ON HAUSDORFF DIMENSIONS RELATED TO SETS WITH GIVEN ASYMPTOTIC AND GAP DENSITIES

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Dedicated to the memory of Professor Pierre Liardet

ABSTRACT. For a set A of positive integers $a_1 < a_2 < \cdots$, let $\underline{d}(A), \overline{d}(A)$ denote its lower and upper asymptotic densities. The gap density is defined as $\lambda(A) = \limsup_{n \to \infty} \frac{a_{n+1}}{a_n}$. The paper investigates the class $\mathcal{G}(\alpha, \beta, \gamma)$ of all sets A with $\underline{d}(A) = \alpha$, $\overline{d}(A) = \beta$ and $\lambda(A) = \gamma$ for given α, β, γ with $0 \le \alpha \le \beta \le 1 \le \gamma$ and $\alpha\gamma \le \beta$. Using the classical dyadic mapping $\varrho(A) = \sum_{n=1}^{\infty} \frac{\chi_A(n)}{2^n}$, where χ_A is the characteristic function of A, the main result of the paper states that the ϱ-image set $\varrho\mathcal{G}(\alpha, \beta, \gamma)$ has the Hausdorff dimension

$$\dim \varrho\mathcal{G}(\alpha, \beta, \gamma) = \min\Big\{\delta(\alpha), \delta(\beta), \frac{1}{\gamma} \max_{\sigma \in [\alpha\gamma, \beta]} \delta(\sigma)\Big\},$$

where δ is the entropy function

$$\delta(x) = -x \log_2 x - (1-x) \log_2(1-x).$$

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1. Introduction

Let $\mathcal{P}$ and $\mathcal{P}_\infty$, respectively, be the classes of all sets of positive integers and of all infinite sets of positive integers. Each $A \in \mathcal{P}_\infty$ can be mapped onto a real number $\varrho(A) \in (0, 1]$ via the map

$$\varrho(A) = \sum_{n=1}^{\infty} \frac{\chi_A(n)}{2^n},$$

χ_A being the characteristic function of A. The mapping is a bijection of $\mathcal{P}_\infty$ onto $(0, 1]$. Having a class $\mathcal{A} \subset \mathcal{P}_\infty$, we will denote by

$$\varrho\mathcal{A} = \{\varrho(A) \mid A \in \mathcal{A}\}$$

the set of all images of elements of $\mathcal{A}$.

By log we will always mean the *binary logarithm*. Recall that the entropy function (of one variable) is the function defined on the interval $[0, 1]$ by

$$\delta(x) = -x \log x - (1 - x) \log(1 - x) \qquad (0 < x < 1)$$

and $\delta(0) = \delta(1) = 0$, rendering the function continuous. It should be noted that the function δ is monotonically increasing from 0 to 1 in the interval $\left[0, \frac{1}{2}\right]$ and that $\delta(1 - x) = \delta(x)$ for all $x \in [0, 1]$.

For any set $A \in \mathcal{P}$ and any $n \in \mathbb{N} = \{1, 2, \dots\}$ we let

$$A^n = A \cap \{1, \dots, n\},$$

calling A^n the *nth initial subset* of A. Furthermore, we use the notation

$$A(n) = \operatorname{card} A^n$$

and

$$A(x) = A(\lfloor x \rfloor) \qquad \text{for real } x \geq 1. \tag{1}$$

By $\lfloor x \rfloor$ we mean the integer part of x.

We denote the lower and upper asymptotic density, respectively, by

$$\underline{d}(A) = \liminf_{n \to \infty} \frac{A(n)}{n} \quad \text{and} \quad \overline{d}(A) = \limsup_{n \to \infty} \frac{A(n)}{n}.$$

Now we recall the concept of Hausdorff dimension of sets of real numbers. Let $A \subset \mathbb{R}$. For $s \geq 0$ and $\varepsilon > 0$ denote

$$H_\varepsilon^s(A) = \inf\left\{\sum_{n=1}^\infty (b_n - a_n)^s \ \middle| \ A \subset \bigcup_{n=1}^\infty (a_n, b_n) \text{ and } b_n - a_n < \varepsilon \text{ for all } n \in \mathbb{N}\right\}.$$

Note that $H_\varepsilon^s(A)$ is decreasing with respect to ε which yields that the limit

$$H^s(A) = \lim_{\varepsilon \to 0+} H_\varepsilon^s(A)$$

exists and is called the *s-dimensional Hausdorff measure* of A. It is an exercise (see, e.g., [1]) to show that there is a unique real number $D \in [0, 1]$ such that

$$H^s(A) = \infty \text{ for all } s < D \quad \text{and} \quad H^s(A) = 0 \text{ for all } s > D.$$

The number D is called *Hausdorff dimension* of A and is denoted by $\dim A$.

In [3], Theorem 52, it is proved, among other results, that given any α and β, $0 \leq \alpha \leq \beta \leq 1$, the set

$$\mathcal{G}(\alpha, \beta) = \left\{ A \in \mathcal{P} \mid \underline{d}(A) = \alpha, \overline{d}(A) = \beta \right\}$$

has an image set $\varrho\mathcal{G}(\alpha, \beta)$ with Hausdorff dimension

$$\dim \varrho\mathcal{G}(\alpha, \beta) = \min\{\delta(\alpha), \delta(\beta)\}. \tag{2}$$

In this paper we investigate the sets $\mathcal{G}(\alpha, \beta)$ in more detail by introducing the gap density $\lambda(A)$ as an additional parameter. As defined in [2], given any set $A = \{a_1 < a_2 < \ldots\} \in \mathcal{P}_\infty$, we let

$$\lambda(A) = \limsup_{n \to \infty} \frac{a_{n+1}}{a_n},$$

noting that $\lambda(A)$ may be infinite. Obviously, the gap density is related to the lower and upper asymptotic densities by the inequality

$$1 \leq \lambda(A) \leq \frac{\overline{d}(A)}{\underline{d}(A)} \qquad \text{for all } A \in \mathcal{P}_\infty.$$

For all α, β, γ with $0 \leq \alpha \leq \beta \leq 1$ and $1 \leq \gamma \leq \frac{\beta}{\alpha}$ (with $\frac{\beta}{0} = \infty$) we define the class

$$\mathcal{G}(\alpha, \beta, \gamma) = \left\{ A \in \mathcal{P}_\infty \mid \underline{d}(A) = \alpha, \overline{d}(A) = \beta, \lambda(A) = \gamma \right\}.$$

The aim of the present paper is to determine the Hausdorff dimension of the set $\varrho\mathcal{G}(\alpha, \beta, \gamma)$.

2. Homogeneous and saturated classes

2.1. Notation

The following definitions and notation will be needed in our paper.

2.1.1.

For any class $\mathcal{S} \subset \mathcal{P}$ we let

$$\mathcal{S}^n = \{A^n \mid A \in \mathcal{S}\} \qquad (n = 1, 2, \ldots),$$

noting that $\mathcal{S}^n$ is a class of finite sets.

2.1.2.

For any class $\mathcal{S} \subset \mathcal{P}$ and any pair m, n with $m < n$, if $\mathcal{T} \subset \mathcal{S}^m$ is given, we define the class

$$\mathcal{S}_m^n(\mathcal{T}) = \bigcup_{T \in \mathcal{T}} \left\{ S \in \mathcal{S}^n \mid S^m = T^m \right\}.$$

Hence $\mathcal{S}_m^n(\mathcal{T})$ is the class of all sets $S \in \mathcal{S}^n$ for which there exists a set $T \in \mathcal{T}$ having the same mth initial subset. We will call these sets S the *descendants* of $\mathcal{T}$ in $\mathcal{S}^n$. We will write $\mathcal{S}_m^n(T)$ instead of $\mathcal{S}_m^n(\{T\})$.

2.1.3.

We define, for every nonempty class $\mathcal{S} \subset \mathcal{P}$,

$$\iota(\mathcal{S}) = \liminf_{n \to \infty} \frac{\log \operatorname{card} \mathcal{S}^n}{n}.$$

Clearly, we always have $0 \leq \iota(\mathcal{S}) \leq 1$.

2.1.4.

As usual, we consider the *basic dyadic intervals* of order n,

$$\left(\frac{k}{2^n}, \frac{k+1}{2^n} \right] \qquad (k = 0, \dots, 2^n - 1),$$

denoting the set of these 2^n intervals by $\mathcal{B}^n$ and letting

$$\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}^n$$

stand for the set of all basic dyadic intervals.

2.1.5.

For $T \in \mathcal{P}_\infty$ and $n \in \mathbb{N}$ denote by

$$B(T^n) = \left\{ \varrho(U) \mid U \in \mathcal{P}_\infty \text{ and } U^n = T^n \right\} = \left(\sum_{i=1}^n \frac{\chi_T(i)}{2^i}, \sum_{i=1}^n \frac{\chi_T(i)}{2^i} + \frac{1}{2^n} \right]$$

the element of $\mathcal{B}^n$ containing $\varrho(T)$, being uniquely determined by the first n binary digits of T. Similarly, for any $\mathcal{S} \subset \mathcal{P}_\infty$ and $n \in \mathbb{N}$ we let

$$B(\mathcal{S}^n) = \bigcup_{T \in \mathcal{S}} B(T^n).$$

Consequently, for every $\mathcal{S} \subset \mathcal{P}_\infty$ and $n \in \mathbb{N}$, the relation $\varrho\mathcal{S} \subset B(\mathcal{S}^n)$ holds. Thus we have

$$\varrho\mathcal{S} \subset \bigcap_{n=1}^\infty B(\mathcal{S}^n) \,.$$

The classes $\mathcal{S}$ for which also the opposite inclusion holds will be of special interest.

2.1.6.

For a class $\mathcal{R} = \{R_i \mid i \in \mathcal{I}\}$ with $R_i \subset \mathbb{R}$ for every $i \in \mathcal{I}$ we will write, as usually,

$$\bigcup\mathcal{R} = \bigcup_{i\in\mathcal{I}} R_i \,.$$

We will say that a class $\mathcal{R}$ *covers* a set $A \subset \mathbb{R}$ if

$$A \subset \bigcup\mathcal{R} \,.$$

2.1.7.

We will call a class $\mathcal{S} \subset \mathcal{P}_\infty$ *saturated* if $\varrho\mathcal{S} = \bigcap_{n=1}^\infty B(\mathcal{S}^n)$.

It will turn out that, for saturated classes $\mathcal{G}$, it is relatively simple to determine $\dim \varrho\mathcal{G}$. But unfortunately, our classes $\mathcal{G}(\alpha, \beta, \gamma)$ are not saturated. Indeed, since the asymptotic densities (lower, upper and gap density) of a set A do not depend on its initial subsets, each set $\varrho\mathcal{G}(\alpha, \beta, \gamma)$ is everywhere dense in $[0, 1]$. Therefore, we always have

$$B(\mathcal{S}^n) = (0, 1] \quad \text{if } \mathcal{S} = \mathcal{G}(\alpha, \beta, \gamma).$$

2.1.8.

A class $\mathcal{S} \subset \mathcal{P}_\infty$ will be called *homogeneous* if there exist a positive integer n_0, a constant $\kappa > 1$ and a nondecreasing function

$$\varphi \colon (0, \infty) \to (0, \infty)$$

such that

$$\log \varphi(n) = o(n) \tag{3}$$

and for all $m, n \in \mathbb{N}$ with $n_0 \le m < n \le \kappa m$, the inequality

$$\operatorname{card} \mathcal{S}_m^n(S) \le \varphi(m) \operatorname{card} \mathcal{S}_m^n(T) \tag{4}$$

holds for any choice of the sets $S, T \in \mathcal{S}^m$.

Roughly speaking, this means the following: Considering any two subintervals, I and J, of $[0, 1]$, the number of basic intervals of order n containing elements from $\varrho\mathcal{S}$ within both of them, increases, as n tends to infinity, with comparable speed. (For explicit details see (12) below.)

EXAMPLE 1. Consider the system

$$\mathcal{S} = \left\{ A \in \mathcal{P}_\infty \ \middle| \ \text{if} \ \frac{A(n_0)}{n_0} > \frac{1}{2} \ \text{for some} \ n_0, \ \text{then} \ n \in A \ \text{for all} \ n > n_0 \right\}.$$

This system is not homogeneous.

2.2. Lemmas

The first lemma of this section is well-known and its simple proof follows from Stirling's formula.

LEMMA 1. *For any $\alpha \in [0,1]$,*

$$\binom{n}{\lfloor \alpha n \rfloor} = 2^{(\delta(\alpha)+o(1))n} \qquad as \ n \to \infty.$$

We will also need a simple modification of the previous lemma.

LEMMA 2. *For any $\alpha, \beta \in [0,1]$ with $\alpha \le \beta$ we have*

$$\sum_{j=\lfloor \alpha n \rfloor}^{\lfloor \beta n \rfloor} \binom{n}{j} = 2^{(\omega+o(1))n} \qquad as \ n \to \infty,$$

where $\omega = \max_{\sigma \in [\alpha,\beta]} \delta(\sigma)$.

P r o o f. In the case $\alpha \le \frac{1}{2} \le \beta$ the result follows from Lemma 1 and inequalities

$$2^{(\delta(\frac{1}{2})+o(1))n} = \binom{n}{\lfloor n/2 \rfloor} \le \sum_{j=\lfloor \alpha n \rfloor}^{\lfloor \beta n \rfloor} \binom{n}{j} \le 2^n.$$

In the case $\alpha \le \beta < \frac{1}{2}$ the result follows from Lemma 1 and inequalities

$$2^{(\delta(\beta)+o(1))n} = \binom{n}{\lfloor \beta n \rfloor} \le \sum_{j=\lfloor \alpha n \rfloor}^{\lfloor \beta n \rfloor} \binom{n}{j} \le \beta n \binom{n}{\lfloor \beta n \rfloor} = 2^{(\delta(\beta)+o(1))n}.$$

The case $\frac{1}{2} < \alpha \le \beta$ is analogous. $\square$

LEMMA 3. *Let $\varphi \colon (0,\infty) \to (0,\infty)$ be a non-decreasing function such that (3) holds, and let r, s with $0 < r < s$ be real numbers. Then there exists a real number $q > 0$ and an $n_0 \in \mathbb{N}$ such that*

$$(1+q)^k n_0 \big(s - (1+q)r\big) > \sum_{i=0}^{k-1} \log \varphi\big((1+q)^i n_0\big) + k + 3 \tag{5}$$

holds for all $k \in \mathbb{N}_0$. The number q can be chosen arbitrarily small and the number n_0 can be chosen arbitrarily large.

P r o o f. Choose $q > 0$ such that $c = s - (1+q)r > 0$, and $n_0 \in \mathbb{N}$ such that

$$n_0 > \frac{3}{c} \quad \text{and} \quad \frac{\log \varphi(\nu_k) + 1}{\nu_k} < qc \,, \tag{6}$$

where $\nu_k = (1+q)^k n_0$ for every $k \in \mathbb{N}_0$. In this notation, denoting the left-hand and right-hand sides of (5) by $L(k)$ and $R(k)$, respectively, we have to prove that

$$L(k) = c\nu_k > \sum_{i=0}^{k-1} \log \varphi(\nu_i) + k + 3 = R(k) \,.$$

Evidently,

$$L(0) = cn_0 > 3 = R(0) \quad \text{by (6).}$$

Furthermore, for $k \in \mathbb{N}_0$, we have

$$\Delta L(k) = L(k+1) - L(k) = c(\nu_{k+1} - \nu_k) = cq\nu_k$$

and

$$\Delta R(k) = R(k+1) - R(k) = 1 + \log \varphi(\nu_k) \,.$$

The second relation in (6) implies $\Delta R(k) < \Delta L(k)$ for all $k \in \mathbb{N}_0$, which proves (5) by induction. $\qquad\square$

3. Computation of dimensions

LEMMA 4. *For each nonempty* $\mathcal{S} \subset \mathcal{P}_\infty$,

$$\dim \varrho \mathcal{S} \le \iota(\mathcal{S}) \,.$$

P r o o f. It is sufficient to show that for any $r > \iota(\mathcal{S})$ and $\eta > 0$ there exists a covering $\mathcal{J}$ of $\varrho \mathcal{S}$, consisting of countably many intervals J with length $|J| < \eta$, and furthermore,

$$\sum_{J \in \mathcal{J}} |J|^r < 1 \,.$$

To see this, first choose $n \in \mathbb{N}$ such that, in the sense of 2.1.1,

$$n > -\log \eta \quad \text{and} \quad \operatorname{card} \mathcal{S}^n < 2^{rn} \,. \tag{7}$$

Let $\mathcal{J} = \{B(T^n) \mid T \in \mathcal{S}\}$. Then $\mathcal{J}$ covers $\varrho \mathcal{S}$, $|J| = 2^{-n} < \eta$ for every $J \in \mathcal{J}$ and using (7), we obtain

$$\sum_{J \in \mathcal{J}} |J|^r < 2^{rn} 2^{-rn} = 1 \,,$$

which proves the lemma. $\qquad\square$

Our next goal is to show that under some additional conditions the opposite inequality $\dim \varrho \mathcal{S} \geq \iota(\mathcal{S})$ also holds.

THEOREM 1. *Let a nonempty class $\mathcal{S} \subset \mathcal{P}_\infty$ be saturated and homogeneous. Then*

$$\dim \varrho \mathcal{S} = \iota(\mathcal{S}).$$

P r o o f. **1.** Let us define the set

$$X = \left\{ \frac{a}{2^b} \,\middle|\, b \in \mathbb{N}, a = 0, \ldots, 2^b \right\}$$

of all dyadic rational points in $[0, 1]$. By Lemma 4 it suffices to prove the inequality

$$\dim \varrho \mathcal{S} \geq \iota(\mathcal{S}).$$

For technical reasons we are going to establish this at first for $\dim(\varrho \mathcal{S} \cup X)$ instead of $\dim \varrho \mathcal{S}$.

2. If $\iota(\mathcal{S}) > 0$, let $r < \iota(\mathcal{S})$ be an arbitrary positive real number. To prove Theorem 1 it is sufficient to find an $\eta > 0$ such that no family $\mathcal{J} = (J_n)$ of intervals with $|J_n| < \eta$ for every $n \in \mathbb{N}$ and

$$\sum_{i=1}^\infty |J_i|^r < 1 \tag{8}$$

covers $\varrho \mathcal{S} \cup X$.

Choose $s \in \big(r, \iota(\mathcal{S})\big)$. Let $q > 0$ and $n_0 \in \mathbb{N}$ be numbers guaranteed by Lemma 3 with $1 + q < \kappa$ (κ being the number from 2.1.8) and with n_0 large enough such that

$$\operatorname{card} \mathcal{S}^n > 2^{sn} \tag{9}$$

for every $n \geq n_0$.

Now let a family $\mathcal{J}$, satisfying (8) with $\eta = 2^{-n_0}$, be fixed, and suppose that its elements $J_1, J_2, \ldots$ are numbered so that $|J_1| \geq |J_2| \geq \cdots$ We use the notation

$$n_k = \big\lfloor (1+q)^k n_0 \big\rfloor \qquad (k = 0, 1, 2, \ldots).$$

For each k we define the symbols

$$j_k = \max\big\{ j \in \mathbb{N} \,\big|\, |J_j| \geq 2^{-n_k} \big\} \quad \text{and} \quad \mathcal{J}_k = \big\{ J_1, J_2, \ldots, J_{j_k} \big\}.$$

Then

$$1 > \sum_{i=1}^{j_k} |J_i|^r \geq j_k 2^{-rn_k},$$

and thus we have, for each $k \in \mathbb{N}$,

$$\operatorname{card} \mathcal{J}_k = j_k < 2^{rn_k}. \tag{10}$$

3. We are going to prove that $\mathcal{J}$ cannot be an η-covering of the set $\varrho\mathcal{S} \cup X$. To do so we use the following procedure. For each index k we show that there is a nonempty closed subset V_k of $\varrho\mathcal{S} \cup X$ which is disjoint from all elements of $\mathcal{J}_{j_{k+1}}$ and satisfies $V_k \subset V_{k-1}$. Therefore, the intersection $\bigcap_{k=1}^{\infty} V_k$ is nonempty and disjoint from $\bigcup \mathcal{J}$.

In order to accomplish this we construct the sequence of systems of intervals,

$$\mathcal{I}_k = \{\overline{B(S)} \mid S \in \mathcal{S}^{n_k}\} \qquad (k = 0, 1, 2, \dots),$$

where $\overline{B(S)}$ denotes the closure of the interval $B(S)$. Since there is a one-to-one correspondence between the elements of $\mathcal{S}^{n_k}$ and the elements of $\mathcal{I}_k$, we have

$$\operatorname{card} \mathcal{I}_k = \operatorname{card} \mathcal{S}^{n_k}. \tag{11}$$

Define the sequence of systems of intervals,

$$\mathcal{U}_k = \{M \in \mathcal{I}_k(\mathcal{U}_{k-1}) \mid M \text{ is disjoint from all elements of } \mathcal{J}_{k+1}\}$$

$(k = 0, 1, 2, \dots)$. Here we let $\mathcal{I}_0(\mathcal{U}_{-1}) = \mathcal{I}_0$ and denote the system of those descendants (see 2.1.2) of $\mathcal{U}_{k-1}$ which are contained in $\mathcal{I}_k$ by

$$\mathcal{I}_k(\mathcal{U}_{k-1}) = \{M \in \mathcal{I}_k \mid \text{there exists } U \in \mathcal{U}_{k-1} \text{ such that } M \subset U\}.$$

4. One can easily prove that the condition (4) implies, for any $\mathcal{T} \subset \mathcal{S}^{n_k}$, inequality

$$\operatorname{card} \mathcal{S}_{n_k}^{n_{k+1}}(\mathcal{T}) \geq \frac{1}{\varphi(n_k)} \frac{\operatorname{card} \mathcal{T}}{\operatorname{card} \mathcal{S}^{n_k}} \operatorname{card} \mathcal{S}^{n_{k+1}}. \tag{12}$$

Now we prove by induction that

$$\operatorname{card} \mathcal{U}_k > \frac{1}{2^{k+1} \prod\limits_{i=0}^{k-1} \varphi(n_i)} \operatorname{card} \mathcal{I}_k . \tag{13}$$

Since every element of $\mathcal{I}_0$ has length 2^{-n_0} and every element of $\mathcal{J}_1$ has length less than $\eta = 2^{-n_0}$, it follows that every element of $\mathcal{J}_1$ can overlap with at most two elements of $\mathcal{I}_0$. Inasmuch as inequality (5) with $k = 0$ implies the relation

$$(sn_0 - 1) - (rn_1 + 1) \geq sn_0 - rn_0(1 + q) - 2 > 0\,,$$

it follows from (9) and (10) that

$$\operatorname{card} \mathcal{U}_0 \geq \operatorname{card} \mathcal{I}_0 - 2 \operatorname{card} \mathcal{J}_1 > \frac{1}{2} \operatorname{card} \mathcal{I}_0 + 2^{sn_0 - 1} - 2^{rn_1 + 1} > \frac{1}{2} \operatorname{card} \mathcal{I}_0\,,$$

which proves (13) for $k = 0$.

Assume (13) for some fixed $k \in \mathbb{N}_0$. Relations (11), (12) and (13) imply

$$\operatorname{card}\mathcal{I}_{k+1}(\mathcal{U}_k) \geq \frac{1}{\varphi(n_k)}\frac{\operatorname{card}\mathcal{U}_k}{\operatorname{card}\mathcal{I}_k}\operatorname{card}\mathcal{I}_{k+1} > \frac{1}{2^{k+1}\prod\limits_{i=0}^{k}\varphi(n_i)}\operatorname{card}\mathcal{I}_{k+1}.$$

Since every element of $\mathcal{I}_{k+1}$ has length $2^{-n_{k+1}}$ and every element of $\mathcal{J}_{k+2}\setminus\mathcal{J}_{k+1}$ has length less than $2^{-n_{k+1}}$, it follows that every element of $\mathcal{J}_{k+2}$ can overlap with at most two elements of $\mathcal{I}_{k+1}(\mathcal{U}_k)$. Thus (9) and (10) imply

$$\operatorname{card}\mathcal{U}_{k+1} \geq \operatorname{card}\mathcal{I}_{k+1}(\mathcal{U}_k) - 2\operatorname{card}\mathcal{J}_{k+2}$$

$$> \frac{1}{2^{k+1}\prod\limits_{i=0}^{k}\varphi(n_i)}\operatorname{card}\mathcal{I}_{k+1} - 2^{rn_{k+2}+1}$$

$$> \frac{1}{2^{k+2}\prod\limits_{i=0}^{k}\varphi(n_i)}\operatorname{card}\mathcal{I}_{k+1} + \frac{1}{2^{k+2}\prod\limits_{i=0}^{k}\varphi(n_i)}2^{sn_{k+1}} - 2^{rn_{k+2}+1}$$

$$= \frac{1}{2^{k+2}\prod\limits_{i=0}^{k}\varphi(n_i)}\operatorname{card}\mathcal{I}_{k+1} + 2^{sn_{k+1}-k-2-\sum\limits_{i=0}^{k}\log\varphi(n_i)} - 2^{rn_{k+2}+1}$$

$$> \frac{1}{2^{k+2}\prod\limits_{i=0}^{k}\varphi(n_i)}\operatorname{card}\mathcal{I}_{k+1}.$$

For the last inequality we have used (5) for $k+1$ if k is the index under consideration. Indeed,

$$\left(sn_{k+1} - k - 2 - \sum_{i=0}^{k}\log\varphi(n_i)\right) - \left(rn_{k+2} + 1\right)$$

$$= \lfloor(1+q)^{k+1}n_0\rfloor s - \lfloor(1+q)^{k+2}n_0\rfloor r - \sum_{i=0}^{k}\log\varphi\bigl(\lfloor(1+q)^i n_0\rfloor\bigr) - k - 3$$

$$> (1+q)^{k+1}n_0 s - s - (1+q)^{k+2}n_0 r - \sum_{i=0}^{k}\log\varphi\bigl((1+q)^i n_0\bigr) - k - 3$$

$$= (1+q)^{k+1}n_0\bigl(s - (1+q)r\bigr) - \sum_{i=0}^{k}\log\varphi\bigl((1+q)^i n_0\bigr) - (k+1) - 3 + (1-s)$$

$$> 1 - s > 0.$$

This finishes the inductive proof of (13).

5. The system $\mathcal{S}$ being saturated by assumption, we have

$$\bigcap_{n=1}^{\infty} \bigcup_{S \in \mathcal{S}^n} B(S) = \varrho\mathcal{S} \,.$$

This and (13) imply that the sets

$$V_k = \bigcup \mathcal{U}_k \subset \varrho\mathcal{S} \cup X \qquad (k = 1, 2, \dots) \tag{14}$$

form a decreasing sequence of nonempty closed sets in the compact $[0, 1]$. Hence $V = \bigcap_{k=1}^{\infty} V_k$ is nonempty. In view of the definition of the classes $\mathcal{U}_k$ and the relation (14) we have

$$\emptyset \neq V \subset (\varrho\mathcal{S} \cup X) \setminus \bigcup \mathcal{J} \,.$$

Thus

$$\dim(\varrho\mathcal{S} \cup X) \geq r \quad \text{for every} \quad r < \iota(\mathcal{S})$$

and hence, since the set X is countable,

$$\dim \varrho\mathcal{S} = \dim(\varrho\mathcal{S} \cup X) \geq \iota(\mathcal{S}) \,,$$

which finishes the proof of Theorem 1. $\qquad\qquad\square$

EXAMPLE 2. Let $A \in \mathcal{P}_{\infty}$. We will show that the system

$$\mathcal{S} = \{B \in \mathcal{P}_{\infty} \mid B \subset A\}$$

is homogeneous and saturated. The homogeneity follows immediately from the fact that for any $m < n$ and for every pair $S, T \in \mathcal{S}^m$ we have

$$\operatorname{card} \mathcal{S}_m^n(S) = \operatorname{card} \mathcal{S}_m^n(T) = 2^{\operatorname{card}(A \cap \{m+1, m+2, \dots, n\})} \,.$$

To prove that $\mathcal{S}$ is saturated, choose any $x \notin \bigcap_{n=1}^{\infty} B(\mathcal{S}^n)$. Then there is $n \in \mathbb{N}$ such that $x \notin B(\mathcal{S}^n)$. But this means that $\varrho^{-1}(x)$ is not a subset of A, consequently $x \notin \varrho\mathcal{S}$.

Now the application of Theorem 1 implies that

$$\dim \varrho\mathcal{S} = \liminf_{n \to \infty} \frac{\log \operatorname{card} \mathcal{S}^n}{n} = \liminf_{n \to \infty} \frac{\log 2^{A(n)}}{n} = \underline{d}(A) \,.$$

This result was first proved in [4].

4. Main result

The main theorem of this paper is the following. (For definitions see the introduction.)

THEOREM 2. *Let α, β, γ with $0 \leq \alpha \leq \beta \leq 1 \leq \gamma$ and $\alpha\gamma \leq \beta$ be given. Then*

$$\dim \varrho\mathcal{G}(\alpha, \beta, \gamma) = \min\Big\{\delta(\alpha), \delta(\beta), \frac{1}{\gamma} \max_{\sigma \in [\alpha\gamma, \beta]} \delta(\sigma)\Big\}.$$

The result can be stated more explicitly in the following way.

$$\dim \varrho\mathcal{G}(\alpha, \beta, \gamma) = \begin{cases} \min\big\{\delta(\alpha), \frac{\delta(\beta)}{\gamma}\big\} & \text{if } \alpha\gamma \leq \beta \leq \frac{1}{2}, \\ \min\big\{\delta(\alpha), \delta(\beta), \frac{1}{\gamma}\big\} & \text{if } \alpha\gamma \leq \frac{1}{2} \leq \beta, \\ \min\big\{\delta(\beta), \frac{1}{\gamma}\delta(\alpha\gamma)\big\} & \text{if } \frac{1}{2} \leq \alpha\gamma \leq \beta. \end{cases}$$

Before proving the main theorem we construct a system of auxiliary sets inductively. First we note that, using (2) and the fact that $\mathcal{G}(\alpha, \beta, \gamma) \subset \mathcal{G}(\alpha, \beta)$, it follows immediately that the theorem holds if either $\alpha = 0$ or $\beta = 1$. Thus we assume $0 < \alpha$ and $\beta < 1$ in the sequel, so having $1 \leq \gamma < \infty$.

We will use the special classes $\mathcal{Z}(\alpha, \beta, \gamma, \sigma) \subset \mathcal{P}_\infty$ defined as follows. Let $\alpha, \beta, \gamma, \sigma$ be real numbers with $0 < \alpha \leq \alpha\gamma \leq \sigma \leq \beta < 1$. Construct the sequences

$$a_1 = 1, \qquad b_1 = 2, \qquad c_1 = \lfloor \gamma b_1 \rfloor, \qquad d_1 = 2c_1,$$
$$a_m = (m+1)d_{m-1}, \quad b_m = (m+1)a_m, \quad c_m = \lfloor \gamma b_m \rfloor, \quad d_m = (m+1)c_m$$

$(m = 2, 3, \dots)$. Let $\mathcal{Z}(\alpha, \beta, \gamma, \sigma)$ denote the class of all sets $A \in \mathcal{P}_\infty$ such that, for every $m \in \mathbb{N}$,

$$\mathrm{card}\big(A \cap (d_{m-1} + (k-1)m, d_{m-1} + km]\big) = \lfloor \alpha m \rfloor, \qquad k = 1, \dots, d_{m-1},$$
$$\mathrm{card}\big(A \cap (a_m + (k-1)m, a_m + km]\big) = \lfloor \sigma m \rfloor, \qquad k = 1, \dots, a_m,$$
$$A \cap (b_m, c_m] = \emptyset$$

and

$$\mathrm{card}\big(A \cap (c_m + (k-1)m, c_m + km]\big) = \lfloor \beta m \rfloor, \qquad k = 1, \dots, c_m.$$

We call W the set of all endpoints of intervals mentioned in these equations.

LEMMA 5. *All classes $\mathcal{Z}(\alpha, \beta, \gamma, \sigma)$ are homogeneous and saturated; moreover,*

$$\dim \varrho\mathcal{Z}(\alpha, \beta, \gamma, \sigma) = \min\Big\{\delta(\alpha), \delta(\beta), \frac{\delta(\sigma)}{\gamma}\Big\}.$$

P r o o f. **1.** We consider $\alpha, \beta, \gamma, \sigma$ to be fixed, writing $\mathcal{Z}(\alpha, \beta, \gamma, \sigma) = \mathcal{Z}$. The cardinality of the class $\mathcal{Z}^{d_{m-1}}$ satisfies

$$1 \leq \operatorname{card} \mathcal{Z}^{d_{m-1}} \leq 2^{d_{m-1}},$$

hence we have, as $m \to \infty$,

$$\operatorname{card} \mathcal{Z}^{d_{m-1}} = 2^{o(a_m)}.$$

Lemma 1 implies that

$$\operatorname{card} \mathcal{Z}^{a_m} = \operatorname{card} \mathcal{Z}^{d_{m-1}} \cdot \binom{m}{\lfloor \alpha m \rfloor}^{d_{m-1}} = 2^{o(a_m) + (\delta(\alpha) + o(1))m d_{m-1}}$$

and

$$\lim_{m \to \infty} \frac{\log \operatorname{card} \mathcal{Z}^{a_m}}{a_m} = \lim_{m \to \infty} \left(\frac{o(a_m)}{a_m} + (\delta(\alpha) + o(1)) \frac{m d_{m-1}}{a_m} \right) = \delta(\alpha).$$

Similarly,

$$\operatorname{card} \mathcal{Z}^{b_m} = \operatorname{card} \mathcal{Z}^{a_m} \cdot \binom{m}{\lfloor \sigma m \rfloor}^{a_m} = 2^{o(b_m) + (\delta(\sigma) + o(1))m a_m}$$

and

$$\lim_{m \to \infty} \frac{\log \operatorname{card} \mathcal{Z}^{b_m}}{b_m} \lim_{m \to \infty} \left(\frac{o(b_m)}{b_m} + (\delta(\sigma) + o(1)) \frac{m a_m}{b_m} \right) = \delta(\sigma).$$

One easily obtains the relation

$$\lim_{m \to \infty} \frac{\log \operatorname{card} \mathcal{Z}^{c_m}}{c_m} = \lim_{m \to \infty} \frac{\log \operatorname{card} \mathcal{Z}^{b_m}}{\lfloor \gamma b_m \rfloor} = \frac{\delta(\sigma)}{\gamma}.$$

Finally, we have

$$\lim_{m \to \infty} \frac{\log \operatorname{card} \mathcal{Z}^{d_m}}{d_m} = \delta(\beta).$$

Let

$$F(n) = \frac{\log \operatorname{card} \mathcal{Z}^n}{n}.$$

If restricted to the elements n of W, the function $F(n)$ is monotone in each of the intervals $[d_{m-1}, a_m]$, $[a_m, b_m]$, $[b_m, c_m]$ and $[c_m, d_m]$ $(m = 2, 3, \dots)$.

Therefore, it suffices to consider the points a_m, b_m, c_m, d_m in order to determine the $\liminf_{n\to\infty} F(n)$. We obtain

$$\iota(\mathcal{Z}) = \liminf_{n\to\infty} \frac{\log \operatorname{card} \mathcal{Z}^n}{n} = \min\left\{\delta(\alpha), \delta(\sigma), \frac{\delta(\sigma)}{\gamma}, \delta(\beta)\right\}$$
$$= \min\left\{\delta(\alpha), \delta(\beta), \frac{\delta(\sigma)}{\gamma}\right\}.$$

2. Now we prove that the class $\mathcal{Z}$ is homogeneous. Choose positive sufficiently large integers m, n such that $m < n < 2m$. Then there is a uniquely determined integer k such that $n \in (d_{k-1}, d_k]$. The construction of the sequence (d_k) implies $n \geq (k!)^3$, and consequently,

$$k \leq \frac{1}{3}\log n < \log m.$$

Now look at the construction of the class $\mathcal{Z}$. Each of the intervals $(d_{k-1}, a_k]$, $(a_k, b_k]$ and $(c_k, d_k]$ consists of many small blocks of k elements. The number $\operatorname{card} \mathcal{Z}^p$ increases in each such block by a factor $\binom{k}{\lfloor \tau k \rfloor}$, where τ is one of the real numbers α, σ, β. Choose $S, T \in \mathcal{Z}^m$ arbitrarily. Then, letting an integer p increase from m to n, both numbers $\operatorname{card} \mathcal{Z}_m^p(S)$ and $\operatorname{card} \mathcal{Z}_m^p(T)$ increase by the same factor in each small block, except the first one (containing m). Their increase may differ at most by the factor 2^k. Thus

$$\operatorname{card} \mathcal{Z}_m^n(S) \leq \operatorname{card} \mathcal{Z}_m^n(T)\, 2^k \leq \operatorname{card} \mathcal{Z}_m^n(T)\, 2^{\log m} = \operatorname{card} \mathcal{Z}_m^n(T)\, m.$$

Consequently, the class $\mathcal{Z}$ satisfies the definition of being homogeneous with a suitable n_0, $\kappa = 2$ and $\varphi(m) = m$.

3. We finish the proof of the lemma by showing that the class $\mathcal{Z}$ is saturated. To do so, it is sufficient to show that for each $\xi \in (0,1] \setminus \varrho\mathcal{Z}$ there exists a positive integer n such that $\xi \notin B(\mathcal{Z}^n)$. Choose an arbitrary $S \in \mathcal{Z}$. Then a set $T \in \mathcal{P}_\infty$ belongs to $\mathcal{Z}$ if and only if

$$\operatorname{card} T^n = \operatorname{card} S^n \qquad \text{holds for all} \quad n \in W.$$

Choose a real number $\xi \in (0,1] \setminus \varrho\mathcal{Z}$ and let $Y = \varrho^{-1}(\xi)$. Then there exists an integer $n \in W$ so that $\operatorname{card} Y^n \neq \operatorname{card} S^n$. Consequently $\xi \notin B(\mathcal{Z}^n)$ and thus $\mathcal{Z}$ is saturated. $\qquad\square$

P r o o f   o f   T h e o r e m 2. First we use Lemma 5 to establish a lower bound for $\dim \varrho\mathcal{G}(\alpha, \beta, \gamma)$. For every $\sigma \in [\alpha\gamma, \beta]$ and $A \in \mathcal{Z}(\alpha, \beta, \gamma, \sigma)$ we have

$$\underline{d}(A) = \alpha, \quad \overline{d}(A) = \beta \quad \text{and} \quad \lambda(A) = \gamma.$$

Therefore, $\mathcal{Z}(\alpha, \beta, \gamma, \sigma) \subset \mathcal{G}(\alpha, \beta, \gamma)$ and

$$\dim \varrho \mathcal{G}(\alpha, \beta, \gamma) \geq \dim \varrho \mathcal{Z}(\alpha, \beta, \gamma, \sigma) = \min\left\{\delta(\alpha), \delta(\beta), \frac{\delta(\sigma)}{\gamma}\right\}.$$

This holds for every $\sigma \in [\alpha\gamma, \beta]$, hence

$$\dim \varrho \mathcal{G}(\alpha, \beta, \gamma) \geq \min\left\{\delta(\alpha), \delta(\beta), \frac{1}{\gamma} \max_{\sigma \in [\alpha\gamma, \beta]} \delta(\sigma)\right\}. \quad (15)$$

Now we prove the opposite inequality to (15). Since the relation

$$\dim \varrho \mathcal{G}(\alpha, \beta, \gamma) \leq \min\{\delta(\alpha), \delta(\beta)\}$$

follows from (2), it is sufficient to show that, if $0 < \alpha \leq \alpha\gamma \leq \beta < 1$, then

$$\dim \varrho \mathcal{G}(\alpha, \beta, \gamma) \leq \frac{1}{\gamma} \max_{\sigma \in [\alpha\gamma, \beta]} \delta(\sigma) = \mu. \quad (16)$$

Properties of the function δ, the assertion (2) and the relation

$$\mathcal{G}(\alpha, \beta, \gamma) \subset \mathcal{G}(\alpha, \beta)$$

imply that this clearly holds in the case $\gamma = 1$. Thus we assume $\gamma > 1$ in the rest of the proof.

To establish (16) it is sufficient to prove that, for each $r \in (\mu, 1)$, there exists a positive constant $C(r)$ such that, for every $\eta > 0$, there exists a sequence (J_n) of intervals such that $|J_n| < \eta$ for every $n \in \mathbb{N}$, satisfying

$$\varrho \mathcal{G}(\alpha, \beta, \gamma) \subset \bigcup_{n=1}^{\infty} J_n \quad \text{and} \quad \sum_{n=1}^{\infty} |J_n|^r < C(r).$$

So let $r \in (\mu, 1)$ be given. Choose the real numbers

$$s \in (\mu, r), \quad \alpha' \in (0, \alpha), \quad \beta' \in (\beta, 1), \quad \gamma' \in (1, \gamma) \quad \text{and} \quad q > 0$$

such that $1 + q < \gamma'$,

$$\frac{q}{1+q} - r + s < 0 \quad (17)$$

and

$$\max_{\sigma \in [\frac{\alpha'\gamma'}{1+q}, \beta' + \frac{q}{1+q}]} \delta(\sigma) < \gamma's. \quad (18)$$

The possibility of this choice is guaranteed by the definition of μ and the continuity of the function δ.

Now letting

$$C(r) = \sum_{k=1}^{\infty} 2^{2 + (\frac{q}{1+q} - r + s)(1+q)^k},$$

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we see by (17) that $C(r) < \infty$. Let $\eta > 0$ be given. Take $n_0 > -\log\eta$. For each $A \in \mathcal{G}(\alpha, \beta, \gamma)$ denote by $n_A \geq n_0$ the least integer n such that

$$A \cap (n, \gamma'n] = \emptyset \quad \text{and} \quad \alpha'\gamma' < \frac{A(n)}{n} < \beta'. \tag{19}$$

For each $k \in \mathbb{N}$ we let $\nu_k = n_0(1+q)^k$ and

$$\mathcal{G}_k = \left\{ A \in \mathcal{G}(\alpha, \beta, \gamma) \mid \gamma'n_A \in [\nu_{k-1}, \nu_k) \right\}.$$

Note that $\mathcal{G}(\alpha, \beta, \gamma) = \bigcup_{k=1}^{\infty} \mathcal{G}_k$. By definition there exists, for each set $A \in \mathcal{G}_k$, an integer

$$m = n_A \in \left[\frac{\nu_{k-1}}{\gamma'}, \frac{\nu_k}{\gamma'} \right) \tag{20}$$

such that

$$\alpha'\gamma'm < A(m) < \beta'm.$$

Consequently, in the sense of (1),

$$\alpha'\gamma'\frac{\nu_{k-1}}{\gamma'} \leq \alpha'\gamma'm < A(m) = A\left(\frac{\nu_k}{\gamma'}\right) \leq A(m) + \left(\frac{\nu_k}{\gamma'} - m\right) < \beta'm + \frac{\nu_k}{\gamma'} - m.$$

Dividing by $\frac{\nu_k}{\gamma'}$ we obtain

$$\frac{\alpha'\gamma'\frac{\nu_{k-1}}{\gamma'}}{\frac{\nu_k}{\gamma'}} = \frac{\alpha'\gamma'}{1+q} < \frac{A\left(\frac{\nu_k}{\gamma'}\right)}{\frac{\nu_k}{\gamma'}} < \frac{\beta'\frac{\nu_k}{\gamma'} + \frac{\nu_{k-1}}{\gamma'}q}{\frac{\nu_k}{\gamma'}} = \beta' + \frac{q}{1+q}.$$

Hence

$$A\left(\frac{\nu_k}{\gamma'}\right) = \sigma_{A,k} \qquad \text{for some } \sigma_{A,k} \in \left(\frac{\alpha'\gamma'}{1+q}, \beta' + \frac{q}{1+q}\right).$$

This yields, using (18) and Lemma 2,

$$\text{card}\,\mathcal{G}_k^{\lfloor \frac{\nu_k}{\gamma'} \rfloor} \leq 2^{\gamma's\frac{\nu_k}{\gamma'}} = 2^{s\nu_k}. \tag{21}$$

First relation in (19) with (20) imply that for every $A \in \mathcal{G}_k$ we have

$$A \cap \left(\frac{\nu_k}{\gamma'}, \nu_{k-1}\right] = \emptyset.$$

Then (21) implies that

$$\text{card}\,\mathcal{G}_k^{\lfloor \nu_k \rfloor} \leq 2^{\lfloor \nu_k \rfloor - \lfloor \nu_{k-1} \rfloor}\,\text{card}\,\mathcal{G}_k^{\lfloor \frac{\nu_k}{\gamma'} \rfloor} \leq 2^{\nu_k - \nu_{k-1}+1}\,2^{s\nu_k} = 2^{1+(s+\frac{q}{1+q})\nu_k}.$$

Thus we can cover the set $\varrho\mathcal{G}_k$ by at most $2^{1+(s+\frac{q}{1+q})\nu_k}$ intervals of length $2^{-\lfloor \nu_k \rfloor}$ forming a family $\mathcal{J}_k$. Hence we have

$$\sum_{J \in \mathcal{J}_k} |J|^r \leq 2^{1+(s+\frac{q}{1+q})\nu_k}\,2^{-r\lfloor \nu_k \rfloor} \leq 2^{2+(\frac{q}{1+q}-r+s)n_0(1+q)^k}.$$

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The family $\mathcal{J} = \bigcup_{k=1}^{\infty} \mathcal{J}_k$ covers $\varrho\mathcal{G}(\alpha, \beta, \gamma)$ and satisfies the inequality

$$\sum_{J \in \mathcal{J}} |J|^r \leq \sum_{k=1}^{\infty} 2^{2+(\frac{q}{1+q}-r+s)n_0(1+q)^k} < C(r).$$

This implies (16) and together with (15) finishes the proof. $\qquad\square$

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