+jb} \right|,$$

where the maximum is taken over all $a, b, t \in \mathbb{N}$ such that

$$0 \le a \le a + (t-1)b < N,$$

and the *correlation measure of order ℓ* of E_N by

$$C_\ell(E_N) = \max_{M,D} \left| \sum_{n=0}^{M-1} e_{n+d_1} e_{n+d_2} \dots e_{n+d_\ell} \right|,$$

where the maximum is taken over all $D = (d_1, \dots, d_\ell)$ and M such that

$$0 \le d_1 < d_2 < \dots < d_\ell \le N - M.$$

We will prove the following bounds on the well-distribution measure and the correlation measure of order ℓ for S_{p^d}.

Theorem 2. *We have the following bound on the well-distribution measure*:

$$W(S_{p^d}) = O(dp^{d-1/2}\log p), \qquad d \ge 2.$$

Theorem 3. *We have the following bound on the correlation measure of order ℓ*:

$$C_\ell(S_{p^d}) = O(\ell^2 dp^{d-1/2}\log p), \quad d \ge 2.$$

Theorem 3 allows us to derive a lower bound on the linear complexity of the binary sequence $S'_{p^d} = (s'_1, s'_2, \dots, s'_{p^d})$ defined by the relation $s_n = (-1)^{s'_n}$.

For an integer $N \geq 1$ the *Nth linear complexity* $L(r, N)$ of a sequence $r = (r_0, \dots, r_{T-1})$ of length T over the finite field $\mathbb{F}_2$ is the smallest positive integer L such that there are constants $c_1, \dots, c_L \in \mathbb{F}_2$ satisfying the linear recurrence relation

$$r_{n+L} = c_{L-1} r_{n+L-1} + \cdots + c_0 r_n, \quad \text{for } 0 \leq n < N - L.$$

If r starts with $N-1$ zeros, then we define

$$L(r, N) = 0 \quad \text{if } r_{N-1} = 0, \quad \text{and} \quad L(r, N) = N \quad \text{if } r_{N-1} = 1.$$

The sequence $(L(r,N))_{1 \leq N \leq T}$ is the *linear complexity profile* of r.

Brandstätter and Winterhof [1] proved a lower bound on the Nth linear complexity $L(E'_N, N)$ of a sequence $E'_N = (e'_0, \dots, e'_{N-1})$ over $\mathbb{F}_2$ terms of the correlation measure $C_\ell(E_N)$ of the finite sequence $E_N = (e_0, \dots, e_{N-1}) \in \{-1, +1\}^N$ defined by $e_n = (-1)^{e'_n}$.

Lemma 1. *Let $E'_N = (e'_0, \dots, e'_{N-1})$ be a finite sequence over $\mathbb{F}_2$ of length N. Writing $e_n = (-1)^{e'_n}$ for $0 \leq n \leq N-1$, we have*

$$L(E'_N, N) \geq N - \max_{2 \leq \ell \leq L(e'_n, N)+1} C_\ell(E_N).$$

By Theorem 3 and Lemma 1 we immediately get the following lower bound.

Corollary 2. *For fixed $d \geq 2$ and any $1 \leq N < p^d$ we have*

$$L(S'_N, N) \gg \frac{N^{1/2}}{d^{1/2} p^{d/2-1/4} (\log p)^{1/2}}$$

for the sequence $S'_N = (s'_0, \dots, s'_{N-1})$, where s'_n $(0 \leq n < N)$ is defined by $s_n = (-1)^{s'_n}$.

2. Proofs

As in [3] we start with Pellet's formula, see [11],

$$\lambda(F) = \mu(F) = \left(\frac{D(F)}{p}\right) \quad \text{if } D(F) \neq 0,$$

where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol and $D(F)$ the discriminant of F. (See also Stickelberger [15] and Skolem [14] as well as [6, 16] for a short proof.)

Moreover, $(-1)^{d(d-1)/2}D(F_n)$ equals the following determinant of a $(2d-1)\times(2d-1)$ matrix,

$$\begin{vmatrix}
1 & n_{d-1} & \cdots & n_1 & n_0 & 0 & \cdots & 0\\
0 & 1 & n_{d-1} & \cdots & n_1 & n_0 & \cdots & 0\\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0\\
0 & 0 & 0 & 1 & n_{d-1} & \cdots & n_1 & n_0\\
d & (d-1)n_{d-1} & \cdots & n_1 & 0 & 0 & \cdots & 0\\
\vdots & \ddots & \ddots & \dots & \ddots & \ddots & \ddots & \vdots\\
0 & 0 & 0 & d & (d-1)n_{d-1} & \cdots & n_1 & 0\\
0 & 0 & 0 & 0 & d & (d-1)n_{d-1} & \cdots & n_1
\end{vmatrix}.$$

Note that

$$D(F_n) = (-1)^{d+1}d^d n_0^{d-1} + (-1)^d(d-1)^{d-1}n_1^d + h(n_0, n_1, n - n_0 - n_1 p),$$

where $h(X_0, X_1, X_2)$ is a polynomial over $\mathbb{F}_p$ of relative degrees in X_0 at most $d-2$ and in X_1 at most $d-1$.

Proof of Theorem 1. Put $N-1 = N_0 + N_1 p + N_2 p^2$ with $0 \le N_0, N_1 < p$. Then we have

$$\left|\sum_{n=0}^{N-1} s_n\right| \le S_1 + S_2 + S_3,$$

where

$$S_1 = \sum_{n_2=0}^{N_2-1}\left|\sum_{n_0,n_1=0}^{p-1} s_{n_0+n_1p+n_2p^2}\right|,$$

$$S_2 = \left|\sum_{n_1=0}^{N_1-1}\sum_{n_0=0}^{p-1} s_{n_0+n_1p+N_2p^2}\right|,$$

$$S_3 = \left|\sum_{n_0=0}^{N_0} s_{n_0+N_1p+N_2p^2}\right|.$$

In the first case (d even), write

$$s_{n_0+n_1p+n_2p^2} = \left(\frac{D(F_{n_0+n_1p+n_2p^2})}{p}\right) \quad \text{if } D(F_{n_0+n_1p+n_2p^2}) \ne 0.$$

Note that there are at most $d-1$ different n_0 with $0 \le n_0 < p$ for any fixed n_1 and n_2 with $D(F_{n_0+n_1p+n_2p^2}) = 0$.

Since now $D(F_{n_0+n_1p+n_2p^2})$ has odd degree in n_0, for any pair (n_1, n_2) the monic polynomial $f(X) = -d^{-d}D(F_{X+n_1p+n_2p^2})$ is not a square and we can apply the Weil bound (for complete character sums)

$$\left|\sum_{n\in\mathbb{F}_p}\left(\frac{af(n)}{p}\right)\right| \le \big(\deg(f)-1\big)p^{1/2}, \quad a\neq 0,$$

(see for example [13, Theorem 2G] or [8, Theorem 5.41]) directly to estimate S_1 and S_2 and the standard method for reducing incomplete character sums to complete ones, see for example [7, Chapter 12] or [18, Theorem 2], to estimate S_3,

$$\begin{aligned}
S_1 &\le \sum_{n_2=0}^{N_2-1}\sum_{n_1=0}^{p-1}\left(\left|\sum_{n_0=0}^{p-1}\left(\frac{D(F_{n_0+n_1p+n_2p^2})}{p}\right)\right| + d - 1\right)\\
&\le N_2p\big((d-2)p^{1/2}+d-1\big),\\
S_2 &\le \sum_{n_1=0}^{N_1-1}\left(\left|\sum_{n_0=0}^{p-1}\left(\frac{D(F_{n_0+n_1p+N_2p^2})}{p}\right)\right| + d - 1\right)\\
&\le N_1\big((d-2)p^{1/2}+d-1\big),\\
S_3 &\le \left|\sum_{n_0=0}^{N_0}\left(\frac{D(F_{n_0+N_1p+N_2p^2})}{p}\right)\right| + d - 1\\
&\le (d-1)p^{1/2}\log p + d - 1,
\end{aligned}$$

and hence the result since $N_1 + N_2p < N/p$. In the second case (d odd) the sums over n_0 can be trivial but not the sums over n_1. Hence, we get

$$S_1 + S_2 + S_3 \le N_2p\big((d-1)p^{1/2}+d\big) + \big(dp^{1/2}\log p + d\big)p + N_0$$

and the result follows, since $N_2p < N/p$. □

Proof of Theorem 2. We can assume without loss of generality, that $d < p^{1/2}$, since otherwise the theorem is trivial. Fix a, b, t with $0 \le a \le a + (t-1)b \le p^d - 1$. If $t < p^{d-1} + 1$, then we use the trivial bound

$$\left|\sum_{j=0}^{t-1} s_{a+bj}\right| \le t.$$

Now we assume $t \geq p^{d-1} + 1$ and thus $b < p$. Put

$$T = p \left\lfloor \frac{t}{p} \right\rfloor .$$

Then we have $t - T = O(p)$ and

$$\sum_{j=0}^{t-1} s_{a+bj} = \sum_{j=0}^{T-1} s_{a+bj} + O(p). \tag{5}$$

For $0 \leq a \leq a + bj \leq p^d - 1$ let

$$a = a_0 + a_1 p + a_2 p^2, \quad 0 \leq a_0,\ a_1 < p,\ 0 \leq a_2 < p^{d-2}$$

and

$$j = j_0 + j_1 p + j_2 p^2, \quad 0 \leq j_0,\ j_1 < p,\ 0 \leq j_2 < p^{d-2}.$$

Put

$$w_0 = \left\lfloor \frac{a_0 + bj_0}{p} \right\rfloor \quad \text{and} \quad w_1 = \left\lfloor \frac{a_1 + bj_1 + w_0}{p} \right\rfloor .$$

Then we have

$$a + bj = z_0 + z_1 p + z_2 p^2, \quad 0 \leq z_0, z_1 < p,\ 0 \leq z_2 < p^{d-2},$$

with

$$z_0 = a_0 + bj_0 - w_0 p, \quad z_1 = a_1 + bj_1 + w_0 - w_1 p, \quad z_2 = a_2 + bj_2 + w_1,$$

and

$$s_{a+bj} = \left(\frac{D(F_{z_0 + z_1 p + z_2 p^2})}{p} \right) \quad \text{if } D\big(F_{z_0 + z_1 p + z_2 p^2}\big) \neq 0.$$

Note that we have at most $(b+1)$ possible choices for w_0 and for w_1 since $0 \leq w_0,\ w_1 \leq b$.

We define

$$S_{w_0,w_1} = \left\{ a + jb : 0 \leq j < T,\ \left\lfloor \frac{a_0 + bj_0}{p} \right\rfloor = w_0, \left\lfloor \frac{a_1 + bj_1 + w_0}{p} \right\rfloor = w_1 \right\}$$

and note that these sets define a partition of $\{a + jb : 0 \leq j < T\}$. For each (w_0, w_1) the set S_{w_0,w_1} is of the form

$$S_{w_0,w_1} = \Big\{ a_0 - w_0 p + bj_0 + \big(w_0 + a_1 - w_1 p + bj_1\big)p + \big(w_1 + a_2 + bj_2\big)p^2 : \\ k_i \leq j_i < K_i,\ i = 0, 1, 2 \Big\},$$

where $k_i = k_i(w_0, w_1)$ and $K_i = K_i(w_0, w_1)$ $(i = 0, 1, 2)$ defined as

$$k_0 = \max \left\{ 0, \left\lfloor \frac{w_0 p - a_0}{b} \right\rfloor \right\}, \quad K_0 = \min \left\{ p, \left\lfloor \frac{(w_0 + 1)p - a_0}{b} \right\rfloor \right\},$$

$$k_1 = \max\left\{0, \left\lfloor \frac{w_1 p - a_0 - w_0}{b} \right\rfloor\right\}, \quad K_1 = \min\left\{p, \left\lfloor \frac{(w_1+1)p - a_0 - w_0}{b} \right\rfloor\right\},$$

$$k_2 = 0, \quad K_2 = \frac{T-1}{p^2} - \left\lceil \frac{w_1}{p} \right\rceil.$$

We remark, that both $K_0 - k_0$ and $K_1 - k_1$ are $O(p/b)$.

If d is even, the absolute value of (5) is at most

$$\sum_{w_0,w_1} \sum_{j_1=k_1(w_0,w_1)}^{K_1(w_0,w_1)} \sum_{j_2=k_2(w_0,w_1)}^{K_2(w_0,w_1)} \left| \sum_{j_0=k_0(w_0,w_1)}^{K_0(w_0,w_1)} \left(\frac{D(F_{a_0-w_0p+bj_0+(w_0+a_1-w_1p+bj_1)p+(w_1+a_2+bj_2)p^2})}{p} \right) \right| \tag{6}$$

As before,

$$D(F_{X+(w_0+a_1-w_1p+bj_1)p+(w_1+a_2+bj_2)p^2}) \in \mathbb{F}_p[X]$$

has odd degree, thus we can apply the Weil-bound after using the standard technique to reduce incomplete sums to complete ones and get, that (6) is

$$O\left(b^2 \frac{p}{b} \frac{T}{p^2} dp^{1/2} \log p\right) = O\left(bTdp^{-1/2} \log p\right).$$

Since $bT = O\left(p^d\right)$ we get the result for even d. For odd d, the proof is similar. $\square$

The proof of Theorem 3 is based on the following form of [3, Proposition 2.1].

Lemma 3. *For given $0 \leq d_1 < d_2 < \cdots < d_\ell < p^d$ let $G \subset \{1, 2, \ldots, p^{d-1}\}$ the set of integers a such that $D(F_{X+ap+d_1}) \in \mathbb{F}_p[X]$ is squarefree and coprime to $D(F_{X+ap+d_i}) \in \mathbb{F}_p[X]$ for $i = 2, 3, \ldots, \ell$. Then, for the complement of G we have*

$$|G^c| = \left|\{1, 2, \ldots, p^{d-1}\} \setminus G\right| \leq 3\ell d^2 p^{d-2}.$$

Proof of Theorem 3. We can assume without loss of generality, that $d < p^{1/2}$, since otherwise the theorem is trivial.

Let $M \in \mathbb{N}$ and let $0 \leq d_1 < d_2 < \cdots < d_\ell < p^d - M$ be integers. If $M \leq p^{d-1}$ we use the trivial bound

$$\left| \sum_{n=0}^{M-1} s_{n+d_1} s_{n+d_2} \ldots s_{n+d_\ell} \right| \leq M.$$

Now, we assume $M \geq p^{d-1} + 1$. Let

$$T = p \left\lfloor \frac{M}{p} \right\rfloor.$$

Then we have $M - T = O(p)$ and

$$\left|\sum_{n=0}^{M-1} s_{n+d_1} s_{n+d_2} \dots s_{n+d_\ell}\right| = \left|\sum_{n=0}^{T-1} s_{n+d_1} s_{n+d_2} \dots s_{n+d_\ell}\right| + O(p).$$

As it has already been written

$$n = n_0 + n_1 p, \quad 0 \le n_0 < p, \quad 0 \le n_1 < p^{d-1}$$

and

$$d_i = d_{i,0} + d_{i,1} p, \quad 0 \le d_{i,0} < p, \quad 0 \le d_{i,1} < p^{d-1}, \quad i = 1, 2, \dots, \ell.$$

If

$$w_i = \left\lfloor \frac{n_0 + d_{i,0}}{p} \right\rfloor \in \{0, 1\}, \quad i = 1, 2, \dots, \ell,$$

then

$$n + d_i = z_{i,0} + z_{i,1} p, \quad 0 \le z_{i,0} < p, \quad 0 \le z_{i,1} < p^{d-1}, \quad i = 1, 2, \dots, \ell,$$

with

$$\begin{aligned} z_{i,0} &= n_0 + d_{i,0} - w_i p, \\ z_{i,1} &= n_1 + d_{i,1} + w_i, \end{aligned} \qquad i = 1, 2, \dots, \ell,$$

and

$$s_{n+d_i} = \left(\frac{D(F_{z_{i,0}+z_{i,1}p})}{p}\right) \quad \text{if } D(F_{z_{i,0}+z_{i,1}p}) \ne 0, \quad i = 1, 2, \dots, \ell.$$

For $(w_1, w_2, \dots, w_\ell) \in \{0, 1\}^\ell$ write

$$\begin{aligned} S_{w_i, d_i} &= \left\{ n : 0 \le n < T, \left\lfloor \frac{n_0 + d_{i,0}}{p} \right\rfloor = w_i \right\} \\ &= \left\{ j_0 + j_1 p : k_{i,0} \le j_0 < K_{i,0},\ k_{i,1} \le j_1 < K_{i,1} \right\}, \end{aligned}$$

where

$$\begin{aligned} k_{i,0} &= k_{i,0}(w_i) = \max\left\{0, p w_i - d_{i,0}\right\}, \\ K_{i,0} &= K_{i,0}(w_i) = \min\left\{p, p(w_i + 1) - d_{i,0}\right\} \end{aligned}$$

and

$$\begin{aligned} k_{i,1} &= k_{i,1}(w_i) = 0, \\ K_{i,1} &= K_{i,1}(w_i) = T/p. \end{aligned}$$

As $(w_1, w_2, \ldots, w_\ell)$ runs in $\{0,1\}^\ell$, the intersections $S_{w_1,d_1} \cap \ldots \cap S_{w_\ell,d_\ell}$ are a partition of integers $0 \le n < T$. However, it can be shown in the same way as in [10], that there are at most $\ell+1$ non-empty intersections. More precisely, let us reorder the integers $d_1 < d_2 < \ldots < d_\ell$ and the carries $(w_1, w_2, \ldots, w_\ell)$ by the first components of

$$\begin{aligned} d_i \ : \ \{\, d_1,\, d_2, \ldots, d_\ell\} &= \{\, d_1',\, d_2', \ldots,\, d_\ell'\}, \\ \{w_1, w_2, \ldots, w_\ell\} &= \{w_1', w_2', \ldots, w_\ell'\}, \\ d_{1,0}' \le\, d_{2,0}' &\le \cdots \le d_{\ell,0}'. \end{aligned}$$

Then writing $d_{0,0}' = 0$ and $d_{0,\ell+1}' = p$ we have

$$\begin{aligned} &\left| \sum_{n=0}^{T-1} s_{n+d_1} s_{n+d_2} \ldots s_{n+d_\ell} \right| \\ &\le \sum_{(w_1,w_2,\ldots,w_\ell)\in\{0,1\}^\ell} \left| \sum_{n\in S_{w_1,d_1}\cap\cdots\cap S_{w_\ell,d_\ell}} s_{n+d_1} s_{n+d_2} \ldots s_{n+d_\ell} \right| \\ &\le \sum_{i=1}^{\ell+1} \sum_{j_1=0}^{T/p-1} \left| \sum_{j_0=p-d_{i,0}'-1}^{p-d_{i-1,0}'} s_{j_0+j_1p+d_1} s_{j_0+j_1p+d_2} \ldots s_{j_0+j_1p+d_\ell} \right| \\ &\le \sum_{i=1}^{\ell+1} \sum_{j_1=0}^{T/p-1} \\ &\left(\left| \sum_{j_0=p-d_{i,0}'}^{p-d_{i-1,0}'-1} \left(\frac{D(F_{j_0+j_1p+d_1}) D(F_{j_0+j_1p+d_2}) \cdots D(F_{j_0+j_1p+d_\ell})}{p} \right) \right| + \ell(d-1) \right) \qquad (7) \end{aligned}$$

For a fixed i, if $j_1 \in G$, then the innermost sum is non-trivial. On the other hand we estimate the inner sum of (7) trivially by p if $j_1 \notin G$. Then we get that (7) is less than

$$(\ell+1)\Big(3\ell d^2 p^{d-1} + \tfrac{T}{p}\big(\ell(d-1)p^{1/2}\log p + \ell(d-1)\big)\Big) = O\big(\ell^2 d p^{d-\frac{1}{2}} \log p\big)$$

and the result follows. $\square$

Final Remarks

- Cassaigne, Ferenzi, Mauduit, Rivat and Sárközy [4, 5] studied the pseudorandomness of the Liouville function for integers.
- Our results as well as the results of [3] are based on Pellet's result which is not true for characteristic 2. Finding analog results for characteristic 2 would be very interesting.
- In this paper as well as in [3] d is fixed and p has to be large with respect to d to get nontrivial bounds. It would be interesting to study the same problems if p is fixed and d goes to infinity.

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ON STRONG NORMALITY

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Dedicated to the memory of Professor Pierre Liardet

ABSTRACT. We introduce the concept of strong normality by defining strong normal numbers and provide various properties of these numbers, including the fact that almost all real numbers are strongly normal.

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Given a fixed integer $q \geq 2$, an irrational number α is said to be a *normal number* in base q (or a q-normal number) if any preassigned sequence of k digits (in base q) appears in the q-ary expansion of α at the expected frequency, namely $1/q^k$.

Normal numbers have been studied since Borel [1] in 1909. Hence the vast literature concerning normal numbers (see for instance Champernowne [3], Copeland and Erdős [4], Davenport and Erdős [5], and the recent book of Bugeaud [2]). In a series of papers (see [6] through [11]), the first two authors obtained new results concerning normal numbers, including various ways of constructing new families of normal numbers.

In this paper, we will identify a very special family of normal numbers—that we will call *strongly normal numbers*—which are connected with arithmetical functions that have a local normal distribution, such as the function $\omega(n)$ which counts the number of distinct prime factors of n.

Let us first recall some definitions.

A sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers is said to be *uniformly distributed modulo 1* (or *mod 1*) if for every interval $[\alpha, \beta) \subseteq [0, 1)$,

$$\lim_{N\to\infty} \frac{1}{N} \#\big\{n \leq N : \{x_n\} \in [a, b)\big\} = b - a\,.$$

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In other words, a sequence of real numbers is said to be uniformly distributed mod 1 if every subinterval of the unit interval gets its fair share of the fractional parts of the elements of this sequence.

Recall also that, given a set of N real numbers $x_1, \ldots, x_N$, the *discrepancy* of this set is defined as the quantity

$$D(x_1, \ldots, x_N) := \sup_{[a,b)\subseteq[0,1)} \left| \frac{1}{N} \sum_{\substack{n\le N \\ \{x_n\}\in[a,b)}} 1 - (b-a) \right| .$$

It is known that a sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers is uniformly distributed mod 1 if $D(x_1, \ldots, x_N) \to 0$ as $N \to \infty$ (see Theorem 1.1 in the book of Kuipers and Niederreiter [15]).

Also, given an integer $q \ge 2$, it can be shown (see Theorem 8.1 in the book of Kuipers and Niederreiter [15]) that a real number α is normal in base q if and only if the sequence $(\{q^n\alpha\})_{n\in\mathbb{N}}$ is uniformly distributed mod 1.

We are now ready to introduce the concept of *strong normality*. For each positive integer N, let

$$M = M_N := \lfloor \delta_N \sqrt{N} \rfloor, \text{ where } \delta_N \to 0 \text{ and } \delta_N \log N \to \infty \text{ as } N \to \infty . \qquad (1)$$

We shall say that an infinite sequence of real numbers $(x_n)_{n\ge1}$ is *strongly uniformly distributed* mod 1 if

$$D(x_{N+1}, \ldots, x_{N+M}) \to 0 \quad \text{as} \quad N \to \infty$$

for every choice of δ_N satisfying (1).

Remark 1. Observe that if a sequence of real numbers $(x_n)_{n\in\mathbb{N}}$ is strongly uniformly distributed mod 1, then it must be uniformly distributed mod 1 as well. The proof goes as follows. Assume that $(x_n)_{n\in\mathbb{N}}$ is strongly uniformly distributed mod 1 and define the sequence $(\epsilon_k)_{k\in\mathbb{N}}$ by

$$\epsilon_k = \begin{cases} 1 & \text{if} \quad k \le e \,, \\ 1/\log k & \text{if} \quad k > e \,. \end{cases}$$

Also, for each integer $k \ge 1$, let $U_k = \lfloor k^2 \epsilon_k \rfloor$ and $V_k = U_{k+1} - U_k - 1$. Moreover, setting $N = U_k$ and $M = M_N = V_k$, one can verify that (1) is satisfied as $k \to \infty$. To see this, observe that

$$\begin{aligned} V_k &= (k+1)^2\epsilon_{k+1} - k^2\epsilon_k + O(1) = 2k\epsilon_{k+1} + k^2(\epsilon_{k+1} - \epsilon_k) + O(1) \\ &= 2k\epsilon_{k+1} + O\left(\frac{k}{\log^2 k}\right) = \big(1 + o(1)\big) 2k\epsilon_k \qquad \text{as } k \to \infty . \end{aligned} \qquad (2)$$

Now, for each $k \in \mathbb{N}$, define δ_{U_k} implicitly by $V_k = \lfloor \delta_{U_k} \sqrt{U_k} \rfloor$. Using this in (2), it follows that

$$2k\epsilon_k\big(1+o(1)\big) = \delta_{U_k} k\sqrt{\epsilon_k}\big(1+o(1)\big) \qquad (k \to \infty)\,,$$

from which we obtain that

$$\delta_{U_k} = \big(1+o(1)\big)2\sqrt{\epsilon_k} \qquad (k \to \infty)\,.$$

Hence, it follows that, as $k \to \infty$,

$$\delta_N = \delta_{U_k} \to 0 \text{ and } \delta_N \log N = \big(1+o(1)\big)2\sqrt{\epsilon_k}\log U_k = \big(1+o(1)\big)4\sqrt{\log k} \to \infty\,,$$

implying that condition (1) is satisfied and also, using the fact that $(x_n)_{n\in\mathbb{N}}$ is strongly uniformly distributed mod 1, that

$$D(x_{U_k}, \dots, x_{U_{k+1}-1}) = D(x_N, \dots, x_{N+M}) \to 0 \qquad (k \to \infty)\,. \tag{3}$$

We shall now use this result to prove that

$$D(x_1, \dots, x_N) \to 0 \qquad (N \to \infty)\,. \tag{4}$$

To do so, for each $N \in \mathbb{N}$, let t_N be the unique integer k for which $U_k \le N < U_{k+1}$, from which it follows that

$$\frac{N - U_{t_N}}{N} \le \frac{U_{t_N+1} - U_{t_N}}{N} \to 0 \qquad (N \to \infty)\,. \tag{5}$$

With this set up, we have

$$ND(x_1, \dots, x_N) \le \sum_{\ell=1}^{t_N-1} (U_{\ell+1} - U_\ell) D(x_{U_\ell}, \dots, x_{U_{\ell+1}-1}) + (N - U_{t_N})\,. \tag{6}$$

Applying (3) successively with $k = \ell$ for $\ell = 1, \dots, t_N - 1$, it follows, in light of (5), that the right hand side of (6) is $o(N)$ as $N \to \infty$. From this, (4) follows immediately, thus proving our claim.

Remark 2. It follows from the above that if α is a strongly normal number, then it must also be a normal number. Indeed, by definition, the sequence $(\{\alpha q^n\})_{n\in\mathbb{N}}$ is strongly uniformly distributed mod 1 and therefore, in light of Remark 1, then it must be uniformly distributed mod 1, which in turn (as we saw above) is equivalent to the statement that α is a normal number.

Given a fixed integer $q \ge 2$, we say that an irrational number α is a *strongly normal number* in base q (or a *strongly q-normal number*) if the sequence $(x_n)_{n\in\mathbb{N}}$, defined by $x_n = \{q^n\alpha\}$, is strongly uniformly distributed mod 1. First, observe that there exist normal numbers which are not strongly normal. For instance, consider the Champernowne number

$$\theta := 0.1\,10\,11\,100\,101\,110\,111\,1000\,1001\,1010\,1011\,1100\,1101\,1110\,1111\,\dots$$

that is the number made up of the concatenation of the positive integers written in base 2. It is known since Champernowne [3] that θ is normal. However, one can show that θ is not a strongly normal number. Indeed, given a positive integer n, let $S_n = \lfloor 2^n/(\sqrt{n}\log n)\rfloor$ and consider the sequence

$$2^{2n}+1, 2^{2n}+2, 2^{2n}+3, \dots, 2^{2n}+S_n\,, \tag{7}$$

writing each of the above S_n integers in binary. Each of the resulting binary integers contains $2n+1$ digits, implying that the total number of digits appearing in the sequence (7) is equal to $(2n+1)S_n$.

Now, letting $\lambda(m)$ stand for the number of digits in the integer m, the total number N of digits of the concatenated integers preceding the number $2^{2n}+1$ is, as n becomes large,

$$N = \sum_{m\le 2^{2n}} \lambda(m) = 2n+1+\sum_{m\le 2^{2n}} \left\lceil \frac{\log m}{\log 2}\right\rceil = \big(1+o(1)\big)2n\cdot 2^{2n}. \tag{8}$$

We can write the first digits of the Champernowne number as

$$\begin{aligned}\theta &= 0.\epsilon_1\epsilon_2\dots\epsilon_N\overline{2^{2n}+1}\,\overline{2^{2n}+2}\,\dots\,\overline{2^{2n}+S_n}\dots\\ &= 0.\epsilon_1\epsilon_2\dots\epsilon_N\,\rho\;\dots,\end{aligned}$$

say, where in fact, $\rho = \overline{2^{2n}+1}\,\overline{2^{2n}+2}\,\dots\,\overline{2^{2n}+S_n} = \epsilon_{N+1}\dots\epsilon_{N+\lambda(\rho)}$. (Here, $\overline{n_1}\,\overline{n_2}\dots\overline{n_r}$ stands for the concatenation of all the digits appearing successively in the integers $n_1, n_2, \dots, n_r$.) We will first show that the proportion of zeros in the word ρ is too large. For this we shall first count the number of 1's in ρ. Setting $\beta(m)$ as the number of 1's in the integer m, the total number of 1's in ρ is equal to

$$\sum_{m\le S_n}\beta(m) = \frac{1}{2}\frac{S_n\log S_n}{\log 2}+O(S_n)\,,$$

from which we can deduce that the total number of zeros in ρ is

$$\sum_{m=1}^{S_n} n + \sum_{m=1}^{S_n}\big(n-\beta(m)\big) = 2nS_n - \frac{1}{2}\frac{S_n\log S_n}{\log 2}+O(S_n)\,. \tag{9}$$

Since $\lambda(\rho) = (2n+1)S_n$ and recalling that $S_n = \lfloor 2^n/(\sqrt{n}\log n)\rfloor$, it follows from (9) that the proportion of zeros in ρ is equal to, as $n\to\infty$,

$$\begin{aligned}\frac{1}{\lambda(\rho)}\times\text{the number of zeros in }\rho &= \frac{2n}{2n+1}-\frac{1}{2}\frac{\log S_n}{(2n+1)\log 2}+o(1)\\ &= 1+o(1)-\frac{1}{2}\frac{n\log 2-\frac{1}{2}\log n}{(2n+1)\log 2}+o(1)\\ &= 1-\frac{1}{4}+o(1) = \frac{3}{4}+o(1)\,.\end{aligned}$$

Then, since

$$\left| \sum_{\substack{N+1 \le \nu \le N+M \\ \{2^\nu \theta\} < \frac{1}{2}}} 1 - \frac{1}{2}(2n+1)S_n \right| \ge \frac{1}{4}(2n+1)S_n\,,$$

it follows that, setting $x_n := \{2^n\theta\}$ and choosing

$$M = M_N = (2n+1)S_n \approx \sqrt{N}/\log\log N$$

(where we used (8)), thereby complying with condition (1), the discrepancy of the sequence of numbers $x_{N+1}, \dots, x_{N+M}$ is

$$\begin{aligned} & D(x_{N+1}, \dots, x_{N+M}) \\ & = \sup_{[a,b) \subseteq [0,1)} \frac{1}{(2n+1)S_n} \left| \sum_{\substack{N+1 \le \nu \le N+M \\ \{2^\nu \theta\} \in [a,b)}} 1 - (b-a)\big((2n+1)S_n\big) \right| \\ & \ge \frac{\frac{1}{4}(2n+1)S_n}{(2n+1)S_n} = \frac{1}{4} \end{aligned}$$

and therefore does not tend to 0, thereby implying that θ is not strongly normal.

Remark 3. Observe that instead of choosing $M_N = \lfloor \delta_N \sqrt{N} \rfloor$ as we did in (1), we could have set $M_N = \lfloor \delta_N N^\gamma \rfloor$, where γ is fixed real number belonging to the interval $(0,1)$, and then introduce the corresponding concept of a γ-strongly uniformly distributed sequence mod 1, with corresponding γ-strong normal numbers. In this case, one could easily show that if $0 < \gamma_1 < \gamma_2 < 1$, then any γ_1-strong normal number is also be a γ_2-strong normal number.

Remark 4. A further discussion on appropriate choices of M_N in the definition of strong normality is exposed in Section 9.

Identifying which real numbers are normal is not an easy task. For instance, no one has been able to prove that any of the classical constants π, e, $\sqrt{2}$ and $\log 2$ is normal, even though numerical evidence indicates that all of them are. Even constructing normal numbers is not an easy task. Hence, one might believe that constructing strongly normal numbers will even be more difficult. So, here we first show how one can construct large families of strongly normal numbers. On the other hand, it has been shown by Borel [1] that almost all real numbers are normal. Although the set of strongly normal numbers is “much smaller” than the whole set of normal numbers, in this paper, we will prove that almost all numbers are strongly normal. After studying the multidimensional case, we examine the relation between arithmetic functions with local normal distribution and strong normality.

1. A simple criteria for strong normality

Our first two propositions provide a simple criteria for strong uniform distribution mod 1 and for strong normality. They are direct consequences of the definition of strong normality.

Proposition 1. *Let $\mathcal{D}$ be the set of all continuous functions $f : [0,1] \to [0,1)$ such that $\int_0^1 f(x)\,dx = 0$. Then, the sequence $(x_n)_{n\geq 1}$ is strongly uniformly distributed mod 1 if and only if, for all $f \in \mathcal{D}$, letting $M = M_N$ be as in* (1),

$$\frac{1}{M}\sum_{j=1}^{M} f(\{x_{N+j}\}) \to 0 \qquad \text{as } N \to \infty\,.$$

Given a positive real number $\alpha < 1$ whose q-ary expansion is written as $\alpha = 0.\epsilon_1\epsilon_2\ldots$, where each $\epsilon_j \in \mathcal{A}_q := \{0,1,\ldots,q-1\}$. For an arbitrary word $\beta = \delta_1\ldots\delta_k \in \mathcal{A}_q^k$, let $R_{N,M}(\beta)$ stand for the number of times that the word β appears as a subword of the word $\epsilon_{N+1}\ldots\epsilon_{N+M}$.

Proposition 2. *A positive real number $\alpha < 1$ is strongly q-normal if and only if, given an arbitrary word $\beta = \delta_1\ldots\delta_k \in \mathcal{A}_q^k$ and $M = M_N$ as in* (1),

$$\lim_{N\to\infty}\frac{R_{N,M}(\beta)}{M} = \frac{1}{q^k}\,.$$

2. The construction of strongly normal numbers

We first show how one can go about constructing strongly normal numbers. One way is as follows. First, we start with a normal number in base $q \geq 2$, say $\alpha = 0.\epsilon_1\epsilon_2\ldots$, and then for each positive integer T, we consider the corresponding word $\alpha_T = \epsilon_1\epsilon_2\ldots\epsilon_T$. One can show that, if the sequences of integers $T_1 < T_2 < \cdots$ and $m_1 < m_2 < \cdots$ are chosen appropriately, and if, for short, we write γ^m for the concatenation of m times the word γ, that is $\gamma^m = \underbrace{\gamma\ldots\gamma}_{m\ \text{times}}$, then the number

$$\beta = 0.\alpha_{T_1}^{m_1}\alpha_{T_2}^{m_2}\ldots$$

is a strongly normal number in base q.

We first show that the choice $T_\ell = \ell$ and $m_\ell = \ell$ is an appropriate one and in fact we state this as a proposition.

Proposition 3. *Let α be a q-normal number. Then, using the above notation, the number*

$$\beta = 0.\alpha_1^1\alpha_2^2\alpha_3^3\ldots$$

is a strongly normal number in base q.

Proof. Given a word $\gamma = c_1 \ldots c_r \in \mathcal{A}_q^r$ and an arbitrary word $\delta_1 \ldots \delta_h \in \mathcal{A}_q^h$, let $E_\gamma(\delta_1 \ldots \delta_h)$ be the number of occurrences of γ as a subword in $\delta_1 \ldots \delta_h$ and let

$$\Delta_\gamma(\delta_1 \ldots \delta_h) := \left| E_\gamma(\delta_1 \ldots \delta_h) - \frac{h}{q^r} \right|.$$

On the other hand, let $\kappa_1, \kappa_2, \ldots$ stand for the q-ary digits of β, so that $\beta = 0.\kappa_1\kappa_2\ldots$, and let $M = M_N$ be as in (1). Finally set

$$\mu := \kappa_{N+1}\kappa_{N+2}\ldots\kappa_{N+M}.$$

We will count how many times the word γ occurs as a subword of μ.

Denoting by $\lambda(\gamma)$ the length of the word γ, observe that

$$\lambda(\alpha_1^1\alpha_2^2\ldots\alpha_R^R) = \sum_{\ell=1}^{R} \ell^2 = \frac{R(R+1)(2R+1)}{6} \qquad (R \in \mathbb{N}).$$

Setting

$$K_R := \frac{R(R+1)(2R+1)}{6} \qquad (R \in \mathbb{N}),$$

it is easily seen that no more than one K_ν is located in the interval $[N+1, N+M]$. Indeed, let us show that

$$\text{if} \quad N < K_\nu \le N+M, \quad \text{then} \quad K_{\nu+1} > N+M. \tag{10}$$

Indeed, it is clear that

$$K_{\nu+1} = K_\nu + (\nu+1)^2 > N + (\nu+1)^2 \qquad (\nu \ge 1) \tag{11}$$

and that, since $\nu^3 > K_\nu > N$, it follows that $\nu > N^{1/3}$, so that $(\nu+1)^2 > N^{2/3}$, which combined with (11) implies that

$$K_{\nu+1} > N + N^{2/3} > N + \lfloor \delta_N \sqrt{N} \rfloor = N + M,$$

thus proving our claim (10).

Now, assume that N is large and let R be the largest integer such that $K_R \le N + M$. We then have two distinct possibilities:

Case I : $N \le K_R$;

Case II : $K_R < N$.

If Case II holds, then

$$E_\gamma(\mu) = \frac{M}{R+1} E_\gamma(\alpha_{R+1}) + O(R) + O\left(\frac{M}{R}\right),$$

from which it follows that

$$\Delta_\gamma(\mu) \le \frac{M}{R+1} \Delta_\gamma(\alpha_{R+1}) + O\left(R + \frac{M}{R}\right). \tag{12}$$

Because α is normal, we have that $\dfrac{\Delta_\gamma(\alpha_R)}{R} \to 0$ as $R \to \infty$, while on the other hand, $\left(R + \dfrac{M}{R}\right) \cdot \dfrac{1}{M} \to 0$ as $M \to \infty$. Using this in (12), it follows that

$$\frac{1}{M}\left| E_\gamma(\mu) - \frac{M}{q^r} \right| \to 0 \qquad \text{as } M \to \infty\,,$$

so that in light of Proposition 2, the number β is strongly normal in base q.

Since Case I can be handled in a similar way, the proposition is proved. □

Remark 5. Other choices of T_ℓ and m_ℓ can also lead to the construction of strongly normal numbers. For instance, let $R > 0$ be a fixed integer and, for each real number $x > 0$, define

$$x_1 := \log_+ x = \max(1, \log x), \qquad x_{\ell+1} = \log_+ x_\ell \qquad (\ell = 1, 2, \ldots)\,.$$

Given a real number

$$\alpha = 0.\epsilon_1\epsilon_2 \ldots \in \mathcal{A}_q^{\mathbb{N}}\,,$$

set

$$F(\alpha; \beta) = \#\{(\gamma_1, \gamma_2) : \alpha = \gamma_1\beta\gamma_2\},$$

that is the number of occurrences of the word β in the digits of the word α. One can construct a real number α such that, for every integer $k \ge 1$,

$$\max_{\beta \in \mathcal{A}_q^k} \left| \frac{1}{M_N} F(\epsilon_{N+1} \ldots \epsilon_{N+M_N}; \beta) - \frac{1}{q^k} \right| \to 0 \quad \text{as } N \to \infty\,. \tag{13}$$

Indeed, for each integer $\ell \ge 1$, let us choose $T_\ell = \ell$ and $m_\ell = 2^{2^{\cdot^{\cdot^{\cdot^{2^\ell}}}}}$, that is

$$\ell = \underbrace{\log_2 \log_2 \ldots \log_2}_{R+1 \text{ times}} m_\ell\,.$$

Now, starting with a q-ary normal number $\gamma = 0.\epsilon_1\epsilon_2 \ldots$, and, for each positive integer T, set $\gamma_T = 0.\epsilon_1\epsilon_2 \ldots \epsilon_T$. Then, one can show that the number

$$\beta = 0.\gamma_1^{m_1}\gamma_2^{m_2} \ldots$$

does indeed satisfy condition (13) and is therefore a strongly q-normal number.

3. Preliminary lemmas

The classical *Borel-Cantelli lemma* can be stated as follows.

Lemma 1. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $A_1, A_2, \dots$ be a list of the elements of $\mathcal{F}$. Let $E = \{x : x$ belongs to infinitely many A_j's$\}$. Assuming that*

$$\sum_{j=1}^{\infty} P(A_j) < \infty\,,$$

then $P(E) = 0$.

Given a probability space $(\Omega, \mathcal{F}, P)$, we say that $A_1, A_2, \dots$ is a list of *completely independent elements* of $\mathcal{F}$ if, given any finite increasing sequence of integers, say $i_1 < i_2 < \cdots < i_k$, we have

$$P(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_k})\,.$$

The second Borel-Cantelli lemma can be considered as the converse of the classical Borel-Cantelli lemma. It can be stated as follows.

Lemma 2. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $A_1, A_2, \dots$ be a list of completely independent elements of $\mathcal{F}$. Letting E be as in Lemma 1 and assuming that*

$$\sum_{j=1}^{\infty} P(A_j) = \infty\,,$$

then $P(E) = 1$.

A real number is *simply normal* in base q if in its base q expansion, every digit $0, 1, \dots, q-1$ occurs with the same frequency $1/q$. The following lemma offers a simple way of establishing if a given real number is a normal number.

Lemma 3. *Let $q \geq 2$ be an integer. If a real number α is simply normal in base q^r for each $r \in \mathbb{N}$, then α is normal in base q.*

Proof. A proof of this result can be found in the book of Kuipers and Niederreiter [15]. □

In the spirit of Proposition 2, we will say that a real number $\alpha < 1$ is a *simply strong normal number in base q* if for every digit $d \in \mathcal{A}_q$,

$$\lim_{N\to\infty} \frac{R_{N,M}(d)}{M} = \frac{1}{q}\,.$$

Lemma 4. *Let $q \geq 2$ be an integer. If a real number α is a simply strong normal in base q^r for each $r \in \mathbb{N}$, then α is strongly normal in base q.*

Proof. This result can be proved along the same lines as one would use to prove Lemma 3. □

Lemma 5. *For each integer $k \geq 1$, let*

$$\pi_k(x) := \#\{n \leq x : \omega(n) = k\}\,.$$

Then, the relation

$$\pi_k(x) = \big(1 + o(1)\big) \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!} \qquad (x \to \infty)$$

holds uniformly for

$$|k - \log\log x| \leq \frac{1}{\delta_x} \sqrt{\log\log x}\,, \tag{14}$$

where δ_x is some function of x chosen appropriately and which tends to 0 as $x \to \infty$.

Proof. This follows from Theorem 10.4 stated in the book of De Koninck and Luca [12]. □

Lemma 6. *Letting δ_x be as in the statement of Lemma 5,*

$$\max_{\substack{k \ \text{satisfying (14)} \\ \ell \in [0, \lceil \delta_x^{3/2} \sqrt{\log\log x} \rceil]}} \left| \frac{\pi_{k+\ell}(x)}{\pi_k(x)} - 1 \right| \to 0 \qquad \text{as } x \to \infty\,.$$

Proof. Given k satisfying (14), let θ_k be defined implicitly by $k = \log\log x + \theta_k$, and let $\ell \in [0, \lceil \delta_x^{3/2} \sqrt{\log\log x} \rceil]$. Then, in light of Lemma 5, we have, as $x \to \infty$,

$$\begin{aligned}
\frac{\pi_{k+\ell}(x)}{\pi_k(x)} &= \big(1 + o(1)\big) \frac{(\log\log x)^\ell}{k^\ell \prod_{\nu=0}^{\ell-1} \left(1 + \frac{\nu}{k}\right)} \\
&= \big(1 + o(1)\big) \left(\frac{\log\log x}{k}\right)^\ell \exp\left\{-\frac{\ell(\ell-1)}{2k} + O\left(\frac{\ell^3}{k^2}\right)\right\} \\
&= \big(1 + o(1)\big) \left(\frac{1}{1 + \theta_k/\log\log x}\right)^\ell \big(1 + o(1)\big) \\
&= \big(1 + o(1)\big) \exp\left\{-\frac{\ell\theta_k}{\log\log x} + O\left(\frac{\ell\theta_k^2}{(\log\log x)^2}\right)\right\} \\
&= 1 + o(1),
\end{aligned}$$

thereby completing the proof of Lemma 6. □

For any particular set of primes $\mathcal{P}$, we introduce the expressions

$$\Omega_{\mathcal{P}}(n) := \sum_{\substack{p^a \,\|\, n \\ p \in \mathcal{P}}} a \quad \text{and} \quad E(x) := \sum_{\substack{p \le x \\ p \in \mathcal{P}}} \frac{1}{p} \,. \tag{15}$$

The following two results, which we also state as lemmas, are due respectively to Halász [13] and Kátai [14].

Lemma 7. (Halász) *Let $0 < \delta \le 1$ and let $\mathcal{P}$ be a set of primes with corresponding functions $\Omega_{\mathcal{P}}(n)$ and $E(x)$ given in (15). Then, the estimate*

$$\sum_{\substack{n \le x \\ \Omega_{\mathcal{P}}(n) = k}} 1 = \frac{xE(x)^k}{k!} e^{-E(x)} \left\{ 1 + O\left(\frac{|k - E(x)|}{E(x)} \right) + O\left(\frac{1}{\sqrt{E(x)}} \right) \right\}$$

holds uniformly for all integers k and real numbers $x \ge 3$ satisfying

$$E(x) \ge \frac{8}{\delta^3} \quad \text{and} \quad \delta \le \frac{k}{E(x)} \le 2 - \delta \,.$$

Lemma 8. (Kátai) *For $1 \le h \le x$, let*

$$\begin{aligned} A_k(x,h) &:= \sum_{\substack{x \le n \le x+h \\ \omega(n) = k}} 1 \,, \\ \delta_k(x,h) &:= \frac{A_k(x,h)}{h} - \frac{\pi_k(x)}{x} \,, \\ E(x,h) &:= \sum_{k=1}^{\infty} \delta_k^2(x,h) \,. \end{aligned}$$

Letting $\varepsilon > 0$ be an arbitrarily small number and $x^{7/12+\varepsilon} \le h \le x$, then

$$E(x,h) \ll \frac{1}{\log^2 x \cdot \sqrt{\log\log x}} \,.$$

4. Main results

Theorem 1. *The Lebesgue measure of the set of all those real numbers $\alpha \in [0,1]$ which are not strongly q-normal is equal to 0.*

Let r be a fixed positive integer and set $E := [0,1)^r$. Consider an r dimensional sequence $(\underline{x}_n)_{n \in \mathbb{N}} := \big(x_1^{(n)}, \ldots, x_r^{(n)} \big)_{n \in \mathbb{N}}$ in $\mathbb{R}^r$. This sequence is said to be *uniformly distributed mod E* if, for all intervals $[a_j, b_j) \subseteq [0,1)$, $j = 1, \dots, r$, we have

$$\lim_{N\to\infty} \frac{1}{N} \# \left\{ n \le N : \left\{ x_j^{(n)} \right\} \in [a_j, b_j) \text{ for } j = 1, \dots, r \right\} = \prod_{j=1}^{r} (b_j - a_j) .$$

Accordingly, the *discrepancy* of the finite sequence $\underline{x}_1, \dots, \underline{x}_N$ in $\mathbb{R}^r$ is defined as

$$D(\underline{x}_1, \dots, \underline{x}_N) = \sup_{\substack{[a_j,b_j)\subseteq[0,1) \\ j=1,\dots,r}} \left| \frac{1}{N} \sum_{\substack{\{x_j^{(n)}\}\in[a_j,b_j) \\ j=1,\dots,r}} 1 - \prod_{j=1}^{r} (b_j - a_j) \right| .$$

Then, we shall say that an infinite sequence $(\underline{x}_n)_{n\in\mathbb{N}}$ is *strongly uniformly distributed mod E* if

$$D(\underline{x}_N, \dots, \underline{x}_{N+M}) \to 0 \qquad \text{as } N \to \infty$$

for every choice of δ_N satisfying (1).

In what follows, we let $q_1, \dots, q_r$ be fixed integers ≥ 2.

Theorem 2. *The Lebesgue measure of the set of all those r-tuples $(\alpha_1, \dots, \alpha_r) \in [0,1)^r$ for which the sequence $(\underline{x}_n)_{n\in\mathbb{N}}$, where $\underline{x}_n := (\{\alpha_1 q_1^n\}, \dots, \{\alpha_r q_r^n\})$, is not strongly uniformly distributed in $[0,1)^r$ is equal to 0.*

Theorem 3. *Assume that for each $i = 1, 2, \dots, r$, the number α_i is strongly q_i-normal. Let $E = [0,1)^r$ and assume that f is a continuous periodic function mod E and that it satisfies $\int_0^1 \cdots \int_0^1 f(x_1, \dots, x_r)\, dx_1 \cdots dx_r = 0$. Further set*

$$y_n = f\left(\alpha_1 q_1^{\omega(n)}, \dots, \alpha_r q_r^{\omega(n)} \right) \qquad (n = 1, 2, \dots) .$$

Then,

$$\frac{1}{x} \sum_{n\le x} y_n \to 0 \qquad \text{as } x \to \infty . \tag{16}$$

Moreover, further defining $\underline{z}_n := \left(\{\alpha_1 q_1^{\omega(n)}\}, \dots, \{\alpha_r q_r^{\omega(n)}\} \right)$ for $n = 1, 2, \dots$, we have that $(\underline{z}_n)_{n\in\mathbb{N}}$ is uniformly distributed in E.

The following result is a direct consequence of Theorem 3 and is related to the result stated in Lemma 7.

Theorem 4. *Let g be any one of the arithmetic functions*

$$\omega(n) := \sum_{p|n} 1, \qquad \Omega(n) := \sum_{p^a \| n} a, \qquad \Omega_{\mathcal{P}}(n) := \sum_{\substack{p^a \| n \\ p \in \mathcal{P}}} a$$

and let

$$\underline{x}_n := \left(\{\alpha_1 q_1^{g(n)}\}, \ldots, \{\alpha_r q_r^{g(n)}\} \right).$$

Then, for almost all $(\alpha_1, \ldots, \alpha_r) \in [0,1)^r$, the sequence $(\underline{x}_n)_{n\geq 1}$ is uniformly distributed in $[0,1)^r$.

The following result is a consequence of Lemma 8 and we shall omit its proof since it is essentially along the same lines as that of Theorem 3.

Theorem 5. *For each integer $i = 1, \ldots, r$, assume that α_i is strongly q_i-normal and set*

$$\underline{x}_n := \left(\{\alpha_1 q_1^{\omega(n)}\}, \ldots, \{\alpha_r q_r^{\omega(n)}\} \right).$$

Then, with $M = \lfloor N^{7/12+\varepsilon} \rfloor$,

$$D(\underline{x}_{N+1}, \ldots, \underline{x}_{N+M}) \to 0 \qquad \text{as } N \to \infty\,.$$

5. Proof of Theorem 1

Theorem 1 will follow immediately from the following lemma.

Lemma 9. *Let $(\Omega, \mathcal{A}, P)$ be a probability space, where $\Omega = [0,1)$, let $\mathcal{A}$ be the ring of Borel sets and let P be the Lebesgue measure. Let $q \geq 2$ be a fixed integer and set $\mathcal{A}_q := \{0, 1, \ldots, q-1\}$. Let $\epsilon_n \in \mathcal{A}_q$, $n = 1, 2, \ldots$, be independent random variables such that $P(\epsilon_n = a) = 1/q$ for each $a \in \mathcal{A}_q$. For each $\omega \in \Omega$, let*

$$\alpha(\omega) := 0.\epsilon_1(\omega)\epsilon_2(\omega)\ldots$$

For an arbitrary $\delta > 0$, let

$$E_\delta := \left\{ \omega \in \Omega : \limsup_{N\to\infty} \max_{d \in \mathcal{A}_q} \left| \frac{1}{M} \sum_{\substack{n=N+1 \\ \epsilon_n = d}}^{N+M} 1 - \frac{1}{q} \right| > \delta \right\},$$

where M satisfies (1). *Then,*

$$P(E_\delta) = 0 \quad \text{for every } \delta > 0\,. \tag{17}$$

Moreover, setting

$$E^* := \left\{ \omega \in \Omega : \limsup_{N\to\infty} \max_{d\in\mathcal{A}_q} \left| \frac{1}{M} \sum_{\substack{n=N+1 \\ \epsilon_n = d}}^{N+M} 1 - \frac{1}{q} \right| \neq 0 \right\} ,$$

we have $P(E^*) = 0$.

P r o o f o f L e m m a 9. Let $U \in \mathbb{N}$ and given any $d \in \mathcal{A}_q$, let

$$\alpha_d(\epsilon_1, \dots, \epsilon_U) = \sum_{\substack{i\in\{1,\dots,U\} \\ \epsilon_i = d}} 1 .$$

leftline It is clear that

$$P\big(\alpha_d(\epsilon_1, \dots, \epsilon_U) = j\big) = \frac{1}{q^U} \binom{U}{j} (q-1)^{U-j}.$$

For each $0 < \delta < 1/q$, set

$$S = S(\delta) := \left\{ \omega \in \Omega : \max_{d\in\mathcal{A}_q} \left| \alpha_d(\epsilon_1, \dots, \epsilon_U) - \frac{U}{q} \right| > \delta U \right\} . \tag{18}$$

If $\omega \in S$, then clearly the inequality

$$\alpha_d(\epsilon_1, \dots, \epsilon_U) < \frac{U}{q} - \delta \frac{U}{q}$$

holds for at least one $d \in \mathcal{A}_q$, in which case we have

$$\begin{aligned} P(S) &\leq \frac{q}{q^U} \sum_{0\leq j\leq (1-\delta)U/q} \binom{U}{j} \cdot (q-1)^{U-j} \\ &= q \left(1 - \frac{1}{q}\right)^U \sum_{0\leq j\leq V} \binom{U}{j} \frac{1}{(q-1)^j} , \quad \text{where } V = \lfloor (1-\delta)U/q \rfloor . \end{aligned} \tag{19}$$

Now let

$$t_j = \binom{U}{j} \frac{1}{(q-1)^j} \qquad (j = 0, 1, \dots, V) .$$

Then, for each integer $j \geq 1$, we have

$$\frac{t_{j-1}}{t_j} = (q-1) \frac{j}{U-j+1} < \frac{(q-1)(1-\delta)U/q}{U+1-(1-\delta)U/q} < \frac{(q-1)(1-\delta)}{q-(1-\delta)} < 1 - \delta ,$$

so that $t_{j-1} < (1-\delta) t_j$, thus implying that

$$\sum_{0\leq j\leq V} t_j \leq t_V \left(1 + (1-\delta) + (1-\delta)^2 + \cdots\right) = \frac{t_V}{\delta} .$$

Using the Stirling formula in the form

$$\log n! = n\log(n/e) + \frac{1}{2}\log(2\pi n) + \theta_n \qquad \text{with } \theta_n \to 0$$

and setting $V = \kappa U$, where $\kappa = \dfrac{\lfloor \frac{1-\delta}{q} U \rfloor}{U} = \dfrac{1-\delta}{q} + O\left(\dfrac{1}{U}\right)$, we then have

$$\begin{aligned}\log t_V &= U\log U - \kappa U\log(\kappa U) - (1-\kappa)U\log\big((1-\kappa)U\big) - \kappa U\log(q-1)\\ &\quad + \frac{1}{2}\log\frac{1}{\kappa(1-\kappa)} - \frac{1}{2}\log(2\pi) + O(\theta_V)\\ &= \big(-\kappa\log\kappa - (1-\kappa)\log(1-\kappa) - \kappa\log(q-1)\big)U\\ &\quad + \frac{1}{2}\log\frac{1}{\kappa(1-\kappa)} - \frac{1}{2}\log(2\pi) - \frac{1}{2}\log U + O(\theta_V)\,.\end{aligned}$$

Letting $h(\kappa) = \kappa\log\dfrac{1}{(q-1)\kappa} + (1-\kappa)\log\dfrac{1}{1-\kappa}$, it follows that

$$\log t_V = Uh(\kappa) + \frac{1}{2}\log\frac{1}{\kappa(1-\kappa)} - \frac{1}{2}\log(2\pi) - \frac{1}{2}\log U + O(\theta_V)\,.$$

Observe that

$$h(1/q) = \log\frac{q}{q-1} \quad \text{and} \quad h(\kappa) < \big(1-c(\delta)\big)\log\frac{q}{q-1}\,,$$

where $c(\delta) > 0$ provided $\delta > 0$.

Using this in (19), we obtain that

$$\begin{aligned}P(S) &\le q\exp\left\{U\log(1-1/q) + \log V + U\big(1-c(\delta)\big)\log\frac{q}{q-1}\right\}\\ &< \exp\{-c_1(\delta)U\}\,,\end{aligned} \tag{20}$$

where $c_1(\delta) > 0$ is some constant depending only on δ and q.

For each integer $r \ge 1$, let $N_r = q^r$ and consider the interval $\mathcal{L}_r = [N_r, N_{r+1}-1]$. Let us cover a given interval $\mathcal{L}_r$ by the union of $K_r := 1 + \left\lfloor \frac{(q-1)q^r}{r^2} \right\rfloor$ consecutive intervals $\mathcal{T}_1^{(r)}, \mathcal{T}_2^{(r)}, \ldots, \mathcal{T}_{K_r+1}^{(r)}$, each of length $U_r := r^2$. Now, we define the sets $S_i^{(r)}$, for $i = 1, \ldots, K_r+1$, as we did for the set S in (18), but this time with the independent variables

$$\epsilon_{N_r+(i-1)U_r+\ell} \qquad (\ell = 1, 2, \ldots, U_r)\,.$$

For these new independent variables, if we proceed as we did to obtain (20), then we have

$$P\big(S_i^{(r)}\big) \le q^r \exp\left\{-c_1(\delta)r^2\right\} \quad (i = 1, \ldots, K_r+1)\,,$$

so that

$$P\left(\bigcup_{i=1}^{K_r+1} S_i^{(r)}\right) \ll K_r q^r \exp\left\{-c_1(\delta)r^2\right\} \leq \frac{q^{2r+1}}{r^2}\exp\left\{-c_1(\delta)r^2\right\}$$
$$= \exp\left\{-c_1(\delta)r^2 + (2r+1)\log q - 2\log r\right\} < \frac{1}{r^3},$$

provided r is sufficiently large.

Since the series $\sum 1/r^3$ converges, we may apply Lemma 1 and conclude that the set

$$E_\delta := \#\left\{\omega : \omega \in \bigcup_{i=1}^{K_r+1} S_i^{(r)} \text{ for infinitely many } r\right\}$$

is such that $P(E_\delta) = 0$, thus establishing (17). From this result, then it follows also that $P(E^*) = 0$. $\square$

6. Proof of Theorem 2

The proof is quite straightforward. Indeed, first set $Q := q_1 q_2 \cdots q_r$ and let $\alpha = 0.a_1 a_2 \ldots$ be a strongly Q-normal number, where each a_j satisfies

$$a_j \in \mathcal{A}_Q, \qquad a_j \equiv b_j^{(\ell)} \pmod{q_\ell} \quad \text{with } b_j^{(\ell)} \in \mathcal{A}_{q_\ell} \qquad (\ell = 1, \ldots, r).$$

Writing

$$\alpha_\ell = 0.b_1^{(\ell)} b_2^{(\ell)} \ldots \qquad (\ell = 1, \ldots, r),$$

then each α_ℓ is a strongly q_ℓ-normal number. This means that the sequence $(x_n)_{n\in\mathbb{N}}$ defined by

$$x_n := \left(\{\alpha_1 q_1^n\}, \ldots, \{\alpha_r q_r^n\}\right) \qquad (n = 1, 2\ldots)$$

is strongly uniformly distributed mod $[0,1)^r$, thus completing the proof of Theorem 2.

7. Proof of Theorem 3

Let x be a large number and let us set $S := \lfloor \delta_x^{3/2}\sqrt{\log\log x}\rfloor$, where δ_x is as in Lemma 5. Moreover, for each positive integer $m \leq 1/\delta_x^{5/2}$, let us consider the interval

$$U_m := \left[\lfloor \log\log x\rfloor + mS, \lfloor \log\log x\rfloor + (m+1)S - 1\right] = [t_m, t_{m+1} - 1],$$

say.

For each integer $k \geq 1$, let $u_k := f(\alpha_1 q_1^k, \ldots, \alpha_r q_r^k)$. Observe that

$$\begin{aligned}
\sum_{n\leq x} y_n &= \sum_{k\geq 1} u_k \pi_k(x) \\
&= \sum_{|k-\log\log x|\leq\sqrt{\log\log x}/\delta_x} u_k \pi_k(x) + \sum_{|k-\log\log x|>\sqrt{\log\log x}/\delta_x} u_k \pi_k(x) \\
&= S_1(x) + S_2(x)\,, \qquad (21)
\end{aligned}$$

say.

It follows from the Turán-Kubilius inequality that

$$\frac{1}{x} S_2(x) \to 0 \qquad \text{as } x \to \infty\,. \qquad (22)$$

For the evaluation of $S_1(x)$, we proceed as follows. Let x be a large number. Then, for each positive integer $m \leq 1/\delta_x^{5/2}$, let us consider the interval

$$U_m := \big[\lfloor \log\log x\rfloor + mS, \lfloor \log\log x\rfloor + (m+1)S - 1\big] = [t_m, t_{m+1} - 1]\,,$$

say. We then have

$$S_1(x) = \sum_{|m|\leq 1/\delta_x^{5/2}} \sum_{k\in U_m} u_k \pi_k(x) = \sum_{|m|\leq 1/\delta_x^{5/2}} S^{(m)}(x)\,, \qquad (23)$$

say.

Using Lemma 6, it follows that, as x becomes large,

$$\left| S^{(m)}(x) - \pi_{t_m}(x) \sum_{k\in U_m} u_k \right| \leq o(1)\pi_{t_m}(x) \sum_{k\in U_m} 1\,. \qquad (24)$$

Since

$$\pi_{t_m}(x) \sum_{k\in U_m} 1 = \big(1 + o(1)\big) \sum_{k\in U_m} \pi_k(x)\,,$$

it follows that

$$\pi_{t_m}(x) \sum_{k\in U_m} u_k = \big(1 + o(1)\big) \left\{ \sum_{k\in U_m} \pi_k(x) \right\} \frac{1}{\sum_{k\in U_m} 1} \sum_{k\in U_m} u_k\,. \qquad (25)$$

Now the fact that each α_i is strongly q_i-normal for $i = 1, \ldots, r$ implies that

$$\frac{1}{\sum_{k\in U_m} 1} \sum_{k\in U_m} u_k = o(1) \qquad \text{for each } m \leq 1/\delta_x^{5/2}.$$

Combining this with (24) and (25), it follows that

$$\left| S^{(m)}(x) \right| \leq o(1) \sum_{k\in U_m} \pi_k(x)\,,$$

which substituted in (23) yields

$$S_1(x) \le o(1) \sum_{|m| \le 1/\delta_x^{5/2}} \sum_{k \in U_m} \pi_k(x) = o(x)\,. \tag{26}$$

Using (22) and (26) in (21) completes the proof of (16). The second part of Theorem 3 then immediately follows from (16).

8. Final remarks

When we introduced the notion of *strongly normal number in base* q, we chose for simplicity to consider intervals $[N+1, N+M]$ with $M = \lfloor \delta_N \sqrt{N} \rfloor$. However, it is interesting to observe that we could have chosen much smaller intervals, namely with $M = \lfloor \log^2 N \rfloor$, and nevertheless still preserve the property that almost all real numbers are strongly normal. Indeed, following the proof used in Lemma 9, as we consider an arbitrary sequence of digits $\epsilon_{N+1}\epsilon_{N+2}\dots\epsilon_{N+M}$, with $M = \lfloor \log^2 N \rfloor$, and examine the occurrence of an arbitrary digit $d \in \mathcal{A}_q$ in this sequence, we could define r as the unique integer such that $q^r \le n < q^{r+1}$, in which case we would have

$$r^2 \le \left(\frac{\log n}{\log q} \right)^2 < (r+1)^2.$$

In the end, we would see that

$$\left| \frac{1}{\log^2 n} \sum_{\substack{\nu = n+1 \\ \epsilon_\nu = d}}^{n + \lfloor \log^2 n \rfloor} 1 - \frac{1}{q} \right| > \delta$$

holds only for finitely many n's and that this is true for each $\delta > 0$. We can conclude from this that, for almost all α,

$$\lim_{n \to \infty} \max_{d \in \mathcal{A}_q} \left| \frac{1}{\log^2 n} \sum_{\substack{\nu = n+1 \\ \epsilon_\nu = d}}^{n + \lfloor \log^2 n \rfloor} 1 - \frac{1}{q} \right| = 0\,,$$

thus also establishing that we could have defined the notion of *strongly normal numbers* with $M = \lfloor \log^2 N \rfloor$ instead of with $M = \lfloor \delta_N \sqrt{N} \rfloor$.

Now, could we have chosen M even smaller, say $M = \lfloor \log N \rfloor$? Not really! Indeed, assume that $(\epsilon_n)_{n \ge 1}$ are independent random variables such that

$P(\epsilon_n = a) = 1/q$ for each $a \in \mathcal{A}_q$. For $N \in \mathbb{N}$, let $H = H_N = \left\lfloor \dfrac{q^{N+1} - q^N}{N} \right\rfloor$ and the set

$$B_\ell^{(N)} := \{\omega : \epsilon_{q^N + \ell N + \nu} = 0,\ \nu = 0, 1, \ldots, N-1\} \qquad (\ell = 0, 1, \ldots, H-1)\,.$$

The events $B_\ell^{(N)}$ $(\ell = 0, 1, \ldots, H-1)$ are independent and $P\left(B_\ell^{(N)}\right) = 1/q^N$. Hence, with $D_N = \bigcup_{\ell=0}^{H-1} B_\ell^{(N)}$, we have

$$P(D_N) = \frac{H}{q^N} \geq \frac{1}{2^N}\,.$$

On the other hand, $D_1, D_2, \ldots$ are independent and $\sum_{N=1}^{\infty} P(D_N) = \infty$. Hence, by the second Borel-Cantelli lemma (see Lemma 2), we may conclude that for almost all events ω, there exists an infinite sequence of N's, say $n_1, n_2, \ldots$ such that

$$\epsilon_{n_\nu+1} = 0, \quad \epsilon_{n_\nu+2} = 0, \ldots, \epsilon_{n_\nu+m_\nu} = 0, \quad \text{where} \quad m_\nu \geq c\frac{\log n_\nu}{\log q}\,.$$

We have thus shown that one could encounter a normal number α with sequences of digits covering intervals of the form $[N+1, N+M]$, with $M \approx \log N$, made up only of zeros.

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ON THE CONJECTURE OF LEHMER, LIMIT MAHLER MEASURE OF TRINOMIALS AND ASYMPTOTIC EXPANSIONS

Jean-Louis Verger-Gaugry

Dedicated to the memory of Professor Pierre Liardet.

ABSTRACT. Let $n \geq 2$ be an integer and denote by θ_n the real root in $(0,1)$ of the trinomial $G_n(X) = -1 + X + X^n$. The sequence of Perron numbers $(\theta_n^{-1})_{n\geq 2}$ tends to 1. We prove that the Conjecture of Lehmer is true for $\{\theta_n^{-1} \mid n \geq 2\}$ by the direct method of Poincaré asymptotic expansions (divergent formal series of functions) of the roots $\theta_n, z_{j,n}$, of $G_n(X)$ lying in $|z| < 1$, as a function of n, j only. This method, not yet applied to Lehmer's problem up to the knowledge of the author, is successfully introduced here. It first gives the asymptotic expansion of the Mahler measures $\mathrm{M}(G_n) = \mathrm{M}(\theta_n) = \mathrm{M}(\theta_n^{-1})$ of the trinomials G_n as a function of n only, without invoking Smyth's Theorem, and their unique limit point above the smallest Pisot number. Comparison is made with Smyth's, Boyd's and Flammang's previous results. By this method we obtain a direct proof that the conjecture of Schinzel-Zassenhaus is true for $\{\theta_n^{-1} \mid n \geq 2\}$, with a minoration of the house $\boxed{\theta_n^{-1}}$, and a minoration of the Mahler measure $\mathrm{M}(G_n)$ better than Dobrowolski's one. The angular regularity of the roots of G_n, near the unit circle, and limit equidistribution of the conjugates, for n tending to infinity (in the sense of Bilu, Petsche, Pritsker), towards the Haar measure on the unit circle, are described in the context of the Erdős-Turán-Amoroso-Mignotte theory, with uniformly bounded discrepancy functions.

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1. Introduction

We consider the family $\big(G_n(X)\big)_{n\geq 2}$ of trinomials $G_n(X) = X^n + X - 1$ and their reciprocal polynomials $G_n^*(X) = X^n G_n(X^{-1}) = -(X^n - X^{n-1} - 1)$. Let $\theta_n, n \geq 2$, be the unique real root in $(0, 1)$ of the trinomial $G_n(X) = -1+X+X^n$. By Smyth's Theorem [Sy1] (Dubickas [Ds1]), since the trinomials $G_n(X)$ are not reciprocal, the Mahler measure of G_n satisfies

$$\mathrm{M}(\theta_n) = \mathrm{M}(G_n) \geq \Theta = 1.3247\ldots, \qquad n \geq 2, \tag{1.1}$$

where $\Theta = \theta_5^{-1}$ is the smallest Pisot number, dominant root of the Pisot polynomial $X^3 - X - 1 = -G_5^*(X)/(X^2 - X + 1)$. Lehmer's Conjecture [Le] (Smyth [Sy3]) asserts that there exists a constant $c > 0$ such that $\mathrm{M}(\alpha) \geq 1 + c$ for every nonzero algebraic number α which is not a root of unity (Blansky and Montgomery [ByM], Dobrowolsky [Do2], Stewart [St]). Since the sequence of Perron numbers $\theta_n^{-1}, n \geq 2$, tends to 1 when n tends to infinity (Lemma 2.2), the inequalities (1.1) imply that Lehmer's Conjecture is true for the family $\{\theta_n^{-1} \mid n \geq 2\}$ with $c = \Theta - 1$.

Lehmer's problem has a long history (Smyth [Sy3]). The search of a proof of Lehmer's Conjecture has led many authors to develop various technics for obtaining a minoration of the Mahler measure, or equivalently the Weil height in the classical context, or in a generalized context: e.g., polynomials (Amoroso [A1], Borwein, Dobrowolski and Mossinghoff [BDM], Flammang [F], Rhin and Smyth [RS], Rhin and Wu [RW], Silverman [Sn]), multivariate polynomials (Boyd [Bo2], Schinzel [Sc3], Smyth [Sy2]), algebraic numbers (Louboutin [Lt]), in abelian extensions (Amoroso and Dvornicich [AD], Amoroso and Zannier [AZ2], Baker [Bk]), limit points (Bertin et al [B-S], Boyd and Mossinghoff [BM], Flammand, Grandcolas and Rhin [FGR]), using auxiliary functions and interpolation determinants (Cantor and Strauss [CS], Laurent [La2], Stewart [St], Waldschmidt [W2]), elliptic curves and elliptic Mahler measures (Galateau and Mahé [GM], Hindry and Silverman [HS], Laurent [La], Masser [Ma], Ratazzi [Rz]), abelian varieties (David and Hindry [DH], Rémond [Rd]), p-adic Mahler measures (Besser and Deninger [BD]).

The objective of this note is to give a direct proof of Lehmer's Conjecture for the family $\{\theta_n^{-1} \mid n \geq 2\}$ by using the (Poincaré) asymptotic expansions of the roots $\theta_n, z_{j,n}$, of $G_n, n \geq 2$ (Borel [Bl], Copson [C], Dingle [Di], Erdélyi [E]), as a function of n, j only (cf. Section 3). Once these asymptotic expansions obtained, other conjectures can readily be investigated (Section 5): (i) we will consider Smyth conjecture for the trinomials G_n (of height 1) of small Mahler measure,

(ii) we will obtain an effective minoration of the Mahler measures $\mathrm{M}(G_n)$, and (iii) we will give a proof of Schinzel-Zassenhaus conjecture for $\{\theta_n^{-1} \mid n \geq 2\}$ with an explicit value of the constant in the minoration of the house $\overline{|\theta_n^{-1}|} = \theta_n^{-1}$.

Namely, we write

$$\theta_n = \mathrm{D}(\theta_n) + \mathrm{tl}(\theta_n),$$
$$\mathrm{Re}(z_{j,n}) = \mathrm{D}\big(\mathrm{Re}(z_{j,n})\big) + \mathrm{tl}\big(\mathrm{Re}(z_{j,n})\big),$$
$$\mathrm{Im}(z_{j,n}) = \mathrm{D}\big(\mathrm{Im}(z_{j,n})\big) + \mathrm{tl}\big(\mathrm{Im}(z_{j,n})\big),$$

where "D" stands for *"development"* (or *"limited expansion"*, or *"lowest order terms"*) and "tl" for *"tail"* (or "remainder", or *"terminant"* in [Di]), and consider the products

$$\Pi_{G_n} := \mathrm{D}\big(\mathrm{M}(G_n)\big) = \mathrm{D}(\theta_n)^{-1} \times \prod_{z_{j,n} \text{ in } |z|<1} \mathrm{D}(|z_{j,n}|)^{-2}$$

instead of $\mathrm{M}(G_n)$, as approximant value of $\mathrm{M}(G_n)$. Ideally each quantity should be represented by "D" as an infinite sum, of infinitely many terms, for an exact matching, and then with a tail "tl" equal to zero; but, in the way they are obtained here (cf. in the proofs of Proposition 3.1 and Proposition 3.4), these terms "D", if they were infinite, as asymptotic expansions of the variables n and j, were divergent expansions. Divergence would be a problem. But the terminants are readily obtained by the method; to overcome the difficulty of divergence, the objective is to show that the control of the *terminants* (Dingle [Di] Chap. XXI and XXII, "Theory of Terminants") at a sufficiently high order, but not too high, makes sense.

Proving Lehmer's Conjecture for $\{\theta_n^{-1} \mid n \geq 2\}$ amounts to (i) show that the limited expansions $\mathrm{D}(\theta_n)$, $\mathrm{D}\big(\mathrm{Re}(z_{j,n})\big)$ and $\mathrm{D}\big(\mathrm{Im}(z_{j,n})\big)$ can be "given a push" up till an order (i.e., can be represented as sums of a sufficiently high number of lower order terms) for which the tails

$$\mathrm{tl}_{G_n} := \mathrm{M}(G_n) - \Pi_{G_n}$$

satisfy $\lim_{n\to+\infty} \mathrm{tl}_{G_n} = 0$, (ii) give an uniform lower bound > 1 to the approximant values Π_{G_n}. We will show more, namely that the two limits

$$\lim_{n\to+\infty} \Pi_{G_n} \quad \text{and} \quad \lim_{n\to+\infty} \mathrm{M}(G_n)$$

exist, are equal and greater than Θ.

Roughly speaking the n roots of G_n are considered as n "bodies" as in celestial mechanics (in the so-called "*n-body problem*"), not linked by differential equations of movement but only by the polynomial equation $G_n(z) = 0$; and we mimic here the strategy of H. Poincaré [P] to describe them in an "n-body problem" using a sort or "theory of perturbations", by divergent asymptotic expansions. It seems that this strategy has never been introduced as such, though divergent formal series were already shown to play a major role into many problems [Bl]: cf. the remarkable book "Divergent Series" by G. H. Hardy for instance.

This method of Poincaré asymptotic expansions is a direct attack of the conjectures (Lehmer, Schinzel-Zassenhaus), which does not make use of Smyth's Theorem (Amoroso [A2]). It amounts to study the geometry of the set of values of Mahler measures $\mathrm{M}(G_n)$ and its limit points (Amoroso [A1], Boyd and Mossinghoff [BM], Dixon and Dubickas [DDs], Langevin [Lg], Smyth [Sy3]) by "controlled" approximants. Theorem 1.1 is obtained by this method of asymptotic expansions (in Section 4.2) and shows that the *first derived set* of $\{\mathrm{M}(G_n) \mid n \geq 2\}$ is reduced to one element, $1.38135\ldots$

To confirm the relevance of the present method of asymptotic expansions, with a control of the tails of the expansions, we prove Theorem 1.1 in two ways: by this method (Section 4.2) and by using a method initiated by Boyd with bivariate Mahler measures (Section 4.1). In both cases, we obtain the same value of the limit, called the *limit Mahler measure* of the trinomials G_n.

Theorem 1.1. *Let χ_3 be the uniquely specified odd character of conductor 3 ($\chi_3(m) = 0, 1$ or -1 according to whether $m \equiv 0, 1$ or $2 \pmod 3$, equivalently $\chi_3(m) = \left(\frac{m}{3}\right)$ the Jacobi symbol), and denote $L(s, \chi_3) = \sum_{m\geq 1} \frac{\chi_3(m)}{m^s}$ the Dirichlet L-series for the character χ_3. Then*

$$\lim_{n\to+\infty} \mathrm{M}(G_n) = \exp\left(\frac{3\sqrt{3}}{4\pi}\mathrm{L}(2,\chi_3)\right)$$
$$= \exp\left(\frac{-1}{\pi}\int_0^{\pi/3}\mathrm{Log}\left(2\sin\left(\frac{x}{2}\right)\right)\mathrm{d}x\right) = 1.38135\ldots =: \Lambda. \quad (1.2)$$

Whether Λ is algebraic or transcendental is unkown (Boyd [Bo1]). Another question of Boyd [Bo0] is whether Λ belongs to the second derived set of the set $\{\mathrm{M}(\beta)\}$ of Mahler measures of algebraic numbers. The value $\mathrm{Log}\,\Lambda$, as Clausen's integral, is obtained by the Bloch-Wigner dilogarithm [BM] [Ln]. In Section 5.1 we obtain the following asymptotic expansions of $\mathrm{M}(G_n)$ in a neighbourhood of the limit Mahler measure $\Lambda = 1.38135\ldots$ of Theorem 1.1.

Theorem 1.2. *Let n_0 be an integer such that $\frac{\pi}{3} > 2\pi\frac{\mathrm{Log}\,n_0}{n_0}$, and let $n \geq n_0$. Then*

$$\mathrm{M}(G_n) = \left(\lim_{m\to+\infty}\mathrm{M}(G_m)\right)\left(1 + r(n)\,\frac{1}{\mathrm{Log}\,n} + O\left(\frac{\mathrm{Log}\,\mathrm{Log}\,n}{\mathrm{Log}\,n}\right)^2\right) \quad (1.3)$$

with the constant $1/6$ involved in the Big O, and with $r(n)$ real, $|r(n)| \leq 1/6$.

In the sequel, we take $n_0 = 18$. Whether the coefficients $r(n), n \geq n_0$ are negative or positive they would allow to prove Smyth's Conjecture for the trinomials G_n; we recall Smyth's Conjecture below, for general trinomials of (naïve) height 1 (Flammang [F]). However the coefficients $r(n)$ do not seem to be easily computed by this method. We report in Theorem 1.3 the distribution of the values $\mathrm{M}(G_n)$ in a neighbourhood of Λ provided by the original method introduced by Smyth and Boyd in [Bo3], in which the signs of the coefficients $s(n)$ of the second-order terms are accessible and computed.

Conjecture (Smyth). *For all integers $n \geq 4,\ k \geq 1$ such that*

$$\gcd(n,k) = 1, \qquad k < n/2,$$

- $\mathrm{M}\big(z^n + z^k + 1\big) < \Lambda$ *if and only if* 3 *divides* $n + k$,
- $\mathrm{M}\big(z^n - z^k + 1\big) < \Lambda$ *with* n *odd if and only if* 3 *does not divide* $n + k$,
- $\mathrm{M}\big(z^n - z^k - 1\big) < \Lambda$ *with* n *even if and only if* 3 *does not divide* $n + k$.

Smyth's conjecture was recently proved by Flammang [F] for large n, using similar techniques; see also Borwein and Straub [BS]. Even though the Lerch transcendent function (or other transcendental functions) is used in the proof of Theorem 1.3, the values $\mathrm{M}(G_n)$ are all Perron numbers (Adler and Marcus [AM], Dubickas [Ds5]).

Theorem 1.3 is obtained by a method dedicated to the trinomials G_n while Theorem 1.2, obtained by a totally different type of proof, independently, can be generalized to other families of polynomials than $\{G_n\}$.

Theorem 1.3. *Let $n \geq 2$ be an integer. Then*

$$\mathrm{M}(-1 + X + X^n) \;=\; \Big(\lim_{m \to +\infty} \mathrm{M}(G_m)\Big)\left(1 + \frac{s(n)}{n^2} + O\big(n^{-3}\big)\right) \tag{1.4}$$

with, for n odd:

$$s(n) = \begin{cases} \sqrt{3}\pi/18 = +0.3023\ldots & \textit{if } n \equiv 1 \textit{ or } 3 \pmod 6, \\ -\sqrt{3}\pi/6 = -0.9069\ldots & \textit{if } n \equiv 5 \pmod 6, \end{cases}$$

for n even:

$$s(n) = \begin{cases} -\sqrt{3}\pi/36 = -0.1511\ldots & \textit{if } n \equiv 0 \textit{ or } 4 \pmod 6, \\ +\sqrt{3}\pi/12 = +0.4534\ldots & \textit{if } n \equiv 2 \pmod 6. \end{cases}$$

In the search of smallest limit Mahler measures, Smyth [Sy4] obtained that the set of Mahler measures $\{\mathrm{M}(\alpha) \mid \alpha \neq 0 \text{ nonreciprocal algebraic number}\}$ admits Θ as isolated infimum. Comparatively, the distribution of values of Mahler's measure for polynomials having coefficients in $\mathbb{Z}$ of bounded Mahler measure is in general spread over larger intervals than a neighbourhood of an unique limit point (Chern and Vaaler [CV], Dixon and Dubickas [DDs], Sinclair [Si]). Corollary 1.4, obtained by Smyth and Boyd's method i.e., as deduced from Theorem 1.3, or by the method of asymptotic expansions, shows that the constant of Lehmer $c = \Theta - 1$ is reached only once.

Corollary 1.4. $\mathrm{M}(G_2) = \theta_2^{-1} = \frac{1+\sqrt{5}}{2} = 1.618\ldots$, $\mathrm{M}(G_n) \geq \mathrm{M}(G_5) = \theta_5^{-1} = \Theta$ *for all $n \geq 3$, with equality if and only if $n = 5$.*

Corollary 1.5. *Smyth's conjecture is true for $\{G_n \mid n \geq 4\}$.*

The method of asymptotic expansions does not seem to be powerful enough for solving Smyth's conjecture (i.e., Flammang's Theorem) for $k = 1$, neither probably for the generic three cases "n, k" in its statement (covering all cases of trinomials of height 1). Comparing (1.3) and (1.4), these two methods surprisingly give rise to very different speeds of convergence towards the limit, simply meaning that (1.3) underestimates it.

However the method of asymptotic expansions gives a new insight into the problem of the minoration of the Mahler measure $\mathrm{M}(G_n)$. In 1979 Dobrowolski [Do2] proved that

$$\mathrm{M}(\alpha) \; > \; 1 + (1-\epsilon)\left(\frac{\mathrm{Log}\,\mathrm{Log}\,d}{\mathrm{Log}\,d}\right)^3, \quad d > d_1(\epsilon). \tag{1.5}$$

for any nonzero algebraic number α of degree d. From Theorem 1.2 we obtain the following minoration of $\mathrm{M}(\theta_n^{-1})$ which is better than (1.5).

Corollary 1.6.

$$\mathrm{M}(\theta_n^{-1}) \; > \; \Lambda - \frac{\Lambda}{6}\left(\frac{1}{\mathrm{Log}\,n}\right), \quad n \geq n_1 = 2. \tag{1.6}$$

The value of n_1 is effective: for the small values of n, a numerical comparison between Theorem 1.3 and the inequality (1.6) gives $n_1 = 2$. In (1.6) the exponent of the denominator "$\mathrm{Log}\,n$" is now "1" instead of "3", where $n = \deg \theta_n^{-1}$ if $n \not\equiv 5 \pmod 6$, and $n = \deg \theta_n^{-1} + 2$ if $n \equiv 5 \pmod 6$. That n is closely related to the degree, but different in some cases, is common: quoting Waldschmidt in [W2, p. 90], "*we insist that n is only an upper bound for the degree of θ_n^{-1}, and not the actual degree*". Minorations of the Mahler measure obtained by several authors (Waldschmidt [W2, § 3.6]) are discussed in Section 5.2.

Assume that $\mathbb{L}$ is a totally real algebraic number field, or a CM field (a totally complex quadratic extension of a totally real number field); then for any nonzero algebraic integer $\alpha \in \mathbb{L}$, of degree d, not being a root of unity, Schinzel [Sc2] obtained the minoration

$$\mathrm{M}(\alpha) \; \geq \; \theta_2^{-d/2}. \tag{1.7}$$

Here, from Theorem 1.2, the following theorem of Zagier [Za] which claims

$$\mathrm{M}(\alpha)\mathrm{M}(\alpha-1) \geq \theta_2^{-d/2}$$

for algebraic numbers α, of degree d, $\alpha \neq 0$, $\alpha \neq 1$, $\alpha \neq \frac{1+\sqrt{-3}}{2}$, and the improvment of the minoration of the Zhang-Zagier height by Doche [De], we readily obtain the following minoration.

Corollary 1.7. *Let $u = 0$ except if $n \equiv 5 \bmod 6$ in which case $u = -2$. Then*

$$\mathrm{M}(\theta_n^{-1} - 1) \;\geq\; \kappa \, \frac{\theta_2^{-(n+u)/2}}{\Lambda}\Big(1 - \frac{1}{6\,\mathrm{Log}\, n}\Big), \qquad n \geq 2, \tag{1.8}$$

with $\kappa = 1.2817770214/\sqrt{\theta_2^{-1}} = 1.0076708\ldots$

The method of asymptotic expansions also gives a direct proof of the conjecture of Schinzel-Zassenhaus (Section 5.3; Dubickas [Ds1], Schinzel-Zassenhaus [SZ]) for $\{\theta_n^{-1} \mid n \geq 2\}$ as follows

Theorem 1.8. *For all $n \geq 2$,*

$$\boxed{\theta_n^{-1}} = \theta_n^{-1} \geq 1 + \frac{c}{n}, \tag{1.9}$$

with $c = 2(\theta_2^{-1} - 1) = 1.2360\ldots$ reached only for $n = 2$, and,

$$\boxed{\theta_n^{-1}} = \theta_n^{-1} \;>\; 1 + \frac{(\mathrm{Log}\, n)\left(1 - \frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\right)}{n}. \tag{1.10}$$

The proof (Section 5.3) of Theorem 1.8 is a consequence of the asymptotic expansion of θ_n and does not invoke Smyth's Theorem [Sy1], which claims

$$\boxed{\beta} \;>\; 1 + \frac{\mathrm{Log}\,\Theta}{n}$$

for any nonreciprocal algebraic integer β, of degree n. For n large enough, assuming true Lind-Boyd's conjecture and Boyd's conjecture (recalled in Section 5.3), we show that θ_n^{-1} is a nonextremal Perron number: the minimum $\mathrm{m}_h(n)$ of the houses of the algebraic integers of degree n is certainly smaller than the lower bound given by (1.10), and the inequality (1.10) corresponds to the asymptotic excess of θ_n^{-1} off extremality in some sense. Another way for obtaining a minoration of (the house of) θ_n^{-1}, more classical, consists in deducing it from the Mahler measure (1.6) and the inequality $\mathrm{M}(\theta_n^{-1}) \leq \left(\theta_n^{-1}\right)^n$; we obtain the following:

$$\boxed{\theta_n^{-1}} = \theta_n^{-1} > 1 - \frac{1}{6}\left(\frac{1}{\mathrm{Log}\, n}\right)\frac{1}{n}, \quad n > n_2. \tag{1.11}$$

The inequality (1.11) is but less good than (1.9) and (1.10) in Theorem 1.8.

In Section 6, we consider the angular distribution of the conjugates of the Perron numbers $\theta_n^{-1}, n \geq 2$, and their limit distribution on the unit circle.

For any Salem number $\alpha < 1.3$, of degree $2d$, of conjugates $\alpha^{(i)}$, having positive imaginary part, M.-J. Bertin [Bn] showed

$$\frac{1}{d-1}\sum_{j=1}^{d-1} \arg\bigl(\alpha^{(j)}\bigr) \;\geq\; 0.96, \qquad d > d_2.$$

In Subsection 6.1 we show that the same type of slight angular dissymmetry may occur for the conjugates $z_{j,n}^{-1}$ of θ_n^{-1} which lie in the first quadrant, but in fact it disappears when n tends to infinity.

Theorem 1.9.

$$\lim_{n\to+\infty} \frac{1}{\lfloor \frac{n-1}{4} \rfloor} \sum_{j=1}^{\lfloor \frac{n-1}{4} \rfloor} \arg(z_{j,n}) = \frac{\pi}{4} = 0.785\ldots, \tag{1.12}$$

with

$$\frac{1}{\lfloor \frac{n-1}{4} \rfloor} \sum_{j=1}^{\lfloor \frac{n-1}{4} \rfloor} \arg(z_{j,n}) \;\geq\; \frac{\pi}{4} - \frac{2\pi}{n}\left(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\right)^2 \qquad n \geq 5. \tag{1.13}$$

For limit distributions of conjugates, we show that the method of bivariate Mahler measures, or equivalently the method of asymptotic expansions of the roots of G_n, as a function of n, j, allows to determine the limsup of Mignotte's discrepancy functions for the trinomials G_n in the Erdős-Turán-Amoroso-Mignotte (ETAM) theory (Theorem 6.1). On the other hand, from Theorem 1.1 and Theorem 1.2, we deduce the asymptotic expansion of the Weil height of θ_n^{-1} as

$$h(\theta_n^{-1}) = \frac{1}{n}\left(\frac{3\sqrt{3}}{4\pi} L(2,\chi_3) + O\left(\frac{1}{\operatorname{Log} n}\right)\right) \tag{1.14}$$

and therefore $\lim_{n\to\infty} h(\theta_n^{-1}) = 0$. As a consequence the limit distribution of the conjugates of θ_n^{-1} is given by Theorem 6.2, as an analogue of Bilu's equidistribution Theorem of small points on the 1-dimensional torus.

Notations. Let $P(z)$ be a polynomial $\in \mathbb{Z}[z]$, and $n = \deg(P)$. The *reciprocal polynomial* of $P(z)$ is $P^*(z) = z^n P(\frac{1}{z})$. A polynomial P is a reciprocal polynomial if $P^*(z) = P(z)$. The opposite of the reciprocal polynomial: $-G_n^*$, of G_n, is the Parry polynomial of θ_n^{-1} (Verger-Gaugry [VG]). If $P(X) = a_0 \prod_{j=1}^{n}(X - \alpha_j)$ is a polynomial of degree $n \geq 1$ with coefficients in $\mathbb{C}$, and roots α_j, the *Mahler measure* of P is

$$\mathrm{M}(P) := |a_0| \prod_{j=1}^{n} \max\{1, |\alpha_j|\}.$$

The Mahler measure of an algebraic number is the Mahler measure of its minimal polynomial. Related to the Mahler measure of an algebraic integer α is $\lceil\overline{\alpha}\rceil$, the house of α, defined as the maximum modulus of its conjugates, including α itself. A *Perron number* is either 1 or a real algebraic integer $\theta > 1$ such that the Galois conjugates $\theta^{(i)}$, $i \neq 0$, of $\theta^{(0)} := \theta$ satisfy: $\left|\theta^{(i)}\right| < \theta$. A *Pisot number* is a Perron number > 1 for which $\left|\theta^{(i)}\right| < 1$ for all $i \neq 0$.

Let A be a countable subset of the line. The *first derived set* of A is by the definition the set of the limit points of nonstationary infinite sequences of elements of A. The second derived set of A is the first derived set of the first derived of A. Recall that the smallest element in the first derived set of the set of Pisot numbers is the golden mean $\theta_2^{-1} = (1+\sqrt{5})/2$ (Bertin et al. [B-S]). In particular, if A is the set of Mahler measures of algebraic numbers (in the algebraic closure $\overline{\mathbb{Q}} \subset \mathbb{C}$ of $\mathbb{Q}$), then Lehmer's Conjecture implies that 1 does not belong to the first derived set of A. For $x > 0$, $\mathrm{Log}^{+}x$ denotes $\max\{0, \mathrm{Log}\, x\}$. Let $\mathcal{F}$ be an infinite subset of the set of nonzero algebraic numbers which are not a root of unity; we say that *the Conjecture of Lehmer is true for* $\mathcal{F}$ if there exists a constant $c_{\mathcal{F}} > 0$ such that

$$\mathrm{M}(\alpha) \geq 1 + c_{\mathcal{F}} \quad \text{for all} \quad \alpha \in \mathcal{F}.$$

For instance, the Conjecture of Lehmer is true for the set of Pisot numbers, and remains open for the set of Salem numbers. We use the same terminology, i.e., "*true for* ...", for the other conjectures: for the Conjecture of Schinzel--Zassenhaus, the Conjecture of Smyth, ...

2. Factorization of the trinomials $-1 + X + X^n$

The factorization of $G_n(X)$ is more or less known since Selmer [Sr]. For fixing the notation and a simple, convenient, and coherent indexation of the roots, it is needed under the form (2.15) in the sequel. The form (2.15) is justified below by the Lemmas 2.15, 2.2 and the Propositions 2.3, 3.1, 3.4 and 3.7. Summing in pairs over complex conjugate imaginary roots, we write the factorization of $G_n(X)$ as

$$G_n(X) = (X - \theta_n) \left(\prod_{j=1}^{\lfloor \frac{n}{6} \rfloor} (X - z_{j,n})(X - \overline{z_{j,n}}) \right) \times q_n(X), \qquad (2.15)$$

where θ_n is the only (real) root of $G_n(X)$ in the interval $(0,1)$, where

$$q_n(X) = \begin{cases} \left(\prod_{j=1+\lfloor \frac{n}{6} \rfloor}^{\frac{n-2}{2}} (X - z_{j,n})(X - \overline{z_{j,n}}) \right) \times (X - z_{\frac{n}{2},n}) & \text{if } n \text{ is even, with} \\ & z_{\frac{n}{2},n} \text{ real} < -1, \\ \prod_{j=1+\lfloor \frac{n}{6} \rfloor}^{\frac{n-1}{2}} (X - z_{j,n})(X - \overline{z_{j,n}}) & \text{if } n \text{ is odd,} \end{cases}$$

where the index $j = 1, 2, \ldots$ is such that $z_{j,n}$ is a (nonreal) complex zero of $G_n(X)$, except if n is even and $j = n/2$, such that the argument $\arg(z_{j,n})$ of $z_{j,n}$ is roughly equal to $2\pi j/n$ and that the family of arguments $\big(\arg(z_{j,n})\big)_{1 \leq j < \lfloor n/2 \rfloor}$ forms a strictly increasing sequence with j :

$$0 < \arg(z_{1,n}) < \arg(z_{2,n}) < \cdots < \arg(z_{\lfloor \frac{n}{2} \rfloor,n}) \leq \pi.$$

For $n \geq 2$ all the roots of $G_n(X)$ are simple, and the roots of $G_n^*(X)$, as inverses of the roots of $G_n(X)$, are classified in the reversed order (see Figure 1).

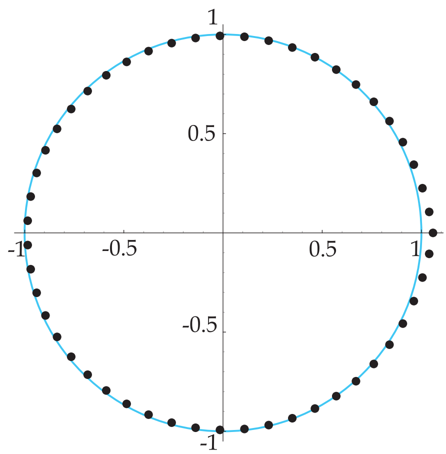

FIGURE 1. The roots (black bullets) of $G_n^*(X)$ (represented here with $n = 51$) are uniformly distributed near $|z| = 1$ according to the theory of Erdős-Turán-Amoroso-Mignotte. A slight bump appears in the half--plane $\Re(z) > 1/2$ in the neighbourhood of 1, at the origin of the different regimes of asymptotic expansions. The dominant root of $G_n^*(X)$ is the Perron number $\theta_n^{-1} > 1$.

Lemma 2.1. *For all $n \geq 2$, all zeros $z_{j,n}$ and θ_n of the polynomials $G_n(X)$ have a modulus in the interval*

$$\left[1 - \frac{2\,\mathrm{Log}\, n}{n},\ 1 + \frac{2\,\mathrm{Log}\, 2}{n}\right]. \tag{2.16}$$

P r o o f. Selmer [Sr, pp. 291–292]. □

The zeros of the trinomials $G_n(X)$ lie at a distance of the unit circle $|z|=1$ which tends to zero with n tending to infinity due to the bounds (2.16). The lower bound is further improved in Proposition 3.1 and Proposition 3.4, as a function of n and the index j of the root only, for those roots lying in the unit disc. The coherency of the indexation of the roots of $G_n(X)$ is made clearer by Proposition 3.7.

Lemma 2.2. (i) The trinomial $G_n(X)$ admits an unique real root θ_n in the interval $(0,1)$. The sequence $(\theta_n)_{n\geq 2}$ is strictly increasing, with

$$\theta_2 = \frac{2}{1+\sqrt{5}} = 0.618\ldots, \quad \text{and} \lim_{n\to+\infty} \theta_n = 1.$$

(ii) The root θ_n is the unique root of smallest modulus among all the roots of $G_n(X)$.

P r o o f. (i) For all $n \geq 2$, the derivative $G_n'(x)$ is strictly positive on $[0,1]$ and the values $G_n(0) = -1$ and $G_n(1) = 1$ are of opposite sign. Now $\theta_{n+1} > \theta_n$ since $G_{n+1}(\theta_n) = (\theta_n)^{n+1} + \theta_n - 1 = -(\theta_n - 1)^2 < 0$. The inequalities $0 < \theta_n < \theta_{n+1} < 1$ imply that the sequence $(\theta_n)_{n\geq 2}$ is increasing. By Lemma 2.1, using (2.16), we deduce that $\lim_{n\to+\infty} \theta_n = 1$.

(ii) Among the moduli of the roots of $G_n(X)$ θ_n is the smallest one: indeed, for all $n \geq 2$ and all z such that $|z| < \theta_n$, $|G_n(z)| \geq 1-|z|-|z|^n = -G_n(|z|) > 0$. The uniqueness comes from Selmer [Sr, pp 291–292]. □

As a consequence of Lemma 2.2 the dominant root of the reciprocal polynomial $G_n^*(X)$ of $G_n(X), n \geq 2$, is $\theta_n^{-1} > 1$: the other roots $z_{j,n}^{-1}, \overline{z_{j,n}^{-1}}$ of $G_n^*(X)$ satisfy $|z_{j,n}^{-1}| < \theta_n^{-1}$ for $j \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$. Therefore, θ_n^{-1} is a Perron number.

From Flatto, Lagarias and Poonen [FLP] the roots $z_{1,n}^{-1}$ and $\overline{z_{1,n}^{-1}}$ are the roots $\neq 1$ the closest to 1 and the alone second-largest roots of $G_n^*(X)$.

Proposition 3.8 extends Lemma 2.2 (ii) and [FLP] by providing a complete classification of the roots of G_n, lying in $|z| < 1$, by strictly increasing modulus.

The Pisot number (golden mean) $\theta_2^{-1} = \frac{1+\sqrt{5}}{2} = 1.618\ldots$ is the largest Perron number in the family $(\theta_n^{-1})_{n\geq 2}$.

Proposition 2.3. *Let $n \geq 2$. If $n \not\equiv 5 \pmod 6$, then $G_n(X)$ is irreducible. If $n \equiv 5pmod6$, then the polynomial $G_n(X)$ admits $X^2 - X + 1$ as irreducible factor in its factorization and $G_n(X)/(X^2 - X + 1)$ is irreducible.*

Proof. Selmer in [Sr, Theorem 3] shows that either $G_n(X)$ is irreducible, or is the product of two irreducible factors, whose one is necessarily of the type $X^2 - X + 1 = (X - e^{i\pi/3})(X - e^{-i\pi/3})$. A modulus equal to 1 for the roots of $G_n(X)$ can occur only for $z = e^{\pm i\pi/3}$. It is easily seen that this is possible only in the case $n \equiv 5 \pmod 6$. $\square$

3. Poincaré asymptotic expansions of the roots

In the calculation of the Mahler measure of the trinomial G_n, $n \geq 2$, only one part of the roots of G_n is required, since $\mathrm{M}(G_n) = \mathrm{M}(\theta_n^{-1}) = \mathrm{M}(\theta_n)$: those which lie inside the closed unit disc, or equivalently, those which lie outside the closed unit disc. We will consider those which lie inside the closed unit disc: they will be characterized by a real part greater than 1/2 (Proposition 3.7). The idea followed here consists in expanding asymptotically θ_n and those roots $z_{j,n}$ of $G_n(X)$ for which $\Re(z_{j,n})$ is strictly greater than 1/2 as a function of n, and j, at an *order* which is sufficiently high, in order to make precise calculations with approximant values. The others roots $z_{j,n}$ for which $0 < \Re(z_{j,n}) \leq 1/2$ will also be considered, in particular for studying how $\{z_{j,n}\}$ are dispatched in the neighbourhood of $e^{i\pi/3}$.

The expressions of the asymptotic expansions given below come from the iterative resolution of the equation $G_n(z) = 0$, starting from the coefficient vector $(-1, 1, 0, 0, \ldots, 1)$ of G_n, with common height equal to 1 (for all $n \geq 2$), of length $n + 1$: cf. Remark 3.3 for precise statements.

The terminology *order* in a given asymptotic expansion comes from the general theory (Borel [Bl], Copson [C], Dingle [Di], Erdélyi [E]); the approximant solutions of $G_n(z) = 0$ which arise naturally correspond to *order* 1. The solutions corresponding to *order* 2 are obtained by inserting the *order* 1 *approximant solutions* into the equation $G_n(z) = 0$, for getting *order* 2 *approximant solutions*, and so on. The order is defined as the number of steps in this iterative convergence process. Ideally, exact expressions of the roots of G_n, as (infinite) asymptotic expansions of the variables n and j, can be obtained by iterating indefinitely this process. However, it would be a formidable task to do so, and stopping at an order sufficiently high is suitable for making calculations, increasing relevance of this iteration over a few steps.

The theory of *terminants* has a long history, refers to the use of divergent series, at least since Euler, and Poincaré for the many-body problem in celestial mechanics, and to asymptotic expansions of many various types (Borel [Bl], Dingle [Di, Chap. XXI]).

For the roots of G_n which lie in $|z| < 1$, we generically write their asymptotic expansions under the following form:

$$\theta_n = \mathrm{D}(\theta_n) + \mathrm{tl}(\theta_n), \tag{3.17}$$

$$\mathrm{Re}(z_{j,n}) = \mathrm{D}\big(\mathrm{Re}(z_{j,n})\big) + \mathrm{tl}\big(\mathrm{Re}(z_{j,n})\big), \tag{3.18}$$

$$\mathrm{Im}(z_{j,n}) = \mathrm{D}\big(\mathrm{Im}(z_{j,n})\big) + \mathrm{tl}\big(\mathrm{Im}(z_{j,n})\big), \tag{3.19}$$

where "D" and "tl" have the meaning given in the introduction. Instead of considering the "exact" product

$$\mathrm{M}(G_n) \; = \; \theta_n^{-1} \prod_{j=1}^{\lfloor n/6 \rfloor} |z_{j,n}|^{-2}$$

we consider the approximant products

$$\Pi_{G_n} \; := \; D(\theta_n)^{-1} \times \prod_{j=1}^{\lfloor n/6 \rfloor} \mathrm{D}(|z_{j,n}|)^{-2} \tag{3.20}$$

in which the Ds' are expanded at a sufficiently high order, and where the index j allowing the enumeration of the roots, and its maximal value $\lfloor n/6 \rfloor$, is justified by Proposition 3.7. We are bound to work with expressions of $\mathrm{D}(\theta_n)$, $\mathrm{D}\big(\mathrm{Re}(z_{j,n})\big)$ and $\mathrm{D}\big(\mathrm{Im}(z_{j,n})\big)$, limited to sums of a few terms. Reaching an high order in the expansions is somehow lengthy but allows to control better the smallness of the tail

$$\mathrm{tl}_{G_n} \; := \; \mathrm{M}(G_n) - \Pi_{G_n}$$

in the asymptotic expansion of the Mahler measure.
Table 1 (the page 92) reports the state of the art for the expressions "D" and "tl".

The basic questions are the following:

(i) Whether the family $(\Pi_{G_n})_{n \geq 2}$ is such that there exists $b > 1$ such that

$$b < \Pi_{G_n} \quad \text{for all } \; n \geq 2. \tag{3.21}$$

(ii) Whether $\liminf_{n \to +\infty} \Pi_{G_n}$ exists.

(iii) Whether the limit $\lim_{n \to +\infty} \mathrm{tl}(G_n)$ is equal to zero.

The expansions of the roots given by Proposition 3.1 and Proposition 3.4 allow to answer positively to these questions (in Section 4.2 and Section 5).

Expanding θ_n, $n \geq 2$, by the method of Poincaré (in Proposition 3.1) leads to asymptotic expansions of θ_n as divergent formal series of functions of only **one variable**: which is "n", the degree of G_n.

Table 1. Expressions of $\mathrm{D}(\theta_n)$, $\mathrm{D}\big(\Re(z_{j,n})\big)$ and $\mathrm{D}\big(\Im(z_{j,n})\big)$ in Selmer [Sr] and Flatto, Lagarias and Poonen [FLP]. They are limited to first order terms, with focus on θ_n and the root $z_{1,n}$ the closest to θ_n. The interest for the other roots $z_{j,n}, 2 \leq j \leq \lfloor n/6 \rfloor$, seems to have been negligible since no result has been found by the author.

D	tail	
$\mathrm{D}(\theta_n) = 1 - \frac{\mathrm{Log}\, n}{n}$	not given	[Sr, p 292]
$\mathrm{D}(\theta_n) = 1 - \frac{(\mathrm{Log}\, n - \mathrm{Log}\,\mathrm{Log}\, n)}{n}$	$\frac{1}{n} O\left(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\right)$	[FLP, Lemma 5.1 (i)]
$\mathrm{D}\big(\Re(z_{1,n})\big) = \theta_n + \frac{2\pi^2}{n\, \mathrm{Log}^2 n}$	$O\left(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{n\, \mathrm{Log}^3 n}\right)$	[FLP, Lemma 5.3 (ii)]
$\mathrm{D}(\Im(z_{1,n})) = \frac{2\pi}{n}$	$o\left(\frac{1}{n}\right)$	[FLP, Lemma 5.3 (ii)]

Proposition 3.1.
Let $n \geq 2$. The root θ_n can be expressed as: $\theta_n = \mathrm{D}(\theta_n) + \mathrm{tl}(\theta_n)$ with

$$\mathrm{D}(\theta_n) = 1 - \frac{\mathrm{Log}\, n}{n}\Bigg(1 - \Big(\frac{n - \mathrm{Log}\, n}{n\,\mathrm{Log}\, n + n - \mathrm{Log}\, n}\Big) \times \Big(\mathrm{Log}\,\mathrm{Log}\, n - n\mathrm{Log}\Big(1 - \frac{\mathrm{Log}\, n}{n}\Big) - \mathrm{Log}\, n\Big)\Bigg) \tag{3.22}$$

and

$$\mathrm{tl}(\theta_n) = \frac{1}{n}\, O\left(\left(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\right)^2\right), \quad \textit{with the constant } 1/2 \textit{ involved in } O\,(\,). \tag{3.23}$$

P r o o f. Let us put $\theta_n = 1 - \frac{t}{n}$ with $0 < t < n$. Then

$$\frac{t}{n} = \left(1 - \frac{t}{n}\right)^n. \tag{3.24}$$

Let us show that $t < \mathrm{Log}\, n$. Let $g(x) = xe^x$ be the increasing function of the variable x on $\mathbb{R}$. The equation (3.24) implies $\frac{t}{n} = (1 - \frac{t}{n})^n < e^{-t} \Leftrightarrow g(t) < n$. Since $n < n\,\mathrm{Log}\, n$ for $n \geq 3$ and $g(\mathrm{Log}\, n) = n\,\mathrm{Log}\, n$, we deduce the claim.

Taking the logarithm of (3.24) we obtain

$$\mathrm{Log}\, t - \mathrm{Log}\, n \;=\; n\,\mathrm{Log}\left(1 - \frac{t}{n}\right) \;=\; -t - \frac{1}{2}\frac{t^2}{n} - \frac{1}{3}\frac{t^3}{n^2} - \cdots$$

The identity

$$t + \operatorname{Log} t + \frac{1}{2}\frac{t^2}{n} + \frac{1}{3}\frac{t^3}{n^2} + \cdots \; = \; \operatorname{Log} n \tag{3.25}$$

has now to be inversed in order to obtain t as a function of n. For doing this, we put $t = \operatorname{Log} n + w$. Equation (3.25) transforms into the following equation in w,

$$(\operatorname{Log} n + w) + \operatorname{Log}(\operatorname{Log} n + w) + \frac{1}{2}\frac{(\operatorname{Log} n + w)^2}{n} + \frac{1}{3}\frac{(\operatorname{Log} n + w)^3}{n^2} + \cdots = \operatorname{Log} n.$$

We deduce

$$\begin{aligned} w + \operatorname{Log}(\operatorname{Log} n) + \operatorname{Log}\left(1 + \frac{w}{\operatorname{Log} n}\right) + \frac{1}{2}\frac{\operatorname{Log}^2 n}{n}\left(1 + \frac{w}{\operatorname{Log} n}\right)^2 \\ + \frac{1}{3}\frac{\operatorname{Log}^3}{n^2}\left(1 + \frac{w}{\operatorname{Log} n}\right)^3 + \cdots \; = \; 0. \end{aligned}$$

Since

$$n \operatorname{Log}\left(1 - \frac{\operatorname{Log} n}{n}\right) + \operatorname{Log} n = -\frac{1}{2}\frac{\operatorname{Log}^2 n}{n} - \frac{1}{3}\frac{\operatorname{Log}^3 n}{n^2} - \cdots$$

and that

$$\operatorname{Log}\left(1 + \frac{w}{\operatorname{Log} n}\right) = \frac{w}{\operatorname{Log} n} - \frac{w^2}{2\operatorname{Log}^2 n} + \frac{w^3}{3\operatorname{Log}^3 n} - \cdots$$

we have

$$\begin{aligned} &\operatorname{Log}\operatorname{Log} n - n\operatorname{Log}\left(1 - \frac{\operatorname{Log} n}{n}\right) - \operatorname{Log} n \\ &\quad = w\left[-1 - \frac{1}{\operatorname{Log} n} - \frac{\operatorname{Log} n}{n} - \frac{\operatorname{Log}^2 n}{n^2} - \frac{\operatorname{Log}^3 n}{n^3} - \cdots\right] \\ &\qquad + w^2\left[\frac{1}{2\operatorname{Log}^2 n} - \frac{1}{2n} - \frac{\operatorname{Log} n}{n^2} - \frac{6}{4}\frac{\operatorname{Log}^2 n}{n^3} - \cdots\right] + \cdots \end{aligned} \tag{3.26}$$

The coefficient coeff(w) of w is

$$-1 - \frac{1}{\operatorname{Log} n} - \frac{\operatorname{Log} n}{n}\left(1 + \frac{\operatorname{Log} n}{n} + \left(\frac{\operatorname{Log} n}{n}\right)^2 + \cdots\right) = \frac{-n\operatorname{Log} n - n + \operatorname{Log} n}{(\operatorname{Log} n)\,(n - \operatorname{Log} n)}.$$

We deduce

$$w = \frac{(\operatorname{Log} n)\,(n - \operatorname{Log} n)\,(\operatorname{Log}\operatorname{Log} n - n\operatorname{Log}(1 - \frac{\operatorname{Log} n}{n}) - \operatorname{Log} n)}{-n\operatorname{Log} n - n + \operatorname{Log} n} + \cdots \tag{3.27}$$

which gives the expression of $\mathrm{D}(\theta_n)$.

Let us write w in (3.27) as $\mathrm{D}(w) + u$, where u denotes the remainding terms. Putting $w = \mathrm{D}(w) + u$ in (3.26) we obtain, for large n,

$$0 \;=\; u \operatorname{coeff}(w) + \mathrm{D}(w)^2 \frac{1}{2\mathrm{Log}^2 n} + \cdots$$

Since, for large n, $\operatorname{coeff}(w) \cong -1$ and $\mathrm{D}(w) \cong -\mathrm{Log}\,\mathrm{Log}\, n$, we deduce

$$u \cong \; O\left(\left(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\right)^2\right),$$

with a constant $1/2$ involved in $O\,(\,)$. We deduce the tail $\mathrm{tl}(\theta_n)$ of θ_n. □

Lemma 3.2. *Given the limited expansion $D(\theta_n)$ of θ_n as in* (3.22)*, denote*

$$\lambda_n := 1 - \big(1 - \mathrm{D}(\theta_n)\big) \frac{n}{\mathrm{Log}\, n}.$$

Then

$$\lambda_n = \mathrm{D}(\lambda_n) + \mathrm{tl}(\lambda_n),$$

with

$$\mathrm{D}(\lambda_n) = \frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\left(\frac{1}{1 + \frac{1}{\mathrm{Log}\, n}}\right), \qquad \mathrm{tl}(\lambda_n) = O\left(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{n}\right) \tag{3.28}$$

with the constant 1 in the Big O.

P r o o f. Indeed, the identity $\mathrm{D}(\theta_n) = 1 - \frac{\mathrm{Log}\, n}{n}(1 - \lambda_n)$, from (3.22), readily gives the result. □

In the sequel, for short, we write λ_n instead of $\mathrm{D}(\lambda_n)$, which amounts to the same in the limited asymptotic expansions.

Remark 3.3. *Angular sectorization of the roots $z_{j,n}$, transition regions and notations used in Proposition 3.4.*

For the roots $z_{j,n}$ lying in the first quadrant, the Poincaré asymptotic expansions of $z_{j,n}$ will be obtained below as divergent formal series of functions of the couple of **two variables** which is:

- $\left(n, \dfrac{j}{n}\right)$, in the angular sector: $\dfrac{\pi}{2} > \arg z > 2\pi \dfrac{\mathrm{Log}\, n}{n}$,
- $\left(n, \dfrac{j}{\mathrm{Log}\, n}\right)$, in the angular sector ("bump" sector)

$$2\pi \frac{\mathrm{Log}\, n}{n} \;>\; \arg z \;\geq\; 0.$$

Moreover expanding the roots at the fourth order has for consequence to dispatch the roots $z_{j,n}$ lying in the bump sector into the two subsectors:

- $2\pi\frac{\sqrt{(\mathrm{Log}\, n)(\mathrm{Log}\,\mathrm{Log}\, n)}}{n} \; > \; \arg z \; > \; 0,$
- $2\pi\frac{\mathrm{Log}\, n}{n} \; > \; \arg z \; > \; 2\pi\frac{\sqrt{(\mathrm{Log}\, n)(\mathrm{Log}\,\mathrm{Log}\, n)}}{n}.$

As compared with the (full) sector $\frac{\pi}{3} > \arg z > 0$, the relative size of the bump sector is negligible, as soon as n is large enough. By transition region, we mean a small neighbourhood of the argument

$$\arg z = 2\pi\frac{\mathrm{Log}\, n}{n}, \quad \text{or of} \quad 2\pi\frac{\sqrt{(\mathrm{Log}\, n)(\mathrm{Log}\,\mathrm{Log}\, n)}}{n}.$$

Outside these two transition regions, one unique dominant asymptotic expansion of $z_{j,n}$ exists. In a transition region an asymptotic expansion contains more second- or third- or fourth- order terms, of the same order of magnitude, which should be taken into account together. Let us define more precisely these neighbourhoods. Let $\epsilon \in (0,1)$. We introduce two strictly increasing sequences of real numbers $(u_n), (v_n)$ which satisfy: for $n \geq n_0 = 18$,

$$\lfloor n/6 \rfloor > v_n > \mathrm{Log}\, n, \qquad \mathrm{Log}\, n > u_n > \sqrt{(\mathrm{Log}\, n)(\mathrm{Log}\,\mathrm{Log}\, n)},$$

such that

$$\lim_{n\to\infty} \frac{v_n}{n} = \lim_{n\to\infty} \frac{\sqrt{(\mathrm{Log}\, n)(\mathrm{Log}\,\mathrm{Log}\, n)}}{u_n} = \lim_{n\to\infty} \frac{u_n}{\mathrm{Log}\, n} = \lim_{n\to\infty} \frac{\mathrm{Log}\, n}{v_n} = 0$$

and

$$v_n - u_n = O\left((\mathrm{Log}\, nt)^{1+\epsilon}\right) \tag{3.29}$$

with the constant 1 involved in the big O.

The roots $z_{j,n}$ lying in the first transition region about $2\pi(\mathrm{Log}\, n)/n$ are such that

$$2\pi\frac{v_n}{n} > \arg z_{j,n} > 2\pi\frac{(2\mathrm{Log}\, n - v_n)}{n}.$$

The roots $z_{j,n}$ lying in the second transition region about

$$2\pi\sqrt{(\mathrm{Log}\, n)(\mathrm{Log}\,\mathrm{Log}\, n)}/n$$

are such that

$$2\pi\,\frac{u_n}{n} > \arg z_{j,n} > 2\pi\,\frac{2\sqrt{(\mathrm{Log}\, n)(\mathrm{Log}\,\mathrm{Log}\, n)} - u_n}{n}.$$

In Proposition 3.4, for simplicity's sake, these two transition regions are schematically denoted by

$$\arg z \asymp 2\pi\,\frac{(\mathrm{Log}\, n)}{n}, \quad \text{respectively}, \quad \arg z \asymp 2\pi\,\frac{\sqrt{(\mathrm{Log}\, n)(\mathrm{Log}\,\mathrm{Log}\, n)}}{n}.$$

By complementarity, the other sectors are schematically written:

$$2\pi\,\frac{\sqrt{(\operatorname{Log} n)(\operatorname{Log}\operatorname{Log} n)}}{n} > \arg z > 0$$

instead of

$$2\pi\,\frac{2\sqrt{(\operatorname{Log} n)(\operatorname{Log}\operatorname{Log} n)} - u_n}{n} > \arg z > 0,$$

respectively,

$$2\pi\,\frac{\operatorname{Log} n}{n} > \arg z 2\pi\,\frac{\sqrt{(\operatorname{Log} n)(\operatorname{Log}\operatorname{Log} n)}}{n}$$

instead of

$$2\pi\,\frac{2\operatorname{Log} n - v_n}{n} > \arg z > 2\pi\,\frac{u_n}{n},$$

respectively,

$$\frac{\pi}{2} > \arg z > 2\pi\,\frac{\operatorname{Log} n}{n} \quad \text{instead of} \quad \frac{\pi}{2} > \arg z > 2\pi\,\frac{v_n}{n}.$$

The asymptotic expansions of the roots $z_{j,n}$ lying in the transition region $\arg z \asymp 2\pi\frac{\operatorname{Log} n}{n}$ are not reported in Proposition 3.4; they are left to the reader, and can be easily obtained by superimposition of those of the two cases (i) and (ii) in Proposition 3.4.

In the following asymptotic expansions, the values of n which will be considered are greater than a critical value n_0 for which $2\pi\,\frac{\operatorname{Log} n}{n}$, is small enough compared to $\pi/3$ for all $n \geq n_0$. Arbitrarily we take $n_0 = 18$ in the sequel and in Theorem 1.2 (cf. Figure 1, on the page 88 to visualize the geometry of the roots of G^*_{51}).

Proposition 3.4. *Let $n \geq n_0 = 18$ and $1 \leq j \leq \lfloor \frac{n-1}{4} \rfloor$. The roots $z_{j,n}$ of $G_n(X)$ have the following asymptotic expansions, $z_{j,n} = \mathrm{D}(z_{j,n}) + \mathrm{tl}(z_{j,n})$, in the following angular sectors:*

(i) *<u>"Bump" sector $2\pi\frac{\operatorname{Log} n}{n} > \arg z > 0$:</u>*

- *<u>Subsector $2\pi\frac{\sqrt{(\operatorname{Log} n)(\operatorname{Log}\operatorname{Log} n)}}{n} > \arg z > 0$:</u>*

$$\mathrm{D}\big(\Re(z_{j,n})\big) = \theta_n + \frac{2\pi^2}{n}\Big(\frac{j}{\operatorname{Log} n}\Big)^2\big(1 + 2\lambda_n\big),$$

$$\mathrm{D}\big(\Im(z_{j,n})\big) = \frac{2\pi\operatorname{Log} n}{n}\Big(\frac{j}{\operatorname{Log} n}\Big)\Big[1 - \frac{1}{\operatorname{Log} n}(1 + \lambda_n)\Big],$$

with

$$\mathrm{tl}\big(\Re(z_{j,n})\big) = \frac{1}{n\operatorname{Log} n}\Big(\frac{j}{\operatorname{Log} n}\Big)^2 O\Big(\Big(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\Big)^2\Big),$$

$$\mathrm{tl}\big(\Im(z_{j,n})\big) = \frac{1}{n\mathrm{Log}\, n}\Big(\frac{j}{\mathrm{Log}\, n}\Big) O\Big(\Big(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\Big)^2\Big),$$

- <u>*Subsector* $2\pi\frac{\mathrm{Log}\, n}{n} > \arg z > 2\pi\frac{\sqrt{(\mathrm{Log}\, n)(\mathrm{Log}\,\mathrm{Log}\, n)}}{n}$:</u>

$$\mathrm{D}\big(\Re(z_{j,n})\big) = \theta_n + \frac{2\pi^2}{n}\Big(\frac{j}{\mathrm{Log}\, n}\Big)^2\Big(1 + \frac{2\pi^2}{3}\Big(\frac{j}{\mathrm{Log}\, n}\Big)^2\Big(1+\lambda_n\Big)\Big)$$

$$\mathrm{D}\big(\Im(z_{j,n})\big) = \frac{2\pi\mathrm{Log}\, n}{n}\Big(\frac{j}{\mathrm{Log}\, n}\Big) \times \Big[1 - \frac{1}{\mathrm{Log}\, n}\Big(1 - \frac{4\pi^2}{3}\Big(\frac{j}{\mathrm{Log}\, n}\Big)^2\Big(1 - \frac{1}{\mathrm{Log}\, n}(1-\lambda_n)\Big)\Big)\Big],$$

with

$$\mathrm{tl}\big(\Re(z_{j,n})\big) = \frac{1}{n} O\Big(\Big(\frac{j}{\mathrm{Log}\, n}\Big)^6\Big), \qquad \mathrm{tl}(\Im(z_{j,n})) = \frac{1}{n} O\Big(\Big(\frac{j}{\mathrm{Log}\, n}\Big)^5\Big),$$

- <u>*Transition region* $\arg z \asymp 2\pi\frac{\sqrt{(\mathrm{Log}\, n)(\mathrm{Log}\,\mathrm{Log}\, n)}}{n}$:</u>

$$\mathrm{D}\big(\Re(z_{j,n})\big) = \theta_n + \frac{2\pi^2}{n}\Big(\frac{j}{\mathrm{Log}\, n}\Big)^2\Big(1 + 2\lambda_n + \frac{2\pi^2}{3}\Big(\frac{j}{\mathrm{Log}\, n}\Big)^2\Big(1+\lambda_n\Big)\Big),$$

$$\mathrm{D}\big(\Im(z_{j,n})\big) = \frac{2\pi\mathrm{Log}\, n}{n}\Big(\frac{j}{\mathrm{Log}\, n}\Big) \times \Big[1 - \frac{1}{\mathrm{Log}\, n} \times \Big(1 + \lambda_n - \frac{4\pi^2}{3}\Big(\frac{j}{\mathrm{Log}\, n}\Big)^2\Big(1 - \frac{1}{\mathrm{Log}\, n}(1-\lambda_n)\Big)\Big)\Big],$$

with

$$\mathrm{tl}\big(\Re(z_{j,n})\big) = \frac{1}{n} O\Big(\Big(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\Big)^3\Big), \quad \mathrm{tl}\big(\Im(z_{j,n})\big) = \frac{1}{n} O\Big(\Big(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\Big)^{5/2}\Big),$$

(ii) <u>*Sector* $\frac{\pi}{2} > \arg z > 2\pi\frac{\mathrm{Log}\, n}{n}$*:*</u>

$$\mathrm{D}\big(\Re(z_{j,n})\big) = \cos\Big(2\pi\frac{j}{n}\Big) + \frac{\mathrm{Log}\,\big(2\,\sin\big(\pi\frac{j}{n}\big)\big)}{n},$$

$$\mathrm{D}\big(\Im(z_{j,n})\big) = \sin\Big(2\pi\frac{j}{n}\Big) + \tan\big(\pi\frac{j}{n}\big)\,\frac{\mathrm{Log}\,\big(2\,\sin\big(\pi\frac{j}{n}\big)\big)}{n}$$

with

$$\mathrm{tl}\big(\Re(z_{j,n})\big) = \mathrm{tl}\big(\Im(z_{j,n})\big) = \frac{1}{n}\,O\Big(\Big(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\Big)^2\Big).$$

P r o o f. Let us write the expression of the jth-root $z_{j,n}$ as

$$z_{j,n} \;=\; \theta_n \, e^{2i\pi(\frac{j}{n}+\phi_{j,n})} \tag{3.30}$$

where $\phi_{j,n}$ a complex number which is assumed to be of small modulus. This assumption on $\phi_{j,n}$ will be confirmed in each case. Let us denote, for short, by $\Re$ the real part of $\phi_{j,n}$, resp. by $\Im$, the imaginary part of $\phi_{j,n}$. The determination of $\phi_{j,n}$ will be done in the sequel by giving expressions of $\mathrm{D}(\Re)$ and $\mathrm{D}(\Im)$ and of their corresponding tails $\mathrm{tl}(\Re)$ and $\mathrm{tl}(\Im)$.

Since $\theta_n^n = 1-\theta_n$, the equation $G_n(z_{j,n}) = 0$ implies

$$0 = 1 - \theta_n e^{2i\pi(\frac{j}{n}+\Re+i\Im)} - (1-\theta_n)e^{2in\pi(\Re+i\Im)}.$$

Then

$$0 = 1 - \theta_n \cos\Big(2\pi\Big(\frac{j}{n}+\Re\Big)\Big)e^{-2\pi\Im} - (1-\theta_n)\cos(2\pi n\Re)e^{-2\pi n\Im}, \tag{3.31}$$

and

$$0 = \; - \theta_n \sin\Big(2\pi\Big(\frac{j}{n}+\Re\Big)\Big)e^{-2\pi\Im} \; - (1-\theta_n)\sin(2\pi n\Re)e^{-2\pi n\Im}. \tag{3.32}$$

We will assume throughout the rest of the proof that $\Re = o(1)$ and $\Im = o(1)$ when n tends to infinity, whatever the value of j in $\{1,2,\ldots,\lfloor\frac{n-1}{2}\rfloor\}$; hence $\sin(2\pi\Re) \cong 2\pi\Re$ and $\cos(2\pi\Re) \cong 1-2\pi^2\Re^2$ for large n. Then (3.32) implies

$$\frac{\theta_n}{1-\theta_n}\Big(\sin\Big(\frac{2\pi j}{n}\Big)(1-2\pi^2\Re^2)+2\pi\Re\cos\Big(\frac{2\pi j}{n}\Big)\Big) = -\sin(2\pi n\Re)e^{-2\pi(n-1)\Im}. \tag{3.33}$$

<u>*"Bump" Sector* $2\pi\,\frac{\mathrm{Log}\,n}{n} \;>\; \arg z \;>\; 0$:</u>

In this case $j/\mathrm{Log}\,n = o(1)$, for all j with $z_{j,n}$ in this sector, and then $n\Re = o(1)$ and $n\Im = o(1)$. In fact, we will consider these two last conditions as assumptions in making the expansions and will show that they are checked at the end. For large n, $\sin(2\pi n\Re) \cong 2\pi n\Re - \frac{4\pi^3}{3}n^3\Re^3$ and $e^{-2\pi(n-1)\Im} \cong 1-2\pi(n-1)\Im+2\pi^2(n-1)^2\Im^2$. From (3.33) we deduce, up the third-order terms,

$$\frac{\theta_n}{2\pi(1-\theta_n)}\,\sin\Big(\frac{2\pi j}{n}\Big) = \Re\Big[-\frac{\theta_n}{1-\theta_n}\,\cos\Big(\frac{2\pi j}{n}\Big) - n\Big] + \Re^2\Big[\pi\frac{\theta_n}{1-\theta_n}\,\sin\Big(\frac{2\pi j}{n}\Big)\Big]$$

$$+\Re\Im\Big[2\pi n(n-1)\Big]+\Re\Im^2\Big[2\pi^2 n(n-1)^2\Big]+\Re^3\Big[\frac{2\pi^2}{3}\frac{\theta_n}{1-\theta_n}\,\cos\Big(\frac{2\pi j}{n}\Big)+\frac{2\pi^2}{3}n^3\Big]+\cdots \tag{3.34}$$

In the rhs of (3.34) only the first term is the dominant one. Then

$$\Re \;=\; -\frac{\sin(\frac{2\pi j}{n})}{2\pi\left(\cos(\frac{2\pi j}{n})\;+\;n\,\frac{1-\theta_n}{\theta_n}\right)} + u, \tag{3.35}$$

where u represents the remaining terms. For large n, since $\sin(\frac{2\pi j}{n}) \cong \frac{2\pi j}{n}$ and $\frac{\mathrm{D}(\theta_n)}{n(1-\mathrm{D}(\theta_n))} \cong \frac{1}{\operatorname{Log} n}$, using (3.22), we obtain the following approximant expression

$$\mathrm{D}\left(-\frac{\sin(\frac{2\pi j}{n})}{2\pi\left(\cos(\frac{2\pi j}{n})\;+\;n\,\frac{1-\theta_n}{\theta_n}\right)}\right) = \mathrm{D}(\Re) = \frac{-j}{n\operatorname{Log} n}, \tag{3.36}$$

with

$$\begin{aligned}\mathrm{tl}\left(-\frac{\sin\left(\frac{2\pi j}{n}\right)}{2\pi\left(\cos(\frac{2\pi j}{n})+n\frac{1-\theta_n}{\theta_n}\right)}\right) &= \frac{-j}{n\operatorname{Log} n}\lambda_n \\ &\quad + \frac{j}{n\operatorname{Log}^2 n}\, O\left(\left(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\right)^2\right).\end{aligned} \tag{3.37}$$

From (3.36) we readily deduce that the approximation conditions $\Re = o(1)$ and $n\Re = o(1)$ we have taken are valid.

Let us now turn to the imaginary part of $\phi_{j,n}$. Multiplying (3.31) by $e^{2\pi\Im}$, with the same assumptions as above: $n\Re = o(1)$ and $n\Im = o(1)$, we obtain

$$\begin{aligned}0 =& \left(1 + 2\pi\Im + 2\pi^2\Im^2 + \frac{4\pi^3}{3}\Im^3\right) \\ &- \theta_n\left[\cos\left(\frac{2\pi j}{n}\right)\left(1 - 2\pi^2\Re^2 + (2\pi^4/3)\Re^4\right) - \sin\left(\frac{2\pi j}{n}\right)\left(2\pi\Re - \frac{4\pi^3}{3}\Re^3\right)\right] \\ &- (1-\theta_n)\left(1 - 2\pi^2 n^2\Re^2 + \frac{2\pi^4}{3}n^4\Re^4\right) \\ &\times\left(1 - 2\pi(n-1)\Im + 2\pi^2(n-1)^2\Im^2 - \frac{4\pi^3}{3}(n-1)^3\Im^3\right) + \cdots\end{aligned} \tag{3.38}$$

Then

$$\begin{aligned}0 =& -\theta_n\left(\cos\left(\frac{2\pi j}{n}\right) - 1\right) + \Re\left[2\pi\theta_n\sin\left(\frac{2\pi j}{n}\right)\right] \\ &+ \Re^2\left[2\pi^2\theta_n\cos\left(\frac{2\pi j}{n}\right) + 2\pi^2 n^2(1-\theta_n)\right] \\ &+ \Im\left[2\pi + 2\pi(n-1)(1-\theta_n)\right] + \Im^2\left[2\pi^2 - 2\pi^2(n-1)^2(1-\theta_n)\right]\end{aligned}$$

$$+ \Re^3 \left[-\frac{4\pi^3}{3} \theta_n \sin\Big(\frac{2\pi j}{n}\Big) \right] + \Re^2 \Im \left[-4\pi^3 (1-\theta_n) n^2 (n-1) \right]$$
$$+ \Im^3 \left[\frac{4\pi^3}{3} + \frac{4\pi^3}{3} (1-\theta_n)(n-1)^3 \right]$$
$$+ \Re^4 \left[-\frac{2\pi^4}{3} \theta_n \cos\Big(\frac{2\pi j}{n}\Big) - \frac{2\pi^4}{3} n^4 (1-\theta_n) \right] + \cdots \tag{3.39}$$

Among the first four terms in the rhs in (3.39), the 3rd and 4th terms are the dominant ones for large n. Except these two dominant terms, all the other terms of order ≥ 2 are negligible for a first-order approximation. Then, replacing $\Re$ as obtained in (3.35), we deduce

$$\Im = -\frac{1}{4\pi} \frac{\left(\sin^2\left(\frac{2\pi j}{n}\right)\right) \left(\cos\left(\frac{2\pi j}{n}\right) + n^2 \frac{1-\theta_n}{\theta_n}\right)}{\left(\cos\left(\frac{2\pi j}{n}\right) + n \frac{1-\theta_n}{\theta_n}\right)^2 \left(1 + n \frac{1-\theta_n}{\theta_n}\right)} + v, \tag{3.40}$$

where v represents the remaining terms. For large n, using (3.22), we obtain the following approximant expression

$$\mathrm{D}\left(-\frac{1}{4\pi} \frac{\left(\sin^2\left(\frac{2\pi j}{n}\right)\right) \left(\cos\left(\frac{2\pi j}{n}\right) + n^2 \frac{1-\theta_n}{\theta_n}\right)}{\left(\cos\left(\frac{2\pi j}{n}\right) + n \frac{1-\theta_n}{\theta_n}\right)^2 \left(1 + n \frac{1-\theta_n}{\theta_n}\right)} \right) = \mathrm{D}(\Im) = \frac{-\pi j^2}{n \operatorname{Log}^2 n}, \tag{3.41}$$

with

$$\mathrm{tl}\left(-\frac{1}{4\pi} \frac{\left(\sin^2\left(\frac{2\pi j}{n}\right)\right) \left(\cos\left(\frac{2\pi j}{n}\right) + n^2 \frac{1-\theta_n}{\theta_n}\right)}{\left(\cos\left(\frac{2\pi j}{n}\right) + n \frac{1-\theta_n}{\theta_n}\right)^2 \left(1 + n \frac{1-\theta_n}{\theta_n}\right)} \right)$$
$$= \frac{-2\pi j^2}{n \operatorname{Log}^2 n} \lambda_n + \frac{j^2}{n \operatorname{Log}^3 n} O\left(\left(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n} \right)^2 \right). \tag{3.42}$$

The expression (3.41) implies that the approximation conditions $\Im = o(1)$ and $n\Im = o(1)$ are valid, and (3.36) together with (3.41) both imply the negligibility of above-mentioned higher-order terms in (3.34) and (3.39). Thus the approximations made are a posteriori justified. Let us now improve the limited expansions $\mathrm{D}(\Re)$ and $\mathrm{D}(\Im)$ given by (3.36) and (3.41) with higher-order terms, namely with some (limited) expansions $\mathrm{D}(u)$ and $\mathrm{D}(v)$ of u and v.

In the rhs of (3.34), except the first one, the 3rd term and the 5th term are the dominant terms among all the others. Replacing the expression (3.35) of $\Re$ in (3.34) gives

$$0 = u \left(-\frac{\theta_n}{1-\theta_n} \cos\Big(\frac{2\pi j}{n}\Big) - n \right) + \frac{4\pi^2}{3} \frac{j^3}{\operatorname{Log}^3 n} + O\left(\frac{j^5}{\operatorname{Log}^5 n} \right). \tag{3.43}$$

Therefore,

$$u = \frac{4\pi^2}{3}\,\frac{j^3}{n\,\mathrm{Log}^{\,3} n}\left(1 - \frac{n}{\mathrm{Log}^{\,2} n}\Big(1 - \mathrm{D}(\theta_n)\Big)\right) + \frac{1}{n} O\left(\left(\frac{j}{\mathrm{Log}\, n}\right)^5\right). \tag{3.44}$$

Denote $\mathrm{D}(\mathrm{tl}_{\Re}) := \frac{-j}{n\,\mathrm{Log}\, n}\lambda_n$ the first term in the rhs of (3.37) and

$$\mathrm{D}(u) := \frac{4\pi^2}{3}\,\frac{j^3}{n\,\mathrm{Log}^{\,3} n}\Big(1 - \frac{1}{\mathrm{Log}\, n}(1-\lambda_n)\Big).$$

According to the range of values taken by j, the expressions $\mathrm{D}(u)$ and $\mathrm{D}(\mathrm{tl}_{\Re})$ for large n are either of similar orders of magnitude, or one is completely dominant over the other. The comparison between both is easy and, with $\mathrm{D}(\theta_n)$ given by (3.22), readily implies the following expression of $\mathrm{D}(\Re)$.

(i) <u>Subsector $2\pi \frac{\sqrt{(\mathrm{Log}\, n)(\mathrm{Log}\,\mathrm{Log}\, n)}}{n} > \arg z > 0$:</u>

in this case, $j/\sqrt{(\mathrm{Log}\, n)(\mathrm{Log}\,\mathrm{Log}\, n)} = o(1)$ for all j such that $z_{j,n}$ belongs to this subsector; $\mathrm{D}(\mathrm{tl}_{\Re})$ is dominant over $\mathrm{D}(u)$ and the new (limited) expansion of $\Re = \Re(\phi_{j,n})$ is

$$\mathrm{D}(\Re) = -\frac{j}{n\,\mathrm{Log}\, n}\big(1 + \lambda_n\big)$$

with

$$\mathrm{tl}(\Re) = \frac{j}{n\,\mathrm{Log}^{\,2} n}\, O\left(\left(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\right)^2\right). \tag{3.45}$$

(ii) <u>Subsector $2\pi \frac{\mathrm{Log}\, n}{n} > \arg z > 2\pi \frac{\sqrt{(\mathrm{Log}\, n)(\mathrm{Log}\,\mathrm{Log}\, n)}}{n}$:</u>

in this case, $\Big(\sqrt{(\mathrm{Log}\, n)(\mathrm{Log}\,\mathrm{Log}\, n)}\Big)/j = o(1)$, for all j such that $z_{j,n}$ belongs to this subsector; $\mathrm{D}(u)$ is dominant over $\mathrm{D}(\mathrm{tl}_{\Re})$, and then

$$\mathrm{D}(\Re) = -\frac{j}{n\,\mathrm{Log}\, n}\left(1 - \frac{4\pi^2}{3}\,\frac{j^2}{\mathrm{Log}^{\,2} n}\Big(1 - \frac{1}{\mathrm{Log}\, n}(1-\lambda_n)\Big)\right) \tag{3.46}$$

with

$$\mathrm{tl}(\Re) = \frac{1}{n} O\left(\left(\frac{j}{\mathrm{Log}\, n}\right)^5\right). \tag{3.47}$$

(iii) <u>Transition region $j \asymp \sqrt{(\mathrm{Log}\, n)(\mathrm{Log}\,\mathrm{Log}\, n)}$:</u>

both expansions have to be considered and

$$\mathrm{D}(\Re) = \frac{-j}{n\,\mathrm{Log}\, n}\left(1 + \lambda_n - \frac{4\pi^2}{3}\,\frac{j^2}{\mathrm{Log}^{\,2} n}\Big(1 - \frac{1}{\mathrm{Log}\, n}(1-\lambda_n)\Big)\right)$$

with

$$\mathrm{tl}(\Re) = \frac{1}{n} O\left(\left(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\right)^{5/2}\right).$$

In the rhs of (3.39) the 5th, 7th and 9th terms are the dominant remaining terms among all the others. Replacing the expressions (3.36) of $\Re$ and (3.41) of $\Im$ in (3.39) gives

$$0 = 2\pi v\,[1 + (n-1)(1-\theta_n)] + \frac{j^4}{n\operatorname{Log}^3 n}\left[-2\pi^4 + 4\pi^4 - \frac{2\pi^4}{3}\right] + \cdots \qquad (3.48)$$

Then

$$v = \frac{-2\pi^3}{3}\frac{j^4}{n\operatorname{Log}^4 n}\left(2 - (1 - \mathrm{D}(\theta_n))\frac{n}{\operatorname{Log} n}\right) + \frac{1}{n}O\left(\left(\frac{j}{\operatorname{Log} n}\right)^6\right). \qquad (3.49)$$

Denote $\mathrm{D}(\mathrm{tl}_{\Im}) := \frac{-2\pi j^2}{n\operatorname{Log}^2 n}\lambda_n$ the first term in the rhs of (3.42) and

$$\mathrm{D}(v) := \frac{-2\pi^3}{4}\,\frac{j^4}{n\operatorname{Log}^4 n}(1+\lambda_n).$$

We now compare the expressions $\mathrm{D}(v)$ and $\mathrm{D}(\mathrm{tl}_{\Im})$ for large n to check which one is dominant over the other. It readily implies the following expression of $\mathrm{D}(\Im)$.

(i) Subsector $2\pi\,\frac{\sqrt{(\operatorname{Log} n)(\operatorname{Log}\operatorname{Log} n)}}{n} > \arg z > 0$:

$\mathrm{D}(\mathrm{tl}_{\Im})$ is dominant over $\mathrm{D}(v)$ and the new (limited) expansion of $\Im = \Im(\phi_{j,n})$ is

$$\mathrm{D}(\Im) = \frac{-\pi j^2}{n\operatorname{Log}^2 n}\big(1 + 2\lambda_n\big)$$

with

$$\mathrm{tl}(\Im) = \frac{j^2}{n\operatorname{Log}^3 n}\,O\left(\left(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\right)^2\right).$$

(ii) Subsector $2\pi\,\frac{\operatorname{Log} n}{n} > \arg z > 2\pi\,\frac{\sqrt{(\operatorname{Log} n)(\operatorname{Log}\operatorname{Log} n)}}{n}$:

$\mathrm{D}(v)$ is dominant over $\mathrm{D}(\mathrm{tl}_{\Im})$, and then

$$\mathrm{D}(\Im) \;=\; -\frac{\pi j^2}{n\operatorname{Log}^2 n}\left(1 + \frac{2\pi^2}{3}\frac{j^2}{\operatorname{Log}^2 n}\,(1+\lambda_n)\right)$$

with

$$\mathrm{tl}(\Im) \;=\; \frac{1}{n}O\left(\left(\frac{j}{\operatorname{Log} n}\right)^6\right).$$

(iii) Transition region: $j \asymp \sqrt{(\mathrm{Log}\, n)(\mathrm{Log}\,\mathrm{Log}\, n)}$:

both expansions have to be considered and

$$\mathrm{D}(\Im) = -\frac{\pi j^2}{n\mathrm{Log}^2 n}\left[1 + 2\lambda_n + \frac{2\pi^2}{3}\frac{j^2}{\mathrm{Log}^2 n}(1+\lambda_n)\right]$$

with

$$\mathrm{tl}(\Im) \;=\; \frac{1}{n} O\left(\left(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\right)^3\right).$$

Sector $\frac{\pi}{2} \;>\; \arg z \;>\; 2\pi\, \frac{\mathrm{Log}\, n}{n}$:

In this case, $(\mathrm{Log}\, n)/j = o(1)$ for all j such that $z_{j,n}$ belongs to this sector. Taking the squares of (3.31) and of (3.32) and summing them up lead to the removal of the terms $\sin(2\pi n\Re)$ and $\cos(2\pi n\Re)$. We obtain

$$(\theta_n - 1)^2 e^{-4\pi n\Im} = \theta_n^2 e^{-4\pi\Im}$$
$$- 2\theta_n e^{-2\pi\Im}\left[\cos\Big(\frac{2\pi j}{n}\Big)\cos(2\pi\Re) - \sin\Big(\frac{2\pi j}{n}\Big)\sin(2\pi\Re)\right] + 1. \quad (3.50)$$

Then, with the assumptions $\Re = o(1)$ and $\Im = o(1)$, (3.50) becomes

$$e^{-4\pi n\Im} = 1 + \frac{2\theta_n}{(\theta_n - 1)^2}\left[-2\pi\Im\Big(\theta_n - \cos\Big(\frac{2\pi j}{n}\Big)\Big)\right.$$
$$\left. + 2\pi\Re\sin\Big(\frac{2\pi j}{n}\Big) + \Big(1-\cos\Big(\frac{2\pi j}{n}\Big)\Big)\right] + \cdots \quad (3.51)$$

Let us observe that $\lim_{n\to+\infty} \theta_n/(\theta_n - 1)^2 = +\infty$, and that the term within brackets is positive and less than (say) 3 in modulus. Then, by taking the logarithm of (3.51), we deduce

$$\Im = \frac{-1}{4\pi n}\left[\mathrm{Log}\left(\frac{\theta_n}{(\theta_n-1)^2}\right) + \mathrm{Log}\left(2\Big(1 - \cos\Big(\frac{2\pi j}{n}\Big)\Big)\right)\right.$$
$$\left. + \mathrm{Log}\left(1 - 2\pi\Im\left(\frac{\theta_n - \cos\big(\frac{2\pi j}{n}\big)}{1-\cos\big(\frac{2\pi j}{n}\big)}\right) + 2\pi\Re\left(\frac{\sin\big(\frac{2\pi j}{n}\big)}{1-\cos\big(\frac{2\pi j}{n}\big)}\right)\right)\right] + \cdots \quad (3.52)$$

Let us give a first-order expansion of $\mathrm{D}(\Im)$ from (3.52) under the assumption that the quantities $2\pi\Im\left(\frac{\theta_n - \cos\left(\frac{2\pi j}{n}\right)}{1-\cos\left(\frac{2\pi j}{n}\right)}\right)$ are $= o(1)$ for all j. This assumption will imply that the quantities $2\pi\Re\left(\frac{\sin\left(\frac{2\pi j}{n}\right)}{1-\cos\left(\frac{2\pi j}{n}\right)}\right)$ also have the same property. We will check below that these assumptions are valid. First, by (3.28), we have

$$\begin{aligned}
\mathrm{Log}\left(\frac{\theta_n}{(\theta_n-1)^2}\right) &= \mathrm{Log}\,\theta_n - 2\mathrm{Log}\,(1-\theta_n)\\
&= -\frac{\mathrm{Log}\,n}{n}(1-\lambda_n) - 2\mathrm{Log}\left(\frac{\mathrm{Log}\,n}{n}(1-\lambda_n)\right) + \cdots\\
&= (2\mathrm{Log}\,n)\left[1 - \frac{\mathrm{Log}\,\mathrm{Log}\,n}{\mathrm{Log}\,n} + \frac{\lambda_n}{\mathrm{Log}\,n}\right] + \lambda_n^2 + \frac{2}{3}\lambda_n^3 + \cdots \qquad (3.53)
\end{aligned}$$

$$= (2\mathrm{Log}\,n)\Big[1 - \frac{\mathrm{Log}\,\mathrm{Log}\,n}{\mathrm{Log}\,n} + \frac{\mathrm{Log}\,\mathrm{Log}\,n}{(\mathrm{Log}\,n)(1+\mathrm{Log}\,n)}\Big] + O\left(\left(\frac{\mathrm{Log}\,\mathrm{Log}\,n}{\mathrm{Log}\,n}\right)^2\right)$$

$$= (2\mathrm{Log}\,n)\big[1-\lambda_n\big] + O\left(\left(\frac{\mathrm{Log}\,\mathrm{Log}\,n}{\mathrm{Log}\,n}\right)^2\right) \qquad (3.54)$$

with the constant 1 involved in the Big O. Expanding the last Logarithm at the first-order in (3.52) implies that the solution of (3.52) is $\Im = \mathrm{D}(\Im) + \mathrm{tl}(\Im)$ with

$$\mathrm{D}(\Im) \;=\; \frac{-1}{2\pi n}\left[(\mathrm{Log}\,n)(1-\lambda_n) + \mathrm{Log}\left(2\,\sin\left(\frac{\pi j}{n}\right)\right)\right] \qquad (3.55)$$

and

$$\mathrm{tl}(\Im) \;=\; +\frac{1}{n}O\left(\left(\frac{\mathrm{Log}\,\mathrm{Log}\,n}{\mathrm{Log}\,n}\right)^2\right), \qquad (3.56)$$

such that the constant involved in the Big O in (3.56) is equal to $1/4\pi$ (and does not depend upon j). Let us show that the assumption about

$$\begin{aligned}
&-2\pi\mathrm{D}(\Im)\left(\frac{\theta_n - \cos\left(\frac{2\pi j}{n}\right)}{1-\cos\left(\frac{2\pi j}{n}\right)}\right)\\
&= \frac{1}{n}\left[(\mathrm{Log}\,n)\Big(1-\mathrm{D}(\lambda_n)\Big) + \mathrm{Log}\left(2\,\sin\left(\frac{\pi j}{n}\right)\right)\right]\left(\frac{\theta_n - 1}{1-\cos\left(\frac{2\pi j}{n}\right)} + 1\right) \qquad (3.57)
\end{aligned}$$

is valid. Indeed, the function $F(x) := (\mathrm{Log}\,(2\sin x))/(1-\cos(2x))$ takes negative values on $[\pi(\mathrm{Log}\,n)/n, \pi/6]$ with positive derivative on this interval. Then checking at $x = \pi(\mathrm{Log}\,n)/n$ is sufficient to obtain an upper bound of the quantity (3.57) in modulus. Taking for $\pi j/n$ the quantity $\pi(\mathrm{Log}\,n)/n$, assuming n large enough, it is easy to check that (3.57) is equivalent to

$$\cong\; -\frac{1}{2\pi^2}\frac{\mathrm{Log}\,\mathrm{Log}\,n}{(\mathrm{Log}\,n)^2} \qquad (3.58)$$

uniformly in j, for all the roots $z_{j,n}$ in this sector.

By truncating the power series $\mathrm{Log}\,(1-\lambda_n) = -\lambda_n - \frac{1}{2}\lambda_n^2 - \frac{1}{3}\lambda_n^3 - \cdots$ in (3.53) into the sum of a polynomial section in λ_n and a remainder, let us observe that, whatever the degree of this polynomial section is, the resulting tail in (3.54) is always the same, as a function of n, and the term $\mathrm{Log}\,\theta_n$ in (3.52) asymptotically negligible. This observation allows to restore (3.52), using only the dominant terms and (3.53) with the full series, as

$$\Im = \frac{-1}{4\pi n}\left[\mathrm{Log}\left(\frac{1}{(\theta_n-1)^2}\right) + 2\mathrm{Log}\left(2\,\sin\Big(\frac{\pi j}{n}\Big)\right)\right] + \frac{1}{n} O\left(\left(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\right)^2\right). \quad (3.59)$$

From (3.59) and from the expansion of the term within brackets [ ] in (3.52), we deduce that the following summation

$$\mathrm{Log}\,\theta_n - 2\pi\Im\left(\frac{\theta_n - \cos\big(\frac{2\pi j}{n}\big)}{1-\cos\big(\frac{2\pi j}{n}\big)}\right) + 2\pi\Re\left(\frac{\sin\big(\frac{2\pi j}{n}\big)}{1-\cos\big(\frac{2\pi j}{n}\big)}\right) \quad (3.60)$$

should be equal to zero, up to second-order terms. Then, from (3.22), (3.59) and (3.60),

$$\mathrm{D}(\Re) \;=\; -\frac{1}{2\pi n}\left[\frac{1-\cos\big(\frac{2\pi j}{n}\big)}{\sin\big(\frac{2\pi j}{n}\big)}\,\mathrm{Log}\left(2\,\sin\Big(\frac{\pi j}{n}\Big)\right)\right] \quad (3.61)$$

with

$$\mathrm{tl}(\Re) \;=\; +\frac{1}{n} O\left(\left(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\right)^2\right), \quad (3.62)$$

and with the constant $\tan(\pi j/n)$ in the Big O. An uniform upper bound of all $\tan(\pi j/n)$ is $\tan(\pi/4) = 1$.

Let us now check the validity of the assumption on

$$2\pi\Re\,\sin\Big(\frac{2\pi j}{n}\Big)\Big/\Big(1-\cos\Big(\frac{2\pi j}{n}\Big)\Big):$$

using (3.58) in the summation (3.60) set equal to 0, we readily obtain

$$+2\pi\Re\left(\frac{\sin\big(\frac{2\pi j}{n}\big)}{1-\cos\big(\frac{2\pi j}{n}\big)}\right) \cong +\frac{1}{2\pi^2}\frac{\mathrm{Log}\,\mathrm{Log}\, n}{(\mathrm{Log}\, n)^2}$$

for n large enough, uniformly in j, for all the roots $z_{j,n}$ in this sector.

To finish up the proof of Proposition 3.4 we have to provide the expansions of the real and imaginary parts of the roots $z_{j,n}$ as functions of $\Re$ and $\Im$. From (3.30), taking the expansions of $\Re$ and $\Im$, we deduce

$$\Re(z_{j,n}) = \cos\Big(\frac{2\pi j}{n}\Big)\Big[\theta_n - 2\pi\mathrm{D}(\Im)\Big] - \sin\Big(\frac{2\pi j}{n}\Big)\Big(2\pi\mathrm{D}(\Re)\Big) + \cdots \quad (3.63)$$

and

$$\Im(z_{j,n}) = \sin\Big(\frac{2\pi j}{n}\Big)\Big[\theta_n - 2\pi \mathrm{D}(\Im)\Big] + \cos\Big(\frac{2\pi j}{n}\Big)\big(2\pi \mathrm{D}(\Re)\big) + \cdots \tag{3.64}$$

We now replace $\mathrm{D}(\Re)$ and $\mathrm{D}(\Im)$ by their respective expressions in (3.63) and (3.64). The expressions of $\mathrm{D}(\Re(z_{j,n}))$ and $\mathrm{D}(\Im(z_{j,n}))$ follow as functions of the two variables n and j/n. □

Proposition 3.5. *The roots $z_{j,n}$ of $G_n(X)$, $n \geq n_0 = 18$, which belong to the angular sector (with the convention taken in Remark 3.3) $\frac{\pi}{2} > \arg z > 2\pi \,\frac{\mathrm{Log}\, n}{n}$, have a modulus whose asymptotic expansion is*

$$|z_{j,n}| = 1 + \frac{1}{n}\,\mathrm{Log}\,\left(2\,\sin\Big(\frac{\pi j}{n}\Big)\right) + \frac{1}{n} O\left(\left(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\right)^2\right), \tag{3.65}$$

where the constant involved in $O(\ )$ in the tail does not depend upon j.

Proof. For the roots $z_{j,n}$ in this sector, since

$$|z_{j,n}| = \theta_n - 2\pi \mathrm{D}(\Im) + \cdots$$

by (3.30) and that $(\mathrm{Log}\, n)\big[1 - \lambda_n\big] = n\,(1 - \mathrm{D}(\theta_n))$ we readily obtain (3.65), from (3.22), (3.23), (3.55) and (3.56). □

Proposition 3.5 will be used in particular to study the geometry of the roots which lie in a small neighbourhood of $e^{i\pi/3}$.

Corollary 3.6. *Let $\psi_{j,n} := \arg(z_{j,n}) \in (0, \pi/3)$ for $1 \leq j \leq \lfloor n/6 \rfloor$. Then, for j/n close to $1/6$ and n large enough, $\psi_{j,n}$ admits the following (two variables-) asymptotic expansion*

$$\begin{aligned}\psi_{j,n} = \frac{\pi}{3} - 2\pi\left(1 - \frac{1}{2n}\right)\left(\frac{1}{6} - \frac{j}{n}\right) - \frac{2\sqrt{3}\pi^2}{3n}\left(\frac{1}{6} - \frac{j}{n}\right)^2 \\ + \frac{1}{n} O\left(\left(\frac{1}{6} - \frac{j}{n}\right)^3\right) + \frac{1}{n} O\left(\left(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\right)^2\right).\end{aligned} \tag{3.66}$$

Proof. By Proposition 3.4 (ii), more precisely using (3.30), (3.61), (3.62), since

$$\mathrm{Log}\,\left(2\,\sin\Big(\frac{\pi j}{n}\Big)\right) = -\sqrt{3}\pi\Big(\frac{1}{6} - \frac{j}{n}\Big) - 2\pi^2\Big(\frac{1}{6} - \frac{j}{n}\Big)^2 + O\left(\Big(\frac{1}{6} - \frac{j}{n}\Big)^3\right), \tag{3.67}$$

we readily obtain the asymptotic expansion of $\psi_{j,n}$ in the two variables "j/n" and "n":

$$\psi_{j,n} = 2\pi\left(\frac{j}{n} - \frac{1}{2\pi n}\frac{\sin(\frac{\pi j}{n})}{\cos(\frac{\pi j}{n})}\mathrm{Log}\,\left(2\sin\Big(\frac{\pi j}{n}\Big)\right)\right) + \frac{1}{n} O\left(\left(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\right)^2\right) = \frac{2\pi j}{n}$$

$$+\frac{\pi}{n}\left(\frac{1}{6}-\frac{j}{n}\right)-\frac{2\sqrt{3}\pi^2}{3n}\left(\frac{1}{6}-\frac{j}{n}\right)^2+\frac{1}{n}O\left(\left(\frac{1}{6}-\frac{j}{n}\right)^3\right)+\frac{1}{n}O\left(\left(\frac{\mathrm{Log}\,\mathrm{Log}\,n}{\mathrm{Log}\,n}\right)^2\right). \tag{3.68}$$

Up to the error terms $\psi_{j,n}$ behaves as $\frac{2\pi j}{n}$. From (3.68) we deduce (3.66). □

Proposition 3.7. *Let* $n \geq 2$. *Then* (i) the number p_n of roots of $G_n(X)$ which lie inside the open sector $\mathcal{S} = \{z \mid |\arg(z)| < \pi/3\}$ is equal to

$$1 + 2\left\lfloor \frac{n}{6} \right\rfloor, \tag{3.69}$$

(ii) the correlation between the geometry of the roots of $G_n(X)$ which lie inside the unit disc and the upper half-plane and their indexation is given by

$$j \in \left\{1, 2, \ldots, \left\lfloor \frac{n}{6} \right\rfloor\right\} \iff \Re(z_{j,n}) > \frac{1}{2} \iff |z_{j,n}| < 1, \tag{3.70}$$

and the Mahler measure $\mathrm{M}(G_n)$ of the trinomial $G_n(X)$ is

$$\mathrm{M}(G_n) \;=\; \mathrm{M}(G_n^*) \;=\; \theta_n^{-1} \prod_{j=1}^{\lfloor n/6 \rfloor} |z_{j,n}|^{-2}. \tag{3.71}$$

Proof. The formula $p_n = 1 + 2\lfloor \frac{n}{6} \rfloor$ is true for $n = 2$ to 51 as it can be easily checked. For larger n we now prove it recursively.

First let us observe that the open unit disc $\{z \in \mathbb{C} \mid |z - 1| < 1\}$ centred at 1 contains all the roots of $G_n(X)$ of modulus < 1 since

$$|z_{j,n}| < 1 \iff |z_{j,n}|^n = |1 - z_{j,n}| < 1. \tag{3.72}$$

Then it suffices to understand the geometry of the roots in a neighbourhood of $e^{i\pi/3}$. The integer j is ≥ 1 in the sequel. Let us show the following equivalences

$$\arg(z_{j,n}) < \pi/3 \iff |z_{j,n}| < 1 \iff \Re(z_{j,n}) > \frac{1}{2}. \tag{3.73}$$

For proving (3.73) we make use of the expansions $\mathrm{D}(\Re(z_{j,n}))$ and $\mathrm{D}(\Im(z_{j,n}))$ for $z_{j,n}$ in a neighbourhood of $e^{i\pi/3}$, belonging to the main sector "$\pi/2 > \arg z > 2\pi \mathrm{Log}\, n/n$" (Proposition 3.4 (ii)): since

$$\begin{aligned}
\Re(z_{j,n}) &= \cos\Big(\frac{2\pi j}{n}\Big) + \frac{\mathrm{Log}\,\big(2\,\sin\big(\frac{\pi j}{n}\big)\big)}{n} + \cdots \\
&= \cos\Big(\Big(\frac{2\pi j}{n} - \frac{\pi}{3}\Big) + \frac{\pi}{3}\Big) + \frac{\mathrm{Log}\,\big(2\,\sin\big(\big(\frac{\pi j}{n} - \frac{\pi}{6}\big) + \frac{\pi}{6}\big)\big)}{n} + \cdots \\
&= \frac{1}{2} - \frac{\sqrt{3}}{2}\left(\frac{2\pi j}{n} - \frac{\pi}{3}\right) + \cdots ,
\end{aligned}$$

we obtain the equivalence

$$\arg(z_{j,n}) < \pi/3 \Longleftrightarrow \Re(z_{j,n}) > \frac{1}{2}$$

when $n \not\equiv 0 \pmod 6$, and it holds, in this case, if and only if $\frac{2\pi j}{n} - \frac{\pi}{3} < 0$, that is $j \leq \lfloor n/6 \rfloor$. By Proposition 3.5,

$$|z_{j,n}| = 1 + \frac{1}{n} \operatorname{Log}\left(2 \sin\Big(\frac{\pi j}{n}\Big)\right) + \frac{1}{n} O\left(\left(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\right)^2\right).$$

Since, for $x \in [-1/2, +1/2]$, $\operatorname{Log}\big(2\,|\sin(\pi x)|\big) < 0 \Longleftrightarrow |x| < 1/6$, we deduce that the inequality $|z_{j,n}| < 1$ is approximately equivalent to $\frac{\pi j}{n} < \frac{\pi}{6}$, i.e., j approximately less than $n/6$. Obviously it is equivalent to $j \leq \lfloor n/6 \rfloor$ when $n \not\equiv 0 \pmod 6$, and therefore (3.73) holds if and only if $1 \leq j \leq \lfloor n/6 \rfloor$ when $n \not\equiv 0 \pmod 6$. Let us now prove that (3.73) is equivalent to $1 \leq j \leq \lfloor n/6 \rfloor$ for all integers $n \geq 2$.

By Proposition 2.3 the trinomial $G_n(X)$ admits $e^{\pm i\pi/3} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ as roots if and only if $n \equiv 5 \pmod 6$. For $n \equiv 5 \pmod 6$ the root $e^{i\pi/3}$ is necessarily one of the roots $z_{j,n}$ for which j is close to $n/6$, by Proposition 3.4 (ii), whose real part is

$$\mathrm{D}\big(\Re(z_{j,n})\big) = \cos\Big(\frac{2\pi j}{n}\Big) + \frac{\operatorname{Log}\big(2 \sin\big(\frac{\pi j}{n}\big)\big)}{n}.$$

Therefore, if $n = 6q+5$, with $q \geq 1$, the corresponding value of j is $q+1$, so that

$$z_{q+1,6q+5} \;=\; z_{q+2,6q+11} \;=\; e^{i\pi/3} \qquad \text{for all } q \geq 0.$$

Let us assume that $p_{6q+5} = 1 + 2\lfloor \frac{6q+5}{6} \rfloor$ (with $q \geq 1$) holds and let us prove that the formula $p_{6q+6} = 1 + 2\lfloor \frac{6q+6}{6} \rfloor$ also holds. From Proposition 3.4 (see (3.68)) the distance between two successive roots of $G_n(X)$ in $\Re(z) > 0$ is $= 2\pi/n + \cdots$, and therefore, when j is fixed, the jth root $z_{j,n}$ "rotates to the right towards 1 along the unit circle", to become the jth root $z_{j,n+1}$ of $G_{n+1}(X)$. Thus the number of roots of $G_{6q+5+1}(X)$ within the sector $\mathcal{S}$ is the number of roots of $G_{6q+5}(X)$ within $\mathcal{S}$ plus 2, and $p_{6q+6} = 1 + 2\lfloor \frac{6q+6}{6} \rfloor = 1 + 2\lfloor \frac{6q+5}{6} \rfloor + 2$ is also true.

Let us continue the recursion. For $n = 6q + m$ and $m = 6$ to 11, when n is increased by 1 to $n+1$, the collection of roots $\{z_{j,n}\}$ which lie near $e^{i\pi/3}$ slightly rotates to the right to become the collection $\{z_{j,n+1}\}$. Using the expressions of $\mathrm{D}\big(\Re(z_{j,n})\big)$ in Proposition 3.4, for j close to $n/6$, we deduce

$$\begin{aligned}
&\Re(z_{q+2,6q+5}) \\
&\quad < \Re(z_{q+2,6q+6}) < \Re(z_{q+2,6q+7}) < \Re(z_{q+2,6q+8}) < \Re(z_{q+2,6q+9}) \\
&\quad < \Re(z_{q+2,6q+10}) < \frac{1}{2} = \Re(z_{q+2,6q+11}) = \Re(z_{q+1,6q+5}) < \Re(z_{q+1,6q+6}) \\
&\quad < \Re(z_{q+1,6q+7}) < \Re(z_{q+1,6q+8}) < \Re(z_{q+1,6q+9}) < \Re(z_{q+1,6q+10}) \,.
\end{aligned}$$

Counting the complex conjugates, these inequalities show that

$$p_{6q+6} = p_{6q+7} = p_{6q+8} = p_{6q+9} = p_{6q+10} = p_{6q+11}.$$

We deduce the general formula (3.69) for p_n, and the equivalences (3.70) and (3.73), for all integers $n \geq 2$.

To obtain (3.71) we first invoke the general fact that $\mathrm{M}(G_n^*) = \mathrm{M}(G_n)$ for $n \geq 2$. Then the roots of $G_n^*(X)$ which possess a real part $> 1/2$ are those which lie outside the closed unit disc. These roots are the inverses of those of $G_n(X)$ which lie inside the unit disc, characterized by (3.70). The expression (3.71) of $\mathrm{M}(G_n)$ follows; it is valid for n odd or n even, whatever the form (2.15) the trinomial $G_n(X)$ takes. □

Proposition 3.8. *Let $n \geq 6$. The roots of modulus < 1 of $G_n(z)$ in the closed upper half-plane have the following properties:*

(i) $\theta_n < |z_{1,n}|$,

(ii) *for any pair of successive indices $j, j+1$ in $\{1, 2, \ldots, \lfloor n/6 \rfloor\}$,*

$$|z_{j,n}| \; < \; |z_{j+1,n}|.$$

Proof. (i) is proved in [FLP]. Let us prove (ii). Assume the contrary: $|z_{j+1,n}| < |z_{j,n}|$ for a certain j in $\{1, 2, \ldots, \lfloor n/6 \rfloor - 1\}$. Then

$$-1 + z_{j,n} + z_{j,n}^n \; = \; -1 + z_{j+1,n} + z_{j+1,n}^n.$$

We deduce

$$\frac{|z_{j+1,n}|}{|z_{j,n}|} \; = \; \frac{|1 + z_{j,n}^{n-1}|}{|1 + z_{j+1,n}^{n-1}|}. \tag{3.74}$$

The assumption $|z_{j+1,n}| < |z_{j,n}|$ implies that the ratio $\frac{|z_{j+1,n}|}{|z_{j,n}|}$ is < 1.

Now there are two cases: either (ii-a) j and $j+1$ together are such that $z_{j,n}$ and $z_{j+1,n}$ belong to the "bump" sector (i.e., for which $j/\mathrm{Log}\, n = o(1)$) or to the "main sector" (i.e., for which $(\mathrm{Log}\, n)/j = o(1)$), (the two cases (i) and (ii) of Proposition 3.4), or (ii-b) j and $j+1$ are both such that $z_{j,n}$ and $z_{j+1,n}$ in the transition region $j \asymp \mathrm{Log}\, n$.

In the first case (ii-a), from Proposition 3.4, we deduce from the asymptotic expansion of $z_{j,n}$ that $\arg(z_{j,n})$ is equal to $\frac{2i\pi j}{n} + \cdots$ Therefore $\arg(z_{j,n}^{n-1}) \cong -\arg(z_{j,n}) = \arg(z_{-j,n})$ and $z_{j,n}^{n-1} \cong |z_{j,n}|^{n-1} e^{-2i\pi j/n}$. The enumeration of the roots $(z_{-j,n})_j$ by the index $-j$ is clockwise and, by Proposition 3.7, both real parts $\Re(|z_{j,n}|^{n-1} e^{-2i\pi j/n})$ and $\Re(|z_{j+1,n}|^{n-1} e^{-2i\pi(j+1)/n})$ are $> 1/2$. Moreover, the assumption $|z_{j+1,n}| < |z_{j,n}|$ implies $|z_{j+1,n}^{n-1}| < |z_{j,n}^{n-1}|$. Therefore, assuming $|z_{j+1,n}| < |z_{j,n}|$ implies that the distance $|1+z_{j,n}^{n-1}|$ between -1 and $z_{j,n}^{n-1}$ is larger than the distance $|1 + z_{j+1,n}^{n-1}|$ between -1 and $z_{j+1,n}^{n-1}$. Contradiction with (3.74) since the ratio $|1 + z_{j,n}^{n-1}|/|1 + z_{j+1,n}^{n-1}|$ is < 1.

In the second case (ii-b), using the notations of Remark 3.3, the two successive roots $z_{j,n}$ and $z_{j+1,n}$ are such that $j \asymp \operatorname{Log} n$, i.e.,

$$\frac{2i\pi u_n}{n} \; < \; \arg(z_{j,n}) \; < \; \arg(z_{j+1,n}) \; < \; \frac{2i\pi(v_n+1)}{n}$$

with

$$\Re(z_{j,n}) > 1/2 \quad \text{and} \quad \Re(z_{j+1,n}) > 1/2.$$

Therefore,

$$\frac{-2i\pi u_n}{n} \; > \; \arg(z_{j,n}^{n-1}) \; > \; \arg(z_{j+1,n}^{n-1}) \; > \; \frac{-2i\pi(v_n+1)}{n}$$

for which $\lim_{n\to\infty} u_n/n = \lim_{n\to\infty} v_n/n = 0$. Hence

$$\Re(z_{j,n}^{n-1}) > 1/2 \quad \text{and} \quad \Re(z_{j+1,n}^{n-1}) > 1/2.$$

We can now conclude as in (ii-a). Assuming $|z_{j+1,n}| < |z_{j,n}|$ implies that the distance $|1 + z_{j,n}^{n-1}|$ between -1 and $z_{j,n}^{n-1}$ is larger than the distance $|1 + z_{j+1,n}^{n-1}|$ between -1 and $z_{j+1,n}^{n-1}$. Therefore we would have $|1 + z_{j,n}^{n-1}|/|1 + z_{j+1,n}^{n-1}| > 1$, which is a contradiction with (3.74). □

4. Limit Mahler measure : proof of Theorem 1.1

4.1. Using bivariate Mahler measures

The Mahler measure of a polynomial $f(X,Y)$ in two variables is defined by

$$\mathrm{M}\big(f(X,Y)\big) = \exp\left(\int_0^1\int_0^1 \operatorname{Log}|f(e^{2\pi is}, e^{2\pi it})| \, \mathrm{d}s \, \mathrm{d}t\right).$$

Boyd [Bo3], [Lw] ([Bo1, p 130]; [Bo2, in Lemma 1]) prove that the Mahler measure of a polynomial in two variables $f(X,Y)$ is the limit of the Mahler measures of polynomials in one variable obtained by replacing the second variable by a power of the first one

$$\mathrm{M}\big(f(X,Y)\big) = \lim_{n\to\infty} \mathrm{M}\big(f(X,X^n)\big).$$

Since

$$\mathrm{M}(G_n) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Log}\,|-1+e^{it}+e^{int}|\,\mathrm{d}t\right),$$

$$\lim_{n\to+\infty} \mathrm{Log}\big(\mathrm{M}(G_n)\big) = \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi}\mathrm{d}s \int_0^{2\pi} \mathrm{Log}|e^{it}+e^{is}-1|\,\mathrm{d}t.$$

From Theorem 2 in Smyth [Sy2] and by Jensen's formula

$$\begin{aligned}\frac{3\sqrt{3}}{4\,\pi}\mathrm{L}(2,\chi_3) &= \mathrm{Log}\,\mathrm{M}(-1+X+Y) = \mathrm{Log}\,\mathrm{M}\big(\max\{|-1+X|,1\}\big)\\ &= \frac{1}{\pi}\int_0^{\pi} \mathrm{Log}^{\,+}|-1+e^{it}|\,\mathrm{d}t = \frac{-1}{\pi}\int_0^{\pi/3} \mathrm{Log}\,\big(2\,\sin(t/2)\big)\,\mathrm{d}t\\ &= \mathrm{Log}\,(1.38135\ldots)\,.\end{aligned}$$

This value is given by the Bloch-Wigner dilogarithm [BM] [Ln].

4.2. Using asymptotic expansions

In the case where G_n is not irreducible, i.e., for $n \equiv 5 \bmod 6$, the Mahler measure $\mathrm{M}(G_n)$ of $G_n(X)$ is equal to the Mahler measure

$$\mathrm{M}\big(G_n(X)/(X^2-X+1)\big)$$

by Proposition 2.3. We will consider the two cases "G_n irreducible" and "G_n reducible" simultaneously, not taking care of reducibility. Indeed, the roots of X^2-X+1 lie on the unit circle, and the indexation of the roots of G_n which is chosen, with j, in Proposition 3.7, is such that the maximal value $\lfloor n/6\rfloor$ of the index j involves both cases.

First let us observe that the Riemann-Stieljes sum

$$S(f,n) := -2\sum_{j=1}^{\lfloor n/6\rfloor} \frac{1}{n}\,\mathrm{Log}\left(2\,\sin\Big(\frac{\pi j}{n}\Big)\right) = \frac{-1}{\pi}\sum_{j=1}^{\lfloor n/6\rfloor}(x_j-x_{j-1})f(x_j)$$

with $x_j = \frac{2\pi j}{n}$ and $f(x) := \mathrm{Log}\left(2\,\sin\big(\frac{x}{2}\big)\right)$ converges to the limit

$$\lim_{n\to\infty} S(f,n) = \frac{-1}{\pi}\int_0^{\pi/3} f(x)dx = \mathrm{Log}\,\Lambda = \mathrm{Log}\,(1.38135\ldots)\,.$$

From Proposition 3.7 we have

$$\operatorname{Log} \mathrm{M}(G_n) = -\operatorname{Log}(\theta_n) - 2 \sum_{j=1}^{\lfloor n/6 \rfloor} \operatorname{Log} |z_{j,n}|. \tag{4.75}$$

Since $\lim_{n_\infty} \theta_n = 1$, by Proposition 3.1, then $\lim_{n_\infty} \operatorname{Log} \theta_n = 0$. Therefore we will just show that

$$\lim_{n\to\infty} S(f,n) = \lim_{n\to\infty} \operatorname{Log} \mathrm{M}(G_n) = -2 \lim_{n\to\infty} \sum_{j=1}^{\lfloor n/6 \rfloor} \operatorname{Log} |z_{j,n}|. \tag{4.76}$$

by a suitable decomposition of the summation in (4.75). In this summation the asymptotic expansion of $\operatorname{Log} |z_{j,n}|$ takes different forms, by Proposition 3.4 and Proposition 3.5, according to the angular sector to which $z_{j,n}$ belongs, i.e., to the asymptotic behaviour of j/n or $j/\operatorname{Log} n$ (cf. Remark 3.3).

For $n \geq n_0 = 18$, the decomposition into angular sectors between $\pi/3$ and 0 follows from Remark 3.3, as:

$$\frac{\pi}{3} \geq 2\pi \frac{v_n}{n} > 2\pi \frac{2\operatorname{Log} n - v_n}{n} > 2\pi \frac{u_n}{n} > 2\pi \frac{2\sqrt{(\operatorname{Log} n)(\operatorname{Log}\operatorname{Log} n)} - u_n}{n} > 0.$$

The corresponding decomposition of the summation $\sum_{j=1}^{\lfloor n/6 \rfloor}$ in (4.76) is

$$= \sum_{j=1}^{\lfloor 2\sqrt{(\operatorname{Log} n)(\operatorname{Log}\operatorname{Log} n)} - u_n \rfloor} + \sum_{\lceil 2\sqrt{(\operatorname{Log} n)(\operatorname{Log}\operatorname{Log} n)} - u_n \rceil}^{\lfloor u_n \rfloor}$$

$$+ \sum_{j=\lceil u_n \rceil}^{\lfloor 2\operatorname{Log} n - v_n \rfloor} + \sum_{\lceil 2\operatorname{Log} n - v_n \rceil}^{\lfloor v_n \rfloor} + \sum_{j=\lceil v_n \rceil}^{\lfloor n/6 \rfloor} \tag{4.77}$$

We now show that, in (4.77), the last summation $\sum_{j=\lceil v_n \rceil}^{\lfloor n/6 \rfloor}$ is asymptotically the dominant one and that the other summations tend to zero when n goes to infinity.

Main sector, dominant contribution: $\pi/3 > \arg z > 2\pi \frac{\operatorname{Log} n}{n}$:

From (3.65) we have

$$\sum_{j=\lceil v_n \rceil}^{\lfloor n/6 \rfloor} \operatorname{Log} |z_{j,n}| = \sum_{j=\lceil v_n \rceil}^{\lfloor n/6 \rfloor} \operatorname{Log} \left(1 + \frac{1}{n} \operatorname{Log}\left(2 \sin\left(\frac{\pi j}{n}\right)\right) + \frac{1}{n} O\left(\left(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\right)^2\right)\right).$$

By the remainder Theorem of alternating series let us recall that, for x a real number, $|x| < 1$, the inequality $|\mathrm{Log}\,(1+x) - x| \leq \frac{x^2}{2}$ holds. Then

$$\left| \sum_{j=\lceil v_n \rceil}^{\lfloor n/6 \rfloor} \mathrm{Log}\,|z_{j,n}| - \sum_{j=\lceil v_n \rceil}^{\lfloor n/6 \rfloor} \frac{1}{n} \mathrm{Log}\left(2\,\sin\Big(\frac{\pi j}{n}\Big)\right) \right| \leq \sum_{j=\lceil v_n \rceil}^{\lfloor n/6 \rfloor} \frac{1}{n} \left| O\left(\Big(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\Big)^2\right) \right|$$

$$+ \frac{1}{2} \sum_{j=\lceil v_n \rceil}^{\lfloor n/6 \rfloor} \frac{1}{n^2} \left[\mathrm{Log}\left(2\,\sin\Big(\frac{\pi j}{n}\Big)\right) + O\left(\Big(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\Big)^2\right) \right]^2. \tag{4.78}$$

For $1 \leq j \leq \lfloor n/6 \rfloor$, the inequalities $0 < 2\sin(\pi j/n) \leq 1$ and $\mathrm{Log}\,(2\sin(\pi j/n)) < 0$ hold. Then $|\mathrm{Log}\,(2\sin(\pi j/n))| \leq |\mathrm{Log}\,(2\sin(\pi/n))| = O(\mathrm{Log}\, n)$. On the other hand, by Proposition 3.5 the two $O(\ )$s in the rhs of the inequality (4.78) involve a constant which does not depend upon j. Therefore the rhs of (4.78) is

$$= O\left(\Big(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\Big)^2\right) + O\left(\frac{\mathrm{Log}^{\,2} n}{n}\right) = O\left(\Big(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\Big)^2\right).$$

From (4.78), using (3.29), we deduce

$$\left| \sum_{j=\lceil v_n \rceil}^{\lfloor n/6 \rfloor} \mathrm{Log}\,|z_{j,n}| - \sum_{j=1}^{\lfloor n/6 \rfloor} \frac{1}{n} \mathrm{Log}\left(2\,\sin\Big(\frac{\pi j}{n}\Big)\right) \right|$$

$$\leq \sum_{j=1}^{\lfloor u_n \rfloor} \frac{1}{n} \left| \mathrm{Log}\left(2\,\sin\Big(\frac{\pi j}{n}\Big)\right) \right| + \sum_{j=\lceil u_n \rceil}^{\lfloor v_n \rfloor} \frac{1}{n} \left| \mathrm{Log}\left(2\,\sin\Big(\frac{\pi j}{n}\Big)\right) \right| + O\left(\Big(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\Big)^2\right)$$

$$= O\left(\frac{\mathrm{Log}^{\,2} n}{n}\right) + O\left(\frac{\mathrm{Log}^{\,2+\epsilon} n}{n}\right) + O\left(\Big(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\Big)^2\right) = O\left(\Big(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\Big)^2\right).$$

Moreover, let us observe that: $\lim_{n\to+\infty} \frac{2\pi}{n} \lfloor \frac{n}{6} \rfloor = \pi/3$. Hence

$$\lim_{n\to\infty} S(f,n) = \lim_{n\to\infty} (-2) \sum_{j=\lceil v_n \rceil}^{\lfloor n/6 \rfloor} \mathrm{Log}\,|z_{j,n}|.$$

"Bump sector", the other contributions in (4.77) *:*

Instead of considering the four other summations of (4.77) independently, it will be sufficient to gather the roots involved into these summations into the following two subcollections indexed by j, as:

- $1 \leq j \leq \lfloor u_n \rfloor$, and
- $\lceil u_n \rceil \leq j \leq \lfloor v_n \rfloor$.

Case (i) : for all j satisfying $1 \leq j \leq \lfloor u_n \rfloor$, by taking the first terms in the asymptotic expansions of the roots $z_{j,n}$, Proposition 3.4 shows that

$$\mathrm{D}\big(\Im(z_{j,n})\big) = \frac{2\pi j}{n} + \frac{1}{n}\, O\left(\frac{u_n}{\operatorname{Log} n}\right),$$

with a j-independent constant involved in $O(\,)$. From Proposition 3.5 the asymptotic expansions of the moduli $|z_{j,n}|$ are given by

$$|z_{j,n}| = \theta_n - 2\pi \mathrm{D}\big(\Im(z_{j,n})\big) + \cdots$$

By the remainder Theorem of alternating series and Proposition 3.1 we deduce

$$\begin{aligned}
\operatorname{Log}|z_{j,n}| &= (\theta_n - 1) - 2\pi \mathrm{D}\big(\Im(z_{j,n})\big) \\
&\quad + \frac{1}{n}\, O\left(\frac{u_n}{\operatorname{Log} n}\right) + O\left(\left((\theta_n - 1) - 2\pi \mathrm{D}\Big(\Im(z_{j,n})\Big)\right)^2\right) \\
&= -\frac{\operatorname{Log} n}{n} - \frac{4\pi^2 j}{n} + O\left(\frac{\operatorname{Log}\operatorname{Log} n}{n}\right) \\
&\quad + \frac{1}{n}\, O\left(\frac{u_n}{\operatorname{Log} n}\right) + O\left(\left(\frac{\operatorname{Log} n}{n}\right)^2\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{j=1}^{\lfloor u_n \rfloor} \operatorname{Log}|z_{j,n}| &= -\frac{\lfloor u_n \rfloor \operatorname{Log} n}{n} - \frac{\lfloor u_n \rfloor (\lfloor u_n \rfloor)\, 2\pi^2}{n} \\
&\quad + O\left(\frac{\lfloor u_n \rfloor\, (\operatorname{Log}\operatorname{Log} n)}{n}\right) = O\left(\frac{\operatorname{Log}^2 n}{n}\right).
\end{aligned}$$

Therefore,

$$\lim_{n \to \infty} \sum_{j=1}^{\lfloor u_n \rfloor} \operatorname{Log}|z_{j,n}| = 0.$$

*C*ase (ii) : for all j satisfying "$\lceil u_n \rceil \leq j \leq \lfloor v_n \rfloor$", Proposition 3.4 does not provide an unique limited expansion of $z_{j,n}$ but the inequalities $1 - |z_{j,n}| \leq 2(\operatorname{Log} n)/n$ hold by Lemma 2.1. We deduce

$$\begin{aligned}
\left| \sum_{j=\lceil u_n \rceil}^{\lfloor v_n \rfloor} \operatorname{Log}|z_{j,n}| \right| &\leq (v_n - u_n)\frac{2\operatorname{Log} n}{n} + O\left((v_n - u_n)\frac{\operatorname{Log}^2 n}{n^2}\right) \\
&= O\left(\frac{\operatorname{Log}^{2+\epsilon} n}{n}\right),
\end{aligned}$$

and the limit

$$\lim_{n\to\infty} \sum_{j=\lceil u_n \rceil}^{\lfloor v_n \rfloor} \mathrm{Log}\,|z_{j,n}| = 0.$$

To finish up the proof of Theorem 1.1 we take the exponential of (4.76).

5. Asymptotics of the Mahler measure

5.1. Second-order terms for the moduli of the roots

The second-order terms in the asymptotic expansions of the moduli $|z_{j,n}|$ of the roots $z_{j,n}$ of G_n are required for obtaining the asymptotic speed of convergence of the Mahler measure $\mathrm{M}(G_n)$ towards the limit Mahler measure Λ. From Subsection 4.2 it is sufficient to obtain them for the roots lying in the main sector: $\pi/3 > \arg z > 2\pi \frac{\mathrm{Log}\, n}{n}$, the other roots lying in the "bump" sector contributing negligibly to the Mahler measure. The expansions used for obtaining (3.65) have to be improved; the first step is to expand further (3.53) by taking into account the term $\lambda_n^2 \cong \left(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\right)^2 \left(1 - \frac{2}{\mathrm{Log}\, n}\right)$, from (3.28). Then (3.55) and (3.56) are replaced by

$$\mathrm{D}(\Im) = \frac{-1}{2\pi n}\left[(\mathrm{Log}\, n)(1-\lambda_n) + \frac{1}{2}\left(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\right)^2 + \mathrm{Log}\left(2\sin\left(\frac{\pi j}{n}\right)\right)\right] \tag{5.79}$$

with

$$\mathrm{tl}(\Im) \;=\; +\frac{1}{n} O\left(\frac{(\mathrm{Log}\,\mathrm{Log}\, n)^2}{(\mathrm{Log}\, n)^3}\right) \tag{5.80}$$

with the constant $1/(2\pi)$ involved in the Big O, independently of j. Then, for $\pi/3 \geq \arg z_{j,n} > 2\pi \frac{\lceil v_n \rceil}{n}$, (3.65) is expanded further as

$$|z_{j,n}| = \mathrm{D}(|z_{j,n}|) + \mathrm{tl}(|z_{j,n}|)$$

with

$$\mathrm{D}(|z_{j,n}|) = 1 + \frac{1}{n}\mathrm{Log}\left(2\sin\left(\frac{\pi j}{n}\right)\right) + \frac{1}{2n}\left(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\right)^2 \tag{5.81}$$

and

$$\mathrm{tl}(|z_{j,n}|) = \frac{1}{n} O\left(\frac{(\mathrm{Log}\,\mathrm{Log}\, n)^2}{(\mathrm{Log}\, n)^3}\right), \tag{5.82}$$

where the constant involved in $O(\ )$ is 1 (does not depend upon j).

5.2. Minoration of the Mahler measure and proof of Theorem 1.2

In 1933, in the search of big prime numbers, Lehmer [Le] asked the following problem: *if ϵ is a positive quantity, to find a polynomial of the form*

$$f(x) = x^r + a_1 x^{r-1} + \cdots + a_r$$

where the a's are integers, such that the absolute value of the product of those roots of f which lie outside the unit circle, lies between 1 *and* $1+\epsilon\ldots$ *Whether or not the problem has a solution for* $\epsilon < 0.176$ *we do not know.*

Lehmer's Conjecture was initially addressed to nonzero algebraic integers which are not roots of unity. Today (2015), the smallest known Mahler measure of nonzero algebraic numbers which are not roots of unity is the one discovered by Lehmer [Le], a Salem number (Lehmer's number)

$$\mathrm{M}\big(X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1\big) = 1.1762\ldots$$

Exhaustive search [Sy3] for the smallest Mahler measures was carried out by Boyd, Poulet, Mossinghoff, Lisonek, Flammang, Grandcolas, Rhin, Sac-Epée. The list of Mossinghoff [MfL] gives primitive, irreducible, noncyclotomic integer polynomials with degree at most 180 having Mahler measure less than 1.3, and this list is complete through degree 44 (Mossinghoff, Rhin and Wu [MRW]).

For α an algebraic integer of degree $d > 1$, not a root of unity, Blansky and Montgomery [ByM] showed, with multivariate Fourier series,

$$\mathrm{m}(\alpha) > 1 + \frac{1}{52}\frac{1}{d\,\mathrm{Log}\,(6\,d)}.$$

By a different approach, using an auxiliary function and a proof of transcendence (Thue's method), Stewart [St] obtained in 1977 the same minoration but with a constant $c \neq 1/52$ instead of $1/52$ (Waldschmidt [W0]). In 1979, Dobrowolski [Do2], using an auxiliary function, obtained the asymptotic minoration

$$\mathrm{M}(\alpha) > 1 + (1-\epsilon)\left(\frac{\mathrm{Log}\,\mathrm{Log}\,d}{\mathrm{Log}\,d}\right)^3, \qquad d > d_5,$$

with the constant $1-\epsilon$ replaced by $1/1200$ for any $d \geq 2$, for an effective version of the minoration. For sufficiently large degree d, Waldschmidt [W2], Theorem 3.17, showed that the constant $1-\epsilon$ could be replaced by $1/250$ with a transcendence proof which uses an interpolation determinant. If α is a nonzero algebraic number of degree $d \geq 2$, Voutier [V] obtained the effective minorations:

$$\mathrm{M}(\alpha) > 1 + \frac{1}{4}\left(\frac{\mathrm{Log}\,\mathrm{Log}\,d}{\mathrm{Log}\,d}\right)^3 \qquad \text{and} \qquad \mathrm{M}(\alpha) > 1 + \frac{2}{(\mathrm{Log}\,(3\,d))^3}.$$

For algebraic integers α, of degree d, of norm $N(\alpha)$, which are totally real, Schinzel's minoration (1.7) was improved by Bertin [Bn] as

$$\mathrm{M}(\alpha) \geq \max \left\{ \theta_2^{-d/2}, \sqrt{|N(\alpha)|}\theta_2^{-\frac{d}{2|N(\alpha)|^{1/d}}} \right\}.$$

Further, following Bertin [Bn], if a ν-Salem is by definition an algebraic integer α having ν conjugates outside the closed unit disc and at least one conjugate of modulus 1, then the measure $\mathrm{M}(\alpha)$ of such an algebraic integer, of degree $2\nu + 2k$, totally real (i.e., for which the conjugates of modulus > 1 are all real), satisfies

$$\mathrm{M}(\alpha) \geq \theta_2^{-\frac{\nu}{2^{k/\nu}}}.$$

Zaimi [Zi] also obtained minorations for the Mahler measures of algebraic integers named K-Pisot numbers.

Lehmer's Conjecture is solved in some cases. If α is an algebraic number of degree d such that there exists a prime number $p \leq d \operatorname{Log} d$ that is not ramified in the field $\mathbb{Q}(\alpha)$, then Mignotte [Mt0] showed: $\mathrm{M}(\alpha) \geq 1.2$. For any noncyclotomic irreducible polynomial P with all odd coefficients, Borwein, Dobrowolski and Mossinghoff [BDM] showed

$$\mathrm{M}(P) \geq 5^{1/4} = 1.4953\ldots$$

In terms of the Weil height, Amoroso and David [ADd2] proved that there exists a constant $c > 0$ such that, for all nonzero algebraic number α, of degree d, not being a root of unity, under the assumption that the extension$\mathbb{Q}(\alpha)/\mathbb{Q}$ is Galois, then

$$h(\alpha) \geq \frac{c}{d}.$$

If $\mathbb{L}/\mathbb{Q}$ is an abelian extension of number fields, Amoroso and Dvornicich [AD] proved

$$h(\alpha) \geq \frac{\operatorname{Log} 5}{12}$$

for any nonzero $\alpha \in \mathbb{L}$ which is not a root of unity. Later, given an abelian extension $\mathbb{L}/\mathbb{Q}$ of number fileds and a nonzero algebraic number α which is not a root of unity, with $D := [\mathbb{L}(\alpha) : \mathbb{L}]$, Amoroso and Zannier [AZ1] proved the relative result, which makes use of Dobrowolski's minoration and the previous minoration

$$h(\alpha) \geq \frac{c(\mathbb{K})}{D} \left(\frac{\operatorname{Log}\operatorname{Log} 5D}{\operatorname{Log} 2D} \right)^{13},$$

where $c(\mathbb{K}) > 0$. Amoroso and Delsinne [ADn] computed a lower bound, depending upon the degree and the discriminant of the number field $\mathbb{K}$, for the constant $c(\mathbb{K})$. In 2010, given $\mathbb{K}/\mathbb{Q}$ an extension of algebraic number fields, of degree d, Amoroso and Zannier [AZ2] showed

$$h(\alpha) \geq 3^{-d^2-2d-6}$$

for any nonzero algebraic number α which is not a root of unity such that $\mathbb{K}(\alpha)/\mathbb{K}$ is abelian. As a corollary they obtained

$$h(\alpha) \geq 3^{-14}$$

for any dihedral extension $\mathbb{L}/\mathbb{Q}$ and any nonzero $\alpha \in \mathbb{L}$ which is not a root of unity. For cyclotomic extensions, they obtained sharper results :

(i) If $\mathbb{K}$ is a number field of degree d, there exists an absolute constant $c_2 > 0$ such that, with $\mathbb{L}$ denoting the number field generated by $\mathbb{K}$ and any given root of unity, then

$$h(\alpha) \geq \frac{c_2}{d}\frac{(\operatorname{Log}\operatorname{Log} 5d)^3}{(\operatorname{Log} 2d)^4},$$

for any nonzero $\alpha \in \mathbb{L}$ which is not a root of unity.

(ii) if $\mathbb{K}$ is a number field of degree d, and α any nonzero algebraic number, not a root of unity, such that $\alpha^n \in \mathbb{K}$ for some integer n under the assumption that $\mathbb{K}(\alpha)/\mathbb{K}$ is an abelian extension, then

$$h(\alpha) \geq \frac{c_3}{d}\frac{(\operatorname{Log}\operatorname{Log} 5d)^2}{(\operatorname{Log} 2d)^4},$$

for some constant $c_3 > 0$.

Proof of Theorem 1.2. We now extend the asymptotic expansions of Section 4.2 with the second-order terms obtained in Section 5.1, inserting them in (4.75).

First, with the new expansions of the moduli of the roots $z_{j,n}$, we have

$$\sum_{j=\lceil v_n \rceil}^{\lfloor n/6 \rfloor} \operatorname{Log}|z_{j,n}| = \sum_{j=\lceil v_n \rceil}^{\lfloor n/6 \rfloor} \operatorname{Log}\left(1 + \frac{1}{n}\operatorname{Log}\left(2\sin\left(\frac{\pi j}{n}\right)\right)\right.$$
$$\left. + \frac{1}{2n}\left(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\right)^2 + \frac{1}{n}O\left(\frac{(\operatorname{Log}\operatorname{Log} n)^2}{(\operatorname{Log} n)^3}\right)\right)$$

with the constant 1 involved in the Big O. Let us apply the remainder Theorem of alternating series: for x real, $|x| < 1$ the folowing inequality holds

$$|\operatorname{Log}(1+x) - x| \leq \frac{x^2}{2}.$$

Then

$$\left|\sum_{j=\lceil v_n\rceil}^{\lfloor n/6\rfloor} \operatorname{Log}|z_{j,n}| - \sum_{j=\lceil v_n\rceil}^{\lfloor n/6\rfloor} \frac{1}{n}\operatorname{Log}\left(2\,\sin\Big(\frac{\pi j}{n}\Big)\right) - \sum_{j=\lceil v_n\rceil}^{\lfloor n/6\rfloor} \frac{1}{2n}\left(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\right)^2\right|$$

$$\leq \sum_{j=\lceil v_n\rceil}^{\lfloor n/6\rfloor} \frac{1}{n}\left|O\left(\frac{(\operatorname{Log}\operatorname{Log} n)^2}{(\operatorname{Log} n)^3}\right)\right| \tag{5.83}$$

$$+\frac{1}{2}\sum_{j=\lceil v_n\rceil}^{\lfloor n/6\rfloor}\frac{1}{n^2}\left[\operatorname{Log}\left(2\,\sin\Big(\frac{\pi j}{n}\Big)\right)+\frac{1}{2}\left(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\right)^2+O\left(\frac{(\operatorname{Log}\operatorname{Log} n)^2}{(\operatorname{Log} n)^3}\right)\right]^2.$$

Similarly as the upper bound in (4.78), we deduce that the upper bound in (5.83) is

$$O\left(\frac{(\operatorname{Log}\operatorname{Log} n)^2}{(\operatorname{Log} n)^3}\right).$$

Therefore,

$$(-2)\sum_{j=\lceil v_n\rceil}^{\lfloor n/6\rfloor}\operatorname{Log}|z_{j,n}|+\sum_{j=\lceil v_n\rceil}^{\lfloor n/6\rfloor}\frac{2}{n}\operatorname{Log}\left(2\,\sin\Big(\frac{\pi j}{n}\Big)\right)=O\left(\left(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\right)^2\right)$$

with the constant 1/6 involved in the Big O. Since $\operatorname{Log}\theta_n = O\left(\frac{\operatorname{Log} n}{n}\right)$ and

$$\sum_{j=\lceil u_n\rceil}^{\lfloor v_n\rfloor}\operatorname{Log}|z_{j,n}| = O\left(\frac{(\operatorname{Log} n)^{2+\epsilon}}{n}\right),\qquad \sum_{j=1}^{\lfloor u_n\rfloor}\operatorname{Log}|z_{j,n}| = O\left(\frac{(\operatorname{Log} n)^{2}}{n}\right),$$

we deduce

$$\operatorname{Log}\mathrm{M}(G_n)-\operatorname{Log}\Lambda+\left(\operatorname{Log}\Lambda+\sum_{j=\lceil v_n\rceil}^{\lfloor n/6\rfloor}\frac{2}{n}\operatorname{Log}\left(2\,\sin\Big(\frac{\pi j}{n}\Big)\right)\right)=O\left(\left(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\right)^2\right) \tag{5.84}$$

with the constant 1/6 involved in the Big O.

In (5.84) the summation $\sum_{j=\lceil v_n\rceil}^{\lfloor n/6\rfloor}$ can be replaced by $\sum_{j=\lceil \operatorname{Log} n\rceil}^{\lfloor n/6\rfloor}$. Indeed, using Remark 3.3,

$$\sum_{j=\lceil \operatorname{Log} n\rceil}^{\lceil v_n\rceil}\frac{2}{n}\operatorname{Log}\left(2\,\sin\Big(\frac{\pi j}{n}\Big)\right)=O\left(\frac{(\operatorname{Log} n)^{2+\epsilon}}{n}\right).$$

Let us now estimate and give an upper bound of

$$\left|\frac{-1}{\pi}\int_0^{\pi/3}\operatorname{Log}\Big(2\sin(x/2)\Big)\,\mathrm{d}x-\sum_{j=\lceil \operatorname{Log} n\rceil}^{\lfloor n/6\rfloor}\frac{-2}{n}\operatorname{Log}\left(2\,\sin\Big(\frac{\pi j}{n}\Big)\right)\right|. \tag{5.85}$$

In (5.85) the sums are truncated Riemann-Stieltjes sums of $\mathrm{Log}\,\Lambda$, the integral being $\mathrm{Log}\,\Lambda$. Referring to Stoer and Bulirsch ([SB], pp 126–128) we now replace $\mathrm{Log}\,\Lambda$ by an approximate value obtained by integration of an interpolation polynomial by the methods of Newton-Cotes; we just need to know this approximate value up to

$$O\left(\left(\frac{\mathrm{Log}\,\mathrm{Log}\,n}{\mathrm{Log}\,n}\right)^2\right).$$

Up to

$$O\left(\left(\frac{\mathrm{Log}\,\mathrm{Log}\,n}{\mathrm{Log}\,n}\right)^2\right),$$

we will show that: (i) an upper bound of (5.85) is $\frac{1}{6\,\mathrm{Log}\,n}$, and (ii) the approximate value of $\mathrm{Log}\,\Lambda$ is independent of the integer m (i.e., step length) used in the Newton-Cotes formulas, assuming the weights $(\alpha_q)_{q=0,1,\ldots,m}$ associated with m all positive. Indeed, if m is arbitrarily large, the estimate of the integral should be very good by these methods, ideally exact at the limit (m“ $=$ ” $+\infty$).

(i) We consider the decomposition of the integration interval as

$$\left(0,\frac{\pi}{3}\right] = \left(0,\frac{2\pi\lceil \mathrm{Log}\,n\rceil}{n}\right] \cup \left(\bigcup_{j=\lceil \mathrm{Log}\,n\rceil}^{\lfloor n/6\rfloor -1}\left[\frac{2\pi j}{n},\frac{2\pi(j+1)}{n}\right]\right) \cup \left[\frac{2\pi}{n}\left\lfloor\frac{n}{6}\right\rfloor,\frac{\pi}{3}\right]$$

and proceed by calculating the estimations of

$$\left|\frac{-1}{\pi}\int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}}\left(\mathrm{Log}\left(2\sin(x/2)\right) - \frac{-2}{n}\mathrm{Log}\left(2\,\sin\left(\frac{\pi j}{n}\right)\right)\right)\mathrm{d}x\right| \tag{5.86}$$

on the intervals $\mathcal{L}_j := \left[\frac{2\pi j}{n},\frac{2\pi(j+1)}{n}\right]$, $j = \lceil \mathrm{Log}\,n\rceil, \lceil \mathrm{Log}\,n\rceil+1,\ldots,\lfloor n/6\rfloor - 1$. On each such $\mathcal{L}_j$, the function $f(x)$ is approximated by its interpolation polynomial $P_m(x)$, where $m \geq 1$ is the number of subintervals forming an uniform partition of $\mathcal{L}_j$ given by

$$y_q = \frac{2\pi j}{n} + q\frac{2\pi}{n}\frac{1}{m}, \qquad q = 0,1,\ldots,m,$$

of step length $h := \frac{2\pi}{n\,m}$, and P_m the interpolating polynomial of degree m or less with

$$P_m(y_q) = f(y_q), \qquad \text{for } q = 0,1,\ldots,m.$$

The Newton-Cotes formulas

$$\int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} P_m(x)\,\mathrm{d}x \;=\; h\sum_{q=0}^{m}\alpha_q f(y_q)$$

provide approximate values of $\int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} f(x)\,\mathrm{d}x$, where the α_q are the weights obtained by integrating the Lagrange's interpolation polynomials. Steffensen [Sff] ([SB], p 127) showed that the approximation error may be expressed as follows

$$\int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} P_m(x)\,\mathrm{d}x - \int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} f(x)\,\mathrm{d}x = h^{p+1} \cdot K \cdot f^{(p+1)}(\xi), \qquad \xi \in \overset{o}{\mathcal{L}}_j,$$

where $p \geq 2$ is an integer related to m, and K a constant.

Using [SB], p. 128, and $m = 1$, the method being the "Trapezoidal rule", we have:

$$p = 2, \quad K = 1/12, \quad \alpha_0 = \alpha_1 = 1/2.$$

Then (5.86) is estimated by

$$\left| \frac{1}{2}\frac{2\pi}{n}\left[\frac{-1}{\pi}\,\mathrm{Log}\left(2\,\sin\Bigl(\frac{\pi j}{n}\Bigr)\right) + \frac{-1}{\pi}\,\mathrm{Log}\left(2\,\sin\Bigl(\frac{\pi(j+1)}{n}\Bigr)\right)\right] \right.$$
$$\left. - \frac{-2}{n}\,\mathrm{Log}\left(2\,\sin\Bigl(\frac{\pi j}{n}\Bigr)\right)\right|$$
$$= \frac{1}{n}\left|\mathrm{Log}\left(2\,\sin\Bigl(\frac{\pi j}{n}\Bigr)\right) - \mathrm{Log}\left(2\,\sin\Bigl(\frac{\pi(j+1)}{n}\Bigr)\right)\right|$$
$$= \frac{2\pi}{n^2}\left|\frac{\cos(\xi/2)}{2\sin(\xi/2)}\right| \leq \frac{1}{n}\,\frac{1}{\mathrm{Log}\,n} \qquad \text{for some} \quad \xi \in \overset{o}{\mathcal{L}}_j, \quad \text{for large} \quad n. \tag{5.87}$$

The (Steffensen's) approximation error "$h^3 \cdot (1/12) \cdot f^{(2)}(\xi)$" is

$$\frac{1}{\pi}\left(\frac{2\pi}{n}\right)^3 \frac{1}{12}\left|\frac{-1}{4\sin^2(\xi/2)}\right| \leq \frac{1}{6n}\,\frac{1}{(\mathrm{Log}\,n)^2}.$$

Summing up the contributions of all the intervals $\mathcal{L}_j$, we obtain the following upper bound of (5.85)

$$\left|\frac{-1}{\pi}\int_0^{(2\pi \mathrm{Log}\,n)/n} \mathrm{Log}\left(2\sin(x/2)\right)\mathrm{d}x\right| + \frac{1}{6\,\mathrm{Log}\,n}. \tag{5.88}$$

with global (Steffensen's) approximation error

$$\frac{1}{36}\,\frac{1}{(\mathrm{Log}\,n)^2} \text{ which is an } \quad O\left(\left(\frac{\mathrm{Log}\,\mathrm{Log}\,n}{\mathrm{Log}\,n}\right)^2\right).$$

By integrating by parts the integral in (5.88), for large n, it is easy to show that this integral is $= O\left(\frac{(\mathrm{Log}\, n)^2}{n}\right)$. We deduce the claim.

(ii) Let us show that the upper bound $1/(6\mathrm{Log}\, n)$ is independent of the integer m used, once assumed the positivity of the weights $(\alpha_q)_{q=0,1,\ldots,m}$. For $m \geq 1$ fixed, this is merely a consequence of the relation between the weights in the Newton-Cotes formulas. Indeed, we have $\sum_{q=0}^{m} \alpha_q = m$, and therefore

$$\left| \int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} P_m(x)\,\mathrm{d}x - hmf(y_0) \right| = h \left| \sum_{q=0}^{m} \alpha_q \big(f(y_q) - f(y_0)\big) \right| \leq h \left(\sum_{q=0}^{m} |\alpha_q| \right) \sup_{\xi \in \mathcal{L}_j} |f'(\xi)|.$$

Since $hm = \frac{2\pi}{n}$ and that the inequality $\sup_{\xi \in \mathcal{L}_j} |f'(\xi)| \leq |f'((2\pi \mathrm{Log}\, n)/n)|$ holds uniformly for all j, we deduce the same upper bound as in (5.87) for the Trapezoidal rule. Summing up the contributions over all the intervals $\mathcal{L}_j$, we obtain the same upper bound (5.88) of (5.85) as before.

As for the (Steffensen's) approximation errors, they make use of the successive derivatives of the function $f(x) = \mathrm{Log}\,(2\sin(x/2))$. We have:

$$f'(x) = \frac{\cos(x/2)}{2\sin(x/2)}, \qquad f''(x) = -\frac{1}{4\sin^2(x/2)}, \qquad f'''(x) = \frac{\cos(x/2)}{4\sin^3(x/2)} \cdots$$

Recursively, it is easy to show that the q-th derivative of $f(x)$, $q \geq 1$, is a rational function of the two quantities $\cos(x/2)$ and $\sin(x/2)$ with bounded numerator on the interval $(0, \pi/3]$, and a denominator which is $\sin^q(x/2)$. For the needs of majoration in the Newton-Cotes formulas over each interval of the collection $(\mathcal{L}_j)$, this denominator takes its smallest value at $\xi = (2\pi\lceil \mathrm{Log}\, n\rceil)/n$. Therefore, for large n, the (Steffensen's) approximation error "$h^{p+1} \cdot K \cdot f^{(p)}(\xi)$" on one interval $\mathcal{L}_j$ is

$$O\left(\left(\frac{2\pi}{nm}\right)^{p+1} \cdot\ K\ \cdot\ \frac{n^p}{(\pi\, \mathrm{Log}\, n)^p} \right) = O\left(\frac{1}{n(\mathrm{Log}\, n)^p} \right).$$

By summing up over the intervals $\mathcal{L}_j$, we obtain the global (Steffensen's) approximation error ($p \geq 2$)

$$O\left(\frac{1}{(\mathrm{Log}\, n)^p} \right) \quad \text{which is an} \quad O\left(\left(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n} \right)^2 \right).$$

□

5.3. Minoration of the house $\boxed{\theta_n^{-1}} = \theta_n^{-1}$, Schinzel-Zassenhaus Conjecture and proof of Theorem 1.8

Denote by $\mathrm{m}_h(n)$ the minimum of the houses of the algebraic integers of degree n which are not a root of unity (to avoid ambiguity with the logarithmic Mahler measure, usually denoted by m, we use m_h with the subscript h for "*houses*"). An algebraic integer α, of degree n, is said *extremal* if $\boxed{\alpha} = \mathrm{m}_h(n)$. An extremal algebraic integer is not necessarily a Perron number.

In 1965 Schinzel and Zassenhaus [SZ] conjectured that

$$\mathrm{m}_h(n) \geq 1 + \frac{c_1}{n} \tag{5.89}$$

for a constant $c_1 > 0$ (i.e.; independent of n). The first result in this direction they obtained is: when $\alpha \neq 0$ is an algebraic integer of degree $n \geq 2$ which is not a root of unity, then $\boxed{\alpha} > 1 + 4^{-(s+2)}$, where $2s$ is the number of nonreal conjugates of α.

For a nonreciprocal algebraic integer α of degree n, Cassels [Ca] obtained the inequality

$$\boxed{\alpha} > 1 + \frac{c_2}{n}, \quad \text{with } c_2 = 0.1; \tag{5.90}$$

Breusch [Br], independently and previously, showed that $c_2 = \mathrm{Log}\,(1.179) = 0.165\ldots$ could be taken; Schinzel [Sc1] noticed that the constant $c_2 = 0.2$ could also be taken. Finally, from Smyth's Theorem [Sy1] which claims that $\mathrm{M}(\alpha) \geq \Theta$, Smyth improved the minoration in 1971, with: $c_2 = \mathrm{Log}\,\Theta = 0.2811\ldots$ On the other hand, Boyd [Bo4] has shown that c_2 cannot exceed $\frac{3}{2}\mathrm{Log}\,\Theta = 0.4217\ldots$ In 1997 Dubickas [Ds3] showed that $c_2 = \omega - \epsilon$ with $\omega = 0.3096\ldots$ the smallest root of an equation in the interval $(\mathrm{Log}\,\Theta, +\infty)$, with $\epsilon > 0$, $n_0(\epsilon)$ an effective constant, and for all $n > n_0(\epsilon)$.

In 1857, for nonzero algebraic integers α, Kronecker showed that $\boxed{\alpha} = 1$ if and only if α is a root of unity. The sufficient condition was weakened by Blansky and Montgomery [ByM] who showed that α, with $\deg \alpha = n$, is a root of unity provided

$$\boxed{\alpha} \leq 1 + \frac{1}{30n^2 \mathrm{Log}\,(6n)}.$$

Dobrowolsky [Do1] sharpened this condition by: if

$$\boxed{\alpha} < 1 + \frac{\mathrm{Log}\, n}{6n^2},$$

then α is a root of unity.

For algebraic integers α, of degree n, Dobrowolski [Do2] showed the inequality (1.5) in 1979. Since $\mathrm{M}(\alpha) \leq \overline{|\alpha|}^n$, (1.5) implies

$$\overline{|\alpha|} > 1 + (1-\epsilon)\left(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\right)^3 \frac{1}{n}, \qquad n > n_3(\epsilon). \tag{5.91}$$

The constant coefficient $1-\epsilon$ in (5.91) has been improved successively by Cantor and Strauss [CS], and Rausch [Ra], to $2-\epsilon$, Louboutin [Lt], and Meyer [Me], to $\frac{9}{4}-\epsilon$, Voutier [V] to $\frac{1}{4}$, Dubickas [Ds1] to $\frac{64}{\pi^2}-\epsilon$. Waldschmidt [W2] showed that this constant $1-\epsilon$ could be replaced by $1/250$, with a proof of transcendence using an interpolation determinant.

Dobrowolski's result is effective [Do2]: he showed that the inequality (5.91) is valid for all $n \geq 3$ with $1-\epsilon$ replaced by $1/1200$. In 1991, Matveev [Mv] obtained a minoration which revealed to be better for the small degrees: if α is an algebraic integer, not a root of unity, with $\deg(\alpha) = n \geq 2$, then

$$\overline{|\alpha|} \geq \exp\frac{\operatorname{Log}(n+\frac{1}{2})}{n^2}. \tag{5.92}$$

In 2007, Rhin and Wu [RW] verified the conjecture of Schinzel-Zassenhaus up to $n = 28$ and improved Matveev's inequality (5.92) to the following: if α is an algebraic integer, not a root of unity, with $\deg(\alpha) = n$, then for $4 \leq n \leq 12$,

$$\overline{|\alpha|} = \exp\frac{3\operatorname{Log}(\frac{n}{3})}{n^2}, \tag{5.93}$$

and, for $n \geq 13$,

$$\overline{|\alpha|} = \exp\frac{3\operatorname{Log}(\frac{n}{2})}{n^2}. \tag{5.94}$$

The lower bound (5.92) is better than Voutier's bound

$$\mathrm{m}_h(n) \geq \left(1 + \frac{1}{4}\left(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\right)^3\right)^{1/n}. \tag{5.95}$$

for $n \leq 1434$, and (5.94) is better than (5.95) for $n \leq 6380$. The two numerical values 1434 and 6380 can be improved if Dubickas's constant [Ds1] $\frac{64}{\pi^2}-\epsilon$ replaces $1/4$ in Voutier's bound (5.95).

For reciprocal nonzero algebraic integers α, of degree $n \geq 2$, which are not roots of unity, the lower bounds of $\overline{|\alpha|}$ deduced from the Mahler measure $\mathrm{M}(\alpha)$ have a factor 2 in the coefficient since $\mathrm{M}(\alpha) \leq \overline{|\alpha|}^{n/2}$ holds, instead of the general inequality: $\mathrm{M}(\alpha) \leq \overline{|\alpha|}^n$. Namely, Dobrowolski's lower bound becomes

$$\overline{|\alpha|} > 1 + (2-\epsilon)\left(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\right)^3 \frac{1}{n}, \qquad n > n_4(\epsilon), \tag{5.96}$$

the constant $2-\epsilon$ could be replaced by $\frac{9}{2}-\epsilon$ from Louboutin [Lt], or better by Dubickas's constant. Actually, the constant $\frac{64}{\pi^2}-\epsilon$ obtained by Dubickas [Ds1] is indifferently valid for reciprocal and nonreciprocal algebraic integers.

For totally real algebraic integers α, of degree n, either the house $\overline{|\alpha|}$ lies in $(1,2]$, in which case there exists a root of unity ω such that $\alpha = \omega + 1/\omega$ from a theorem of Kronecker (1857) [Sy3], or, for $\overline{|\alpha|} > 2$ and $\alpha \neq 2\cos(\pi r)$ for some $r \in \mathbb{Q}$, the following lower bound were obtained by Dubickas [Ds2]

$$\overline{|\alpha|} > 2 + 3.8 \left(\frac{(\operatorname{Log}\operatorname{Log} n)^3}{(\operatorname{Log} n)^4} \right) \frac{1}{n}, \qquad n > n_4(\epsilon),$$

where the constant 3.8 can be replaced by 4.6 (Dubickas [Ds3]).

In Boyd [Bo4] the following conjectures are formulated:

Conjecture (Lind - Boyd). *The smallest Perron number of degree $n \geq 2$ has minimal polynomial:*

$$\begin{array}{rlcr} X^n - X - 1 & \text{if n} & \not\equiv & 3, 5 \bmod 6, \\ (X^{n+2} - X^4 - 1)/(X^2 - X + 1) & \text{if n} & \equiv & 3 \bmod 6, \\ (X^{n+2} - X^2 - 1)/(X^2 - X + 1) & \text{if n} & \equiv & 5 \bmod 6. \end{array}$$

Conjecture (Boyd).

(i) *If α is extremal, then it is always nonreciprocal.*

(ii) *If $n = 3k$, then the extremal α has minimal polynomial*

$$X^{3k} + X^{2k} - 1, \quad \text{or} \quad X^{3k} - X^{2k} - 1.$$

(iii) *The extremal α of degree n has asymptotically a number of conjugates $\alpha^{(i)}$ outside the closed unit disc equal to*

$$\cong \frac{2}{3} n, \qquad n \to \infty.$$

Proposition 5.1. *Assuming Lind-Boyd's conjecture and Boyd's conjecture true, the Perron number θ_n^{-1}, $n \geq 2$, is not extremal, except if $n = 2, 3$.*

Proof. First, by Proposition 3.7 (i), the number of conjugates of θ_n^{-1}, $n \geq 2$, lying outside the closed unit disc, is equal to $1 + 2\lfloor n/6 \rfloor$. Asymptotically it is equal to $n/3$ and not to $2n/3$ as expected from Boyd's conjecture. Second, for the small values of n, the minimal polynomials of θ_n^{-1}, $n \geq 2$, are $X^n - X^{n-1} - 1$ if $n \not\equiv 5 \bmod 6$, and $(X^n - X^{n-1} - 1)/(X^2 - X + 1)$ if $n \equiv 5 \bmod 6$ (Section 2). Then, from Lind-Boyd's conjecture, the only cases of extremality are reached for $n = 2, 3$. □

Let $n \neq 2, 3$. How far is θ_n^{-1} from extremality?

P r o o f o f T h e o r e m 1.8. Let $n \geq 2$. From Proposition 3.1 and Lemma 3.2, the Perron number θ_n^{-1} can be expressed as: $\theta_n^{-1} = \mathrm{D}(\theta_n^{-1}) + \mathrm{tl}(\theta_n^{-1})$ with $\mathrm{tl}(\theta_n^{-1}) = \mathrm{tl}(\theta_n)$, and

$$\mathrm{D}(\theta_n^{-1}) = 1 + \frac{\operatorname{Log} n}{n}(1 - \lambda_n) = 1 + \frac{\operatorname{Log} n}{n}\left(1 - \frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\left(\frac{1}{1 + \frac{1}{\operatorname{Log} n}}\right)\right)$$

so that, for sufficiently large n,

$$\theta_n^{-1} > 1 + \frac{\operatorname{Log} n(1 - \frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n})}{n}.$$

From this strict inequality we deduce (1.10); the Schinzel-Zassenhaus lower bound (1.9) is obtained by computing the values of

$$n(\theta_n^{-1} - 1) \quad \text{for} \quad 2 \leq n \leq 200. \qquad \square$$

5.4. Proofs of Theorem 1.3, Corollary 1.4 and Corollary 1.5

P r o o f o f T h e o r e m 1.3. We will use the method introduced by Smyth and Boyd in [Bo3], Appendix 2, to deduce the expansions (1.4). The case "n odd" is merely a consequence of Appendix 2 in [Bo3], since

$$\mathrm{M}(G_n) = \mathrm{M}(-1 + X + X^n) = \mathrm{M}(-1 - X - X^n) = \mathrm{M}(1 + X + X^n),$$

in this case. Let us now assume that n is even. Let $z = e^{it}$ for $-\pi < t < \pi$. then

$$\operatorname{Log}(1 + z - z^n) = \begin{cases} \operatorname{Log}(1+z) - \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{z^n}{1+z}\right)^m & \text{if } |t| < \frac{2\pi}{3}, \\ \operatorname{Log}(-z^n) - \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{1+z}{z^n}\right)^m & \text{if } |t| > \frac{2\pi}{3}. \end{cases}$$

We have

$$\begin{aligned} \operatorname{Log}\mathrm{M}(-1 + z + z^n) &= \operatorname{Log}\mathrm{M}(1 - z - z^n) = \operatorname{Log}\mathrm{M}(1 + z - z^n) \\ &= \frac{1}{\pi}\int_0^\pi \operatorname{Log}|1 + e^{it} - e^{int}|\, dt \\ &= \frac{1}{\pi}\int_0^\pi \Re\left(\operatorname{Log}(1 + e^{it} - e^{int})\right)\, dt \end{aligned}$$

and, by Jensen's formula and in Boyd [Bo3, Appendix 1],

$$\begin{aligned} \operatorname{Log}\mathrm{M}(1 + z_1 - z_2) &= \operatorname{Log}\mathrm{M}(\max\{|1 + z_1|, 1\}) \\ &= \frac{1}{\pi}\int_0^\pi \operatorname{Log}^+|1 + e^{it}|\, dt = \frac{1}{\pi}\int_0^{2\pi/3} \operatorname{Log}|1 + e^{it}|\, dt \\ &= 1\pi\int_0^{2\pi/3} \Re\left(\operatorname{Log}(1 + e^{it})\right)\, dt. \end{aligned}$$

Then the difference $\Delta := \mathrm{Log}\,\mathrm{M}(1+z-z^n) - \mathrm{Log}\,\mathrm{M}(1+z_1-z_2)$ is

$$\Delta \;=\; \frac{-1}{\pi}\,\Re\left[\sum_{m=1}^{\infty}\frac{1}{m}\big(c_1(m)+c_2(m)\big)\right]$$

with

$$c_1(m) = \int_0^{2\pi/3}(1+e^{it})^{-m}e^{inmt}\ \mathrm{d}t, \qquad c_2(m) = \int_{2\pi/3}^{\pi}(1+e^{it})^{m}e^{-inmt}\ \mathrm{d}t.$$

Let $\omega = e^{i\pi/3}$. Using $1+\omega^2=\omega$, we now integrate by parts three times. Then

$$c_1(m) = \frac{\omega^{m(2n-1)}-2^{-m}}{inm} + \frac{\omega^{-m(1-2n)+1}-2^{-m-1}}{in(nm+1)} + O\Big(\frac{1}{n^3m}\Big)$$

and

$$c_2(m) = \frac{\omega^{m(1-2n)}}{inm} + \frac{\omega^{m(1-2n)+1}}{in(nm-1)} + O\Big(\frac{1}{n^3m}\Big).$$

We deduce

$$\Re\big(c_1(m)+c_2(m)\big) = \frac{1}{n^2m}\Re\left[(-i)\omega\, 2\cos\big(m(1-2n)\pi/3\big)\right] + O\Big(\frac{1}{n^3m}\Big)$$

and

$$\Delta = -\frac{\sqrt{3}}{\pi n^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\cos\big(m(1-2n)\pi/3\big) + O\big(n^{-5}\big).$$

Since $\exp(\Delta) = 1 - \frac{\sqrt{3}}{\pi n^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\cos\big(m(1-2n)\pi/3\big) + O(n^{-3})$, the coefficient $s(n)$ is

$$s(n) = -\frac{\sqrt{3}}{\pi}\sum_{m=1}^{\infty}\frac{1}{m^2}\cos(m(1-2n)\pi/3).$$

Obviously, $s(n+6) = s(n)$ for all (even) integer $n \geq 2$, and $s\,(4) = s\,(6)$. Denote by

$$\Phi(z,s,a) = \sum_{m=0}^{\infty}\frac{z^m}{(m+a)^s}$$

the Lerch transcendent function [Al] [Ln], and by $\zeta(2) = \sum_{m\geq 1} m^{-2} = \pi^2/6$ the value of the Riemann zeta function at $z=2$. Let us compute $s\,(4)$ and $s\,(8)$. For $n=4$,

$$s(4) = -\frac{\sqrt{3}}{\pi}\sum_{m=1}^{\infty}\frac{1}{m^2}\cos(m\pi/3)$$

The sum $\sum_{m=1}^{\infty} \frac{1}{m^2} \cos(m\pi/3)$ is equal to

$$\sum_{\substack{m=1\\ m\in 6\mathbb{Z}}}^{\infty} \frac{1}{m^2} - \sum_{\substack{m=1\\ m\in 6\mathbb{Z}+3}}^{\infty} \frac{1}{m^2} + \frac{1}{2}\left[\sum_{\substack{m=1\\ m\in 6\mathbb{Z}+1}}^{\infty} \frac{1}{m^2} + \sum_{\substack{m=1\\ m\in 6\mathbb{Z}+5}}^{\infty} \frac{1}{m^2}\right] - \frac{1}{2}\left[\sum_{\substack{m=1\\ m\in 6\mathbb{Z}+4}}^{\infty} \frac{1}{m^2} + \sum_{\substack{m=1\\ m\in 6\mathbb{Z}+2}}^{\infty} \frac{1}{m^2}\right]$$

$$= \frac{1}{36}\left(\zeta(2) - \Phi\left(1,2,\frac{1}{2}\right) + \frac{1}{2}\left[\Phi\left(1,2,\frac{1}{6}\right) + \Phi\left(1,2,\frac{5}{6}\right)\right]\right.$$
$$\left. - \frac{1}{2}\left[\Phi\left(1,2,\frac{1}{3}\right) + \Phi\left(1,2,\frac{2}{3}\right)\right]\right).$$

The decomposition

$$\sum_{m\geq 1} m^{-2} = \pi^2/6 = \sum_{\substack{m=1\\ m \text{ even}}}^{\infty} m^{-2} + \sum_{\substack{m=1\\ m \text{ odd}}}^{\infty} m^{-2}$$

implies $\Phi(1,2,\frac{1}{2}) = 3\zeta(2) = \pi^2/2$. Then the decomposition

$$\frac{\pi^2}{8} = \sum_{\substack{m=1\\ m \text{ odd}}}^{\infty} \frac{1}{m^2} = \sum_{\substack{q=0\\ q\in 3\mathbb{Z}+1}}^{\infty} \frac{1}{(2q+1)^2} + \sum_{\substack{q=0\\ q\in 3\mathbb{Z}}}^{\infty} \frac{1}{(2q+1)^2} + \sum_{\substack{q=0\\ q\in 3\mathbb{Z}+2}}^{\infty} \frac{1}{(2q+1)^2}$$

implies

$$\Phi\Big(1,2,\frac{1}{6}\Big) + \Phi\Big(1,2,\frac{5}{6}\Big) = 36\left(\frac{\pi^2}{8} - \frac{1}{36}\Phi\Big(1,2,\frac{1}{2}\Big)\right) = 4\pi^2.$$

Finally, the decomposition

$$\frac{\pi^2}{6} = \sum_{m\geq 1} \frac{1}{m^2} = \sum_{q=0}^{\infty} \frac{1}{(3q+3)^2} + \sum_{q=0}^{\infty} \frac{1}{(3q+2)^2} + \sum_{q=0}^{\infty} \frac{1}{(3q+1)^2}$$

implies

$$\Phi\Big(1,2,\frac{1}{3}\Big) + \Phi\Big(1,2,\frac{2}{3}\Big) = 9\Big(\frac{\pi^2}{6} - \frac{1}{9}\zeta(2)\Big) = \frac{4\pi^2}{3}.$$

Summing up the contributions, we obtain $s(4) = -\sqrt{3}\pi/36$.

For $n = 8$,

$$s\,(8) = -\frac{\sqrt{3}}{\pi}\sum_{m=1}^{\infty} \frac{1}{m^2}\cos(m\pi) = -\frac{\sqrt{3}}{\pi}\left(\sum_{\substack{m=1\\ m \text{ even}}}^{\infty} m^{-2} - \sum_{\substack{m=1\\ m \text{ odd}}}^{\infty} m^{-2}\right)$$

$$= -\frac{\sqrt{3}}{4\pi}\Big(\zeta(2) - \Phi\Big(1,2,\frac{1}{2}\Big)\Big) = \frac{\sqrt{3}\pi}{12}. \qquad \square$$

P r o o f o f C o r o l l a r y 1.4. The expansion of $\mathrm{M}(G_n)$ in Theorem 1.3, is such that the coefficient $s(n)$ is bounded by 1 in modulus for all integers $n \geq 2$, that is uniformly in n. We deduce the existence of some integer n_0 such that $\mathrm{M}(G_n) \geq 1.35$ for all $n \geq n_0$ (take $n_0 = 51$ for instance, which corresponds to Figure 1) since $\lim_{m\to+\infty} \mathrm{M}(G_m) = 1.38\cdots > 1.35$. The claim of the uniqueness, for $n = 5$, comes from the checking of the values $\mathrm{M}(G_n)$, for all the integers n between 3 and n_0, by hand and on the computer. The similar claim can also be obtained using (1.3) with a larger value of n_0 (which underestimates the speed of convergence in some sense). $\square$

P r o o f o f C o r o l l a r y 1.5. From Theorem 1.3 the function $s(n)$ takes a negative sign in the two cases:

(i) $n \geq 2$ odd and $n \equiv 5 \pmod 6$; in this case, 3 divides $n+1$ and $\mathrm{M}(-1+z+z^n) = \mathrm{M}(1+z+z^n) < \Lambda$ as soon as n is large enough,

(ii) $n \geq 2$ even and $n \equiv 0$ or $4 \pmod 6$; in this case, 3 does not divide $n+1$ and $\mathrm{M}(-1+z+z^n) = \mathrm{M}(-1-z+z^n) < \Lambda$ as soon as n is large enough.

The conditions are necessary and sufficient since the coefficients $s(n)$ take positive signs in the other respective cases. In the statement of Smyth's conjecture the trinomials G_n only refer to the cases $k = 1$ of the first and third items. To finish up the proof of Corollary 1.5 we observe that the coefficient $s(n)$ is uniformly bounded by 1 in modulus for all integers $n \geq 4$. We deduce the existence of some integer n_0 such that $\mathrm{M}(G_n) \geq 1.35$ for all $n \geq n_0$ (take $n_0 = 51$ for instance, as in Figure 1 on the 88), and compute the values of $\mathrm{M}(G_n)$ for $4 \leq n \leq 51$ for comparison with Λ. $\square$

6. Erdős-Turán-Amoroso-Mignotte Theory and distribution of conjugates

The purpose of the Erdős-Turán-Amoroso-Mignotte (ETAM) theory is to study the angular regularity of the geometry of the roots of a given polynomial, in sectors all centered at the origin, by establishing best discrepancy functions (Erdős-Turán [ET], Ganelius [G], Amoroso and Mignotte [AM], Mignotte [Mt1] [Mt2]). The ETAM theory is one of the basic ingredient of recent limit equidistribution theorems of conjugates on the unit circle: in Bilu [Bu], Petsche [Pe] and Pritsker [Pr].

6.1. Proof of Theorem 1.9

Let $n \geq n_0 = 18$ and $1 \leq j \leq \lfloor \frac{n-1}{4} \rfloor$.

(i) First, let us prove (1.12). Let us decompose the summation in (1.12) as

$$\sum_{j=1}^{\lfloor \frac{n-1}{4} \rfloor} \arg(z_{j,n}) = \sum_{j=1}^{\lfloor v_n \rfloor} \arg(z_{j,n}) + \sum_{j=\lceil v_n \rceil}^{\lfloor \frac{n-1}{4} \rfloor} \arg(z_{j,n}).$$

Using (3.30), (3.61) and (3.62), the roots $z_{j,n}$ of G_n lying in the main Sector "$\frac{\pi}{2} > \arg z > 2\pi \frac{\operatorname{Log} n}{n}$" have the following arguments:

$$\arg(z_{j,n}) = 2\pi \Big(\frac{j}{n} + \Re\Big) \quad \text{with} \quad \Re = -\frac{1}{2\pi n} \left[\frac{1 - \cos\big(\frac{2\pi j}{n}\big)}{\sin\big(\frac{2\pi j}{n}\big)} \operatorname{Log} \left(2 \sin\Big(\frac{\pi j}{n}\Big)\right)\right]$$

with

$$\operatorname{tl}\big(\arg(z_{j,n})\big) = +\frac{1}{n} O \left(\left(\frac{\operatorname{Log}\operatorname{Log} n}{\operatorname{Log} n}\right)^2\right)$$

with the constant 2π involved in the Big O. Second, using (3.45), (3.46) and (3.47), the roots which lie in the "Bump Sector", in the first quadrant, are such that

$$\left| \frac{1}{\lfloor \frac{n-1}{4} \rfloor} \sum_{j=1}^{\lfloor v_n \rfloor} \arg(z_{j,n}) \right| = \left| \frac{2\pi(1 - \frac{1}{\operatorname{Log} n})}{n \lfloor \frac{n-1}{4} \rfloor} \sum_{j=1}^{\lfloor v_n \rfloor} j + \cdots \right| = O\left(\frac{(\operatorname{Log} n)^{2+\epsilon}}{n^2}\right).$$

Then

$$\lim_{n \to +\infty} \frac{1}{\lfloor \frac{n-1}{4} \rfloor} \sum_{j=1}^{\lfloor \frac{n-1}{4} \rfloor} \arg(z_{j,n}) = \lim_{n \to \infty} \frac{2\pi}{n \lfloor \frac{n-1}{4} \rfloor} \sum_{j=\lceil \operatorname{Log} n \rceil}^{\lfloor \frac{n-1}{4} \rfloor} j = \frac{\pi}{4}.$$

(ii) Now, to prove (1.13), let us observe that the Riemann-Stieljes sum

$$S(a,n) := -\sum_{j=1}^{\lfloor (n-1)/4 \rfloor} \frac{1}{n} \tan\Big(\frac{\pi j}{n}\Big) \operatorname{Log}\left(2 \sin\Big(\frac{\pi j}{n}\Big)\right)$$

$$= \frac{-1}{2\pi} \sum_{j=1}^{\lfloor (n-1)/4 \rfloor} (x_j - x_{j-1}) a(x_j)$$

with $x_j = \frac{2\pi j}{n}$ and $a(x) := \tan(\frac{x}{2}) \operatorname{Log}\big(2\sin(\frac{x}{2})\big)$ converges to the limit

$$\lim_{n \to \infty} S(a,n) = \frac{-1}{2\pi} \int_0^{\pi/2} a(x)\,\mathrm{d}x \;=\; 0.07994\ldots$$

To estimate the second-order terms, we proceed as in Subsection 5.2 to give an upper bound of

$$\left| \frac{-1}{2\pi} \int_0^{\pi/2} \tan\Big(\frac{x}{2}\Big) \mathrm{Log}\left(2\,\sin\Big(\frac{x}{2}\Big)\right) \mathrm{d}x - \sum_{j=\lceil \mathrm{Log}\, n\rceil}^{\lfloor (n-1)/4 \rfloor} \frac{-1}{n}\, \tan\Big(\frac{\pi j}{n}\Big) \mathrm{Log}\left(2\,\sin\Big(\frac{\pi j}{n}\Big)\right) \right|. \tag{6.97}$$

with Newton-Cotes methods applied to the integral. We apply the Trapezoidal rule with the error terms in the tail $\mathrm{tl}\big(\sum \arg(z_{j,n})\big)$ controlled by Steffensen's approximation error and $\frac{1}{n} O\Big(\big(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\big)^2\Big)$. Let us prove that these terms are here negligible. The derivative

$$a'(x) = \frac{1}{2}\left(1 + \frac{\mathrm{Log}\,(2\sin(x/2))}{\cos^2(x/2)}\right) \quad \text{satisfies} \quad \sup_{y \in [\frac{2\pi \mathrm{Log}\, n}{n}, \frac{\pi}{2}]} |a'(y)| \le \mathrm{Log}\, n.$$

Hence an upper bound of (6.97) is

$$\frac{n}{4}\frac{1}{2n}\frac{\pi}{n} \mathrm{Log}\, n + \left| \frac{1}{2\pi} \int_0^{(2\pi \mathrm{Log}\, n)/n} \tan(x/2) \mathrm{Log}\,\big(2\sin(x/2)\big)\, \mathrm{d}x \right|$$

which is negligible, as for (5.88) (by integration by parts for the integral). Therefore,

$$\frac{1}{\lfloor \frac{n-1}{4} \rfloor} \sum_{j=1}^{\lfloor \frac{n-1}{4} \rfloor} \arg(z_{j,n}) - \frac{\pi}{4} = O\left(\frac{1}{n}\left(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\right)^2\right)$$

with the constant 2π in the Big O. We deduce the claim.

6.2. An upper bound for Mignotte's discrepancy functions

Let $\kappa := \sum_0^\infty \frac{(-1)^{m-1}}{(2m+1)^2} = 0.916\ldots$ be the constant of Catalan. Denote

$$C := \int_0^{2\pi} \left(\int_0^{2\pi} \mathrm{Log}^+ |e^{it} + e^{is} - 1|\, \frac{\mathrm{d}t}{2\pi} \right) \frac{\mathrm{d}s}{2\pi} = 0.42627\ldots$$

Mignotte [Mt1] [Mt2] and Amoroso and Mignotte [AM] have improved the discrepancy function previously given by Erdős and Turán in [ET]. Let us recall Mignotte's discrepancy function for a polynomial R in $\mathbb{Z}[X]$ [Mt2]: the "radial" operator defined as ${}^{(r)} : \mathbb{Z}[X] \to \mathbb{R}[X]$,

$$R(X) = a_0 \prod_{j=1}^{m} (X - \alpha_j) \;\to\; R^{(r)}(X) = \prod_{j=1}^{m} \left(X - \frac{\alpha_j}{|\alpha_j|} \right),$$

transforms the polynomial $R(X)$ given in its factored form ($m \ge 1$ being its degree, $a_0 \neq 0$ its leading coefficient and (α_i) its roots in $\mathbb{C}$) to another polynomial $R^{(r)}(X)$, which is monic, of the same degree, and has all its roots on $|z| = 1$. Mignotte's discrepancy function of the polynomial R is defined by

$$\widetilde{h}(R) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Log}^+ |R^{(r)}(e^{i\theta})| \,\mathrm{d}\theta \tag{6.98}$$

and satisfies the inequality, for all $0 \leq a < b \leq 2\pi$,

$$\left| \frac{1}{m} \times \left(\sum_{\alpha_i,\, a < \arg(\alpha_i) < b} 1 \right) - \frac{b-a}{2\pi} \right|^2 \leq \frac{2\pi}{\kappa} \times \frac{\widetilde{h}(R)}{m}. \tag{6.99}$$

Theorem 6.1.

$$\limsup_{n\to\infty} \widetilde{h}(G_n) \ \leq \ C. \tag{6.100}$$

P r o o f. On one hand, from Erdős-Turán [ET, p 112, Schur's remark], we deduce the following inequality for the moduli:

$$\left|G_n^{(r)}(e^{i\theta})\right| \leq \left|G_n(e^{i\theta})\right| \quad \text{for all } \ n \geq 2 \quad \text{on } \ [0, 2\pi]. \tag{6.101}$$

On the other hand,

$$\frac{1}{2\pi} \int_0^{2\pi} \mathrm{Log}^+ \left|G_n(e^{i\theta})\right| \mathrm{d}\theta \ = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Log}^+ \left|e^{int} + e^{it} - 1\right| \,\mathrm{d}t.$$

But Lemma 1 in Boyd [Bo2] implies that the following limit holds

$$\lim_{n\to+\infty} \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Log}^+ |e^{int} + e^{it} - 1| \,\mathrm{d}t = \int_0^{2\pi} \left(\int_0^{2\pi} \mathrm{Log}^+ |e^{is} + e^{it} - 1| \frac{\mathrm{d}t}{2\pi} \right) \frac{\mathrm{d}s}{2\pi} \tag{6.102}$$

using bivariate Mahler measures. We deduce (6.100) from the formulas (6.98), (6.101) and (6.102). □

As a consequence, we deduce from (6.99) the following inequalities relative to the trinomials G_n: for all $\epsilon > 0$, for all $0 \leq a < b \leq 2\pi$, there exists n_0 such that

$$\left| \frac{1}{n} \times \left(\sum_{\alpha_i \text{root of } G_n,\, a < \arg(\alpha_i) < b} 1 \right) - \frac{b-a}{2\pi} \right|^2 \leq \frac{2\pi(C+\epsilon)}{\kappa} \times \frac{1}{n} \quad \text{for all } n \geq n_0. \tag{6.103}$$

The proof of Theorem 6.1 relies upon the limit properties of bivariate Mahler measures in the method developped by Smyth and Boyd [Bo2]. Comparatively, the method of asymptotic expansions of the roots $\theta_n, z_{j,n}$, of G_n can also be applied directly to obtain the analogue of (6.100) with a constant $C' \geq C$ instead of C: it suffices to deduce the asymptotic expansions of the complex numbers $z_{j,n}/|z_{j,n}|$ of modulus 1 from those of $z_{j,n}$, resp.$|z_{j,n}|$, i.e.; from Proposition 3.4 and Proposition 3.5, and insert them into the discrepancy functions $\widetilde{h}(G_n)$ of Mignotte (6.98). Though this method works, the upper bound of the limsup in (6.100) is less good. We leave the details of the calculations to the reader.

6.3. Limit equidistribution of conjugates on the unit circle

Let $h : \overline{\mathbb{Q}} \to [0, +\infty)$ denote the logarithmic Weil height (the notation h is traditional and presents no ambiguity with Mignotte's discrepancy function $\widetilde{h}$). We denote by $\mathbb{C}^{\times}$, resp. $\mathbb{Q}^{\times}$, the multiplicative group of nonzero elements of $\mathbb{C}$, resp. $\mathbb{Q}$. The unit Dirac measure supported at $\omega \in \mathbb{C}$ is denoted by δ_{ω} and $\mu_{\mathbb{T}}$ denotes the Haar measure (unit Borel measure), invariant by rotation, that is supported on the unit circle $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$, compact subgroup of $\mathbb{C}^{\times}$. Given a point $\alpha \in \overline{\mathbb{Q}}^{\times}$, of degree $d = [\mathbb{Q}(\alpha) : \mathbb{Q}]$, we define the unit Borel measure (probability)

$$\mu_{\alpha} = \frac{1}{d} \sum_{\sigma} \delta_{\sigma(\alpha)}$$

on $\mathbb{C}^{\times}$, the sum being taken over all d embeddings $\sigma : \mathbb{Q}(\alpha) \to \mathbb{C}$. A sequence $\{\alpha_k\}$ of points in $\overline{\mathbb{Q}}^{\times}$ is said to be *strict* if any proper algebraic subgroup of $\overline{\mathbb{Q}}^{\times}$ contains α_k for only finitely many values of k. Theorem 6.2 below is stated as a theorem, but is merely a corollary of Theorem 1.1 in Bilu [Bu], or Theorem 1.2 in Petsche [Pe].

Theorem 6.2. *Let $\{\theta_n^{-1} \mid n = 2, 3, 4, \ldots\}$ be the infinite sequence of Perron numbers in $\mathbb{C}^{\times}$ which are the dominant roots of the trinomials G_n^*. Then*

$$\mu_{\theta_n^{-1}} \to \mu_{\mathbb{T}}, \quad n \to +\infty, \quad \textit{weakly}, \tag{6.104}$$

or equivalently,

$$\mu_{\theta_n} \to \mu_{\mathbb{T}}, \quad n \to +\infty, \quad \textit{weakly}, \tag{6.105}$$

i.e., for all bounded, continuous functions $f : \mathbb{C}^{\times} \to \mathbb{C}$,

$$\int f d\mu_{\theta_n^{-1}} \to \int f d\mu_{\mathbb{T}}, \quad n \to +\infty. \tag{6.106}$$

P r o o f. The roots of the trinomials $G_n, n \geq n_0 = 18$, all belong to a fixed compact which contains the unit circle, which is the compact annulus

$$\left\{ z \in \mathbb{C} \mid 1 - \frac{2\mathrm{Log}\, n_0}{n_0} \leq |z| \leq 1 + \frac{2\mathrm{Log}\, 2}{n_0} \right\}.$$

The supports of $\mu_{\theta_n^{-1}}$ all tend to the unit circle if n tends to infinity, by Lemma 2.1. Let us show that the sequence is strict: by Proposition 2.3, the degree $\deg(\theta_n)$ is respectively equal to n if $n \not\equiv 5 \bmod 6$, and to $n-2$ if $n \equiv 5 \bmod 6$. Hence, $\lim \deg(\theta_n) = +\infty$. The Weil heights are respectively $h(\theta_n) = \frac{1}{n} \times \operatorname{Log} \mathrm{M}(G_n)$ if $n \not\equiv 5 \bmod 6$, and $h(\theta_n) = \frac{1}{n-2} \times \operatorname{Log} \mathrm{M}(G_n(X)/(X^2 - X + 1))$ if $n \equiv 5 \bmod 6$, and can be written (1.14) asymptotically. We have $\lim h(\theta_n) = 0$. By Theorem 6.1, Mignotte's discrepancy functions are uniformly bounded. These conditions ensure that Bilu's Theorem [Bu] can be applied. $\square$

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ON HAUSDORFF DIMENSIONS RELATED TO SETS WITH GIVEN ASYMPTOTIC AND GAP DENSITIES

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Dedicated to the memory of Professor Pierre Liardet

ABSTRACT. For a set A of positive integers $a_1 < a_2 < \cdots$, let $\underline{d}(A), \overline{d}(A)$ denote its lower and upper asymptotic densities. The gap density is defined as $\lambda(A) = \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}$. The paper investigates the class $\mathcal{G}(\alpha,\beta,\gamma)$ of all sets A with $\underline{d}(A) = \alpha$, $\overline{d}(A) = \beta$ and $\lambda(A) = \gamma$ for given α,β,γ with $0 \le \alpha \le \beta \le 1 \le \gamma$ and $\alpha\gamma \le \beta$. Using the classical dyadic mapping $\varrho(A) = \sum_{n=1}^{\infty} \frac{\chi_A(n)}{2^n}$, where χ_A is the characteristic function of A, the main result of the paper states that the ϱ-image set $\varrho\mathcal{G}(\alpha,\beta,\gamma)$ has the Hausdorff dimension

$$\dim \varrho\mathcal{G}(\alpha,\beta,\gamma) = \min\Big\{\delta(\alpha), \delta(\beta), \frac{1}{\gamma} \max_{\sigma\in[\alpha\gamma,\beta]} \delta(\sigma)\Big\},$$

where δ is the entropy function

$$\delta(x) = -x\log_2 x - (1-x)\log_2(1-x)\,.$$

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1. Introduction

Let $\mathcal{P}$ and $\mathcal{P}_\infty$, respectively, be the classes of all sets of positive integers and of all infinite sets of positive integers. Each $A \in \mathcal{P}_\infty$ can be mapped onto a real number $\varrho(A) \in (0,1]$ via the map

$$\varrho(A) = \sum_{n=1}^{\infty} \frac{\chi_A(n)}{2^n}\,,$$

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χ_A being the characteristic function of A. The mapping is a bijection of $\mathcal{P}_\infty$ onto $(0,1]$. Having a class $\mathcal{A} \subset \mathcal{P}_\infty$, we will denote by

$$\varrho\mathcal{A} = \{\varrho(A) \mid A \in \mathcal{A}\}$$

the set of all images of elements of $\mathcal{A}$.

By log we will always mean the *binary logarithm.* Recall that the entropy function (of one variable) is the function defined on the interval $[0,1]$ by

$$\delta(x) = -x\log x - (1-x)\log(1-x) \qquad (0 < x < 1)$$

and $\delta(0) = \delta(1) = 0$, rendering the function continuous. It should be noted that the function δ is monotonically increasing from 0 to 1 in the interval $\left[0, \frac{1}{2}\right]$ and that $\delta(1-x) = \delta(x)$ for all $x \in [0,1]$.

For any set $A \in \mathcal{P}$ and any $n \in \mathbb{N} = \{1, 2, \dots\}$ we let

$$A^n = A \cap \{1, \dots, n\},$$

calling A^n the *nth initial subset* of A. Furthermore, we use the notation

$$A(n) = \operatorname{card} A^n$$

and

$$A(x) = A(\lfloor x \rfloor) \qquad \text{for real } x \geq 1. \tag{1}$$

By $\lfloor x \rfloor$ we mean the integer part of x.

We denote the lower and upper asymptotic density, respectively, by

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{A(n)}{n} \quad \text{and} \quad \overline{d}(A) = \limsup_{n\to\infty} \frac{A(n)}{n}\,.$$

Now we recall the concept of Hausdorff dimension of sets of real numbers. Let $A \subset \mathbb{R}$. For $s \geq 0$ and $\varepsilon > 0$ denote

$$H^s_\varepsilon(A) = \inf\left\{ \sum_{n=1}^{\infty} (b_n - a_n)^s \;\middle|\; A \subset \bigcup_{n=1}^{\infty} (a_n, b_n) \text{ and } b_n - a_n < \varepsilon \text{ for all } n \in \mathbb{N} \right\}.$$

Note that $H^s_\varepsilon(A)$ is decreasing with respect to ε which yields that the limit

$$H^s(A) = \lim_{\varepsilon\to 0+} H^s_\varepsilon(A)$$

exists and is called the *s-dimensional Hausdorff measure* of A. It is an exercise (see, e.g., [1]) to show that there is a unique real number $D \in [0,1]$ such that

$$H^s(A) = \infty \text{ for all } s < D \quad \text{and} \quad H^s(A) = 0 \text{ for all } s > D\,.$$

The number D is called *Hausdorff dimension* of A and is denoted by $\dim A$.

In [3], Theorem 52, it is proved, among other results, that given any α and β, $0 \le \alpha \le \beta \le 1$, the set

$$\mathcal{G}(\alpha,\beta) = \left\{ A \in \mathcal{P} \;\middle|\; \underline{d}(A) = \alpha, \overline{d}(A) = \beta \right\}$$

has an image set $\varrho\mathcal{G}(\alpha,\beta)$ with Hausdorff dimension

$$\dim \varrho\mathcal{G}(\alpha,\beta) = \min\{\delta(\alpha), \delta(\beta)\}\,. \tag{2}$$

In this paper we investigate the sets $\mathcal{G}(\alpha,\beta)$ in more detail by introducing the gap density $\lambda(A)$ as an additional parameter. As defined in [2], given any set $A = \{a_1 < a_2 < \dots\} \in \mathcal{P}_\infty$, we let

$$\lambda(A) = \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}\,,$$

noting that $\lambda(A)$ may be infinite. Obviously, the gap density is related to the lower and upper asymptotic densities by the inequality

$$1 \le \lambda(A) \le \frac{\overline{d}(A)}{\underline{d}(A)} \qquad \text{for all } A \in \mathcal{P}_\infty.$$

For all α,β,γ with $0 \le \alpha \le \beta \le 1$ and $1 \le \gamma \le \frac{\beta}{\alpha}$ (with $\frac{\beta}{0} = \infty$) we define the class

$$\mathcal{G}(\alpha,\beta,\gamma) = \left\{ A \in \mathcal{P}_\infty \;\middle|\; \underline{d}(A) = \alpha, \overline{d}(A) = \beta, \lambda(A) = \gamma \right\}.$$

The aim of the present paper is to determine the Hausdorff dimension of the set $\varrho\mathcal{G}(\alpha,\beta,\gamma)$.

2. Homogeneous and saturated classes

2.1. Notation

The following definitions and notation will be needed in our paper.

2.1.1.

For any class $\mathcal{S} \subset \mathcal{P}$ we let

$$\mathcal{S}^n = \{A^n \mid A \in \mathcal{S}\} \qquad (n = 1, 2, \dots)\,,$$

noting that $\mathcal{S}^n$ is a class of finite sets.

2.1.2.

For any class $\mathcal{S} \subset \mathcal{P}$ and any pair m, n with $m < n$, if $\mathcal{T} \subset \mathcal{S}^m$ is given, we define the class

$$\mathcal{S}_m^n(\mathcal{T}) = \bigcup_{T \in \mathcal{T}} \{S \in \mathcal{S}^n \mid S^m = T^m\} .$$

Hence $\mathcal{S}_m^n(\mathcal{T})$ is the class of all sets $S \in \mathcal{S}^n$ for which there exists a set $T \in \mathcal{T}$ having the same mth initial subset. We will call these sets S the *descendants* of $\mathcal{T}$ in $\mathcal{S}^n$. We will write $\mathcal{S}_m^n(T)$ instead of $\mathcal{S}_m^n(\{T\})$.

2.1.3.

We define, for every nonempty class $\mathcal{S} \subset \mathcal{P}$,

$$\iota(\mathcal{S}) = \liminf_{n \to \infty} \frac{\log \operatorname{card} \mathcal{S}^n}{n} .$$

Clearly, we always have $0 \leq \iota(\mathcal{S}) \leq 1$.

2.1.4.

As usual, we consider the *basic dyadic intervals* of order n,

$$\left(\frac{k}{2^n}, \frac{k+1}{2^n} \right] \qquad (k = 0, \ldots, 2^n - 1) ,$$

denoting the set of these 2^n intervals by $\mathcal{B}^n$ and letting

$$\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}^n$$

stand for the set of all basic dyadic intervals.

2.1.5.

For $T \in \mathcal{P}_\infty$ and $n \in \mathbb{N}$ denote by

$$B(T^n) = \left\{ \varrho(U) \;\middle|\; U \in \mathcal{P}_\infty \text{ and } U^n = T^n \right\} = \left(\sum_{i=1}^{n} \frac{\chi_T(i)}{2^i}, \sum_{i=1}^{n} \frac{\chi_T(i)}{2^i} + \frac{1}{2^n} \right]$$

the element of $\mathcal{B}^n$ containing $\varrho(T)$, being uniquely determined by the first n binary digits of T. Similarly, for any $\mathcal{S} \subset \mathcal{P}_\infty$ and $n \in \mathbb{N}$ we let

$$B(\mathcal{S}^n) = \bigcup_{T \in \mathcal{S}} B(T^n) .$$

Consequently, for every $\mathcal{S} \subset \mathcal{P}_\infty$ and $n \in \mathbb{N}$, the relation $\varrho\mathcal{S} \subset B(\mathcal{S}^n)$ holds. Thus we have

$$\varrho\mathcal{S} \subset \bigcap_{n=1}^{\infty} B(\mathcal{S}^n)\,.$$

The classes $\mathcal{S}$ for which also the opposite inclusion holds will be of special interest.

2.1.6.

For a class $\mathcal{R} = \{R_i \mid i \in \mathcal{I}\}$ with $R_i \subset \mathbb{R}$ for every $i \in \mathcal{I}$ we will write, as usually,

$$\bigcup \mathcal{R} = \bigcup_{i \in \mathcal{I}} R_i\,.$$

We will say that a class $\mathcal{R}$ *covers* a set $A \subset \mathbb{R}$ if

$$A \subset \bigcup \mathcal{R}\,.$$

2.1.7.

We will call a class $\mathcal{S} \subset \mathcal{P}_\infty$ *saturated* if $\varrho\mathcal{S} = \bigcap_{n=1}^{\infty} B(\mathcal{S}^n)$.

It will turn out that, for saturated classes $\mathcal{G}$, it is relatively simple to determine $\dim \varrho\mathcal{G}$. But unfortunately, our classes $\mathcal{G}(\alpha,\beta,\gamma)$ are not saturated. Indeed, since the asymptotic densities (lower, upper and gap density) of a set A do not depend on its initial subsets, each set $\varrho\mathcal{G}(\alpha,\beta,\gamma)$ is everywhere dense in $[0,1]$. Therefore, we always have

$$B(\mathcal{S}^n) = (0,1] \quad \text{if } \mathcal{S} = \mathcal{G}(\alpha,\beta,\gamma).$$

2.1.8.

A class $\mathcal{S} \subset \mathcal{P}_\infty$ will be called *homogeneous* if there exist a positive integer n_0, a constant $\kappa > 1$ and a nondecreasing function

$$\varphi\colon (0,\infty) \to (0,\infty)$$

such that

$$\log \varphi(n) = o(n) \tag{3}$$

and for all $m, n \in \mathbb{N}$ with $n_0 \le m < n \le \kappa m$, the inequality

$$\operatorname{card} \mathcal{S}_m^n(S) \le \varphi(m) \operatorname{card} \mathcal{S}_m^n(T) \tag{4}$$

holds for any choice of the sets $S, T \in \mathcal{S}^m$.

Roughly speaking, this means the following: Considering any two subintervals, I and J, of $[0,1]$, the number of basic intervals of order n containing elements from $\varrho\mathcal{S}$ within both of them, increases, as n tends to infinity, with comparable speed. (For explicit details see (12) below.)

Example 1. Consider the system

$$\mathcal{S} = \left\{ A \in \mathcal{P}_\infty \;\middle|\; \text{if } \frac{A(n_0)}{n_0} > \frac{1}{2} \text{ for some } n_0, \text{ then } n \in A \text{ for all } n > n_0 \right\}.$$

This system is not homogeneous.

2.2. Lemmas

The first lemma of this section is well-known and its simple proof follows from Stirling's formula.

Lemma 1. *For any $\alpha \in [0,1]$,*

$$\binom{n}{\lfloor \alpha n \rfloor} = 2^{(\delta(\alpha)+o(1))n} \qquad \text{as } n \to \infty.$$

We will also need a simple modification of the previous lemma.

Lemma 2. *For any $\alpha, \beta \in [0,1]$ with $\alpha \le \beta$ we have*

$$\sum_{j=\lfloor \alpha n \rfloor}^{\lfloor \beta n \rfloor} \binom{n}{j} = 2^{(\omega+o(1))n} \qquad \text{as } n \to \infty,$$

where $\omega = \max_{\sigma \in [\alpha,\beta]} \delta(\sigma)$.

Proof. In the case $\alpha \le \frac{1}{2} \le \beta$ the result follows from Lemma 1 and inequalities

$$2^{(\delta(\frac{1}{2})+o(1))n} = \binom{n}{\lfloor n/2 \rfloor} \le \sum_{j=\lfloor \alpha n \rfloor}^{\lfloor \beta n \rfloor} \binom{n}{j} \le 2^n .$$

In the case $\alpha \le \beta < \frac{1}{2}$ the result follows from Lemma 1 and inequalities

$$2^{(\delta(\beta)+o(1))n} = \binom{n}{\lfloor \beta n \rfloor} \le \sum_{j=\lfloor \alpha n \rfloor}^{\lfloor \beta n \rfloor} \binom{n}{j} \le \beta n \binom{n}{\lfloor \beta n \rfloor} = 2^{(\delta(\beta)+o(1))n} .$$

The case $\frac{1}{2} < \alpha \le \beta$ is analogous. $\square$

Lemma 3. *Let $\varphi\colon (0,\infty) \to (0,\infty)$ be a non-decreasing function such that* (3) *holds, and let r, s with $0 < r < s$ be real numbers. Then there exists a real number $q > 0$ and an $n_0 \in \mathbb{N}$ such that*

$$(1+q)^k n_0 \big(s - (1+q)r\big) > \sum_{i=0}^{k-1} \log \varphi\big((1+q)^i n_0\big) + k + 3 \tag{5}$$

holds for all $k \in \mathbb{N}_0$. The number q can be chosen arbitrarily small and the number n_0 can be chosen arbitrarily large.

P r o o f. Choose $q > 0$ such that $c = s - (1+q)r > 0$, and $n_0 \in \mathbb{N}$ such that

$$n_0 > \frac{3}{c} \quad \text{and} \quad \frac{\log \varphi(\nu_k) + 1}{\nu_k} < qc\,, \tag{6}$$

where $\nu_k = (1+q)^k n_0$ for every $k \in \mathbb{N}_0$. In this notation, denoting the left-hand and right-hand sides of (5) by $L(k)$ and $R(k)$, respectively, we have to prove that

$$L(k) = c\nu_k > \sum_{i=0}^{k-1} \log \varphi(\nu_i) + k + 3 = R(k)\,.$$

Evidently,

$$L(0) = cn_0 > 3 = R(0) \quad \text{by (6)}.$$

Furthermore, for $k \in \mathbb{N}_0$, we have

$$\Delta L(k) = L(k+1) - L(k) = c(\nu_{k+1} - \nu_k) = cq\nu_k$$

and

$$\Delta R(k) = R(k+1) - R(k) = 1 + \log \varphi(\nu_k)\,.$$

The second relation in (6) implies $\Delta R(k) < \Delta L(k)$ for all $k \in \mathbb{N}_0$, which proves (5) by induction. □

3. Computation of dimensions

Lemma 4. *For each nonempty $\mathcal{S} \subset \mathcal{P}_\infty$,*

$$\dim \varrho\mathcal{S} \le \iota(\mathcal{S})\,.$$

P r o o f. It is sufficient to show that for any $r > \iota(\mathcal{S})$ and $\eta > 0$ there exists a covering $\mathcal{J}$ of $\varrho\mathcal{S}$, consisting of countably many intervals J with length $|J| < \eta$, and furthermore,

$$\sum_{J \in \mathcal{J}} |J|^r < 1\,.$$

To see this, first choose $n \in \mathbb{N}$ such that, in the sense of 2.1.1,

$$n > -\log \eta \quad \text{and} \quad \operatorname{card} \mathcal{S}^n < 2^{rn}. \tag{7}$$

Let $\mathcal{J} = \{B(T^n) \mid T \in \mathcal{S}\}$. Then $\mathcal{J}$ covers $\varrho\mathcal{S}$, $|J| = 2^{-n} < \eta$ for every $J \in \mathcal{J}$ and using (7), we obtain

$$\sum_{J \in \mathcal{J}} |J|^r < 2^{rn} 2^{-rn} = 1\,,$$

which proves the lemma. □

Our next goal is to show that under some additional conditions the opposite inequality $\dim \varrho\mathcal{S} \geq \iota(\mathcal{S})$ also holds.

Theorem 1. *Let a nonempty class $\mathcal{S} \subset \mathcal{P}_\infty$ be saturated and homogeneous. Then*

$$\dim \varrho\mathcal{S} = \iota(\mathcal{S})\,.$$

Proof. **1.** Let us define the set

$$X = \left\{ \frac{a}{2^b} \,\middle|\, b \in \mathbb{N}, a = 0, \dots, 2^b \right\}$$

of all dyadic rational points in $[0,1]$. By Lemma 4 it suffices to prove the inequality

$$\dim \varrho\mathcal{S} \geq \iota(\mathcal{S})\,.$$

For technical reasons we are going to establish this at first for $\dim(\varrho\mathcal{S} \cup X)$ instead of $\dim \varrho\mathcal{S}$.

2. If $\iota(\mathcal{S}) > 0$, let $r < \iota(\mathcal{S})$ be an arbitrary positive real number. To prove Theorem 1 it is sufficient to find an $\eta > 0$ such that no family $\mathcal{J} = (J_n)$ of intervals with $|J_n| < \eta$ for every $n \in \mathbb{N}$ and

$$\sum_{i=1}^{\infty} |J_i|^r < 1 \tag{8}$$

covers $\varrho\mathcal{S} \cup X$.

Choose $s \in \big(r, \iota(\mathcal{S})\big)$. Let $q > 0$ and $n_0 \in \mathbb{N}$ be numbers guaranteed by Lemma 3 with $1 + q < \kappa$ (κ being the number from 2.1.8) and with n_0 large enough such that

$$\operatorname{card} \mathcal{S}^n > 2^{sn} \tag{9}$$

for every $n \geq n_0$.

Now let a family $\mathcal{J}$, satisfying (8) with $\eta = 2^{-n_0}$, be fixed, and suppose that its elements $J_1, J_2, \dots$ are numbered so that $|J_1| \geq |J_2| \geq \cdots$ We use the notation

$$n_k = \big\lfloor (1+q)^k n_0 \big\rfloor \qquad (k = 0, 1, 2, \dots).$$

For each k we define the symbols

$$j_k = \max\big\{ j \in \mathbb{N} \;\big|\; |J_j| \geq 2^{-n_k} \big\} \quad \text{and} \quad \mathcal{J}_k = \big\{ J_1, J_2, \dots, J_{j_k} \big\}\,.$$

Then

$$1 > \sum_{i=1}^{j_k} |J_i|^r \geq j_k 2^{-r n_k},$$

and thus we have, for each $k \in \mathbb{N}$,

$$\operatorname{card} \mathcal{J}_k = j_k < 2^{r n_k}. \tag{10}$$

3. We are going to prove that $\mathcal{J}$ cannot be an η-covering of the set $\varrho\mathcal{S} \cup X$. To do so we use the following procedure. For each index k we show that there is a nonempty closed subset V_k of $\varrho\mathcal{S} \cup X$ which is disjoint from all elements of $\mathcal{J}_{j_{k+1}}$ and satisfies $V_k \subset V_{k-1}$. Therefore, the intersection $\bigcap_{k=1}^{\infty} V_k$ is nonempty and disjoint from $\bigcup \mathcal{J}$.

In order to accomplish this we construct the sequence of systems of intervals,

$$\mathcal{I}_k = \{\overline{B(S)} \mid S \in \mathcal{S}^{n_k}\} \qquad (k = 0, 1, 2, \dots),$$

where $\overline{B(S)}$ denotes the closure of the interval $B(S)$. Since there is a one-to-one correspondence between the elements of $\mathcal{S}^{n_k}$ and the elements of $\mathcal{I}_k$, we have

$$\operatorname{card} \mathcal{I}_k = \operatorname{card} \mathcal{S}^{n_k}. \tag{11}$$

Define the sequence of systems of intervals,

$$\mathcal{U}_k = \left\{ M \in \mathcal{I}_k(\mathcal{U}_{k-1}) \;\middle|\; M \text{ is disjoint from all elements of } \mathcal{J}_{k+1} \right\}$$

($k = 0, 1, 2, \dots$). Here we let $\mathcal{I}_0(\mathcal{U}_{-1}) = \mathcal{I}_0$ and denote the system of those descendants (see 2.1.2) of $\mathcal{U}_{k-1}$ which are contained in $\mathcal{I}_k$ by

$$\mathcal{I}_k(\mathcal{U}_{k-1}) = \left\{ M \in \mathcal{I}_k \;\middle|\; \text{there exists } U \in \mathcal{U}_{k-1} \text{ such that } M \subset U \right\}.$$

4. One can easily prove that the condition (4) implies, for any $\mathcal{T} \subset \mathcal{S}^{n_k}$, inequality

$$\operatorname{card} \mathcal{S}_{n_k}^{n_{k+1}}(\mathcal{T}) \geq \frac{1}{\varphi(n_k)} \frac{\operatorname{card} \mathcal{T}}{\operatorname{card} \mathcal{S}^{n_k}} \operatorname{card} \mathcal{S}^{n_{k+1}}. \tag{12}$$

Now we prove by induction that

$$\operatorname{card} \mathcal{U}_k > \frac{1}{2^{k+1} \prod_{i=0}^{k-1} \varphi(n_i)} \operatorname{card} \mathcal{I}_k\,. \tag{13}$$

Since every element of $\mathcal{I}_0$ has length 2^{-n_0} and every element of $\mathcal{J}_1$ has length less than $\eta = 2^{-n_0}$, it follows that every element of $\mathcal{J}_1$ can overlap with at most two elements of $\mathcal{I}_0$. Inasmuch as inequality (5) with $k = 0$ implies the relation

$$(sn_0 - 1) - (rn_1 + 1) \geq sn_0 - rn_0(1 + q) - 2 > 0\,,$$

it follows from (9) and (10) that

$$\operatorname{card} \mathcal{U}_0 \geq \operatorname{card} \mathcal{I}_0 - 2 \operatorname{card} \mathcal{J}_1 > \frac{1}{2} \operatorname{card} \mathcal{I}_0 + 2^{sn_0 - 1} - 2^{rn_1 + 1} > \frac{1}{2} \operatorname{card} \mathcal{I}_0\,,$$

which proves (13) for $k = 0$.

Assume (13) for some fixed $k \in \mathbb{N}_0$. Relations (11), (12) and (13) imply

$$\operatorname{card} \mathcal{I}_{k+1}(\mathcal{U}_k) \geq \frac{1}{\varphi(n_k)} \frac{\operatorname{card} \mathcal{U}_k}{\operatorname{card} \mathcal{I}_k} \operatorname{card} \mathcal{I}_{k+1} > \frac{1}{2^{k+1} \prod_{i=0}^{k} \varphi(n_i)} \operatorname{card} \mathcal{I}_{k+1} .$$

Since every element of $\mathcal{I}_{k+1}$ has length $2^{-n_{k+1}}$ and every element of $\mathcal{J}_{k+2} \setminus \mathcal{J}_{k+1}$ has length less than $2^{-n_{k+1}}$, it follows that every element of $\mathcal{J}_{k+2}$ can overlap with at most two elements of $\mathcal{I}_{k+1}(\mathcal{U}_k)$. Thus (9) and (10) imply

$$\begin{aligned}
\operatorname{card} \mathcal{U}_{k+1} &\geq \operatorname{card} \mathcal{I}_{k+1}(\mathcal{U}_k) - 2 \operatorname{card} \mathcal{J}_{k+2} \\
&> \frac{1}{2^{k+1} \prod_{i=0}^{k} \varphi(n_i)} \operatorname{card} \mathcal{I}_{k+1} - 2^{r n_{k+2}+1} \\
&> \frac{1}{2^{k+2} \prod_{i=0}^{k} \varphi(n_i)} \operatorname{card} \mathcal{I}_{k+1} + \frac{1}{2^{k+2} \prod_{i=0}^{k} \varphi(n_i)} 2^{s n_{k+1}} - 2^{r n_{k+2}+1} \\
&= \frac{1}{2^{k+2} \prod_{i=0}^{k} \varphi(n_i)} \operatorname{card} \mathcal{I}_{k+1} + 2^{s n_{k+1} - k - 2 - \sum_{i=0}^{k} \log \varphi(n_i)} - 2^{r n_{k+2}+1} \\
&> \frac{1}{2^{k+2} \prod_{i=0}^{k} \varphi(n_i)} \operatorname{card} \mathcal{I}_{k+1} .
\end{aligned}$$

For the last inequality we have used (5) for $k+1$ if k is the index under consideration. Indeed,

$$\begin{aligned}
&\Big(s n_{k+1} - k - 2 - \sum_{i=0}^{k} \log \varphi(n_i)\Big) - \big(r n_{k+2} + 1\big) \\
&= \big\lfloor (1+q)^{k+1} n_0 \big\rfloor s - \big\lfloor (1+q)^{k+2} n_0 \big\rfloor r - \sum_{i=0}^{k} \log \varphi\big(\big\lfloor (1+q)^i n_0 \big\rfloor\big) - k - 3 \\
&> (1+q)^{k+1} n_0 s - s - (1+q)^{k+2} n_0 r - \sum_{i=0}^{k} \log \varphi\big((1+q)^i n_0\big) - k - 3 \\
&= (1+q)^{k+1} n_0 \big(s - (1+q) r\big) - \sum_{i=0}^{k} \log \varphi\big((1+q)^i n_0\big) - (k+1) - 3 + (1-s) \\
&> 1 - s > 0 .
\end{aligned}$$

This finishes the inductive proof of (13).

5. The system $\mathcal{S}$ being saturated by assumption, we have

$$\bigcap_{n=1}^{\infty} \bigcup_{S\in\mathcal{S}^n} B(S) = \varrho\mathcal{S} .$$

This and (13) imply that the sets

$$V_k = \bigcup \mathcal{U}_k \subset \varrho\mathcal{S} \cup X \qquad (k = 1, 2, \dots) \tag{14}$$

form a decreasing sequence of nonempty closed sets in the compact $[0, 1]$. Hence $V = \bigcap_{k=1}^{\infty} V_k$ is nonempty. In view of the definition of the classes $\mathcal{U}_k$ and the relation (14) we have

$$\emptyset \neq V \subset (\varrho\mathcal{S} \cup X) \setminus \bigcup \mathcal{J} .$$

Thus

$$\dim(\varrho\mathcal{S} \cup X) \geq r \quad \text{for every} \quad r < \iota(\mathcal{S})$$

and hence, since the set X is countable,

$$\dim \varrho\mathcal{S} = \dim(\varrho\mathcal{S} \cup X) \geq \iota(\mathcal{S}) ,$$

which finishes the proof of Theorem 1. □

Example 2. Let $A \in \mathcal{P}_\infty$. We will show that the system

$$\mathcal{S} = \{B \in \mathcal{P}_\infty \mid B \subset A\}$$

is homogeneous and saturated. The homogeneity follows immediately from the fact that for any $m < n$ and for every pair $S, T \in \mathcal{S}^m$ we have

$$\operatorname{card} \mathcal{S}^n_m(S) = \operatorname{card} \mathcal{S}^n_m(T) = 2^{\operatorname{card}(A\cap\{m+1,m+2,\dots,n\})}.$$

To prove that $\mathcal{S}$ is saturated, choose any $x \notin \bigcap_{n=1}^{\infty} B(\mathcal{S}^n)$. Then there is $n \in \mathbb{N}$ such that $x \notin B(\mathcal{S}^n)$. But this means that $\varrho^{-1}(x)$ is not a subset of A, consequently $x \notin \varrho\mathcal{S}$.

Now the application of Theorem 1 implies that

$$\dim \varrho\mathcal{S} = \liminf_{n\to\infty} \frac{\log \operatorname{card} \mathcal{S}^n}{n} = \liminf_{n\to\infty} \frac{\log 2^{A(n)}}{n} = \underline{d}(A) .$$

This result was first proved in [4].

4. Main result

The main theorem of this paper is the following. (For definitions see the introduction.)

Theorem 2. *Let α, β, γ with $0 \le \alpha \le \beta \le 1 \le \gamma$ and $\alpha\gamma \le \beta$ be given. Then*

$$\dim \varrho\mathcal{G}(\alpha, \beta, \gamma) = \min\Big\{\delta(\alpha), \delta(\beta), \frac{1}{\gamma} \max_{\sigma \in [\alpha\gamma, \beta]} \delta(\sigma)\Big\}.$$

The result can be stated more explicitely in the following way.

$$\dim \varrho\mathcal{G}(\alpha, \beta, \gamma) = \begin{cases} \min\big\{\delta(\alpha), \frac{\delta(\beta)}{\gamma}\big\} & \text{if } \alpha\gamma \le \beta \le \frac{1}{2}, \\ \min\big\{\delta(\alpha), \delta(\beta), \frac{1}{\gamma}\big\} & \text{if } \alpha\gamma \le \frac{1}{2} \le \beta, \\ \min\big\{\delta(\beta), \frac{1}{\gamma}\delta(\alpha\gamma)\big\} & \text{if } \frac{1}{2} \le \alpha\gamma \le \beta. \end{cases}$$

Before proving the main theorem we construct a system of auxiliary sets inductively. First we note that, using (2) and the fact that $\mathcal{G}(\alpha, \beta, \gamma) \subset \mathcal{G}(\alpha, \beta)$, it follows immediately that the theorem holds if either $\alpha = 0$ or $\beta = 1$. Thus we assume $0 < \alpha$ and $\beta < 1$ in the sequel, so having $1 \le \gamma < \infty$.

We will use the special classes $\mathcal{Z}(\alpha, \beta, \gamma, \sigma) \subset \mathcal{P}_\infty$ defined as follows. Let $\alpha, \beta, \gamma, \sigma$ be real numbers with $0 < \alpha \le \alpha\gamma \le \sigma \le \beta < 1$. Construct the sequences

$$\begin{aligned} a_1 &= 1\,, & b_1 &= 2\,, & c_1 &= \lfloor \gamma b_1 \rfloor\,, & d_1 &= 2c_1\,, \\ a_m &= (m+1)d_{m-1}\,, & b_m &= (m+1)a_m\,, & c_m &= \lfloor \gamma b_m \rfloor\,, & d_m &= (m+1)c_m \end{aligned}$$

$(m = 2, 3, \dots)$. Let $\mathcal{Z}(\alpha, \beta, \gamma, \sigma)$ denote the class of all sets $A \in \mathcal{P}_\infty$ such that, for every $m \in \mathbb{N}$,

$$\operatorname{card}\big(A \cap (d_{m-1} + (k-1)m, d_{m-1} + km]\big) = \lfloor \alpha m \rfloor\,, \qquad k = 1, \dots, d_{m-1}\,,$$

$$\operatorname{card}\big(A \cap (a_m + (k-1)m, a_m + km]\big) = \lfloor \sigma m \rfloor\,, \qquad k = 1, \dots, a_m\,,$$

$$A \cap (b_m, c_m] = \emptyset$$

and

$$\operatorname{card}\big(A \cap (c_m + (k-1)m, c_m + km]\big) = \lfloor \beta m \rfloor\,, \qquad k = 1, \dots, c_m\,.$$

We call W the set of all endpoints of intervals mentioned in these equations.

Lemma 5. *All classes $\mathcal{Z}(\alpha, \beta, \gamma, \sigma)$ are homogeneous and saturated; moreover,*

$$\dim \varrho\mathcal{Z}(\alpha, \beta, \gamma, \sigma) = \min\Big\{\delta(\alpha), \delta(\beta), \frac{\delta(\sigma)}{\gamma}\Big\}.$$

P r o o f. **1.** We consider $\alpha, \beta, \gamma, \sigma$ to be fixed, writing $\mathcal{Z}(\alpha, \beta, \gamma, \sigma) = \mathcal{Z}$. The cardinality of the class $\mathcal{Z}^{d_{m-1}}$ satisfies

$$1 \le \operatorname{card} \mathcal{Z}^{d_{m-1}} \le 2^{d_{m-1}},$$

hence we have, as $m \to \infty$,

$$\operatorname{card} \mathcal{Z}^{d_{m-1}} = 2^{o(a_m)}.$$

Lemma 1 implies that

$$\operatorname{card} \mathcal{Z}^{a_m} = \operatorname{card} \mathcal{Z}^{d_{m-1}} \cdot \binom{m}{\lfloor \alpha m \rfloor}^{d_{m-1}} = 2^{o(a_m)+(\delta(\alpha)+o(1))m d_{m-1}}$$

and

$$\lim_{m\to\infty} \frac{\log \operatorname{card} \mathcal{Z}^{a_m}}{a_m} = \lim_{m\to\infty} \left(\frac{o(a_m)}{a_m} + (\delta(\alpha)+o(1)) \frac{m d_{m-1}}{a_m} \right) = \delta(\alpha).$$

Similarly,

$$\operatorname{card} \mathcal{Z}^{b_m} = \operatorname{card} \mathcal{Z}^{a_m} \cdot \binom{m}{\lfloor \sigma m \rfloor}^{a_m} = 2^{o(b_m)+(\delta(\sigma)+o(1))m a_m}$$

and

$$\lim_{m\to\infty} \frac{\log \operatorname{card} \mathcal{Z}^{b_m}}{b_m} \lim_{m\to\infty} \left(\frac{o(b_m)}{b_m} + (\delta(\sigma)+o(1)) \frac{m a_m}{b_m} \right) = \delta(\sigma).$$

One easily obtains the relation

$$\lim_{m\to\infty} \frac{\log \operatorname{card} \mathcal{Z}^{c_m}}{c_m} = \lim_{m\to\infty} \frac{\log \operatorname{card} \mathcal{Z}^{b_m}}{\lfloor \gamma b_m \rfloor} = \frac{\delta(\sigma)}{\gamma}.$$

Finally, we have

$$\lim_{m\to\infty} \frac{\log \operatorname{card} \mathcal{Z}^{d_m}}{d_m} = \delta(\beta).$$

Let

$$F(n) = \frac{\log \operatorname{card} \mathcal{Z}^n}{n}.$$

If restricted to the elements n of W, the function $F(n)$ is monotone in each of the intervals $[d_{m-1}, a_m]$, $[a_m, b_m]$, $[b_m, c_m]$ and $[c_m, d_m]$ $(m = 2, 3, \dots)$.

Therefore, it suffices to consider the points a_m, b_m, c_m, d_m in order to determine the $\liminf_{n\to\infty} F(n)$. We obtain

$$\iota(\mathcal{Z}) = \liminf_{n\to\infty} \frac{\log \operatorname{card} \mathcal{Z}^n}{n} = \min\left\{\delta(\alpha), \delta(\sigma), \frac{\delta(\sigma)}{\gamma}, \delta(\beta)\right\}$$
$$= \min\left\{\delta(\alpha), \delta(\beta), \frac{\delta(\sigma)}{\gamma}\right\}.$$

2. Now we prove that the class $\mathcal{Z}$ is homogeneous. Choose positive sufficiently large integers m, n such that $m < n < 2m$. Then there is a uniquely determined integer k such that $n \in (d_{k-1}, d_k]$. The construction of the sequence (d_k) implies $n \geq (k!)^3$, and consequently,

$$k \leq \frac{1}{3} \log n < \log m\,.$$

Now look at the construction of the class $\mathcal{Z}$. Each of the intervals $(d_{k-1}, a_k]$, $(a_k, b_k]$ and $(c_k, d_k]$ consists of many small blocks of k elements. The number $\operatorname{card} \mathcal{Z}^p$ increases in each such block by a factor $\binom{k}{\lfloor \tau k \rfloor}$, where τ is one of the real numbers α, σ, β. Choose $S, T \in \mathcal{Z}^m$ arbitrarily. Then, letting an integer p increase from m to n, both numbers $\operatorname{card} \mathcal{Z}^p_m(S)$ and $\operatorname{card} \mathcal{Z}^p_m(T)$ increase by the same factor in each small block, except the first one (containing m). Their increase may differ at most by the factor 2^k. Thus

$$\operatorname{card} \mathcal{Z}^n_m(S) \leq \operatorname{card} \mathcal{Z}^n_m(T)\, 2^k \leq \operatorname{card} \mathcal{Z}^n_m(T)\, 2^{\log m} = \operatorname{card} \mathcal{Z}^n_m(T)\, m\,.$$

Consequently, the class $\mathcal{Z}$ satisfies the definition of being homogeneous with a suitable n_0, $\kappa = 2$ and $\varphi(m) = m$.

3. We finish the proof of the lemma by showing that the class $\mathcal{Z}$ is saturated. To do so, it is sufficient to show that for each $\xi \in (0,1] \setminus \varrho\mathcal{Z}$ there exists a positive integer n such that $\xi \notin B(\mathcal{Z}^n)$. Choose an arbitrary $S \in \mathcal{Z}$. Then a set $T \in \mathcal{P}_\infty$ belongs to $\mathcal{Z}$ if and only if

$$\operatorname{card} T^n = \operatorname{card} S^n \qquad \text{holds for all} \quad n \in W.$$

Choose a real number $\xi \in (0,1] \setminus \varrho\mathcal{Z}$ and let $Y = \varrho^{-1}(\xi)$. Then there exists an integer $n \in W$ so that $\operatorname{card} Y^n \neq \operatorname{card} S^n$. Consequently $\xi \notin B(\mathcal{Z}^n)$ and thus $\mathcal{Z}$ is saturated. □

P r o o f o f T h e o r e m 2. First we use Lemma 5 to establish a lower bound for $\dim \varrho\mathcal{G}(\alpha, \beta, \gamma)$. For every $\sigma \in [\alpha\gamma, \beta]$ and $A \in \mathcal{Z}(\alpha, \beta, \gamma, \sigma)$ we have

$$\underline{d}(A) = \alpha\,, \quad \overline{d}(A) = \beta \quad \text{and} \quad \lambda(A) = \gamma\,.$$

Therefore, $\mathcal{Z}(\alpha,\beta,\gamma,\sigma)\subset\mathcal{G}(\alpha,\beta,\gamma)$ and

$$\dim\varrho\mathcal{G}(\alpha,\beta,\gamma)\geq\dim\varrho\mathcal{Z}(\alpha,\beta,\gamma,\sigma)=\min\Big\{\delta(\alpha),\delta(\beta),\frac{\delta(\sigma)}{\gamma}\Big\}\,.$$

This holds for every $\sigma\in[\alpha\gamma,\beta]$, hence

$$\dim\varrho\mathcal{G}(\alpha,\beta,\gamma)\geq\min\Big\{\delta(\alpha),\delta(\beta),\frac{1}{\gamma}\max_{\sigma\in[\alpha\gamma,\beta]}\delta(\sigma)\Big\}\,.\qquad(15)$$

Now we prove the opposite inequality to (15). Since the relation

$$\dim\varrho\mathcal{G}(\alpha,\beta,\gamma)\leq\min\big\{\delta(\alpha),\delta(\beta)\big\}$$

follows from (2), it is sufficient to show that, if $0<\alpha\leq\alpha\gamma\leq\beta<1$, then

$$\dim\varrho\mathcal{G}(\alpha,\beta,\gamma)\leq\frac{1}{\gamma}\max_{\sigma\in[\alpha\gamma,\beta]}\delta(\sigma)=\mu\,.\qquad(16)$$

Properties of the function δ, the assertion (2) and the relation

$$\mathcal{G}(\alpha,\beta,\gamma)\subset\mathcal{G}(\alpha,\beta)$$

imply that this clearly holds in the case $\gamma=1$. Thus we assume $\gamma>1$ in the rest of the proof.

To establish (16) it is sufficient to prove that, for each $r\in(\mu,1)$, there exists a positive constant $C(r)$ such that, for every $\eta>0$, there exists a sequence (J_n) of intervals such that $|J_n|<\eta$ for every $n\in\mathbb{N}$, satisfying

$$\varrho\mathcal{G}(\alpha,\beta,\gamma)\subset\bigcup_{n=1}^{\infty}J_n\quad\text{and}\quad\sum_{n=1}^{\infty}|J_n|^r<C(r)\,.$$

So let $r\in(\mu,1)$ be given. Choose the real numbers

$$s\in(\mu,r),\quad\alpha'\in(0,\alpha),\quad\beta'\in(\beta,1),\quad\gamma'\in(1,\gamma)\quad\text{and}\quad q>0$$

such that $1+q<\gamma'$,

$$\frac{q}{1+q}-r+s<0\qquad(17)$$

and

$$\max_{\sigma\in[\frac{\alpha'\gamma'}{1+q},\beta'+\frac{q}{1+q}]}\delta(\sigma)<\gamma's\,.\qquad(18)$$

The possibility of this choice is guaranteed by the definition of μ and the continuity of the function δ.

Now letting

$$C(r)=\sum_{k=1}^{\infty}2^{2+(\frac{q}{1+q}-r+s)(1+q)^k}\,,$$

we see by (17) that $C(r) < \infty$. Let $\eta > 0$ be given. Take $n_0 > -\log\eta$. For each $A \in \mathcal{G}(\alpha,\beta,\gamma)$ denote by $n_A \geq n_0$ the least integer n such that

$$A \cap (n, \gamma' n] = \emptyset \quad \text{and} \quad \alpha'\gamma' < \frac{A(n)}{n} < \beta' \,. \tag{19}$$

For each $k \in \mathbb{N}$ we let $\nu_k = n_0(1+q)^k$ and

$$\mathcal{G}_k = \left\{ A \in \mathcal{G}(\alpha,\beta,\gamma) \;\middle|\; \gamma' n_A \in [\nu_{k-1}, \nu_k) \right\}.$$

Note that $\mathcal{G}(\alpha,\beta,\gamma) = \bigcup_{k=1}^{\infty} \mathcal{G}_k$. By definition there exists, for each set $A \in \mathcal{G}_k$, an integer

$$m = n_A \in \left[\frac{\nu_{k-1}}{\gamma'}, \frac{\nu_k}{\gamma'} \right) \tag{20}$$

such that

$$\alpha'\gamma' m < A(m) < \beta' m \,.$$

Consequently, in the sense of (1),

$$\alpha'\gamma' \frac{\nu_{k-1}}{\gamma'} \leq \alpha'\gamma' m < A(m) = A\left(\frac{\nu_k}{\gamma'}\right) \leq A(m) + \left(\frac{\nu_k}{\gamma'} - m\right) < \beta' m + \frac{\nu_k}{\gamma'} - m \,.$$

Dividing by $\frac{\nu_k}{\gamma'}$ we obtain

$$\frac{\alpha'\gamma' \frac{\nu_{k-1}}{\gamma'}}{\frac{\nu_k}{\gamma'}} = \frac{\alpha'\gamma'}{1+q} < \frac{A\left(\frac{\nu_k}{\gamma'}\right)}{\frac{\nu_k}{\gamma'}} < \frac{\beta' \frac{\nu_k}{\gamma'} + \frac{\nu_{k-1}}{\gamma'} q}{\frac{\nu_k}{\gamma'}} = \beta' + \frac{q}{1+q} \,.$$

Hence

$$A\left(\frac{\nu_k}{\gamma'}\right) = \sigma_{A,k} \qquad \text{for some } \sigma_{A,k} \in \left(\frac{\alpha'\gamma'}{1+q}, \beta' + \frac{q}{1+q} \right).$$

This yields, using (18) and Lemma 2,

$$\operatorname{card} \mathcal{G}_k^{\lfloor \frac{\nu_k}{\gamma'} \rfloor} \leq 2^{\gamma' s \frac{\nu_k}{\gamma'}} = 2^{s\nu_k} \,. \tag{21}$$

First relation in (19) with (20) imply that for every $A \in \mathcal{G}_k$ we have

$$A \cap \left(\frac{\nu_k}{\gamma'}, \nu_{k-1} \right] = \emptyset \,.$$

Then (21) implies that

$$\operatorname{card} \mathcal{G}_k^{\lfloor \nu_k \rfloor} \leq 2^{\lfloor \nu_k \rfloor - \lfloor \nu_{k-1} \rfloor} \operatorname{card} \mathcal{G}_k^{\lfloor \frac{\nu_k}{\gamma'} \rfloor} \leq 2^{\nu_k - \nu_{k-1} + 1}\, 2^{s\nu_k} = 2^{1 + (s + \frac{q}{1+q})\nu_k}.$$

Thus we can cover the set $\varrho\mathcal{G}_k$ by at most $2^{1+(s+\frac{q}{1+q})\nu_k}$ intervals of length $2^{-\lfloor \nu_k \rfloor}$ forming a family $\mathcal{J}_k$. Hence we have

$$\sum_{J \in \mathcal{J}_k} |J|^r \leq 2^{1+(s+\frac{q}{1+q})\nu_k}\, 2^{-r\lfloor \nu_k \rfloor} \leq 2^{2+(\frac{q}{1+q} - r + s) n_0 (1+q)^k}.$$

The family $\mathscr{J} = \bigcup_{k=1}^{\infty} \mathscr{J}_k$ covers $\varrho\mathcal{G}(\alpha, \beta, \gamma)$ and satisfies the inequality

$$\sum_{J \in \mathscr{J}} |J|^r \leq \sum_{k=1}^{\infty} 2^{2+(\frac{q}{1+q}-r+s)n_0(1+q)^k} < C(r)\,.$$

This implies (16) and together with (15) finishes the proof. □

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PAIR CORRELATIONS AND RANDOM WALKS ON THE INTEGERS

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Dedicated to the memory of Professor Pierre Liardet.

ABSTRACT. The paper gives conditions for a sequence of fractional parts of real numbers $\big(\{a_n x\}\big)_{n=1}^{\infty}$ to satisfy a pair correlation estimate. Here x is a fixed non-zero real number and $(a_n)_{n=1}^{\infty}$ is a random walk on the integers.

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Let X be a $\mathbb{Z}$-valued function defined on the probability space (Ω, β, P) with characteristic function $\phi(\xi) = \mathbb{E}\big(e^{iX(.)\xi}\big)$ and let $\chi = \{X_n : n \geq 1\}$ be a sequence of independent copies of X. For a positive integer $n > 0$ let $a_n = X_1 + \cdots + X_n$ and let $a_0 \equiv 0$. The sequence of integers $(a_n)_{n\geq 1}$ is the random walk which we assume to satisfy $|\phi(t) - 1| \geq C|t|$, some $C > 0$. This last property follows, for instance, if the random walk and its absolute value have finite non-zero mean [Sp, p. 62]. In [W1] the condition $|\phi(t)-1| \geq C|t|$ is said to follow from the assumption that the random is aperiodic and transient—a claim the authors was unable to verify. This is then used to deduce a discrepancy estimate for the sequence $\big(X_n(x)\big)_{n=1}^{\infty}$. This is so, for instance (as a consequence of the law of large numbers) if $\mathbb{E}|X| < \infty$ and $\mathbb{E}X \neq 0$ or if X is centred and $\frac{a_n}{n^{\frac{1}{\alpha}}}$ converges in distribution to F_α, a stable law of index $\alpha \in (0,1)$. This second class of examples can be deduced using a local limit theorem of Stone [St]. For a real number x let $X_n(x) = a_n x$. For an interval I let $\chi_I(x)$ denote the characteristic function of the set I. This means that we have

$$\chi_I(x) = 1 \quad \text{if } x \in I \quad \text{and} \quad \chi_I(x) = 0 \quad \text{otherwise.}$$

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For a real number y let $\{y\}$ denote its fractional part. Set

$$V_N(I)(x) \ = \sum_{1\le n<m\le N} \chi_I\big(\{X_n(x)-X_m(x)\}\big)$$

and then define

$$\Delta_N(x) = \sup_{I\subseteq \mathbb{T}} \left| V_N(I) - \frac{N(N-1)}{2}\mathrm{leb}(I)\right|,$$

where the supremum is over all intervals I in the one dimensional torus $\mathbb{T}$. Let $||x|| = \min_{n\in\mathbb{Z}} |x-n| = \min(\{x\},\{1-x\})$. Let η be a positive real number or infinity. The irrational number x is said to be of type η if η is the supremum of all γ for which $\liminf_{q\to\infty} q^{\gamma}||qx|| = 0$. Using Dirichlet's theorem on diophantine approximation we can deduce for all irrational x that $\liminf_{q\to\infty} q^{\gamma}||qx|| = 0$ so $\eta \ge 1$. On the other hand, the Thue-Siegel-Roth theorem tells us that for every irrational algebraic x and every $\epsilon > 0$ there exists a constant $c(x,\epsilon) > 0$ such that

$$\left| x - \frac{p}{q}\right| \ge \frac{c(x,\epsilon)}{q^{2+\epsilon}}$$

holds for all coprime integers $q > 0$ and p, so that algebraic η must be of type 1. Liouville numbers can easily be used to show constructively that there exist real numbers that of type strictly greater than 1.

Our theorem is the following.

Theorem. *Suppose* $\big(X_n(x)\big)_{n=1}^{\infty}$ *is as described above that* x *has type* $\eta > 1$. *Then given* $\epsilon > 0$,

$$\Delta_N(x) = o\big(N^{2-\frac{1}{\eta}+\epsilon}\big)$$

for P *almost all* $\omega \in \Omega$.

Let $D_N(x)$ denote the N-term discrepancy of the sequence $\big(X_n(x)\big)_{n\ge 1}$. See [KN, p 88] for the definition. M. Weber [W2, p 411] has given an estimate for almost everywhere behaviour of $D_N(x)$ as N tends to infinity in terms of the type of x and the properties of the the function ϕ. The formulation is however somewhat involved and forgone here. Results like our theorem, where $(a_n)_{n=1}^{\infty}$ is fixed and deterministic and x is random are now known. See [NP] for details and further background.

We proceed by a series of lemmas. For real x let $e(x) = e^{2\pi i x}$ and let

$$\theta_N(h) = \sum_{n=1}^{N} e(ha_n x) \qquad (N = 1, 2, \cdots).$$

We need the following lemma taken for [W1].

Lemma 1. *For integers $N \geq R \geq 1$ one has*

$$\mathbb{E}|\theta_N(m) - \theta_R(m)|^2 \leq \min\left(\frac{7(N-R)}{|\phi(mx)-1|}, N-R\right).$$

Let $(Y_t)_{t=1}^{\infty}$ be a sequence of measurable functions defined on a measure space Ω and then write

$$S_j = \sum_{1\leq t\leq\ j} Y_t, \qquad \text{for } j{=}1,2,\cdots$$

We can define

$$Y_{rs} = \sum_{r\leq t<s} Y_t (= S_s - S_r), \quad \text{for } r{<}s,$$

and let

$$M_n = \sup_{1\leq j\leq n} |S_j|.$$

We have the following elementary lemma proved in [NP].

Lemma 2. *For $K \geq 1$,*

$$\int_\Omega M_{2^K}^2(\omega)\,\mathrm{d}\omega \leq (K+1)\left(\sum_{i=1}^{K+1}\sum_{\nu=1}^{2^i-1}\int_\Omega |Y_{\nu 2^{(K+1)-i},\,(\nu+1)2^{(K+1)-i}}|^2(\omega)\,\mathrm{d}\omega\right).$$

Thus if $K = 1,2,\cdots$

$$\begin{aligned}
\mathbb{E}|\max_{1\leq j\leq 2^K}\theta_j(m)|^2 &\leq (K+1)\left(\sum_{i=1}^{K+1}\sum_{\nu=1}^{2^i-1}\mathbb{E}|\theta_{\nu 2^{(K+1)-i}}(m) - \theta_{(\nu+1)2^{(K+1)-i}}(m)|^2\right)\\
&\leq (K+1)\left(\sum_{i=1}^{K+1}\sum_{\nu=1}^{2^i-1}\left(\frac{7.2^{(K+1)-i}}{|\phi(mx)-1|}\right)\right)\\
&\leq (K+1)\left(\sum_{i=1}^{K+1}2^{i-1}\left(\frac{7.2^{(K+1)-i}}{|\phi(mx)-1|}\right)\right)\\
&\leq \frac{7}{2}(K+1)^2\left(\frac{2^{(K+1)}}{|\phi(mx)-1|}\right).
\end{aligned}$$

Thus, using the Erdős-Turan inequality [KN, p 112-4], we can show that for $L \geq 1$, there exists $C > 0$,

$$\mathbb{E}|\max_{1\le j\le 2^K}\Delta_j(x)| \le C\left(\frac{2^{2(K+1)}}{L}+\sum_{h=1}^{L}\frac{1}{h}\left(2^{K+1}+\mathbb{E}\max_{1\le j\le 2^K}|\theta_j(h)|^2\right)\right)$$

$$\le C\left(\frac{2^{2(K+1)}}{L}+\sum_{h=1}^{L}\frac{1}{h}\left(2^{K+1}+\frac{7}{2}(K+1)^2\left(\frac{2^{(K+1)}}{|\phi(hx)-1|}\right)\right)\right). \tag{1}$$

Let $\Lambda:\mathbb{R}_+\to\mathbb{R}_+$ be a non-decreasing and such that for any $m\in\mathbb{N}$,

$$\sum_{h=1}^{L}\frac{1}{h|\phi(hx)-1|}\le\Lambda(L).$$

Then the left hand side of (1)

$$\ll\left(\frac{2^{2(K+1))}}{L}+(\log L)2^{(K+1)}+\Lambda(L)(K+1)^2 2^{(K+1)}\right). \tag{2}$$

Here, of course, $\ll$ denotes Vinogradov order notation. Recall that there exists

$$C>0 \quad \text{such that} \quad |1-\phi(t)|\ge C|t|.$$

Since

$$\phi(hx)=\phi(\{hx\}),$$

we therefore have

$$\sum_{h=1}^{L}\frac{1}{h|\phi(hx)-1|}=O\left(\sum_{h=1}^{L}\frac{1}{h||hx||}\right).$$

If x is irrational of type $\eta>1$ [KN, p. 123, Lemma 3.3] for any $\epsilon>0$, then

$$\sum_{h=1}^{L}\frac{1}{h||hx||}=O(L^{\eta-1+\epsilon}).$$

In consequence, we can choose $\Lambda(L)=L^{\eta-1+\epsilon}$ the right hand side of (2) is

$$\ll\frac{2^{2(K+1)}}{L}+(\log L)2^{(K+1)}+\Lambda(L)(K+1)^2 2^{(K+1)}. \tag{3}$$

Choosing L essentially optimally $2^K\approx L\Lambda(L)=L^{\eta+\epsilon}$ the right hand side of (3) is

$$\ll 2^{(K+1)(2-\frac{1}{\eta}+\epsilon)}(K+1)^2. \tag{4}$$

Now we will complete the proof of our Theorem.

Proof of Theorem. Given $\epsilon, \epsilon_0 > 0$, we define

$$E_{\epsilon,\epsilon_0} = \{\omega \in \Omega : \Delta_N(\omega, x) > N^{2-\frac{1}{\eta}+\epsilon}(\log N)^{3+\epsilon_0} \text{ for infinitely many } N\}.$$

We now proceed to show the P measure of E_{ϵ,ϵ_0} is zero. If we denote

$$A_K = \left\{\omega \in \Omega : \max_{2^{K-1} \leq m < 2^K} \Delta_m(\omega, x) > 2^{K(2-\frac{1}{\eta}+\epsilon)} K^{3+\epsilon_0}\right\} \quad \text{for each} \quad K \geq 1,$$

then one easily sees that

$$E_{\epsilon,\epsilon_0} \subseteq \bigcap_{r=1}^{\infty} \bigcup_{K=r}^{\infty} A_K.$$

Using (4) we can bound

$$\begin{aligned}
\mathrm{P}(A_K) &\leq \frac{\mathbb{E}\left|\max_{2^{K-1} \leq m < 2^K} \Delta_m(x)\right|}{2^{K(2-\frac{1}{\eta}+\epsilon)} K^{3+\epsilon_0}} \\
&\leq \frac{C 2^{K(2-\frac{1}{\eta}+\epsilon)} K^2}{K^{3+\epsilon_0} 2^{K(2-\frac{1}{\eta}+\epsilon)}} \\
&\leq C K^{-(1+\epsilon_0)} \quad \text{for sufficiently large } C > 0.
\end{aligned}$$

In particular, now we can observe that

$$\sum_{K=1}^{\infty} \mathrm{P}(A_K) \leq \sum_{K=1}^{\infty} K^{-(1+\epsilon_0)} < +\infty.$$

It follows from the Borel-Cantelli lemma that $\mathrm{P}(E_{\epsilon,\epsilon_0}) = 0$. □

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ON SMALL SETS OF DISTRIBUTION FUNCTIONS OF RATIO BLOCK SEQUENCES

David Krčmarský — Ladislav Mišík — Zuzana Václavíková

Dedicated to the memory of Professor Pierre Liardet

ABSTRACT. There are various methods how to describe and characterize distribution of elements of sets of positive integers. One of the most interesting is that using the set of all distribution functions of the corresponding ratio block sequence introduced in [Strauch, O.—Tóth, J. T.: Publ. Math. Debrecen **58** (2001), no. 4, 751–778]. In the present paper we give some sufficient conditions under which this set is small in a metric sense. As a corollary we obtain a new characterization of the case of asymptotic distribution.

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1. Introduction

Basic definitions and properties

• From now on $X = \{x_1 < x_2 < \cdots\}$ denotes an increasing sequence of positive integers and $x \in [0,1)$.
• For every positive integer n put

$$X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \cdots \frac{x_n}{x_n}\right),$$

the nth term of the ratio block sequence $(X_n)_{n=1}^{\infty}$.
• Denote by $F(X_n, x)$ the step distribution function

$$F(X_n, x) = \frac{\#\{i \leq n; \frac{x_i}{x_n} < x\}}{n},$$

for $x \in [0,1)$ and for $x = 1$ we define $F(X_n, 1) = 1$.

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• Directly from the definition we obtain (see [ST])

$$F(X_m, x) = \frac{n}{m} F\left(X_n, x\frac{x_m}{x_n}\right) \tag{1}$$

for each $m \leq n$ for every $x \in [0, 1)$.

• For any positive t denote by $X(t)$ the counting function

$$X(t) = \#\{n \in \mathbb{N}; x_n < t\}.$$

The lower asymptotic density $\underline{d}$ and the upper asymptotic density $\overline{d}$ of $X = \{x_1 < x_2 < \dots\} \subset \mathbb{N}$ are defined as

$$\underline{d}(X) = \liminf_{t\to\infty} \frac{X(t)}{t} = \liminf_{n\to\infty} \frac{n}{x_n},$$

$$\overline{d}(X) = \limsup_{t\to\infty} \frac{X(t)}{t} = \limsup_{n\to\infty} \frac{n}{x_n}.$$

In the case when $\underline{d}(X) = \overline{d}(X)$ we speak about asymptotic density of X and denote this common value by $d(X)$.

• A non-decreasing function

$$g : [0, 1] \to [0, 1], \qquad g(0) = 0, \;\; g(1) = 1$$

is called distribution function (abbreviating d.f.). We shall identify any two d.f. coinciding at common points of continuity. We will denote the set of all distribution functions by $\mathcal{D}$. Let us note that $\mathcal{D}$ endowed with the metric

$$\varrho(g_1, g_2) = \sqrt{\int_0^1 \big(g_1(x) - g_2(x)\big)^2 \,\mathrm{d}x}$$

is a compact topological space [W].

• A d.f. $g(x)$ is a d.f. of the sequence of blocks $(X_n)_{n=1}^{\infty}$ if there exists an increasing sequence $n_1 < n_2 < \cdots$ of positive integers such that

$$\lim_{k\to\infty} F(X_{n_k}, x) = g(x) \quad \text{a.e. on } [0, 1].$$

This is equivalent to the weak convergence, i.e., the preceding limit holding for every point $x \in [0, 1]$ of continuity of $g(x)$, and it is also equivalent to the convergence with respect to the $\mathcal{L}^2$ metric in $\mathcal{D}$.

• Denote by $G(X_n)$ the set of all d.f.s of a ratio block sequence (X_n). For a singleton $G(X_n) = \{g(x)\}$, the d.f. $g(x)$ is also called asymptotic d.f. (abbreviating a.d.f.) of (X_n).

• It is evident that every $G(X_n)$ is nonempty, by compactness of $\mathcal{D}$, and closed, as a set of all limit points of a sequence in metric space. One of the most challenging open problem of theory of sets of distribution functions of ratio block sequences is that of characterization of all possible sets $G(X_n)$, see [SN]. It can be formulated as: given a nonempty closed subset H of $\mathcal{D}$, decide if there exists a set $X \subset \mathbb{N}$ such that $G(X_n) = H$. The simplest case, when $G(X_n)$ is a singleton was characterized in [ST]. It was shown that the only possible singletons are of the form $G(X_n) = \{x^\lambda\}$, where $\lambda \in [0,1]$ (here x^0 means the distribution function identically equal to 1 for all $x \in (0,1]$). In addition, the case $\lambda = 0$ was completely characterized. The remaining cases for $\lambda \in (0,1]$ were characterized in [FT], where it is shown that for $0 < \lambda \le 1$ the equality $G(X_n) = \{x^\lambda\}$ happens if and only if

$$\lim_{n\to\infty} \frac{x_{kn}}{x_n} = k^{\frac{1}{\lambda}} \quad \text{holds for every } \ k = 1, 2, \dots$$

For more information in the topic see [SP], [ST] and [GS].

• In this paper, we will show that if there exists a function $f \in G(X_n)$ and a sequence of indices (n_k) such that $f = \lim_{k\to\infty} F(X_{n_k}, x)$ and (n_k) is large in some sense, then the set $G(X_n)$, as a subset of the metric space $(\mathcal{D}, \varrho)$, has a small diameter.

As corollaries of the general theorem we give several sufficient conditions under which the set $G(X_n)$ is a singleton. We will show that if there exists a function $f \in G(X_n)$ such that the set of indices n_k with $f = \lim_{k\to\infty} F(X_{n_k}, x)$ is sufficiently large then necessarily $G(X_n) = \{f\}$. Usually the term "sufficiently large set" is related to asymptotic density. We will also use another kind of density, the so called gap density introduced in the paper [GV]. It is defined by

$$\lambda(X) = \limsup_{n\to\infty} \frac{x_{n+1}}{x_n}, \quad \text{where} \quad X = \{x_1 < x_2, \dots\} \subset \mathbb{N}.$$

2. Results

The following simple lemma plays the key role in our investigation.

Lemma 1. *Let $m < n$ be positive integers. Then*

$$F(X_n, x) \ge \frac{m}{n} F(X_m, x) \tag{2}$$

for every $x \in [0,1]$.

P r o o f. Let $m < n$ be positive integers and $x \in (0,1)$. Using the obvious inclusion

$$\left\{i \le m; \frac{x_i}{x_m} < x\right\} \subset \left\{i \le n; \frac{x_i}{x_n} < x\right\}$$

calculate

$$F(X_n, x) = \frac{\#\{i \le n; \frac{x_i}{x_n} < x\}}{n} \ge \frac{\#\{i \le m; \frac{x_i}{x_m} < x\}}{m}\frac{m}{n} = \frac{m}{n}F(X_m, x).$$

□

Recall that the diameter of a set A, a subset of a metric space (X, ϱ), is defined by

$$\operatorname{diam}\ A = \sup\bigl\{\varrho(a, a') \mid a,\ a' \in A\bigr\}.$$

Now we are prepared to formulate our general theorem.

Theorem 1. *Let $X \subset \mathbb{N}$, $f \in G(X_n)$ and let $J = \{j_1 < j_2 < \dots\} \subset \mathbb{N}$ be such that*

$$\lim_{j \in J} F(X_j, x) = f(x) \quad \textit{and} \quad \lambda(J) < \infty.$$

Then

$$\operatorname{diam} G(X_n) \le \bigl(\lambda(J) - 1\bigr)\sqrt{\int_0^1 f^2(x)\,\mathrm{d}x}. \tag{3}$$

P r o o f. Put $\lambda = \lambda(J)$ and let g be any function belonging to $G(X_n)$. A simple compactness argument yields that there exist:

- numbers α and β such that $0 < \frac{1}{\lambda} \le \alpha \le \beta \le 1$,
- a subsequence (u_k) of the sequence (j_n) and
- a sequence of positive integers (v_k) with $u_k < v_k < u_{k+1}$ for all $k \in \mathbb{N}$

suh that

$$\lim_{k\to\infty} \frac{u_k}{u_{k+1}} = \alpha, \quad \lim_{k\to\infty} \frac{u_k}{v_k} = \beta \quad \text{and} \quad \lim_{k\to\infty} F(X_{v_k}, x) = g(x). \tag{4}$$

Two applications of Lemma 1, the first one with $m = u_k$, $n = v_k$ and the next one with $m = v_k$, $n = u_{k+1}$, yield

$$F(X_{v_k}, x) \ge \frac{u_k}{v_k}F(X_{u_k}, x) \quad \text{and} \quad F(X_{u_{k+1}}, x) \ge \frac{v_k}{u_{k+1}}F(X_{v_k}, x),$$

hence

$$\frac{u_k}{v_k}F(X_{u_k}, x) \le F(X_{v_k}, x) \le \frac{u_{k+1}}{v_k}F(X_{u_{k+1}}, x) \quad \text{for all} \quad x \in [0,1]. \tag{5}$$

Taking a limit in (5), by use of (4) we get

$$\beta f(x) \le g(x) \le \frac{\beta}{\alpha} f(x) \quad \text{for all} \quad x \in [0,1]. \tag{6}$$

Consequently,

$$\begin{aligned}
\operatorname{diam} G(X_n) &= \sup\{\varrho(g_1, g_2) \mid g_1, g_2 \in G(X_n)\} \le \varrho\left(\beta f(x), \frac{\beta}{\alpha} f(x)\right) \\
&= \sqrt{\int_0^1 \left(\frac{\beta}{\alpha} f(x) - \beta f(x)\right)^2 \mathrm{d}x} \\
&= \sqrt{\left(\frac{\beta}{\alpha} - \beta\right)^2 \int_0^1 f^2(x)\,\mathrm{d}x} \;\le\; (\lambda - 1)\sqrt{\int_0^1 f^2(x)\,\mathrm{d}x}\,,
\end{aligned}$$

where the last inequality follows from the fact that the function $\left(\frac{t}{\alpha} - t\right)^2$ attains its maximum $\left(\frac{1}{\alpha} - 1\right)^2 \le (\lambda - 1)^2$ in the interval $t \in [\alpha, 1]$ at point $t = 1$.

$\square$

It is evident, that the preceding theorem is valuable if either $\lambda(J)$ is not too much greater than 1, or when the integral $\int_0^1 f^2(x)\,\mathrm{d}x$ is small [1]. Two extremal cases, $\int_0^1 f^2(x)\,\mathrm{d}x = 0$ and $\lambda(J) = 1$ will be discussed in the rest of the paper.

First, let us suppose that $\int_0^1 f^2(x)\,\mathrm{d}x = 0$. This means that $f = c_1$, where c_1 is the distribution function with total jump at 1, i.e., $c_1(x) = 0$ for all $x \in [0,1)$. The following statement says that if c_1 belongs to $G(X_n)$, then the corresponding subsequence of step distribution functions converging to c_1 has arbitrary large gaps.

Proposition 1. *Let* $X \subset \mathbb{N}$ *and* $J \subset \mathbb{N}$ *be such that* $\lim_{j \in J} F(X_j, x) = c_1(x)$. *Then* $\lambda(J) = \infty$.

P r o o f. Suppose that $\lambda(J) < \infty$. Then, by Theorem 1, it would be that

$$G(X_n) = \{c_1\}.$$

But, by the result in [ST] characterising all possible singletons $G(X_n)$ and mentioned in the Introduction, this happens for no $X \subset \mathbb{N}$, a contradiction. $\square$

The extremal case $\lambda(J) = 1$ is presented by the next theorem.

[1] Another possible kind of application of the previous theorem can be the following one. Suppose that we know $G(X_n)$, i.e., also $\operatorname{diam} G(X_n)$ for some $X \subset \mathbb{N}$. Then the inequality (3) provides a lower bound for $\lambda(J)$ for arbitrary $f(x) \in G(X_n)$, in special cases also for $f(x) = x$.

Theorem 2. *Let $X \subset \mathbb{N}$. Then $G(X_n) = \{f\}$ if and only if there exists a set $J \subset \mathbb{N}$ such that*

$$\lim_{j \in J} F(X_j, x) = f(x) \quad \textit{and} \quad \lambda(J) = 1.$$

P r o o f. The implication $\Rightarrow$ is evident as $\lim_{n\to\infty} F(X_n, x) = f(x)$. The opposite implication follows from Theorem 1 and the fact that for every subset A of a metric space $\operatorname{diam} A = 0$ if and only if A is a singleton. $\square$

Example 1. Let $k \in \mathbb{N}$ and suppose that there exists $f \in G(X_n)$ such that $\lim_{n\to\infty} F(X_{n^k}, x) = f$. Then $G(X_n) = \{f\}$.

Remark 1. Roughly spoken, Theorem 2 says that if there exists a distribution function in $G(X_n)$ so that it is a limit along a "quite large and regular" set of indices, then $G(X_n)$ is a singleton.

Now we will introduce two important special cases of the preceding theorem. For the first one we need the following definition. It was introduced by Pólya in [Po] in a bit more general context.

Definition 1. Let $X \subset \mathbb{N}$. The lower and upper Pólya densities of X are defined by

$$\underline{p}(X) = \lim_{\delta\to 0^+} \liminf_{n\to\infty} \frac{\#\bigl(X \cap ((1-\delta)n, n]\bigr)}{\delta n}$$

and

$$\overline{p}(X) = \lim_{\delta\to 0^+} \limsup_{n\to\infty} \frac{\#\bigl(X \cap ((1-\delta)n, n]\bigr)}{\delta n},$$

respectively.

Remark 2. It is known [Po] that for every set X of positive integers the following inequalities

$$0 \le \underline{p}(X) \le \underline{d}(X) \le \overline{d}(X) \le \overline{p}(X) \le 1 \tag{7}$$

hold. In addition, the equality $\underline{d}(X) = \overline{d}(X)$ implies the equality $\underline{p}(X) = \overline{p}(X)$.

Corollary 1. *Let $X \subset \mathbb{N}$ be such that there exists a function $f \in G(X_n)$ and a set $K \subset \mathbb{N}$ with $\underline{p}(K) > 0$ so that $\lim_{k\in K} F(X_k, x) = f$. Then $G(X_n) = \{f\}$.*

P r o o f. Let $K = \{k_1 < k_2 < \cdots\}$. By Theorem 1 it is sufficient to prove that $\lim_{n\to\infty} \frac{k_{n+1}}{k_n} = 1$. Suppose the contrary, i.e., $\limsup_{n\to\infty} \frac{k_{n+1}}{k_n} = h > 1$. Choose any $q \in (\frac{1}{h}, 1)$. Then there exists an increasing sequence (n_m) of positive integers such that $\frac{k_{n_m - 1}}{k_{n_m}} < q$. Put $\delta = 1 - q$ and notice that $K \cap \bigl((1-\delta)k_{n_m}, k_{n_m}\bigr] = \emptyset$ for every $m \in \mathbb{N}$, contradicting $\underline{p}(K) > 0$. $\square$

The next statement follows from the previous one and the last sentence in Remark 2.

Corollary 2. *Let $X \subset \mathbb{N}$ be such that there exists a function $f \in G(X_n)$ and a set $K \subset \mathbb{N}$ with $d(K) > 0$ so that* $\lim_{k\in K} F(X_k, x) = f$. *Then $G(X_n) = \{f\}$.*

In the previous corollary "quite large and regular" set of indices (see Remark 1) means that it possesses a positive asymptotic density. One can wonder what happens if we lose a part of this information, i.e., the set of indices is either "quite large" or "quite regular". The first case is discussed in Example 2. In fact, it says that there are very large sets of indices, with respect to asymptotic densities, such that the convergence along them does not guarantee that $G(X_n)$ is a singleton. To introduce the example we will need some notation and a simple lemma. For every $\gamma \in [0,1]$ let us denote by h_γ the distribution function such that $h_\gamma(x) = \gamma$ for all $x \in (0,1)$.

Lemma 2. *For every $n \in \mathbb{N}$ let a finite sequence $T_n = (t_1 < t_2 < \cdots < t_{k_n})$ in $[0,1]$ be given and put $F_n(x) = \frac{\#\{i \le k_n \mid t_i < x\}}{k_n}$. Suppose that there exists a sequence of non-negaitive integers (m_n) with $m_n \le k_n$ and such that*

$$t_{m_n} \to 0, \qquad t_{m_n+1} \to 1 \qquad \textit{and} \qquad \frac{m_n}{k_n} \to \gamma \in [0,1].$$

Then $F_n(x) \to h_\gamma(x)$ for every $x \in [0,1]$.

P r o o f. The statement of lemma definitely holds for $x = 0$ and $x = 1$, thus suppose that $x \in (0,1)$. Then there exists n_0 such that for all $n > n_0$ inequalities $t_{m_n} < x < t_{m_n+1}$ hold. Thus

$$F_n(x) = \frac{\#\{i \le k_n \mid t_i < x\}}{k_n} = \frac{m_n}{k_n} \to \gamma.$$

□

Example 2. For every $\varepsilon \in (0,1)$ there exist sets $X \subset \mathbb{N}$, $K \subset \mathbb{N}$ such that

$$\underline{d}(K) = 1 - \varepsilon, \qquad \overline{d}(K) = 1 \tag{8}$$

and $\lim\limits_{k\in K} F(X_k, x) = h_1(x)$ for almost all $x \in [0,1]$, but

$$G(X_n) = \big\{h_\gamma \mid \gamma \in [1-\varepsilon, 1]\big\},$$

consequently $\operatorname{diam} G(X_n) = \varepsilon$.

P r o o f. First put $\eta = \frac{\varepsilon}{1-\varepsilon}$, i.e., $\varepsilon = \frac{\eta}{1+\eta}$. To construct the example set

$$J = \bigcup_{n=1}^{\infty} \big[n! + 2, (1+\eta)n!\big) \cap \mathbb{N} \quad \text{and} \quad K = \mathbb{N} \setminus J.$$

Then for every sufficiently large n

$$J\big((1+\eta)n!\big) \approx \sum_{k=1}^{n} \eta k! = \eta \sum_{k=1}^{n} k! \approx \eta n!$$

and consequently,

$$\underline{d}(K) = 1 - \overline{d}(J) = 1 - \lim_{n\to\infty} \frac{\eta n!}{(1+\eta)n!} = 1 - \frac{\eta}{1+\eta} = 1 - \varepsilon.$$

Similarly,

$$\overline{d}(K) = \lim_{n\to\infty} \frac{K(n!)}{n!} = \lim_{n\to\infty} \frac{n! - \eta(n-1)!}{n!} = 1$$

which verifies (8). Now define

$$x_k = k! \quad \text{for all} \;\; k \in K, \qquad x_j = x_{j-1} + 1 \quad \text{for all} \;\; j \in J$$

and let

$$X = \{x_1, x_2, \dots\}.$$

Notice that

$$x_n \le n! \quad \text{holds for every} \quad n \in \mathbb{N}. \tag{9}$$

Now we show that

$$\lim_{k\in K} F(X_k, x) = h_1(x) \quad \text{for all} \quad x \in [0,1].$$

Let (k_n) be an increasing sequence in K. Then, by (9), for every n all terms of the block

$$X_{k_n} = \left(\frac{x_1}{x_{k_n}}, \frac{x_2}{x_{k_n}}, \dots, \frac{x_{k_n-1}}{x_{k_n}}, \frac{x_{k_n}}{x_{k_n}} \right),$$

except the last one are less than or equal to $\frac{x_{k_n-1}}{x_{k_n}} \le \frac{(k_n-1)!}{(k_n)!} \le \frac{1}{k_n} \to 0$. Using Lemma 2 with $m_n = k_n - 1$, giving $\gamma = \lim\limits_{n\to\infty} \frac{m_n}{k_n} = 1$ yields

$$\lim_{k\in K} F(X_k, x) = h_1(x) \quad \text{for all} \quad x \in [0,1].$$

Now suppose that (j_n) is an increasing sequence in J such that there exists

$$\lim_{n\to\infty} F(X_{j_n}, x) \quad \text{for almost all} \quad x \in [0,1].$$

Then, passing to a suitable subsequence, if necessary, we can assume that there exists an increasing sequence of positive integers (k_n) such that

(i) $(k_n)! < j_n \le (1+\eta)(k_n)!$

(ii) $\lim\limits_{n\to\infty} \frac{(k_n)!}{j_n} = \gamma \in \left[\frac{1}{1+\eta}, 1\right]$.

For simplicity put $K_n = (k_n)!$ and look at the terms of

$$X_{j_n} = \left(\frac{x_1}{x_{j_n}}, \dots, \frac{x_{K_n}}{x_{j_n}}, \frac{x_{K_n+1}}{x_{j_n}}, \dots, \frac{x_{j_n}}{x_{j_n}}\right).$$

By definition of sets J and K, the elements K_n and $K_n + 1$ belong to K, while $K_n + 2 \in J$. Consequently, for $m \le K_n$ we have

$$\frac{x_m}{x_{j_n}} \le \frac{x_{K_n}}{x_{j_n}} \le \frac{(K_n)!}{(K_n+1)!} = \frac{1}{K_n+1} \to 0.$$

On the other hand, for $K_n < m \le j_n$ and putting

$$m - K_n - 1 = a, \qquad j_n - K_n - 1 = b,$$

we have

$$\frac{x_m}{x_{j_n}} = \frac{(K_n+1)! + a}{(K_n+1)! + b} \to 1$$

as both a and b are less than or equal to

$$\eta K_n = o\big((K_n)!\big).$$

Now application of Lemma 2 yields

$$\lim_{n\to\infty} F(X_{j_n}, x) \to h_\gamma(x) \quad \text{for almost all} \quad x \in [0,1].$$

On the other hand, for each $\gamma \in [\frac{1}{1+\eta}, 1]$ such a sequence (j_n) exists, it is sufficient to put $j_n = \lfloor \frac{1}{\gamma} n! \rfloor$. This proves

$$G(X_n) = \big\{h_\gamma \mid \gamma \in [1-\varepsilon, 1]\big\} \quad \text{and} \quad \operatorname{diam} G(X_n) = \varepsilon. \qquad \square$$

In the context of the previous example, let us note that in [GS] an example of an increasing sequence of positive integers $x_n,\ n = 1, 2, \dots$ such that there is given

$$G(x_n) = \big\{h_\alpha(x);\ \alpha \in [0,1]\big\}.$$

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LINEAR RECURSIVE ODOMETERS AND BETA-EXPANSIONS

Maria Rita Iacò—Wolfgang Steiner—Robert F. Tichy

Dedicated to the memory of Professor Pierre Liardet

ABSTRACT. The aim of this paper is to study the connection between different properties related to β-expansions. In particular, the relation between two conditions, both ensuring purely discrete spectrum of the odometer, is analyzed. The first one is the so-called Hypothesis B for the G-odometers and the second one is denoted by (QM) and it has been introduced in the framework of tilings associated to Pisot β-numerations.

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1. Introduction

In the early 1990's Pierre Liardet visited Graz several times and there with Peter Grabner and the third author started an intensive cooperation on dynamic properties of digital expansions. Their main results from that period where published in [8]. In this work, the basic theory of odometers was developed. In the subsequent years, these aspects of arithmetic dynamics were extended by various authors. Several PhD students of Pierre worked in this field, in particular we want to mention Guy Barat who received his PhD in Marseilles 1995 and his habilitation at Graz University of Technology 2006.

A special focus lies on arithmetic conditions which guarantee purely discrete spectrum of the odometer. In [10], and before in [8], the authors posed the question whether Hypothesis B, introduced in [8], and the finiteness property (F), introduced in [7], are equivalent. Hypothesis B is a condition on the carries

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of the digits in the expansion of positive integers in a base system defined by a linear recurrence.

Let $(G_k)_{k\geq 0}$ be an increasing sequence of positive integers, with initial value $G_0 = 1$. Then every positive integer can be expanded as

$$n = \sum_{k=0}^{\infty} \varepsilon_k(n) G_k \ ,$$

where $\varepsilon_k(n) \in \{0, \ldots, \lceil G_{k+1}/G_k \rceil - 1\}$ and $\lceil x \rceil$ denotes the smallest integer not less than $x \in \mathbb{R}$. This expansion (called G-expansion) is uniquely determined and finite, provided that for every K,

$$\sum_{k=0}^{K-1} \varepsilon_k(n) G_k < G_K. \tag{1}$$

For short we write ε_k for the k-th digit of the G-expansion; $G = (G_k)_{k\geq 0}$ is called numeration system and the digits ε_k can be computed by the greedy algorithm.

We denote by $\mathcal{K}_G$ the subset of sequences that satisfy (1) and we call its elements G-admissible. More precisely,

$$\mathcal{K}_G = \left\{ \epsilon_0 \epsilon_1 \epsilon_2 \cdots \, : \, \forall j \geq 0 \, , \quad \epsilon_0 G_0 + \cdots + \epsilon_j G_j < G_{j+1} \right\}.$$

In order to extend the addition-by-one map τ defined on $\mathbb{N}$ to $\mathcal{K}_G$ the following subset of $\mathcal{K}_G$ is introduced:

$$\mathcal{K}_G^0 = \left\{ x \in \mathcal{K}_G \ : \exists M_x, \forall j \geq M_x, \ \sum_{k=0}^{j} \varepsilon_k G_k \ < G_{j+1} - 1 \right\}. \tag{2}$$

Put $x(j) = \sum_{k=0}^{j} \varepsilon_k G_k$, and set

$$\tau(x) = \Big(\varepsilon_0\big(x(j)+1\big) \, \ldots \, \varepsilon_j\big(x(j)+1\big) \Big) \varepsilon_{j+1}(x) \varepsilon_{j+2}(x) \ldots \, , \tag{3}$$

for every $x \in \mathcal{K}_G^0$ and $j \geq M_x$. This definition does not depend on the choice of $j \geq M_x$. We extend the definition of τ to sequences x in $\mathcal{K}_G \setminus \mathcal{K}_G^0$ by $\tau(x) = 0 = (0^\infty)$; in this way, the transformation τ is defined on $\mathcal{K}_G$ and it is called G-odometer.

As in [8], we consider sequences $(G_k)_{k\geq 0}$ associated to real numbers $\beta > 1$, defined by

$$G_k = \sum_{j=1}^{k} a_j G_{k-j} + 1, \tag{4}$$

where $a_1 a_2 \cdots$ is the quasi-greedy β-expansion of 1, i.e., the smallest sequence (w.r.t. the lexicographical order) containing infinitely many non-zero digits and satisfying $1 = \sum_{j=1}^{\infty} a_j \beta^{-j}$. By [8], the G-odometer $(\mathcal{K}_G, \tau)$ is continuous if and only if $a_1 a_2 \cdots$ is purely periodic. Note that, when $a_1 a_2 \cdots$ has period length d, we have

$$G_k = \sum_{j=1}^{d} a_j G_{k-j} + G_{k-d} \quad \text{for all } k \geq d.$$

For purely periodic $a_1 a_2 \cdots$, it was shown in [8] also that $(\mathcal{K}_G, \tau)$ is uniquely ergodic, an explicit formula for the unique invariant measure μ defined on $\mathcal{K}_G$ is provided. For more general G-expansions, unique ergodicity follows from the work of Barat and Grabner [3]. The following condition is used to prove that $\mathcal{K}_G$ has purely discrete spectrum.

Hypothesis B. *There exists an integer $b \geq 0$ such that for all k and*

$$N = \sum_{i=0}^{k-1} \epsilon_i(N) G_i + \sum_{j=k+b}^{\infty} \epsilon_j(N) G_j,$$

the addition of G_m to N, where $m \geq k+b$, does not change the first k digits in the greedy representation, i.e.,

$$N + G_m = \sum_{i=0}^{k-1} \epsilon_i(N) G_i + \sum_{j=k}^{\infty} \epsilon_j(N + G_m) G_j.$$

The finiteness property (F) is defined in the framework of β-expansions. Let $\beta > 1$ be a fixed real number. A β-expansion of a real number $x \in [0,1)$ is a representation of the form

$$x = \sum_{i=1}^{\infty} \epsilon_i \beta^{-i},$$

where $\epsilon_i \in \{0, 1, \ldots, \lceil \beta \rceil - 1\}$ and $\lceil x \rceil$ denotes the smallest integer not less than x. Beta-expansions were introduced by Rényi [14] and generalize standard representations in an integral base. These expansions can be obtained via the iteration of the so-called β-transformation T_β defined by

$$T_\beta \colon [0,1) \to [0,1)\ , \quad x \mapsto \beta x - \lfloor \beta x \rfloor\ ,$$

where $\lfloor x \rfloor$ is the largest integer not exceeding x. Taking, at each iteration of T_β, $\epsilon_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor$, we obtain the following greedy expansion of x

$$x = \sum_{k=1}^{\infty} \epsilon_k \beta^{-k} = 0.\epsilon_1 \epsilon_2 \epsilon_3 \ldots$$

To obtain the quasi-greedy β-expansions, one can use the transformation

$$\tilde{T}_\beta : \ (0,1] \to (0,1], \quad x \mapsto \beta x - \lceil \beta x \rceil + 1,$$

which differs from T_β only at the points of discontinuity. Then the quasi-greedy β-expansion of 1 is given by $a_j = \lceil \beta \tilde{T}_\beta^{j-1}(1) \rceil - 1$. Let

$$V_\beta = \big\{ \tilde{T}_\beta^k(1) : k \geq 0 \big\}.$$

If V_β is finite, then β is called a Parry number. As for G-adic expansions, not all strings of digits in $\{0, 1, \ldots, \lceil \beta \rceil - 1\}$ are admissible. Parry [13] observed that a sequence $\epsilon_1 \epsilon_2 \cdots$ is admissible if and only if

$$\epsilon_j \epsilon_{j+1} \cdots < a_1 a_2 \cdots \qquad \text{for all } j \geq 1 \ . \tag{5}$$

A sequence is the β-expansion of some $x \in [0,1)$ if and only if it is admissible.

A significant question in this setting is for which β is the expansion in base β of x finite, i.e., it is important to provide a description of the set

$$\mathrm{Fin}(\beta) = \{x \in [0,1) : \exists k \geq 0, \ T_\beta^k(x) = 0\}.$$

Note that many authors rather consider $x \in [0, \infty)$ in the definition of $\mathrm{Fin}(\beta)$, with the condition that $T_\beta^k(\beta^{-n} x) = 0$ for $x \in [0, \beta^n)$. A number β is said to have the *finiteness property* if

$$\mathrm{Fin}(\beta) = \mathbb{Z}[\beta^{-1}] \cap [0,1) \tag{F}$$

holds. This property was introduced by Frougny and Solomyak [7], and they proved [7, Lemma 1] that if (F) holds, then β is a Pisot number. An algebraic integer $\beta > 1$ is called a Pisot number if all its Galois conjugates have modulus less than 1. However, there exist Pisot numbers that do not fullfill (F), such as all numbers with non purely periodic quasi-greedy β-expansion of 1. In [7], it is also shown that if $a_1 \geq a_2 \geq a_3 \geq \cdots$, then

$$\mathbb{Z}_+[\beta^{-1}] \cap [0,1) \subseteq \mathrm{Fin}(\beta) \ , \tag{PF}$$

where $\mathbb{Z}_+ = \mathbb{Z} \cap [0, \infty)$. This condition is usually referred to as the *positive finiteness* condition (PF), and it is equivalent to say that $\bigcup_{n \geq 0} \beta^n \mathrm{Fin}(\beta)$ is closed under addition. Akiyama [2, Theorem 1] proved that if $\beta > 1$ is a real number satisfying (PF), then β satisfies (F) or $a_1 \geq a_2 \geq \cdots$. This result will be used in the proof of Lemma 4.

In the present paper we show that Property (F) does not imply Hypothesis B. We show that we also need the so-called *quotient mapping condition*

$$\mathrm{rank}\big(\langle V_\beta - V_\beta \rangle_{\mathbb{Z}}\big) = \deg(\beta) - 1, \tag{QM}$$

where $\langle V_\beta - V_\beta \rangle_{\mathbb{Z}}$ denotes the $\mathbb{Z}$-module spanned by differences of elements of V_β.

This condition was introduced by Siegel and Thuswaldner [15] in the framework of tilings associated to Pisot substitutions, and for β-expansions in the present form in [12]. Moreover, if we allow sequences $a_1 a_2 \cdots$ that are not purely periodic, then Hypothesis B does not imply (F) but only (PF).

Theorem 1. *Let $\beta > 1$. Hypothesis B holds for the sequence $(G_k)_{k \geq 0}$ associated to β if and only if conditions* (PF) *and* (QM) *hold.*

Since $\langle V_\beta \rangle_{\mathbb{Z}} = \mathbb{Z}[\beta]$, condition (QM) holds when $\# V_\beta = \deg(\beta)$, i.e., when G satisfies a linear recurrence with the minimal polynomial of β as characteristic polynomial. A class of non-trivial examples of bases satisfying (QM) was given in [15, 12] by $\beta^3 = t\beta^2 - \beta + 1$, $t \geq 2$; in this case, we have $\# V_\beta = \deg(\beta) + 1$. The following theorem gives a characterization of (QM) for $\beta > 1$ satisfying

$$\# V_\beta = \deg(\beta) + 1 \, .$$

Theorem 2. *Let $\beta > 1$ be such that*

$$\# V_\beta = \deg(\beta) + 1, \quad \text{with} \quad \tilde{T}_\beta^{\deg(\beta)+1}(1) = \tilde{T}_\beta^{k}(1), \quad 0 \leq k \leq \deg(\beta),$$

i.e.,

$$a_1 a_2 \cdots = a_1 \cdots a_k \, (a_{k+1} \cdots a_{\deg(\beta)+1})^\infty .$$

Then β satisfies (QM) *if and only if* $\deg(\beta) - k$ *is even.*

In particular, when β is a simple Parry number with $\# V_\beta = \deg(\beta) + 1$, we have $k = 0$ and thus (QM) holds if and only if $\deg(\beta)$ is odd, e.g., for $\beta^3 = 3\beta^2 - 2\beta + 2$ or $\beta^3 = 3\beta^2 - \beta + 1$. Of course, it would be interesting to know what happens if we drop the condition $\# V_\beta = n = \deg(\beta) + 1$ and if there still exist numbers β for which (QM) holds.

We conclude with a theorem showing that the odometer has purely discrete spectrum when β is a Pisot number satisfying (QM), even when (PF) does not hold. Its proof is based on recent results by Barge [5].

Theorem 3. *Let β be a Pisot number satisfying* (QM)*. Then the odometer $(\mathcal{K}_G, \tau_G)$ associated to β has purely discrete spectrum (with respect to the unique invariant measure).*

We do not know whether (QM) is a necessary condition for purely discrete spectrum.

2. Quotient mapping condition

We first explain the relation of the condition (QM) above to the quotient mapping condition defined in [15] for substitutions. Let σ be a primitive substitution on a finite alphabet A and $M_\sigma = \big(|\sigma(j)|_i\big)_{i,j\in A}$ its incidence matrix, where $|\sigma(j)|_i$ denotes the number of occurrences of the letter i in $\sigma(j)$.
Let $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ be a left eigenvector of M_σ to the dominant eigenvalue $\beta > 1$, with $v_i \in \mathbb{Q}(\beta)$, and

$$L_\sigma = \langle v_i - v_j : i, j \in A\rangle_{\mathbb{Z}}$$

be the $\mathbb{Z}$-module generated by the differences of coordinates of $\mathbf{v}$. Note that $\mathbf{v}$ and L_σ are only defined up to a constant factor, which plays no role in the following. The substitution σ satisfies the quotient mapping condition if

$$\mathrm{rank}(L_\sigma) = \deg(\beta) - 1.$$

This definition is equivalent to Definition 3.13 in [15].

Let now $\beta > 1$ be a Parry number, with $\#V_\beta = n$, and $a_1 \cdots a_k (a_{k+1} \cdots a_n)^\omega$ its quasi-greedy β-expansion of 1. Then the *β-substitution* σ_β is defined on $A = \{1, 2, \ldots, n\}$ by

$$\sigma_\beta : \quad i \mapsto \underbrace{1\,1\,\cdots\,1}_{a_i \text{ times}}(\,i+1) \quad \text{if } 1 \le i < n,$$
$$n \mapsto \underbrace{1\,1\,\cdots\,1}_{a_n \text{ times}}(k+1).$$

Then $\big(1, \tilde{T}_\beta(1), \ldots, \tilde{T}_\beta^{n-1}(1)\big)$ is a left eigenvector of σ_β, and $L_{\sigma_\beta} = \langle V_\beta - V_\beta\rangle_{\mathbb{Z}}$, thus (QM) holds if and only if the β-substitution satisfies the quotient mapping condition of [15].

For an algebraic number β with r real and s complex conjugates $\beta_1, \ldots, \beta_r$, $\beta_{r+1}, \ldots, \beta_{r+s}$, set

$$\delta_\infty : \mathbb{Q}(\beta) \to \mathbb{R}^r \times \mathbb{C}^s, \quad x \mapsto \Big(x^{(1)}, \ldots, x^{(r)}, x^{(r+1)}, \ldots, x^{(r+s)}\Big),$$

where $x \mapsto x^{(i)}$ is the Galois embedding $\mathbb{Q}(\beta) \to \mathbb{R}$ or $\mathbb{C}$ that maps β to $\beta^{(i)}$.

Proposition 1. *A primitive substitution σ satisfies the quotient mapping condition if and only if there exists $c \in \mathbb{Q}(\beta)$ such that, for the scalar product, $\delta_\infty(c) \cdot \delta_\infty(v_i) = 1$ for all $i \in A$.*

Proof. If the quotient mapping condition holds, then there exists $c \in \mathbb{Q}(\beta)$ such that $\delta_\infty(c) \cdot \delta_\infty(x) = 0$ for all $x \in L_\sigma$. This implies that

$$\delta_\infty(c) \cdot \delta_\infty(v_i) = \delta_\infty(c) \cdot \delta_\infty(v_j) \text{ for all } i, j \in A, \quad \text{with} \quad q = \delta_\infty(c) \cdot \delta_\infty(v_i) \in \mathbb{Q}.$$

Then we have $\delta_\infty(c/q) \cdot \delta_\infty(v_i) = 1$ for all $i \in A$.

For the other direction, suppose that $\delta_\infty(c) \cdot \delta_\infty(v_i) = 1$ for all $i \in A$, thus $\delta_\infty(c) \cdot \delta_\infty(x) = 0$ for all $x \in L_\sigma$. Then L_σ has rank at most $\deg(\beta) - 1$. Since $\langle v_i : i \in A\rangle_{\mathbb{Z}}$ has full rank $\deg(\beta)$, the rank of L_σ is also at least $\deg(\beta) - 1$. □

Note that $\delta_\infty(c) \cdot \delta_\infty(v_i) = 1$ for all $i \in A$ means that the vector $(1, 1, \ldots, 1) \in \mathbb{Z}^n$ lies in the subspace spanned by the left eigenvectors of M_σ to the eigenvalues that are Galois conjugates of β.

We can now prove the characterization of (QM) for β with $\#V_\beta = \deg(\beta) + 1$.

Proof of Theorem 2. If $\#V_\beta = n = \deg(\beta)+1$, then the eigenvalues of M_{σ_β} are the conjugates of β and -1. Note that 1 cannot be an eigenvalue of M_{σ_β} because the characteristic polynomial of M_{σ_β} is

$$\left(x^n - a_1 x^{n-1} - a_2 x^{n-2} - \cdots - a_n\right) - \left(x^k - a_1 x^{k-1} - a_2 x^{k-2} - \cdots - a_k\right). \quad (6)$$

The right eigenspace to the eigenvalue -1 is spanned by $\mathbf{w} = {}^t(w_1, w_2, \ldots, w_n)$ with

$$w_i = \begin{cases} (-1)^i & \text{if} \quad k < i \le n, \\ (-1)^i \left(1 - (-1)^{n-k}\right) & \text{if} \quad 1 \le i \le k. \end{cases}$$

Indeed, we have

$$w_i + w_{i+1} = 0 \quad \text{for } 1 \le i < k \quad \text{and} \quad k < i < n, \quad \text{and} \quad w_k + w_{k+1} + w_n = 0.$$

By Proposition 1, (QM) is equivalent to the vector $\mathbf{1} = (1, 1, \ldots, 1)$ lying in the subspace spanned by the left eigenvectors corresponding to the conjugates of β. This means that $\mathbf{1}$ is orthogonal to $\mathbf{w}$, i.e., that $n - k$ is even. □

3. Equivalence of Hypothesis B and (PF) & (QM)

In this section, let $(G_k)_{k\ge0}$ be a sequence associated to $\beta > 1$, as defined in (4). If β is a Parry number, then we can write

$$G_k = (1, 1, \ldots, 1)\, M_{\sigma_\beta}^k\, {}^t(1, 0, \ldots, 0). \quad (7)$$

Lemma 1. *Property* (QM) *holds if and only if* $(G_k)_{k\ge0}$ *satisfies a recurrence with the minimal polynomial of* β *as characteristic polynomial.*

Proof. If (QM) holds, then, by (7) and Proposition 1, G_k satisfies a recurrence with the minimal polynomial of β as characteristic polynomial.

If (QM) does not hold, then $\mathbf{1} = (1, 1, \ldots, 1)$ does not lie in the subspace spanned by the eigenvectors corresponding to the conjugates of β. Let $\mathbf{1} = \mathbf{b} + \mathbf{c}$ be the decomposition in a vector $\mathbf{b}$ lying in this subspace and $\mathbf{c}$ lying in the complementary invariant subspace. By the structure of M_{σ_β}, we obtain that $\mathbf{c}\, M_{\sigma_\beta}^k\, {}^t(1, 0, \ldots, 0) \ne 0$ for some $k \ge 0$, thus G_j does not satisfy a recurrence with the minimal polynomial of β as characteristic polynomial. □

Lemma 2. *Hypothesis B implies* (PF)*, in particular β is a Pisot number.*

P r o o f. The proof is done by contradiction. Assume that (PF) does not hold. Then there is some $y \in \mathbb{Z}_+[\beta^{-1}] \cap [0,1)$ with $y \notin \mathrm{Fin}(\beta)$. We can choose y minimal in the sense that $x = y - \beta^{-n} \in \mathbb{Z}_+[\beta^{-1}] \cap \mathrm{Fin}(\beta)$ for some $n > 0$. Let $x = \sum_{j=1}^{\ell} x_j \beta^{-j}$ be the (finite) β-expansion of x, $y = \sum_{j=1}^{\infty} y_j \beta^{-j}$ the (infinite) β-expansion of y. Suppose that Hypothesis B holds for some $b > 0$. Choose $h > \ell + b$ such that $y_h \neq 0$, and set $N_k = \sum_{j=1}^{\ell} x_j G_{k-j}$, $N'_k = N_k + G_{k-n}$ for $k \geq \ell$. We show that $y_j = \epsilon_{k-j}(N'_k)$ for all $1 \leq j \leq h$ and sufficiently large k, contradicting Hypothesis B since

$$N_k = \sum_{j=k-\ell}^{\infty} \epsilon_j(N_k) G_j \quad \text{and} \quad \epsilon_{k-h}(N_k + G_{k-n}) = y_h \neq 0 = \epsilon_{k-h}(N_k).$$

To find the G-expansion of N'_k, recall that $G_j = c\,\beta^j + \mathcal{O}(\alpha^j)$ for some constant $c > 0$ and $0 < \alpha < \beta$; see, e.g., [6, 11]. We have thus $N'_k = c\,y\,\beta^k + \mathcal{O}(\alpha^k) + \mathcal{O}(1)$, and

$$N'_k - \sum_{j=1}^{i} y_j G_{k-j} = c\,T_\beta^i(y)\,\beta^{k-i} + \mathcal{O}(\alpha^{k-i}) + \mathcal{O}(1)$$

for all $1 \leq i \leq k$. As $0 < T_\beta^i(y) < 1$ for all $i \geq 0$, we obtain that

$$0 \leq N'_k - \sum_{j=1}^{i} y_j G_{k-j} < G_{k-i}$$

for all $1 \leq i \leq h$, provided that k is sufficiently large. This proves that $y_1, \ldots, y_h$ are the digits of the greedy G-expansion of N'_k, i.e., $y_j = \epsilon_{k-j}(N'_k)$ for $1 \leq j \leq h$.

Finally, by [2, Theorem 1] and [7, Lemma 1], the condition (PF) implies that β is a Pisot number. □

Remark 1. Condition (PF) does not imply Hypothesis B. Moreover, even (F) would not be sufficient. As an example let β be the smallest Pisot number, $\beta^3 = \beta + 1$. Then we have $a_1 a_2 \cdots = (10000)^\infty$, and G satisfies the linear recurrence $G_k = G_{k-1} + G_{k-5}$. Its associated characteristic polynomial is $x^5 - x^4 - 1$, which is reducible in the product $(x^3 - x - 1)(x^2 - x + 1)$. We know that (F) holds by [1]. However, Hypothesis B does not hold because of the following relation among the elements of the recurrence.

$$G_{k+3} = G_{k+1} + G_k + \begin{cases} 0 & \text{if } k \equiv 1 \mod 3, \\ -1 & \text{if } k \equiv -1, 0 \mod 6, \\ 1 & \text{if } k \equiv 2, 3 \mod 6\,. \end{cases}$$

This property, easily provable by induction, shows that Hypothesis B does not hold, since if we sum up $\tilde{N} = G_n$ and G_{n+1}, then in the second case considered above the first digit will change. More generally, we have the following lemma.

Lemma 3. *Hypothesis B implies* (QM).

Proof. We know from Lemma 2 that β is a Pisot number; let $\sum_{i=0}^{d} p_i\beta^i$ be its minimal polynomial. Since $G_k = c\,\beta^k + \mathcal{O}(\alpha^k)$, with $c > 0$ and $0 < \alpha < \beta$, we have

$$f_k = \sum_{i=0}^{d} p_i G_{k+i} = \mathcal{O}(\alpha^k).$$

If (QM) does not hold, then Lemma 1 implies that $f_k \neq 0$ for infinitely many k. Assume that $f_k > 0$, the case $f_k < 0$ being symmetric. Hypothesis B implies that the G-expansion of $\sum_{0\le i\le d:\, p_i>0} p_i G_{k+i}$ has no small terms, more precisely it ends with at least $k - \sum_{1\le i\le d:\, p_i>0} p_i b$ zeros. It also implies that the G-expansion of $\sum_{1\le i\le d:\, p_i<0} |p_i| G_{k+i} + f_k$ has small terms equal to f_k for sufficiently large k. This contradicts that

$$\sum_{1\le i\le d:\, p_i>0} p_i G_{k+i} \quad = \sum_{1\le i\le d:\, p_i<0} |p_i| G_{k+i} + f_k. \qquad \square$$

Lemma 4. *The properties* (PF) *and* (QM) *imply Hypothesis B.*

Proof. By (QM) and Lemma 1, arithmetic operations, that is addition and carries, on the strings of digits defining G-expansions can be performed in the same way as for β-expansions. As (PF) implies that β is a Pisot number, there exists by [7, Proposition 2] some L such that, for each $x \in \mathbb{Z}_+[\beta^{-1}] \cap [0, 1 - \beta^{-n})$ with $T_\beta^\ell(x) = 0$, $\ell \ge n$, we have

$$T_\beta^{\ell+L}(x + \beta^{-n}) = 0.$$

This implies that addition of G_m to $N = \sum_{j=k+L}^{\infty} \epsilon_j(N) G_j$, $m \ge k + L$, does not change the first k digits in the G-expansion. Let furthermore L' be the longest run of 0's in the quasi-greedy β-expansion of 1. Then addition of $M < G_{k-L'}$ to $N = \sum_{j=k+L}^{\infty} \epsilon_j(N) G_j$ or $N + G_m = \sum_{j=k}^{\infty} \epsilon_j(N + G_m) G_j$ is performed by concatenating the corresponding G-expansions. Therefore, Hypothesis B holds with $b = L + L'$. $\square$

Lemmas 2, 3 and 4 prove Theorem 1.

4. Purely discrete spectrum

The set $\mathcal{K}_\beta$ of β-admissible sequences is

$$\mathcal{K}_\beta = \left\{ (\epsilon_j)_{j\ge 0} \in \{0,1,\ldots,\lceil\beta\rceil - 1\}^{\mathbb{N}} : \sum_{j=1}^{k} \frac{\epsilon_{k-j}}{\beta^j} \in [0,1) \text{ for all } k \ge 1 \right\}$$

and is equal to $\mathcal{K}_G$. Similarly to δ_∞ in Section 2, we define δ'_∞ for $\beta = \beta_1 \in \mathbb{R}$ as

$$\delta'_\infty : \mathbb{Q}(\beta) \to \mathbb{R}^{r-1} \times \mathbb{C}^s, \quad x \mapsto \left(x^{(2)}, \ldots, x^{(r)}, x^{(r+1)}, \ldots, x^{(r+s)}\right),$$

and

$$\delta'_\beta : \mathbb{Z}[\beta] \to \mathbb{K}'_\beta = \mathbb{R}^{r-1} \times \mathbb{C}^s \times \varprojlim \mathbb{Z}[\beta]/\beta^n\mathbb{Z}[\beta], \quad x \mapsto \big(\delta'_\infty(x), \delta_{\mathrm{f}}(x)\big)$$

with the natural projection δ_{f} from $\mathbb{Z}[\beta]$ to the inverse limit $\varprojlim \mathbb{Z}[\beta]/\beta^n\mathbb{Z}[\beta]$. Setting

$$\varphi_\beta : \mathcal{K}_\beta \to \mathbb{K}'_\beta, \quad (\epsilon_j)_{j\ge 0} \mapsto \sum_{j=0}^{\infty} \delta'_\beta\left(\epsilon_j \beta^j\right),$$

the *Rauzy fractal* or *central tile* is defined by

$$\mathcal{R}_\beta = \varphi_\beta(\mathcal{K}_\beta)$$

(see, e.g., [12]). Note that δ'_β is defined differently in [12]; the relation to our inverse limit definition is described in [9].

Proof of Theorem 3. For each $x \in \mathcal{K}_G$ with $\tau_G(x) \ne (0,0,\ldots)$, we have

$$\varphi_\beta\big(\tau_G(x)\big) - \varphi_\beta(x) = \delta'_\beta\left(\beta^k - a_1\beta^{k-1} - a_2\beta^{k-2} - \cdots - a_k\right) = \delta'_\beta\big(\tilde{T}_\beta^k(1)\big)$$

for some $k \ge 0$, thus

$$\varphi_\beta\big(\tau_G(x)\big) - \varphi_\beta(x) \in \delta'_\beta(V_\beta) \subset \delta'_\beta(1) + \delta'_\beta(L_\beta),$$

with $L_\beta = \langle V_\beta - V_\beta\rangle_{\mathbb{Z}}$. By [12, Theorem 4], (QM) and the weak finiteness property

$$\forall\, x \in \mathbb{Z}[\beta] \cap [0,1) \;\; \exists\, y \in [0, 1-x), \;\; k \in \mathbb{N} : \; T_\beta^k(x+y) = T_\beta^k(y) = 0 \qquad \text{(W)}$$

imply that $\mathcal{R}_\beta$ is a fundamental domain of $Z'/\delta'_\beta(L_\beta)$. Moreover, the unique invariant measure μ of $(\mathcal{K}_G, \tau_G)$ is given by $\mu = \lambda \circ \varphi_\beta$, where λ is the Haar measure on $\mathbb{K}'_\beta$, and φ_β is injective up to a set of μ-measure zero.

By [5], every Pisot number β satisfies (W). Therefore, τ_G is measurably conjugate to the translation by $\delta'_\beta(1)$ on the compact group $Z'/\delta'_\beta(L_\beta)$ and has thus purely discrete spectrum. □

In [10], the spectrum of cartesian products of odometers is investigated. In particular, linear recurrences $G^1, \ldots, G^s$ of the multi-nacci type

$$G_n^i = a^i G_{n-1} + \cdots + a^i G_{d_i}^i$$

of orders $d_1, \ldots, d_s$ with pairwise coprime positive integers a^i $(i = 1, \ldots, s)$ are considered. It is shown that under a certain assumption on the independence of the dominating characteristic roots $\beta_1, \ldots, \beta_s$ of the recurrences $G^1, \ldots, G^s$ the cartesian product of the corresponding odometers is uniquely ergodic. The correct independence condition is $\beta_i \notin \mathbb{Q}(\beta_j)$ (for all $i \neq j$), whereas in [10] a wrong condition is stated. Note that in general such result does not hold.

When the third author gave a seminar talk in Luminy (2013), Pierre Liardet was in the audience and gave interesting comments on the structure of the spectrum of cartesian products of odometers. This was the last time when the third author could meet Pierre. His death is a great loss for mathematics as well as for his family and all his friends. For a detailed obituary, see [4].

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