Uniform Distribution Theory **10** (2015), no.2, 185–203

uniform
distribution
theory

ON THE DISTRIBUTION OF THE ARGUMENT OF THE RIEMANN ZETA-FUNCTION ON THE CRITICAL LINE

Selin Selen Özbek—Jörn Steuding

Dedicated to Prof. H. Niederreiter on the occasion of his 70th birthday
and
to Prof. E. Wegert on the occasion of his 60th birthday

ABSTRACT. We investigate the distribution of the argument of the Riemann zeta-function on arithmetic progressions on the critical line. We prove uniform distribution modulo $\frac{\pi}{2}$ and we show uniform distribution modulo π under certain restrictions. We also discuss continuous uniform distribution.

Communicated by Werner Georg Nowak

1. Introduction and statement of the main results

The Riemann zeta-function ζ plays a prominent role in multiplicative number theory. One of the reasons is the link between its complex zeros and the distribution of prime numbers. The famous yet unsolved Riemann hypothesis states that all so-called nontrivial (non-real) zeros $\rho = \beta + i\gamma$ lie on the critical line $\frac{1}{2} + i\mathbb{R}$. There are infinitely many such zeros; namely, if $N(T)$ counts there number with imaginary part $\gamma \in (0, T)$ (according multiplicities), then the Riemann-von Mangoldt formula states

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T), \tag{1}$$

as T tends to infinity. The main term of this formula had been announced (without proof) by Riemann in 1859 in his path-breaking nine pages article [30]; an asymptotic formula had been proved by von Mangoldt, first with an error term of size $O\big((\log T)^2\big)$ in [24], and in [25] finally with the error term given above.

2010 Mathematics Subject Classification: 11M06, 11M26.
Keywords: Riemann Zeta-Function, Uniform Distribution.

The reasoning behind these formulae is the simple principle of the argument and rather subtle properties of the zeta-function; the most important ingredient is the functional equation

$$\zeta(s) = \Delta(s)\zeta(1-s), \tag{2}$$

where

$$\Delta(s) = \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})},$$

and Γ denotes Euler's gamma-function. This symmetry was conjectured by Euler and first proved by Riemann [30]; see also Titchmarsh's monography [37]. By Stirling's formula, the asymptotic growth of Δ is for fixed real part given by

$$\Delta(\sigma + \mathrm{i}t) = \left(\frac{t}{2\pi}\right)^{\frac{1}{2}-\sigma-\mathrm{i}t} \exp\Big(\mathrm{i}\left(t + \tfrac{\pi}{4}\right)\Big)\Big(1 + O\left(t^{-1}\right)\Big), \tag{3}$$

as $t \to +\infty$. A proof of Stirling's formula can be found in Rudin's wonderful book [31], §8.22; details about its application to Δ can be found in Titchmarsh [37], §7.4. It is essentially this asymptotical formula which yields the main term in (1). Backlund [2, 3] obtained the more precise expression

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + \frac{7}{8} + S(T) + O(T^{-1}), \tag{4}$$

where

$$S(t) := \frac{1}{\pi} \arg \zeta(\tfrac{1}{2} + \mathrm{i}t)$$

is the argument of the zeta-function on the critical line. In view of the multivalued complex logarithm one defines the value of the logarithm $\log \zeta$ at $\frac{1}{2} + \mathrm{i}t$ by continuous variation along the polygon with vertices $2, 2+\mathrm{i}t, \frac{1}{2}+\mathrm{i}t$, provided t is not equal to an ordinate of a nontrivial zero $\beta + \mathrm{i}\gamma$; otherwise, when $t = \gamma$, we define $S(\gamma)$ by

$$S(\gamma) = \tfrac{1}{2} \lim_{\epsilon \to 0} \big(S(\gamma + \epsilon) + S(\gamma - \epsilon)\big). \tag{5}$$

Notice that $\zeta(2)$ is a positive real number (equal to $\frac{\pi^2}{6}$ as Euler proved), hence $\log \zeta(s)$ is the principal branch of the logarithm on the subinterval $(1, \infty)$ of the real axis; however, for complex $\frac{1}{2} + it$ the argument might be very large. For example, Tsang [39] showed that $S(t) = o\big((\log t/ \log\log t)^{\frac{1}{3}}\big)$ cannot be true.[1]

[1]Here the notation $f = o(g)$ means that the limit $\lim f(t)/g(t)$ exists as t tends to infinity and is equal to zero.

On the contrary, we have $S(t) = O(\log t)$ by von Mangoldt's work [25] and, assuming the Riemann hypothesis, the slightly better $S(t) = O(\log t/\log\log t)$ due to Littlewood [23] who also showed unconditionally that

$$\int_0^T S(t)\mathrm{d}t = O(\log T),$$

which implies that $S(t)$ oscillates heavily.

In view of (4) it follows that $S(t)$ is continuous except for ordinates of non-trivial zeros, and that $\pi S(t)$ jumps at each ordinate by an integer multiple of π (according to the multiplicity of the zero). It is conjectured that all (or at least almost all) zeta zeros are simple, so we shall expect that $\pi S(t)$ jumps by $\pm\pi$ at the ordinate of a zero (which is also reflected in Figure 2 below by the curve $t \mapsto \zeta(\frac{1}{2} + \mathrm{i}t)$ passing through the origin).

Here we are concerned about the distribution of values of the argument of the zeta-function. Although a probabilistic limit law for the logarithm of the zeta-function had already been found by Bohr[2] & Jessen [5] there are several phenomena unclarified. In view of the beautiful phase plots for complex-valued functions in general and the zeta-function in particular provided by Wegert and Semmler [32, 41] (see Figure 1) it is an interesting question whether the values of the argument are uniformly distributed in some sense. Taking the Dirichlet series representation $\zeta(s) = \sum_{n\geq 1} n^{-s}$, valid in the half-plane $\mathrm{Re}\, s > 1$, into account, $\zeta(s)$ is almost periodic in $\mathrm{Re}\, s > 1$ which is visually supported by Figure 1 and further computations (which had also been the motivation for Steuding & Wegert [36]). Notice that Bohr [4] introduced the notation of almost-periodicity in order to investigate the Riemann zeta-function and Riemann's hypothesis in particular.

In this note we shall investigate the discrete question of uniform distribution of the argument on arithmetic progressions on the critical line (although our results also provide information for the continuous case; see Corollary 4). Our main result is the following

Theorem 1. *The argument of the Riemann zeta-function on an arbitrary infinite arithmetic progression on the critical line is uniformly distributed modulo* $\frac{\pi}{2}$.

For the sake of completeness we recall the definition of uniform distribution introduced by Weyl [42] in 1916. Let μ be a positive real number. Then a sequence of real numbers x_n is said to be uniformly distributed modulo μ if for all α, β

[2]More precisely, it was Harald Bohr, the younger brother of the physicist and Nobel prize laureate Niels Bohr.

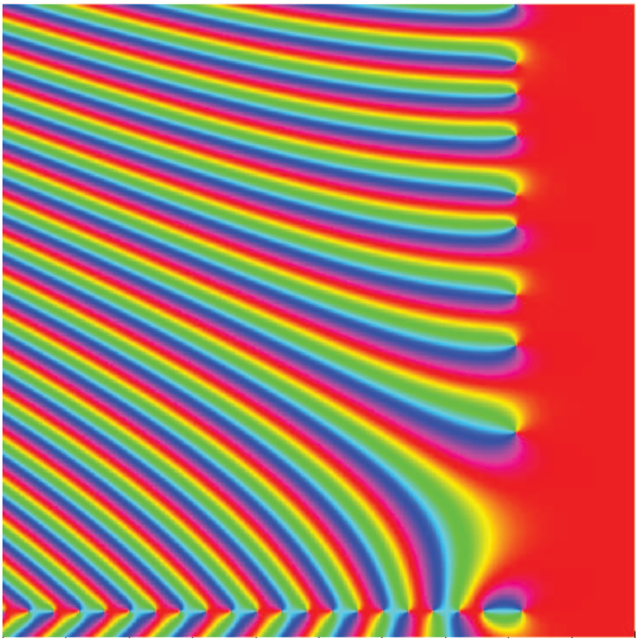

FIGURE 1. This phase plot of the zeta-function is taken from Semmler & Wegert [32] and shows the phase plot of the Riemann zeta function in the rectangle $-40 \leq \operatorname{Re} s \leq 10, -2 \leq \operatorname{Im} s \leq 48$. One can easily identify the pole of $\zeta(s)$ at $s = 1$, several nontrivial and trivial (real) zeros. One also observes a certain regularity in the colours as $\operatorname{Im} s$ increases which corresponds to almost periodicity properties of $\zeta(s)$. Some comments to this phenomenon are given in the final section.

with $0 \leq \alpha < \beta \leq \mu$ the proportion of the fractional parts of the x_n modulo μ in the interval $[\alpha, \beta)$ corresponds to its length in the following sense:

$$\lim_{N\to\infty} \frac{1}{N} \sharp\{1 \leq n \leq N \,:\, x_n \bmod \mu \in [\alpha, \beta)\} = \frac{\beta - \alpha}{\mu}.$$

Here we define $x_n \bmod \mu$ by

$$x_n \bmod \mu := x_n - \left\lfloor \frac{x_n}{\mu} \right\rfloor \mu, \tag{6}$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. Weyl considered the case $\mu = 1$, however, for our purpose a modulus related to the geometry of the complex plane (i.e., a modulus μ such that $\frac{2\pi}{\mu}$ is a positive integer) is more natural.

Unfortunately, Theorem 1 does not give information about uniform distribution modulo π or 2π; for these moduli our approach only allows us to prove conditional or rather non-explicit results.

Theorem 2. *Let τ and δ be real numbers, δ positive. Assume that the number $m(N)$ of zeros of $\zeta(s)$ in the arithmetic progression $\frac{1}{2}+\mathrm{i}(\tau+n\delta)$ with $n \in \mathbb{N}$ and $n \leq N$ satisfies*

$$m(N) = o(N) \tag{7}$$

as $N \to \infty$. Then the argument of the Riemann zeta-function on the arithmetical progression $\frac{1}{2}+\mathrm{i}(\tau+n\delta)$ with $n \in \mathbb{N}$ is uniformly distributed modulo π.

It is exactly the question about nontrivial zeros of the zeta-function in arithmetic progression which makes a difference for our approach. In 1954, Putnam [27, 28] showed that there is no infinite arithmetic progression of nontrivial zeros; a sharpened result due to van Frankenhuijsen [11] states that for a hypothetical arithmetic progression with $\zeta(\frac{1}{2}+\mathrm{i}n\delta)=0$ for $1 \leq n < N$ and fixed positive real δ its length is bounded by $N < 13\delta$. Recently, Martin & Ng [26] and Li & Radziwiłł [22] gave quantitive bounds for the hypothetical number of nontrivial zeros in an arithmetic progression, however, their results are too weak to have any impact on our reasoning. Assuming the Riemann hypothesis in addition with Montgomery's pair correlation conjecture Ford, Soundararajan & Zaharescu [10], have shown that the proportion of zeros in an arbitrary vertical arithmetic progression is indeed zero. Consequently, under this assumption (7) is fulfilled. Probably, one should not expect any zeros in vertical arithmetic progression (of length at least three). Another stronger conjecture about the zeros of $\zeta(s)$ due to Ingham [14] deals with linear independence over $\mathbb{Q}$: if the nontrivial zeros are denoted as $\beta+\mathrm{i}\gamma$, it is widely believed that there are no rational linear relations for the imaginary parts γ apart from the trivial ones (resulting from the fact that with $\beta+\mathrm{i}\gamma$ also its complex conjugate $\beta-\mathrm{i}\gamma$ is a zero). It is expected that the imaginary parts of the nontrivial zeros should not carry any arithmetical information but are distributed like random data (e.g., the eigenangles of the Gaussian Unitary Ensemble in Random Matrix Theory, see Keating Snaith [18]). In view of this difficulty with zeros in an arithmetic progression we shall also show

Theorem 3. *For all pairs $(\tau,\delta) \in \mathbb{R}\times\mathbb{R}_{>0}$ except a possible at most countable set of exceptions, the argument of the Riemann zeta-function on the arithmetical progression $\frac{1}{2}+\mathrm{i}(\tau+n\delta)$ with $n \in \mathbb{N}$ is uniformly distributed modulo π.*

The article is organized as follows. We recall some basic facts from uniform distribution theory in the following section. In Section 3 we outline our method and prove Theorem 1. The proofs of Theorem 2 and 3 are given in Section 4 as well as an application to the case of continuous distribution modulo π. In the final section we discuss related results.

2. Variations of Weyl's criterion

Weyl's celebrated criterion [42] states that a sequence of real numbers x_n is uniformly distributed modulo one if, and only if, for all integers $m \neq 0$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{1\leq n\leq N} \exp(2\pi \mathrm{i} m x_n) = 0.$$

Since we investigate the argument of a complex-valued function it is natural to have moduli related to π, and by our approach (as will be explained in the beginning of the following section) we are forced to consider $\mu = \frac{\pi}{2}$ and $\mu = \pi$, respectively. We shall make use of the following variant of Weyl's criterion: a sequence of real numbers x_n is uniformly distributed modulo μ if, and only if, for all integers $m \neq 0$,

$$\sum_{n\leq N} \exp\left(\tfrac{2\pi}{\mu} \mathrm{i} m x_n\right) = o(N), \tag{8}$$

as $N \to \infty$. For a proof of Weyl's critierion in this form we refer to Weber's monography [40], Theorem 1.6.3.

For technical reasons it is sometimes advantageous to get rid of the first elements of the sequence in question. This leads to an equivalent form of the above criterion where (8) is replaced by

$$\sum_{M<n\leq M+N} \exp\left(\tfrac{2\pi}{\mu} \mathrm{i} m x_n\right) = o(N), \tag{9}$$

which has to hold for all sufficiently large M. The changed range of summation does not influence uniform distribution since this is a property of the infinite tail of a sequence and not of finitely many initial values.

Finally, we notice that in view of the Weyl criterion in whatever form we can omit a sufficiently small set of elements from our sequence. Namely, if $\mathcal{N} \cup \mathcal{M} = \mathbb{N}$ is a disjoint union with

$$\lim_{N\to\infty} \frac{1}{N} \sharp\{n \leq N \,:\, n \in \mathcal{M}\} = 0,$$

then $(x_n)_{n\in\mathbb{N}}$ is uniformly distributed modulo μ if, and only if, the subsequence $(x_n)_{n\in\mathcal{N}}$ is uniformly distributed modulo μ. This follows immediately from Weyl's criterion and the trivial bound

$$\sum_{\substack{n\leq N\\ n\in\mathcal{M}}} \exp\left(\tfrac{2\pi}{\mu}\mathrm{i}mx_n\right) \ll \sharp\{n\leq N\,:\,n\in\mathcal{M}\} \ll \epsilon N$$

for all positive ϵ and sufficiently large N.[3] Such a set $\mathcal{M}$ will be called *negligible*.

3. The main idea of our method and the proof of Theorem 1

We want to study the distribution of the argument of the zeta-function on an infinite vertical arithmetic progression:

$$\zeta\left(\tfrac{1}{2}+\mathrm{i}t_n\right) \qquad \text{with} \qquad t_n := \tau + n\delta \qquad \text{for} \quad n\in\mathbb{N}.$$

Here and in the sequel τ and δ are arbitrary but fixed real numbers, δ being positiv. In view of (9) we put $x_n = \arg\zeta\big(\frac{1}{2}+\mathrm{i}(\tau+n\delta)\big) = \pi S(\tau+n\delta)$ and thus have to study the exponential sum

$$\Sigma_m(N) \;:=\; \sum_{M<n\leq M+N} \exp\Big(\tfrac{2\pi}{\mu}\mathrm{i}m\arg\zeta\big(\tfrac{1}{2}+\mathrm{i}(\tau+n\delta)\big)\Big), \tag{10}$$

where m is a non-zero integer and μ equals either π or $\frac{\pi}{2}$.

Firstly, we may assume that $\zeta(\frac{1}{2}+\mathrm{i}t_n)\neq 0$. By the reflection principle, $\overline{\zeta(s)}=\zeta(\overline{s})$, hence also $\zeta(\frac{1}{2}-\mathrm{i}t_n)$ does not vanish. This implies

$$\exp\big(2m\mathrm{i}\arg\zeta(\tfrac{1}{2}+\mathrm{i}t_n)\big) = \left(\frac{\zeta(\frac{1}{2}+\mathrm{i}t_n)}{|\zeta(\frac{1}{2}+\mathrm{i}t_n)|}\right)^{2m} = \left(\frac{\zeta(\frac{1}{2}+\mathrm{i}t_n)}{\zeta(\frac{1}{2}-\mathrm{i}t_n)}\right)^{m}.$$

Notice that the quantity on the right-hand side determines the argument of the zeta-function only modulo 2π (the period of the exponential function $t\mapsto\exp(\mathrm{i}t)$). Consequently, our reasoning is limited to positive real moduli μ for which $\frac{2\pi}{\mu}$ is an integer. This gives for the corresponding exponential sum in Weyl's criterion (9)

$$\Sigma_m(N) \;=\; \sum_{M<n\leq M+N} \left(\frac{\zeta(\frac{1}{2}+\mathrm{i}t_n)}{\zeta(\frac{1}{2}-\mathrm{i}t_n)}\right)^{\frac{\pi}{\mu}m}. \tag{11}$$

Now, using the functional equation (2), we get

$$\Sigma_m(N) \;=\; \sum_{M<n\leq M+N} \Delta\left(\tfrac{1}{2}+\mathrm{i}t_n\right)^{\frac{\pi}{\mu}m}, \tag{12}$$

[3]Here we have used Vinogradov-notation $f\ll g$ which is equivalent to $f=O(g)$.

which is not too difficult to estimate (see (14) and its proof below). It will become clear soon that by this method the modulus $\mu = 2\pi$ is impossible to combine with odd m.

In view of the above simplification (12) one might be tempted to consider our results as easy consequences of the functional equation only. However, there is a certain obstacle, namely our assumption on the non-vanishing of the zeta-function on the arithmetic progression in question.

Now let $\mu = \frac{\pi}{2}$. If $\zeta(\frac{1}{2} + \mathrm{i}t_n) = 0$, i.e., $t_n = \gamma$ is an ordinate of a nontrivial zero, then the argument of the zeta-function is in view of (5) given by

$$\pi S(\gamma) = \frac{\pi}{2} \lim_{\epsilon \to 0} \big(S(\gamma + \epsilon) + S(\gamma - \epsilon)\big)$$

(and it is this definition which appears the reason for our restriction to $\mu = \frac{\pi}{2}$ in Theorem 1). In this case the values $S(\gamma \pm \epsilon)$ differ by an integer (as follows from (4) and had already been explained in the introductory section). Thus, for every sufficiently small ϵ,

$$\lim_{\epsilon \to 0} \pi S(\gamma + \epsilon) \equiv \lim_{\epsilon \to 0} \pi S(\gamma - \epsilon) \bmod \pi,$$

respectively,

$$\pi S(\gamma) = \frac{1}{2} \lim_{\epsilon \to 0} \big(2\pi S(\gamma - \epsilon) + \pi k\big)$$

for some integer k. Hence, we obtain

$$\begin{aligned}
\exp\Big(4m\mathrm{i} \arg \zeta\left(\tfrac{1}{2} + \mathrm{i}\gamma\right)\Big) &= \exp\Big(4m\mathrm{i} \lim_{\epsilon \to 0} \pi S(\gamma - \epsilon)\Big) \exp\left(4m\mathrm{i}\tfrac{\pi k}{2}\right) \\
&= \lim_{\epsilon \to 0} \exp\Big(4m\mathrm{i} \arg \zeta\left(\tfrac{1}{2} + \mathrm{i}(\gamma - \epsilon)\right)\Big) \\
&= \lim_{\epsilon \to 0} \Delta\big(\tfrac{1}{2} + \mathrm{i}(\gamma - \epsilon)\big)^{2m}
\end{aligned}$$

by the reasoning from above. Since the limit on the right-hand side exists, we get

$$\exp\Big(4m\mathrm{i} \arg \zeta\left(\tfrac{1}{2} + \mathrm{i}\gamma\right)\Big) = \Delta\big(\tfrac{1}{2} + \mathrm{i}\gamma\big)^{2m}.$$

Consequently, for the modulus $\mu = \frac{\pi}{2}$ it does not matter whether $\frac{1}{2} + \mathrm{i}t_n$ is a zero of the zeta-function or not (since $\pi S(t)$ is continuous modulo $\frac{\pi}{2}$). Notice that the above reasoning fails for $\mu = \pi$ because in this case our reasoning would lead to $\exp\big(2m\mathrm{i} \arg \zeta(\frac{1}{2} + \mathrm{i}\gamma)\big) = \pm\Delta(\frac{1}{2} + \mathrm{i}\gamma)^m$ without control of the sign $\pm$.

In view of (12) we arrive at

$$\Sigma_m(N) = \sum_{M < n \leq M+N} \Delta\left(\tfrac{1}{2} + \mathrm{i}t_n\right)^{2m}.$$

Taking (3) into account this expression can be simplified to

$$\Sigma_m(N) = \Big(\exp\left(\tfrac{\pi m}{2}\mathrm{i}\right) + O\big(M^{-|2mk|}\big)\Big) \sum_{M<n\leq M+N} \left(\frac{2\pi e}{t_n}\right)^{2mit_n}. \tag{13}$$

Next we apply van der Corput's method [7] for estimating the appearing exponential sum. For this aim we shall make use of Lemma 8.12 from Iwaniec & Kowealski's monography [15]: let $b - a \geq 1$, let f be a twice differentiable real-valued function on $[a, b]$ with $f''(x) \geq \Lambda > 0$ on $[a, b]$. Then

$$\sum_{a<n<b} \exp\big(2\pi \mathrm{i} f(n)\big) \ll \big(f'(b) - f'(a) + 1\big)\Lambda^{-\frac{1}{2}},$$

where the implicit constant is absolute. Of course, the statement of this lemma is also true if $f''(x) \leq -\Lambda < 0$ on $[a, b]$ (as follows by complex conjugation); in this case we only have to change the upper bound to

$$\big(f'(a) - f'(b) + 1\big)\Lambda^{-\frac{1}{2}}.$$

In view of (13) we consider

$$f(x) = -\frac{m}{\pi}(\tau + x\delta)\log\frac{\tau + x\delta}{2\pi e}$$

with

$$f'(x) = -\frac{m\delta}{\pi}\log\frac{\tau + x\delta}{2\pi} \qquad \text{and} \qquad f''(x) = -\frac{m\delta^2}{\pi(\tau + x\delta)}.$$

Applying the lemma with $a = M$, $b = M + N$ and Λ of approximate size N^{-1} leads to

$$\Sigma_m(N) \ll N^{\frac{1}{2}} \log N, \tag{14}$$

where the implicit constant may depend on τ, δ and m. In view of (9) this estimate is suitable and the theorem is proved. Notice that we do not attempt to prove sharper bounds here.

4. Proof of Theorem 2 and 3 and an application to continuous uniform distribution

First we prove Theorem 2. In view of (7) the set of elements $\frac{1}{2} + \mathrm{i}t_n$ of our arithmetic progression which leads to zeros of the zeta-function is (according to our remark in Section 2) negligible with respect to uniform distribution modulo π. Hence, we may assume that $\zeta(\frac{1}{2} + \mathrm{i}t_n)$ does not vanish for any positive integer n. Since $\arg\zeta(\frac{1}{2} + \mathrm{i}t) = \pi S(t)$ makes jumps at the ordinates which are integer multiples of π, we obtain by the same reasoning as in the previous proof applied to $f(x) = -\frac{m}{2\pi}(\tau + x\delta)\log\frac{\tau+x\delta}{2\pi e}$ uniform distribution modulo π.

The proof of Theorem 3 is just an application of Cantor's notions of countability and uncountability. Since the set of nontrivial zeros is countable and the set of $(\tau,\delta) \in \mathbb{R} \times \mathbb{R}_{>0}$ is not, for almost all pairs of τ and δ (in the sense of Lebesgue measure) the corrseponding arithmetic progression $\frac{1}{2} + \mathrm{i}(\tau + n\delta)$ will avoid nontrivial zeros. Consequently, the statement follows by the same reasoning as in the previous proofs. Of course, the statement of the theorem also holds with either fixed $\delta = 1$ or fixed $\tau = 0$.

In view of the just proven theorem one can hardly expect that the continuous version of uniform distribution differs from the discrete one. To be more precise, since the argument $\pi S(t_n)$ of the zeta-function on almost all vertical arithmetic progressions t_n on the critical line is uniformly distributed modulo π, the values $\pi S(t)$ modulo π have to be equidistributed as well as t ranges continuously through $\mathbb{R}$. In order to prove that we start with a standard notion. A real-valued Lebesgue integrable function f defined on $(0,\infty)$ is said to be continuously uniformly distributed modulo μ if for all $0 \leq \alpha < \beta \leq \mu$ the following limit exists and satisfies

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{1}_{[\alpha,\beta)}(\{f(t)\})\mathrm{d}t = \frac{\beta-\alpha}{\mu},$$

where $\mathbf{1}_{[\alpha,\beta)}(f)$ is the characteristic function of the interval $[\alpha,\beta)$ (being equal to 1 if $f \in [\alpha,\beta)$ and zero otherwise) and $\{f\}$ denotes the fractional part modulo μ defined analogously to (6). This is a straightforward generalization of the corresponding definition of continuous uniform distribution modulo one which can be found, for instance, in the monograph of Kuipers & Niederreiter [19]. Moreover, their Theorem 9.6 states that a real-valued Lebesgue integrable function f is continuously uniformly distributed modulo one if the sequence of real numbers $x_n = f(\tau+n)$ with $n = 1, 2, \ldots$ is uniformly distributed modulo one for almost all τ; the proof follows from the continuous version of Weyl's criterion and incorporating the information about the discrete uniform distribution in form of approximating exponential sums. We leave it to the reader to extend this result to the situation of (continuous) uniform distribution modulo μ and just mention that in combination with Theorem 3 this reasoning leads to

Corollary 4. *The values of the argument of the zeta-function on the critical line are continuously uniformly distributed modulo π.*

A straightforward proof of this result (without the detour via discrete uniform distribution) would follow from the continuous Weyl criterion (which can be found in Kuipers & Niederreiter [19] as well as already in Weyl's original paper [42]).

5. Concluding remarks

Shanks [33] conjectured that the curve $\mathcal{C}\,:\,t\mapsto\zeta(\frac{1}{2}+\mathrm{i}t)$ approaches the origin most of the times from the third or fourth quadrant, i.e., $\zeta'(\frac{1}{2}+\mathrm{i}\gamma)$ is positive real in the mean. This was proved by Fujii [12] and Trudgian [38]. In this sense, Figure 2 (as well as Figure 3 below) provides a good idea about the distribution of the values of the zeta-function on the critical line. In the papers of Kalpokas & Steuding [16] and Kalpokas, Korolev & Steuding [17] the value-distribution of the

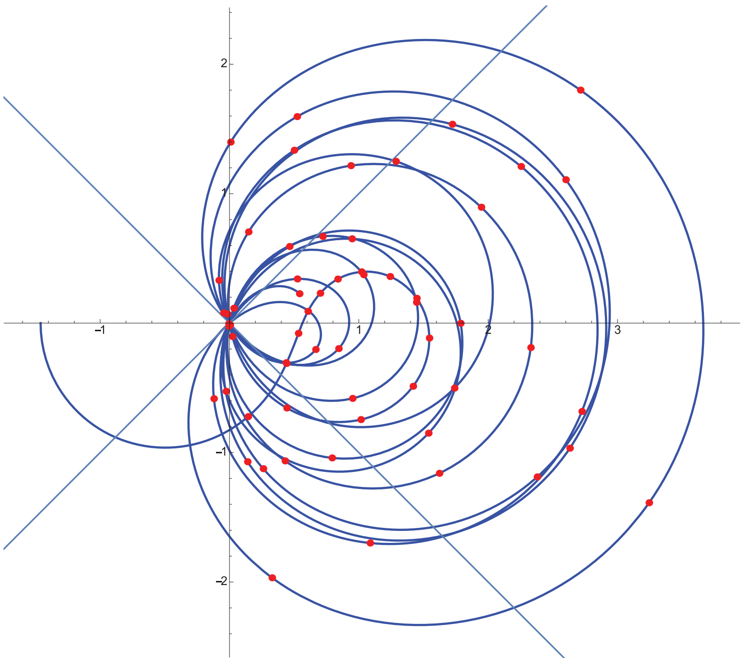

FIGURE 2. The figure shows the graph of the curve $t\mapsto\zeta\big(\frac{1}{2}+\mathrm{i}t\big)$ in the complex plane for the range $0\le t\le\mathbf{60}$ with 13 simple zeros. The values of $\zeta(s)$ on the arithmetic progression $s=\frac{1}{2}+\mathrm{i}n$ for $n=1,2,\ldots,60$ are marked as red points; the values for $n=14,21,25,33,41,48$ and 53 are very close to the origin (and the corresponding $\frac{1}{2}+\mathrm{i}n$ near to nontrivial zeros). In the eight octants we count in sequence $18,7,3,0,1,3,10,18$ starting with the sector in the first quadrant bounded by the positive real axis and continuing counterclockwise. The data matches the statements of the theorems. The graphic resembles a similar one due to Shanks [33]. We do not provide any data about these computations here since nowadays – different to the time of Shanks, Haselgrove and others—it is easy to reproduce the data by modern computer algebra packages.

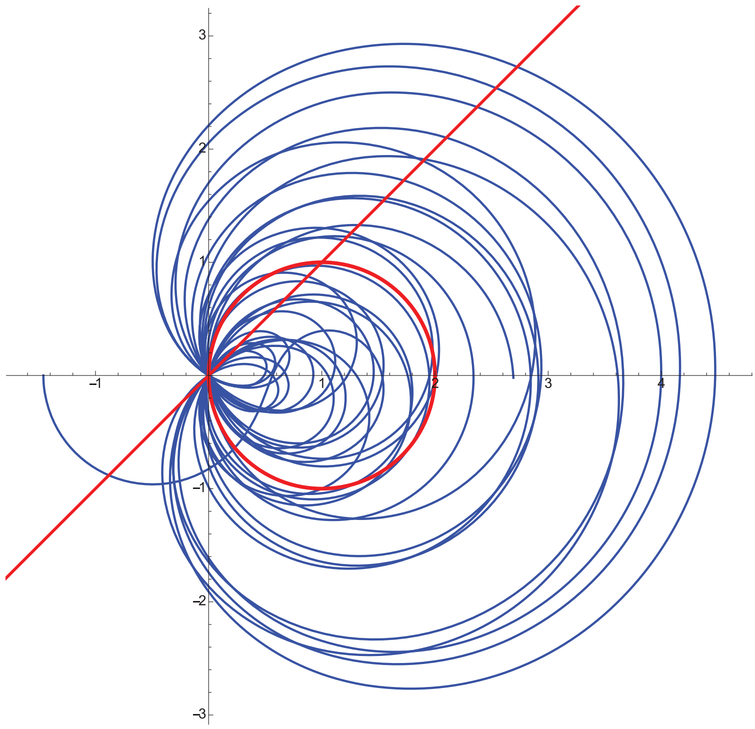

FIGURE 3. The curve $t \mapsto \zeta\big(\frac{1}{2}+\mathrm{i}t\big)$ for $t \in [0, 100]$ in addition with the circle of the mean-values and the bisecting line of the first and third quadrant ($\phi = \frac{\pi}{4}$); here the mean-value is $\frac{1}{2}(1+\mathrm{i})$ which is the intersection point (different from the origin) of the circle with the bisecting line. A different approach is due to Arias de Reyna, Brent & van der Lune [1].

curve $\mathcal{C}$ was investigated in detail. Their results explain why "*the real part of* ζ *has a strong tendency to be positive*" as observed by, for example, Edwards in his monography [8] (page 121), as well as the almost symmetry of $\mathcal{C}$ with respect to the real axis. In particular, they have shown that the mean value of the intersection points of the curve $t \mapsto \zeta(\frac{1}{2}+\mathrm{i}t)$ with an arbitrary straight line $\exp(\mathrm{i}\varphi)$ with $\phi \in [0, \pi)$ exists and is equal to $2\exp(\mathrm{i}\phi)\cos\phi = 1 + \exp(\mathrm{i}\phi)$ (see Figure 3).[4]

[4]Moreover, they showed that the zeta-function on the critical line assumes arbitrarily large positive real values and arbitrarily large negative values (what figures like Figure 2, 4 and 5 cannot show).

Notice that the circle built from these mean values, i.e., $1+\exp(i\phi)$ for $0 \leq \phi < \pi$, reflects the typical shape of the curve made from the zeta-values $\zeta(\frac{1}{2} + it)$ as t ranges through some fixed interval, and taking this circle as a model for the value-distribution of the zeta-function, we observe uniform distribution modulo π for a generic arithmetic progression.[5]

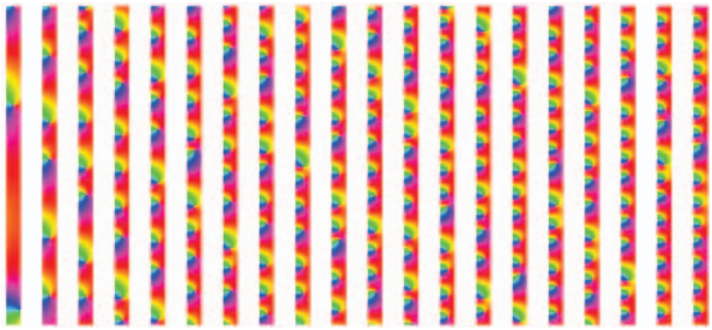

FIGURE 4. These are phase plots from [36] showing $\zeta(s)$ along the critical strip.

In Steuding & Wegert [36] it is written that "*visual inspection of the phase plot of ζ in the critical strip almost immediately reveals a surprising 'stochastic periodicity' of the phase $\psi = \zeta/|\zeta|$. (...) The eye-catching yellow diagonal stripes led us to the conjecture that the phase plot of zeta in the critical strip has sort of stochastic period, and some heuristic arguments suggest that its length equals* $2\pi/\log 2$." According to this observation the authors proved for fixed $s \in \mathbb{C} \setminus \{1\}$ with $\mathrm{Re}\, s \in (0, 1]$, and $d = \frac{2\pi}{\log \ell}$ with $2 \leq \ell \in \mathbb{N}$,

$$\frac{1}{N} \sum_{0 \leq n < N} \zeta(s + ind) = (1 - \ell^{-s})^{-1} + O\left(N^{-\mathrm{Re}\, s} \log N\right), \quad \text{as} N \to \infty.$$

It is worth to notice that the main term equals the factor for a prime number ℓ in the Euler product representation of the zeta-function $\zeta(s) = \prod_\ell (1 - \ell^{-s})^{-1}$, valid for $\mathrm{Re}\, s > 1$. This factor on its own obeys a uniform distribution law modulo 2π with any arithmetic progression $t_n = \tau + \delta n$ for which $\delta \log \ell \notin 2\pi\mathbb{Q}$. Nevertheless, for a generic progression $t_n = \tau + n\frac{2\pi}{\log \ell}$ we have uniform distribution modulo π by Theorem 3 (and Figure 5 provides and illustration).

[5]Something similar can be deduced for the characteristic polynomials to random matrices from certain unitary matrix ensembles which are often used to model the value-distribution of $\zeta(\frac{1}{2} + it)$, see Keating & Snaith [18].

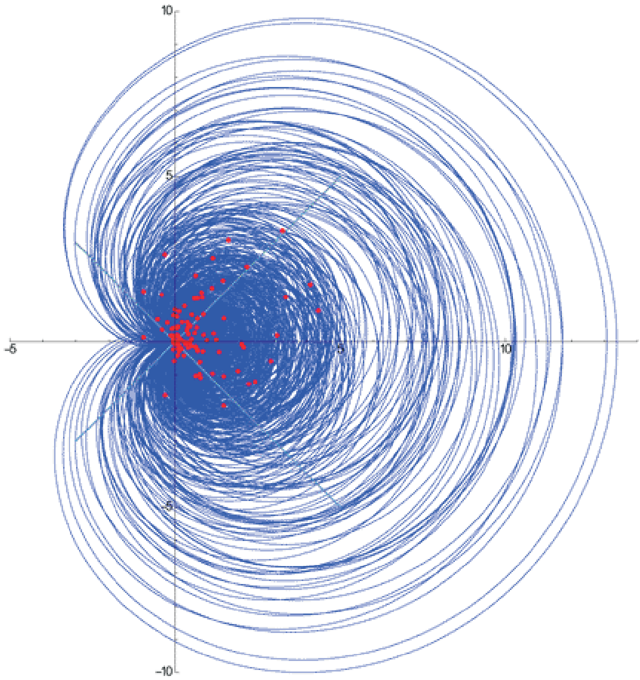

FIGURE 5. The figure shows the graph of the curve $t \mapsto \zeta\left(\frac{1}{2} + it\right)$ in the complex plane for the range $1955 \leq t \leq 2865$. The values of $\zeta(s)$ on the arithmetic progression $s = \frac{1}{2} + i\left(1955 + n\frac{2\pi}{\log 2}\right)$ for $n = 1, 2, \ldots, 100$ are marked as red points. Notice that the range up to 2865 is according to the one hundred points: $1955 + 100 \cdot \frac{2\pi}{\log 2} < 2865 < 1955 + 101 \cdot \frac{2\pi}{\log 2}$. The distributions in the four quadrants is $44, 15, 6, 35$, matching the uniform distribution modulo π very well; in the octants, however, we count $18, 26, 12, 3, 1, 5, 12, 23$ (in the same sequence as before) which shows some deviation. It is an interesting phenomenon that the values taken on this arithmetic progression are *comparably small.*

In view of the graphics and mean-value theorems mentioned above (as, e.g., in [16, 36]) one might be tempted to conjecture that the values of the argument of the zeta-function on an arithmetic progression on the critical line are not uniformly distributed modulo 2π. Actually, we are uncertain about the critical line but expect this to hold for vertical lines to the right. It is easy to show that

$$\operatorname{Re}\zeta(\sigma + it) > 2 - \zeta(\sigma) \quad \text{for} \quad \sigma > 1$$

and

$$2 > \zeta(\sigma) \quad \text{for} \quad \sigma > 1.72865.$$

Hence, the argument of $\zeta(s)$ for fixed $\operatorname{Re} s$ and varying $\operatorname{Im} s$ is definitely not uniformly distributed modulo 2π. It might be interesting to notice that, building on computer experiments, Arias de Reyna, Brent & van der Lune [1] write that it is plausible that

$$\lim_{T\to\infty} \frac{1}{T}\lambda\Big(\{t\in[0,T)\,:\,\operatorname{Re}\zeta\left(\tfrac12+\mathrm{i}t\right)<0\}\Big)=\frac{1}{2},$$

where λ denotes the Lebesgue measure. Unfortunately, their reasoning is limited to vertical lines to the right of the critical line. (See Figure 6 below for an illustration of this question with respect to an arithmetic progression.)

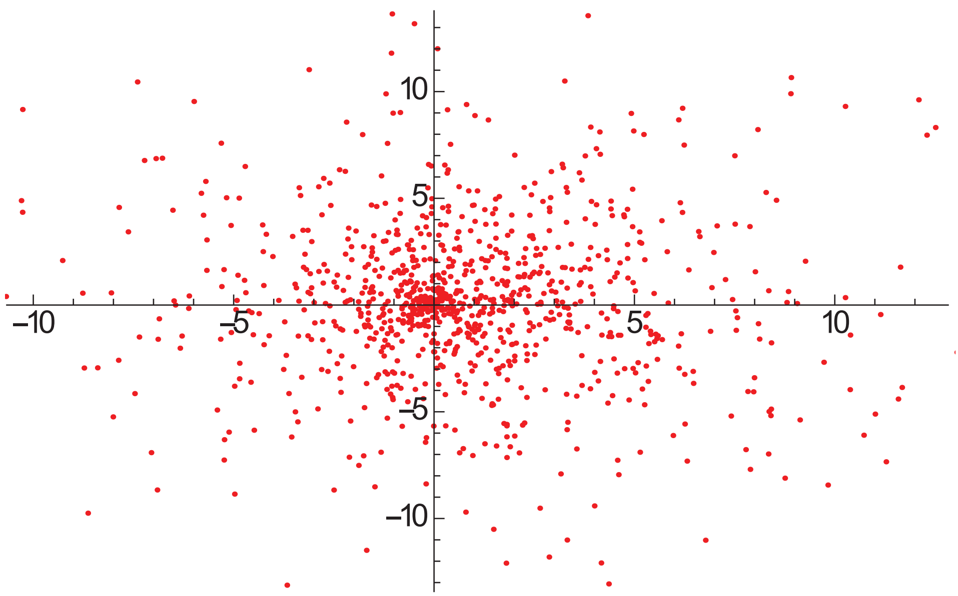

FIGURE 6. The points $\zeta(\frac12+\mathrm{i}n\frac{2\pi}{\log 2})$ for $n=10^{15}+1,\ldots,10^{15}+1\,000$ (without the curve). By this sample one could imagine uniform distribution modulo 2π.

We conclude with a historical detail. The first to investigate uniform distribution in the context of the zeta-function was Rademacher [29] in 1956 who proved that the ordinates of the nontrivial zeros of the zeta-function are uniformly distributed modulo one provided that the Riemann hypothesis is true;

later Elliott [9] remarked that the latter condition can be removed, and (independently) Hlawka [13] obtained the following unconditional theorem: for any real number $\alpha \neq 0$ the sequence $\alpha\gamma$, where γ ranges through the set of positive ordinates of the nontrivial zeros of $\zeta(s)$ in ascending order, is uniformly distributed modulo one. In particular, the ordinates of the nontrivial zeros of the zeta-function are uniformly distributed modulo one. In [34] the second author proved that the same holds true for the ordinates of the roots of the equation

$$\zeta(s) = a,$$

where a is an arbitrary fixed complex number. These so-called a-points had been under consideration ever since Landau's work [20, 6].[6] As a matter of fact, one distinguishes between trivial and nontrivial a-points, and indeed the trivial ones are similarly distributed as the trivial zeros of $\zeta(s)$ as $s = -2n$, $n \in \mathbb{N}$. Also the distribution of the nontrivial a-points shares some patterns with the nontrivial zeros. If $N_a(T)$ denotes the number of nontrivial a-points with imaginary part γ_a satisfying $0 < \gamma_a \leq T$ (counted according multiplicities), then pretty similar to (1) one has

$$N_a(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e c_a} + O(\log T), \quad \text{as } T \to \infty,$$

where

$$c_a = 1 \quad \text{if } a \neq 1, \quad \text{and} \quad c_1 = 2.$$

One observes that the main term is independent of a (which is not too surprising having Nevanlinna's value distribution theory in mind). Moreover, Landau proved that almost all a-points are clustered around the critical line provided the Riemann hypothesis is true, and later Levinson [21] showed that this holds unconditionally. More details can be found in Steuding [35].

Acknowledgements. The authors would like to thank the anonymous referee for his or her valuable corrections and comments.

REFERENCES

[1] ARIAS DE REYNA, J.—BRENT, R. P.—VAN DE LUNE, J.: *On the sign of the real part of the Riemann zeta function*, in: *Number Theory and Related Fields. In memory of Alf van der Poorten*, (J.M. Borwein et al. eds.), Springer, New York, 2013, pp. 75–97.

[2] BACKLUND, R. J.: *Sur les zéros de la fonction $\zeta(s)$ de Riemann*, Comptes Rendus Acad. Sci. Paris **158** (1914), 1979–1981.

[6] The paper [6] of Bohr, Landau & Littlewood consists of three independent chapters, the first one due to Bohr, the second written by Landau, and the third by Littlewood.

[3] BACKLUND, R. J.: *Über die Nullstellen der Riemannschen ζ-Funktion*, Dissertation Helsingfors. Acta Math. **41**, (1916) no. 1, 345—375.

[4] BOHR, H.: *Über eine quasi-periodische Eigenschaft Dirichletscher Reihen mit Anwendung auf die Dirichletschen L-Funktionen,* Math. Ann. **85** (1922), 115–122.

[5] BOHR, H.—JESSEN, B.: , *Über die Werteverteilung der Riemannschen Zeta-funktion. II*, Acta Math. **58** (1932), 1–55.

[6] BOHR, H.—LANDAU, E.—LITTLEWOOD, J. E.: *Sur la fonction $\zeta(s)$ dans le voisinage de la droite $\sigma = \frac{1}{2}$*, Bull. de l'Acad. Royale de Belgique (1913), 3–35.

[7] VAN DER CORPUT, J. G.: *Zahlentheoretische Abschätzungen*, Math. Ann. **84** (1921), 53–79.

[8] EDWARDS, H. M.: *Riemann's Zeta function*, Pure and Applied Mathematics, Vol. 58, Academic Press, New York-London, 1974.

[9] ELLIOTT, P. D. T. A.: *The Riemann zeta function and coin tossing*, J. Reine Angew. Math. **254** (1972), 100–109.

[10] FORD, K.—SOUNDARARAJAN, K.—ZAHARESCU, A.: , *On the distribution of imaginary parts of zeros of the Riemann zeta function, II*, Math. Ann. **343** (2009), 487–505.

[11] VAN FRANKENHUIJSEN, M.: , *Arithmetic progressions of zeros of the Riemann zeta-function*, J. Number Theor. **115** (2005), 360–370

[12] FUJII, A.: *On a conjecture of Shanks*, Proc. Japan Acad. Ser. A Math. Sci. **70** (1994), 109–114

[13] HLAWKA, E.: *Über die Gleichverteilung gewisser Folgen, welche mit den Nullstellen der Zetafunktion zusammenhängen*, Österr. Akad. Wiss., Math.-Naturw. Kl. Abt. II **184** (1975), 459–471

[14] INGHAM, A. E.: *On two conjectures in the theory of numbers*, Amer. J. Math. **64** (1942), 313–319.

[15] IWANIEC, H.—KOWALSKI, E.: *Analytic Number Theory*, AMS, Providence, 2004.

[16] KALPOKAS, J.—STEUDING, J.: *On the value-distribution of the Riemann zeta-function on the critical line*, Moscow J. Combinatorics Number Theory **1** (2011), 26–42.

[17] KALPOKAS, J.—KOROLEV, M.—STEUDING, J.: *Negative values of the Riemann zeta function on the critical line*, Mathematika **59** (2013), 443–462.

[18] KEATING, J. P.—SNAITH, N. C.: *Random matrix theory and $\zeta(1/2 + it)$*, Commun. Math. Phys. **214** (2000), 57–89.

[19] KUIPERS, L.—NIEDERREITER, H.: *Uniform distribution of sequences*, John Wiley & Sons, New York, 1974.

[20] LANDAU, E.: *Über die Nullstellen der Zetafunktion*, Math. Ann. **71** (1912), 548–564.

[21] LEVINSON, N.: *Almost all roots of $\zeta(s) = a$ are arbitrarily close to $\sigma = 1/2$*, Proc. Nat. Acad. Sci. U.S.A. **72** (1975), 1322–1324.

[22] LI, X.—RADZIWIŁŁ, M.: *The Riemann zeta function on vertical arithmetic progressions*, Int. Math. Res. Not. **2** (2015), 325–354.

[23] LITTLEWOOD, J. E.: *On the zeros of the Riemann zeta-function*, Proc. Cambridge Philos. Soc. **22** (1924), 295–318.

[24] VON MANGOLDT, H.: *Zu Riemann's Abhandlung "Ueber die Anzahl der Primzahlen unter einer gegebenen Größe"*, J. Reine Angew. Math. **114** (1895), 255–305.

[25] VON MANGOLDT, H.: *Zur Verteilung der Nullstellen der Riemannschen Funktion* $\xi(t)$, Math. Ann. **60** (1905), 1–19.

[26] MARTIN, G—NG, N.: *Nonzero values of Dirichlet L-functions in vertical arithmetic progressions*, Int. J. Number Theory **9** (2013), 813–843.

[27] PUTNAM, C. R.: *On the non-periodicity of the zeros of the Riemann zeta-function*, Amer. J. Math. **76** (1954), 97–99.

[28] PUTNAM, C. R.: *Remarks on periodic sequences and the Riemann zeta-function*, Amer. J. Math. **76** (1954), 828–830.

[29] RADEMACHER, H. A.: *Fourier Analysis in Number Theory*, in: Collected Papers of Hans Rademacher, Vol. II, Symposium on Harmonic Analysis and Related Integral Transforms (Cornell Univ., Ithaca, N.Y., 1956) Massachusetts Inst. Tech., Cambridge, Mass., 1974, pp. 434–458,.

[30] RIEMANN, R.: *Über die Anzahl der Primzahlen Unterhalb Einer Gegebenen Grösse*, Monatsber. Preuss. Akad. Wiss. Berlin (1859), 671–680.

[31] RUDIN, W.: *Principles of Mathematical Analysis*, the 3rd ed. MacGraw-Hill, 1976.

[32] SEMMLER, G.—WEGERT, E.: *Phase plots of complex functions: a journey in illustration*, Notices Amer. Math. Soc. **58** (2011), no. 6, 768–780.

[33] SHANKS, D.: *Review of 'Tables of the Riemann zeta function' by C. B. Haselgrove in collaboration with J. C. P. Miller*, Math. Comp. **15** (1961), 84–86.

[34] STEUDING, J.: *The roots of the equation* $\zeta(s) = a$ *are uniformly distributed modulo one*, in: Analytic and Probabilistic Methods in Number Theory, (A. Laurinčikas et al. eds.), Proceedings of the Fifth International Conference in Honour of J. Kubilius, Palanga 2011, TEV, Vilnius 2012, 243–249.

[35] STEUDING, J.: *One hundred years uniform distribution modulo one and recent applications to Riemann's zeta-function*, in: Topics in Mathematical Analysis and Applications (T.M. Rassias, L. Tóth eds), Springer Optim. App. **94**, springer Cham (2014), pp.659–698.

[36] STEUDING, J.—WEGERT, E.: *The Riemann zeta function on arithmetic progressions*, Exp. Math. **21** (2012), 235–240.

[37] TITCHMARSH, E. C.: *The theory of the Riemann zeta-function*, the 2nd ed. (D. R. Heath-Brown, ed.), The Clarendon Press, Oxford University Press, New York, 1986.

[38] TRUDGIAN, T. S.: *On a conjecture of Shanks*, J. Number Theory **130** (2010), 2635–2638.

[39] TSANG, K.-M.: , Some Ω-theorems for the Riemann zeta-function, Acta Arith. **46** (1985), 369–395.

[40] WEBER, M.: *Dynamical Systems and Processes*, 14. European Mathematical Society (EMS), Zürich, 2009.

[41] WEGERT, E.: *Visual Complex Functions*, Birkhäuser Basel, 2012.

[42] WEYL, H.: *Über die Gleichverteilung von Zahlen Mod. Eins*, Math. Ann. **77** (1916), 313–352.

Received April 7, 2015
Accepted June 18, 2015

Selin Selen Özbek
Akdeniz University
Department of Mathematics
07058 Antalya
TURKEY
E-mail: s.selenozbek@gmail.com

Jörn Steuding
Department of Mathematics
Würzburg University
Emil-Fischer-Str. 40
97 074 Würzburg
GERMANY
E-mail: steuding@mathematik.uni-wuerzburg.de