Uniform Distribution Theory **10** (2015), no.2, 117–183

uniform distribution theory

SOME APPLICATIONS OF DISTRIBUTION FUNCTIONS OF SEQUENCES

Oto Strauch

Dedicated to Prof. H. Niederreiter on the occasion of his 70th birthday

ABSTRACT. This expository paper presents some old and some new results on distribution functions of sequences $x_n \in [0,1)$, $n = 1, 2, \dots$ Firstly we describe old applications: Statistically independent sequences; statistically convergent sequences; statistical limit points; and uniform maldistributed sequences. Then we give some recent results: Benford's law; copulas; and ratio sequences. Secondly we present some methods for computing the set $G(x_n)$ of all distribution functions of x_n: directly by definition of distribution functions; using connectivity of $G(x_n)$; solving a moment problem $X_1 = \int_0^1 g(x)\mathrm{d}x$, $X_2 = \int_0^1 xg(x)\mathrm{d}x$ and $X_3 = \int_0^1 g^2(x)\mathrm{d}x$ for distribution functions $g(x)$; and mapping x_n to $f(x_n)$, for some function $f : [0,1] \to [0,1]$. Parts of this paper were presented at the UDT conferences in Marseilles 2008, Strobl 2010, Smolenice 2012, and Ostravice 2014 and also in MCQMC conference, Warszawa 2010.

Communicated by Werner Georg Nowak

Contents

2010 Mathematics Subject Classification: 11K06, 11K31.
Keywords: Distribution function, copula, statistical independent sequences, statistical limit, sequence of logarithm, moment problem, Benford's law, prime number, moment problem, Riemann-Stieltjes integral, von Neumann-Kakutani transformation.
Supported by the VEGA Project 2/0146/14.

1. Introduction

Let $\mathbf{x}_n$, $n = 1, 2, \dots$, be an infinite sequence in the s-dimensional unit cube $[0,1)^s$. Denote the step distribution function $F_N(\mathbf{x})$ of $\mathbf{x}_1, \dots, \mathbf{x}_N$ as

$$F_N(\mathbf{x}) = \frac{\#\{n \le N; \mathbf{x}_n \in [\mathbf{0}, \mathbf{x})\}}{N}, \tag{1}$$

where $[\mathbf{0}, \mathbf{x}) = [0, x_1) \times \cdots \times [0, x_s)$. By Riemann-Stieltjes integration we have

$$\frac{1}{N_k}\sum_{n=1}^{N_k} f(\mathbf{x}_n) = \int_{[\mathbf{0},\mathbf{1}]} f(\mathbf{x})\mathrm{d}F_{N_k}(\mathbf{x}). \tag{2}$$

By Helly theorem, for continuous $f(\mathbf{x})$ and for a weak limit

$$\lim_{k\to\infty} F_{N_k}(\mathbf{x}) \ = \ g(\mathbf{x}) \tag{3}$$

we have

$$\lim_{k\to\infty} \frac{1}{N_k}\sum_{n=1}^{N_k} f(\mathbf{x}_n) \Longrightarrow \int_{[\mathbf{0},\mathbf{1}]} f(\mathbf{x})\mathrm{d}g(\mathbf{x}). \tag{4}$$

The function $g(\mathbf{x})$ in (3) is called distribution function (abbreviating d.f.) of $\mathbf{x}_n$ [1] Denote by $G(\mathbf{x}_n)$ the set of all possible limits in (3), for an arbitrary $N_1 < N_2 < \dots$ and the given infinite sequence $\mathbf{x}_n$, $n = 1, 2, \dots$

This expository paper is devoted specially, for employing and calculating $G(\mathbf{x}_n)$ in the dimension $s = 1$ and $s = 2$.

The study of the set of d.f.s of a sequence, still unsatisfactory today, was initiated by J. G. van der Corput [46]. The one-element set $G(\mathbf{x}_n) = \{g(\mathbf{x})\}$ correspond to the notion of asymptotic distribution function (abbreviating a.d.f.) $g(\mathbf{x})$. In the case $g(\mathbf{x}) = \mathbf{x}$ the sequence is called uniformly distributed (abbreviating u.d.) In the monograph L. Kuipers and H. Niederreiter [22] to d.f.s is devoted Chapter 7, pp. 53–68, and in M. Drmota and R.F. Tichy [9] Part 1.5, pp. 138–153. [2] The a.d.f. of a sequence x_n was introduced by I.J. Schoenberg [34]. Main goal of this paper is a propagation of some partial results in the theory of d.f.s.

The outline of our paper is as follows.

In Section 2 we characterize some known classes of sequences x_n, originally defined by some properties of x_n, by using the set $G(x_n)$ of all distribution functions of x_n :

- Statistically independent sequences.
- Statistically convergent sequences.
- Statistical limit points.
- Uniform maldistributed sequences.

Then we give some recent results:

- Benford's law.
- Copulas.
- Ratio sequences.

In Section 3 we present some methods for computing $G(x_n)$, namely:

- Directly by definition of d.f.s.
- Using connectivity of $G(x_n)$.
- Solving moment problem $(X_1, X_2, X_3) = \left(\int_0^1 g(x)\mathrm{d}x, \int_0^1 xg(x)\mathrm{d}x, \int_0^1 g^2(x)\mathrm{d}x\right)$.

[1] The limit (4) generalizes $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{\infty} f(x_n) = \int_0^1 f\mathrm{d}x$ - the fundamental Weyl's limit relation holding for any continuous function $f(x)$ defined on $[0, 1]$ and any uniformly distributed sequence x_n.

[2] Some authors, see R. Winkler [50], instead distribution functions $g(x)$ use measures μ induced by the interval (x, y) measure $\mu((x, y)) = g(y) - g(x)$.

• Mapping x_n to $f(x_n)$.

To clarify methods we add, in some places, sketch of proofs and examples. [3] [4]

2. Examples of applications of $G(x_n)$

We repeat definitions in the Introduction for dimension $s = 1$ following monographs [22], [25], [9] and [44]:

• a sequence x_n, $n = 1, 2, \dots$, $x_n \in [0, 1)$.

• Define step d.f. of x_n as

$$F_N(x) = \frac{\#\{n \le N; x_n \in [0, x)\}}{N}.$$

• A function $g : [0, 1] \to [0, 1]$ is d.f. of x_n if there exists a sequence of indices $N_1 < N_2 < \dots$ such that $F_{N_k}(x) \to g(x)$ for all continuity points x of $g(x)$ as $k \to \infty$.

• The set of all such $g(x)$ we shall denote by $G(x_n)$ and the notion of the distribution of x_n we shall identify with $G(x_n)$, i.e., the distribution of x_n is known if we known the set $G(x_n)$.

2.1. Basic properties of $G(x_n)$

For every sequence $x_n \in [0, 1)$:

• $G(x_n)$ is non–empty, and it is either a singleton or has infinitely many elements.

• $G(x_n)$ is closed and connected in the topology of the weak convergence defined by the metric

$$d(g_1, g_2) = \sqrt{\int_0^1 \big(g_1(x) - g_2(x)\big)^2 \mathrm{d}x}. \tag{5}$$

These properties are characteristic for a set of d.f.s:

• Given a non–empty set H of distribution functions, there exists a sequence x_n in $[0, 1)$ such that $G(x_n) = H$ if and only if H is closed and connected.

• First Helly theorem (or Helly selection principle): Any sequence $g_n(x)$ of d.f. contains a subsequence $g_{k_n}(x)$ such that the sequence $g_{k_n}(x)$ converges for every $x \in [0, 1]$ and its point limit $\lim_{n\to\infty} g_{k_n}(x) = g(x)$ is also a d.f.

• Second Helly theorem (or Helly-Bray theorem): If we have $\lim_{n\to\infty} g_n(x) = g(x)$ a.e. on $[0, 1]$, then for a continuous function $f : [0, 1] \to \mathbb{R}$ we have $\lim_{n\to\infty} \int_0^1 f(x)\mathrm{d}g_n(x) = \int_0^1 f(x)\mathrm{d}g(x)$.

[3] In each paragraph we shall numbering figures starting from 1.

[4] We shall see in many cases that we need solve corresponding functional equations.

• The upper $\underline{g}(x)$ and the lower $\overline{g}(x)$ d.f.s are
$\liminf_{N\to\infty} F_N(x) = \underline{g}(x), \limsup_{N\to\infty} F_N(x) = \overline{g}(x)$.
It is equivalent to
$\underline{g}(x) = \inf_{g\in G(x_n)} g(x), \overline{g}(x) = \sup_{g\in G(x_n)} g(x)$.

A connectivity of $G(x_n)$ can be proved by the following theorem of Barone [4]

Theorem 1 (H.G. Barone (1939)). *If t_n, $n = 1, 2, \dots$ is a sequence in a metric space (X, ρ) and*

- *any subsequence of t_n contains a convergent subsequence;*
- $\lim_{n\to\infty} \rho(t_{n+1}, t_n) = 0$;

then the set of all limit points of t_n is connected in (X, ρ).

Now put X= the set of all d.f.s defined on

$$[0, 1], t_N = F_N(x) \quad \text{and} \quad \rho(t_{N+1}, t_N) = d\big(F_{N+1}(x), F_N(x)\big).$$

The limit $d\big(F_{N+1}(x), F_N(x)\big) \to 0$ follows directly from the definition $F_N(x)$, using the identity

$$\int_0^1 \big(g_1(x) - g_2(x)\big)^2 \mathrm{d}x = \int_0^1 \int_0^1 |x-y| \mathrm{d}g_1(x)\mathrm{d}g_2(y) - \\ - \frac{1}{2}\int_0^1\int_0^1 |x-y|\mathrm{d}g_1(x)\mathrm{d}g_1(y) - \frac{1}{2}\int_0^1\int_0^1 |x-y|\mathrm{d}g_2(x)\mathrm{d}g_2(y) \quad (6)$$

which holds for every d.f.s $g_1(x)$ and $g_2(x)$. Putting $g_1(x) = F_{N+1}(x)$ and $g_2(x) = F_N(x)$ we find exactly

$$\int_0^1 (F_{N+1}(x) - F_N(x))^2 \mathrm{d}x = \\ - \frac{1}{2(N+1)^2N^2} \sum_{m,n=1}^{N} |x_m - x_n| + \frac{1}{(N+1)^2 N} \sum_{n+1}^{N} |x_{N+1} - x_n|, \quad (7)$$

where the right-hand side of (7) tends to zero as $N \to \infty$.

Putting $g_1(x) = F_N(x)$ and $g_2(x) = F_M(x)$ in (6) then we have

Theorem 2. *The sequence $x_n \in [0, 1)$ possesses an a.d.f. if and only if*

$$\lim_{M,N\to\infty} \Bigg(\frac{1}{MN} \sum_{m=1}^{M}\sum_{n=1}^{N} |x_m - x_n| - \frac{1}{2M^2}\sum_{m,n=1}^{M} |x_m - x_n| \\ - \frac{1}{2N^2}\sum_{m,n=1}^{N} |x_m - x_n| \Bigg) = 0.$$

Now, we prove (6) for every two d.f.s $g_1(x)$ and $g_2(x)$.

P r o o f. For given $g_1(x)$ and $g_2(x)$, let $x_n \in [0,1)$, $n = 1, 2, \dots$, be a sequence such that there exist index sequences $N_1 < N_2 < \dots$ and $M_1 < M_2 < \dots$ such that $\lim_{k\to\infty} F_{N_k}(x) = g_1(x)$ and $\lim_{k\to\infty} F_{M_k}(x) = g_2(x)$. Such sequence x_n exists. Put $N_k = N$, $M_k = M$ and express $F_N(x) = \frac{1}{N}\sum_{n=1}^{N} c_{(x_n,1]}(x)$, $F_M(x) = \frac{1}{M}\sum_{m=1}^{M} c_{(x_m,1]}(x)$. Compute [5]

$$
\begin{aligned}
&\int_0^1 \big(F_N(x) - F_M(x)\big)^2 \mathrm{d}x \\
&= \int_0^1 \left(\frac{1}{N}\sum_{n=1}^{N} c_{(x_n,1]}(x) - \frac{1}{M}\sum_{m=1}^{M} c_{(x_m,1]}(x) \right)^2 \mathrm{d}x \\
&= \frac{1}{N^2}\sum_{m,n=1}^{N} \int_0^1 c_{(x_n,1]}(x) c_{(x_m,1]}(x)\mathrm{d}x + \frac{1}{M^2}\sum_{m,n=1}^{M} \int_0^1 c_{(x_n,1]}(x) c_{(x_m,1]}(x)\mathrm{d}x \\
&- 2\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{M} \int_0^1 c_{(x_n,1]}(x) c_{(x_m,1]}(x)\mathrm{d}x. \qquad (8)
\end{aligned}
$$

Since

$$\int_0^1 c_{(x_n,1]}(x) c_{(x_m,1]}(x)\mathrm{d}x = 1 - \max(x_m, x_n)$$

(8) implies

$$
\begin{aligned}
&\int_0^1 \big(F_N(x) - F_M(x)\big)^2 \mathrm{d}x \qquad (9) \\
&= \int_0^1\int_0^1 \big(1 - \max(x,y)\big)\mathrm{d}F_N(x)\mathrm{d}F_N(y) \\
&+ \int_0^1\int_0^1 \big(1 - \max(x,y)\big)\mathrm{d}F_M(x)\mathrm{d}F_M(y) \\
&- 2\int_0^1\int_0^1 \big(1 - \max(x,y)\big)\mathrm{d}F_M(x)\mathrm{d}F_N(y) \\
&= \int_0^1\int_0^1 \big(1 - \max(x,y)\big)\mathrm{d}\big(F_N(x) - F_M(x)\big)\mathrm{d}\big(F_N(y) - F_M(y)\big). \qquad (10)
\end{aligned}
$$

The limit of (9) by Lebesgue theorem of dominant convergence and the limit of (10) by Second Helly theorem, where

$$M = M_k, \quad N = N_k, \quad k \to \infty,$$

[5] $c_A(x)$ is the characteristic function of the set A.

gives

$$\int_0^1 \big(g_1(x)-g_2(x)\big)^2 \mathrm{d}x = \int_0^1\int_0^1 \big(1-\max(x,y)\big)\mathrm{d}\big(g_1(x)-g_2(x)\big)\mathrm{d}\big(g_1(y)-g_2(y)\big). \tag{11}$$

Since

$$\max(x,y) = \frac{x+y+|x-y|}{2},$$

and

$$\int_0^1\int_0^1 1.\mathrm{d}\big(g_1(x)-g_2(x)\big)\mathrm{d}\big(g_1(y)-g_2(y)\big) = 0,$$

$$\int_0^1\int_0^1 (x+y)\mathrm{d}\big(g_1(x)-g_2(x)\big)\mathrm{d}\big(g_1(y)-g_2(y)\big) = 0,$$

we find (6) in the form

$$\int_0^1 \big(g_1(x)-g_2(x)\big)^2 \mathrm{d}x = \int_0^1\int_0^1 -\frac{|x-y|}{2}\mathrm{d}\big(g_1(x)-g_2(x)\big)\mathrm{d}\big(g_1(y)-g_2(y)\big). \tag{12}$$

□

Theorem 3. *Assume that for the sequence $x_n \in [0,1)$ there exists the first moment*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{n} x_n = \alpha.$$

Then either $G(x_n)$ is singleton or $\underline{g} \notin G(x_n)$ or $\overline{g} \notin G(x_n)$.

P r o o f. We have $\int_0^1 x\mathrm{d}F_N(x) = \frac{1}{N}\sum_{n=1}^N x_n$ and by Helly Theorem, if the first moment is constant, then for every $g(x) \in G(x_n)$, we have $\int_0^1 x\mathrm{d}g(x) = \alpha$. Thus if $\underline{g}(x), \overline{g}(x) \in G(x_n)$, then $\int_0^1 x\mathrm{d}\underline{g}(x) = \int_0^1 x\mathrm{d}\overline{g}(x) = \alpha$ and from $\underline{g}(x) \le \overline{g}(x)$ for all $x \in [0,1]$ follows $\underline{g}(x) = \overline{g}(x)$ for common continuity points $x \in [0,1]$. □

Theorem 4. *Let $x_n, y_n \in [0,1)$, $n = 1,2,\dots$. Then*

$$\frac{1}{N}\sum_{n=1}^N |x_n - y_n| \to 0 \Longrightarrow G(x_n) = G(y_n)$$

P r o o f. Put

$$F_N^{(1)}(x) = \frac{1}{N}\sum_{n=1}^N c_{[0,x)}(x_n)$$

and

$$F_N^{(2)}(x) = \frac{1}{N}\sum_{n=1}^N c_{[0,x)}(y_n)$$

and applying (6) we find

$$\int_0^1 \left(F_N^{(1)}(x) - F_N^{(2)}(x)\right)^2 \mathrm{d}x = \frac{1}{N^2}\sum_{m,n=1}^{N} |x_m - y_n| - $$
$$-\frac{1}{2}\frac{1}{N^2}\sum_{m,n=1}^{N} |x_m - x_n| - \frac{1}{2}\frac{1}{N^2}\sum_{m,n=1}^{N} |y_m - y_n|. \quad (13)$$

From

$$x_m - y_n = x_m - x_n + x_n - y_n \quad \text{and} \quad x_m - y_n = y_m - y_n + x_m - y_n$$

follows

$$|x_m - y_n| \le \frac{1}{2}|x_m - x_n| + \frac{1}{2}|y_m - y_n| + \frac{1}{2}|x_n - y_n| + \frac{1}{2}|x_m - y_m|. \quad (14)$$

Substitute (14) to (13) then we find

$$\int_0^1 \left(F_N^{(1)}(x) - F_N^{(2)}(x)\right)^2 \mathrm{d}x \le \frac{1}{N}\sum_{n=1}^{N} |y_n - x_n|. \quad (15)$$

□

Example 1. Let $\{x\}$ be the fractional part of x. For $x_n = \{\log n\}$, $n = 1, 2, \dots$, we have the set of d.f.s

$$G(x_n) = \left\{ g_u(x) = \frac{1}{e^u}\frac{e^x - 1}{e - 1} + \frac{e^{\min(x,u)} - 1}{e^u}; u \in [0,1] \right\}, \quad (16)$$

and

$$\{\log N_k\} \to u \quad \text{implies} \quad F_{N_k}(x) \to g_u(x).$$

The lower and upper d.f. of $\log n \bmod 1$ are

$$\underline{g}(x) = \frac{e^x - 1}{e - 1}, \qquad \overline{g}(x) = \frac{1 - e^{-x}}{1 - e^{-1}},$$

and $\underline{g} \in G(x_n)$ but $\overline{g} \notin G(x_n)$. This set $G(x_n)$ was found by A. Wintner [47], also see Theorem 21.

2.2. Statistically independent sequences

G. Rauzy [33, p. 91, 4.1. Def.]:

Definition 1. Let x_n and y_n be two infinite sequences from the unit interval $[0, 1)$. The pair of sequences (x_n, y_n) is called *statistically independent* if

$$\lim_{N\to\infty} \left(\frac{1}{N}\sum_{n=1}^{N} f_1(x_n) f_2(y_n) - \left(\frac{1}{N}\sum_{n=1}^{N} f_1(x_n)\right)\left(\frac{1}{N}\sum_{n=1}^{N} f_2(y_n)\right)\right) = 0$$

for all continuous real functions f_1, f_2 defined on $[0, 1]$.

Theorem 5 (G. Rauzy (1976) [33]). *Two sequences $x_n \bmod 1$ and $y_n \bmod 1$ are statistically independent if and only if*

$$\lim_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i(hx_n+ky_n)} - \left(\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i hx_n}\right)\left(\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i ky_n}\right)\right) = 0$$

for every integers $(h,k)\neq(0,0)$.

Theorem 6 (G. Rauzy [33, p. 92, 4.2. par.]). *For an arbitrary $(x_n,y_n)\in[0,1)^2$, $n=1,2,\dots,$ the sequences x_n and y_n are statistically independent if and only if*

$$\forall_{g\in G(x_n,y_n)}\ g(x,y)=g(x,1)g(1,y) \quad \textit{a.e. on } [0,1]^2.$$

P r o o f. For given two-dimensional sequence (x_n,y_n) put

$$F_N(x,y)=\frac{\#\{n\le N; (x_n,y_n)\in[0,x)\times[0,y)\}}{N}.$$

By Riemann-Stieltjes integration and Helly theorem, there exist a sequence of indices $N_1<N_2<\dots$ and d.f. $g(x,y)$ such that

$$\frac{1}{N_k}\sum_{n=1}^{N_k} f_1(x_n)f_2(y_n) = \int_0^1\!\!\int_0^1 f_1(x)f_2(y)\mathrm{d}F_{N_k}(x,y) \to \int_0^1\!\!\int_0^1 f_1(x)f_2(x)\mathrm{d}g(x,y),$$

$$\frac{1}{N_k}\sum_{n=1}^{N_k} f_1(x_n) = \int_0^1 f_1(x)\mathrm{d}F_{N_k}(x,1) \to \int_0^1 f_1(x)\mathrm{d}g(x,1),$$

$$\frac{1}{N_k}\sum_{n=1}^{N_k} f_2(y_n) = \int_0^1 f_2(y)\mathrm{d}F_{N_k}(1,y) \to \int_0^1 f_2(y)\mathrm{d}g(1,y)$$

as $k\to\infty$. Assuming statistical independence x_n and y_n we have

$$\int_0^1\int_0^1 f_1(x)f_2(x)\mathrm{d}g(x,y) = \left(\int_0^1 f_1(x)\mathrm{d}g(x,1)\right)\left(\int_0^1 f_2(y)\mathrm{d}g(1,y)\right).$$

The integration by parts gives

$$\int_0^1\int_0^1 f_1(x)f_2(x)\mathrm{d}g(x,y) = f_1(1)f_2(1) - f_2(1)\int_0^1 g(x,1)\mathrm{d}f_1(x)$$
$$- f_1(1)\int_0^1 g(1,y)\mathrm{d}f_2(y) + \int_0^1\int_0^1 g(x,y)\mathrm{d}f_1(x)\mathrm{d}f_2(y)$$

and

$\int_0^1 f_1(x)\mathrm{d}g(x,1) = f_1(1) - \int_0^1 g(x,1)\mathrm{d}f_1(x),$

$\int_0^1 f_1(x)\mathrm{d}g(x,1) = f_2(1) - \int_0^1 g(1,y)\mathrm{d}f_2(y).$

From it follows

$$\int_0^1 \int_0^1 g(x,y)\mathrm{d}f_1(x)\mathrm{d}f_2(y) = \Big(\int_0^1 g(x,1)\mathrm{d}f_1(x)\Big)\Big(\int_0^1 g(1,y)\mathrm{d}f_2(y)\Big)$$

for an arbitrary differentiable $f_1(x)$ and $f_2(y)$. Now, for a continuity point (x_0, y_0) of $g(x,y)$ we can select $f_1(x)$ and $f_2(y)$ such that the above implies $g(x_0,y_0) = g(x_0,1)g(1,y_0)$. □

Note that Grabner and Tichy [16] proved that the extremal discrepancy $\sup_{x,y\in[0,1]} |F_N(x,y) - F_N(x,1)F_N(1,y)|$ does not characterize statistical independence, but the L^2-discrepancy $\int_0^1\int_0^1 (F_N(x,y) - F_N(x,1)F_N(1,y))^2 \mathrm{d}x\mathrm{d}y$ provides a characterization. L^2-discrepancy can be computed also by Wiener's measure $\mathrm{d}f$ (see [38]):

$$\begin{aligned}
&\int_{\mathbf{X}}\int_{\mathbf{X}} \left(\frac{1}{N}\sum_{n=1}^N f(x_n)g(y_n) - \frac{1}{N}\sum_{n=1}^N f(x_n)\frac{1}{N}\sum_{n=1}^N g(x_n)\right)^2 \mathrm{d}f\mathrm{d}g \\
&= \frac{1}{N^2}\sum_{m,n=1}^N \frac{\min(x_m,x_n)}{2}\frac{\min(y_m,y_n)}{2} + \frac{1}{N^4}\sum_{m,n,k,l=1}^N \frac{\min(x_m,x_n)}{2}\frac{\min(y_k,y_l)}{2} \\
&- \frac{2}{N^3}\sum_{m,k,l=1}^N \frac{\min(x_m,x_k)}{2}\frac{\min(y_m,y_l)}{2}. \qquad (17)
\end{aligned}$$

THEOREM 7. *Let x_n and y_n be two sequences in $(0,1)^2$. If*

(i) *x_n and y_n are statistically independent;*

(ii) *x_n is u.d.;*

(iii) *all $g(x) \in G(y_n)$ are continuous;*

then the sequence $x_n + y_n \bmod 1$, $n = 1, 2, \dots$ is u.d.

P r o o f. By (i) and (ii) every $g(x,y) \in G(x_n,y_n)$ has the form $g(x,y) = xg(y)$. Divide unit square $[0,1]^2$ into three parts in Fig. 1

$X_1(t) = \{(x,y) \in [0,1]; x + y < t\}$,

$X_2(t) = \{(x,y) \in [0,1]; 1 < x + y < t + 1, x \le t\}$,

$X_3(t) = \{(x,y) \in [0,1]; 1 < x + y < t + 1, x > t\}$,

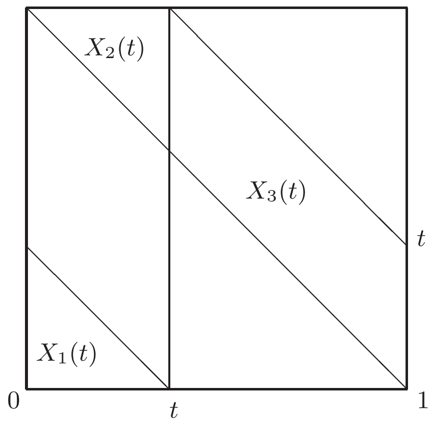


FIGURE 1. Regions $X_i(t)$

By integration

$$\int_{X_1(t)} 1.\mathrm{d}g(x,y) = \int_0^t \mathrm{d}x \int_0^{t-x} 1.\mathrm{d}g(y) = \int_0^t g(t-x)\mathrm{d}x$$

$$\int_{X_2(t)} 1.\mathrm{d}g(x,y) = \int_0^t \mathrm{d}x \int_{1-x}^{1} 1.\mathrm{d}g(y) = \int_0^t (1-g(1-x))\mathrm{d}x$$

$$\int_{X_3(t)} 1.\mathrm{d}g(x,y) = \int_t^1 \mathrm{d}x \int_{1-x}^{t+1-x} 1.\mathrm{d}g(y) = \int_t^1 (g(t+1-x)-g(1-x))\mathrm{d}x.$$

Thus

$$\int_{x+y \bmod 1\in[0,t)} 1\mathrm{d}g(x,y) = \int_0^t 1.\mathrm{d}x - \int_0^1 g(1-x)\mathrm{d}x + \int_0^t g(t-x)\mathrm{d}x + \int_t^1 g(t+1-x)\mathrm{d}x.$$

Now, by integrating

(j) $-\int_0^1 g(1-x)\mathrm{d}x = \int_0^1 g(x)\mathrm{d}x$,

(jj) $\int_0^t g(t-x)\mathrm{d}x = \int_0^t g(x)\mathrm{d}x$,

(jjj) $\int_t^1 g(t+1-x)\mathrm{d}x = \int_t^1 g(x)\mathrm{d}x$,

then finally we have

$$\int_{x+y \bmod 1\in[0,t)} 1\mathrm{d}g(x,y) = t.$$ □

Example 2. Let x_n and y_n be two sequences in $[0,1)$. Assume that

(i) x_n and y_n are u.d.

(ii) x_n and y_n are statistically independent.

Then by Theorem 7 the sequence $x_n + y_n \bmod 1$ is again u.d. It can be proved directly by Weyl's criterion if we prove

$$\frac{1}{N}\sum_{n=1}^{N}(\{x_n+y_n\})^k \to \frac{1}{k+1}, \quad k=1,2,\dots$$

From (ii) follows that the sequence (x_n, y_n) has a.d.f. $g(x,y)=xy$ and by Helly theorem

$$\frac{1}{N}\sum_{n=1}^{N}(\{x_n+y_n\})^k \to \int_0^1\int_0^1(\{x+y\})^k \mathrm{d}x\mathrm{d}y, \quad k=1,2,\dots$$

Now

$$\int_0^1\int_0^1(\{x+y\})^k \mathrm{d}x\mathrm{d}y = \iint_{0\le x+y\le 1}(x+y)^k\mathrm{d}x\mathrm{d}y + \iint_{1\le x+y\le 2}(x+y-1)^k\mathrm{d}x\mathrm{d}y$$

which is $\frac{1}{k+1}$ and the proof is finished.

Theorem 8 (G. Rauzy [33]). *Let $x_n \in [0,1)$, $n=1,2,\dots$, be u.d. sequence. Then x_n and $\log n \bmod 1$ are statistically independent, i.e., every $g(x,y) \in G(x_n, \{\log n\})$ has the form $g(x,y) = x.g(1,y)$.*

Proof. Let $N \in [e^K, e^{K+1})$ i.e., $N = e^{K+\theta_N}$, and divide $n \le N$ to the subsets $n \in [e^k, e^{k+1})$, $k \le K$. For such n we have $\{\log n\} \in [0,y) \iff n \in [e^k, e^{k+y})$. For $n \in [e^k, e^{k+y})$ we ask the number of $x_n \in [0,x)$ which is $x(e^{k+y}-e^k) + O(e^kD_{e^k} + e^{k+y}D_{e^{k+y}})$. Omitting integer parts here we use discrepancy D_M of the initial string $x_1, x_2, \dots, x_M$ (for definition of discsrepancy, see [44, 1–40]) and the formula $A([0,x);M;x_n) = xM + O(MD_M)$. [6] Thus

$$\frac{A\big([0,x)\times[0,y);N;(x_n,\{\log n\})\big)}{N}$$
$$= \frac{\sum_{k=0}^{K-1} x(e^{k+y}-e^k) + x(e^{K+\min(y,\theta_N)} - e^K) + O(\sum_{k=0}^{K} e^kD_{e^k} + e^{k+y}D_{e^{k+y}})}{N}.$$

[6] O-constant can be put $=1$ and in the interval $[e^K, e^{K+\min(y,\theta_N)}]$, for simplification, an error term we put $O(e^kD_{e^K} + e^{K+y}D_{e^{K+y}})$.

As $N \to \infty$ and $\theta_N \to u$, we have

$$\frac{\sum_{k=0}^{K-1} x(e^{k+y} - e^k)}{N} = \frac{\sum_{k=0}^{K-1} x(e^{k+y} - e^k)}{\sum_{k=0}^{K-1}(e^{k+y} - e^k)} \frac{\sum_{k=0}^{K-1}(e^{k+y} - e^k)}{N} \to x\frac{e^y - 1}{e-1}\frac{1}{e^u},$$

$$\frac{x(e^{K+\min(y,\theta_N)} - e^K)}{N} \to x\frac{e^{\min(y,u)} - 1}{e^u},$$

$$\frac{O(\sum_{k=0}^{K} e^k D_{e^k} + e^{k+y} D_{e^{k+y}})}{N} = O\left(\frac{\sum_{k=0}^{K} e^k D_{e^k} + e^{k+y} D_{e^{k+y}}}{\sum_{k=0}^{K}(e^{k+y} - e^k)}\right) \to 0.$$

In the final parenthesis we have used $\frac{e^k D_{e^k} + e^{k+y} D_{e^{k+y}}}{e^{k+y} - e^k} \to 0$ as $k \to \infty$. Collected all above results we have

$$\frac{A([0,x) \times [0,y); N; (x_n, \{\log n\}))}{N} \to x\left(\frac{e^y - 1}{e-1}\frac{1}{e^u} + \frac{e^{\min(y,u)} - 1}{e^u}\right) = x g_u(y),$$

where $g_u(y)$ is the same as in (16). □

In 2011 Y. Ohkubo [27] proved that the function $\log n$ can be replaced by $\log(n \log n)$ in Theorem 8.

Theorem 9. *An arbitrary u.d. sequence* $x_n \bmod 1$ *and* $\log(n \log n) \bmod 1$ *are statistically independent.*

P r o o f. By G. Rauzy Theorem 5 two sequences $x_n \bmod 1$ and $y_n \bmod 1$ are statistically independent if and only if

$$\lim_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i(hx_n + ky_n)} - \left(\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h x_n}\right)\left(\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k y_n}\right)\right) = 0$$

for every integers h and k. Now, by Abel partial summation we obtain

$$\sum_{n=1}^{N} e^{2\pi i(hx_n + k\log(n\log n))}$$
$$= \sum_{n=1}^{N-1}\left(e^{2\pi i k \log(n \log n)} - e^{2\pi i k \log((n+1)\log(n+1))}\right)\sum_{j=1}^{n} e^{2\pi i h x_n}$$
$$+ e^{2\pi i k \log(N \log N)} \sum_{j=1}^{N} e^{2\pi i h x_n}$$

and

$$\left|e^{2\pi i k \log(n\log n)} - e^{2\pi i k \log((n+1)\log(n+1))}\right| \le 2\pi|k|\frac{(\log n) + 1}{n \log n}.$$

Thus

$$\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i(hx_n+k\log(n\log n))}\right|$$
$$\le \frac{1}{N}\sum_{n=1}^{N-1} 2\pi|k|\frac{(\log n)+1}{n\log n}n\left|\frac{1}{n}\sum_{j=1}^{n} e^{2\pi ihx_j}\right| + \left|\frac{1}{N}\sum_{j=1}^{N} e^{2\pi ihx_j}\right|$$

which tends to 0. □

Using Theorem 9 Ohkubo [27] proved that in Theorem 8 the $\log n$ can be instead by $\log p_n$, where p_n are sequence of all primes.

Theorem 10. *Let $x_n \in [0,1)$, $n = 1,2,\dots,$ be u.d. sequence. Then x_n and $\log p_n \bmod 1$ are statistically independent.*

Proof. Firstly he proved that

$$\log p_n = \log(n\log n) + o\left(\frac{\log\log n}{\log n}\right) + O\left(\frac{1}{\log p_n}\right). \tag{18}$$

Then Ohkubo used

Let (x_n, y_n) and (x'_n, y'_n), $n = 1,2,\dots$ be two-dimensional sequences. Assume:

(i) $|x_n - x'_n| \to 0$ and $|y_n - y'_n| \to 0$.

(ii) Every d.f. $g(x,y) \in G\big((x_n,y_n)\big)$ is continuous in $(0,0)$,$(0,1)$, $(1,0)$ and $(1,1)$.

Then $G\big((x_n,y_n)\big) = G\big((x'_n,y'_n)\big)$.

Then the limit

$$\lim_{n\to\infty}\big(\log p_n - \log(n\log n)\big) = 0,$$

given by (18), implies

$$G\big((x_n,\{\log p_n\})\big) = G\big((x_n,\{\log(n\log n)\})\big) = G\big((x_n,\{\log n\})\big). \tag{19}$$

Proof of (18). He starts with the prime number theorem of the form

$$\pi(x) = \frac{x}{\log x \ -1} + O\left(\frac{x}{(\log x)^3}\right). \tag{20}$$

This implies

$$\frac{p_n}{n} = \log p_n \ -1 + O\left(\frac{1}{\log p_n}\right).$$

Then he used (see [43])

$$\frac{p_n}{n} = \log n + (\log\log n \ -1) + o\left(\frac{\log\log n}{\log n}\right)$$

which implies (18). □

Some generalization:

Example 3. J. Coquet and P. Liardet [7]: Given an integer $q \geq 2$, a real number θ and a real polynomial $p(x)$, let

(i) $x_n = \theta q^n \bmod 1$,

(ii) $y_n = p(n) \bmod 1$,

(iii) $\mathbf{x}_n = (x_{n+1}, \dots, x_{n+s})$ and $\mathbf{y}_n = (y_{n+1}, \dots, y_{n+s})$.

If x_n is u.d. (i.e., θ is normal in the base q), then for every $s = 1, 2, \dots$, the sequence

$$(\mathbf{x}_n, \mathbf{y}_n), \quad n = 1, 2, \dots,$$

has d.f.s $g(\mathbf{x}, \mathbf{y}) \in G\big((\mathbf{x}_n, \mathbf{y}_n)\big)$ of the form $g(\mathbf{x}, \mathbf{y}) = g_1(\mathbf{x}) g_2(\mathbf{y})$ for some $g_1(\mathbf{x}) \in G(\mathbf{x}_n)$ and $g_2(\mathbf{y}) \in G(\mathbf{y}_n)$, i.e., the sequences x_n and y_n are *completely statistically independent*.

2.3. Statistical limit

H. Fast [10] and I.J. Schoenberg [34] defined, independently:

Definition 2. The sequence x_n is said to be *statistically convergent* to the number α provided that for each $\varepsilon > 0$,

$$\lim_{N \to \infty} \frac{1}{N} \#\{n \leq N; |x_n - \alpha| \geq \varepsilon\} = 0.$$

• Fast [10] mentioned: A sequence x_n is statistically convergent to α if and only if there exists a sequence of indices k_n of the asymptotic density $d(k_n) = 1$ such that $\lim_{n \to \infty} x_{k_n} = \alpha$ in the standard sense.

• Let us consider *one-jump function* $c_\alpha(x)$ which has a jump of size 1 for α.

Theorem 11 (I.J. Schoenberg [34])**.** *The sequence $x_n \in [0, 1)$ is statistically convergent to the number $\alpha \in [0, 1]$ if and only if the sequence x_n admits the asymptotic distribution function $c_\alpha(x)$.*

Example 4. [44, p. 2–192, 2.20.18]: Let $\mathrm{ord}_p(n) = \alpha$ for $p^\alpha \| n$. If p stands for a prime, then the sequence

$$\log p \frac{\mathrm{ord}_p(n)}{\log n}, \quad n = 2, 3, \dots,$$

is dense in $[0, 1]$ and has the a.d.f. $c_0(x)$, and thus statistically converge to zero.

Theorem 12 ([36]). *The sequence $x_n \in [0,1)$ possesses a statistical limit if and only if*

$$\lim_{M,N\to\infty} \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} |x_m - x_n| = 0.$$

Proof. Let $F_{M_{i_k}}(x) \to g_1(x)$ and $F_{N_{j_k}}(x) \to g_2(x)$. Applying the Helly-Bray theorem we find

$$\lim_{k\to\infty} \int_0^1 \int_0^1 |x-y| \mathrm{d}F_{M_{i_k}}(x) \mathrm{d}F_{N_{j_k}}(y) = \int_0^1 \int_0^1 |x-y| \mathrm{d}g_1(x) \mathrm{d}g_2(y).$$

By Riemann-Stieltjes integration we obtain

$$\int_0^1 \int_0^1 |x-y| \mathrm{d}F_{M_{i_k}}(x) \mathrm{d}F_{N_{j_k}}(y) = \frac{1}{M_{i_k} N_{j_k}} \sum_{m=1}^{M_{i_k}} \sum_{n=1}^{N_{j_k}} |x_m - x_n|.$$

Thus

$$\int_0^1 \int_0^1 |x-y| \mathrm{d}g_1(x) \mathrm{d}g_2(y) = 0$$

which gives $g_1(x) = g_2(x) = c_\alpha(x)$ a.e. for some $\alpha \in [0,1]$. □

2.4. Statistical limit points

Following the concept od statistical convergence J. A. Fridy [14] introduced:

Definition 3. A real number x is said to be *a statistical limit point* of the sequence x_n if there exists a subsequence x_{k_n}, $n = 1, 2, \dots$, such that $\lim_{n\to\infty} x_{k_n} = x$ and the set of indices k_n has a positive upper asymptotic density.

Fridy studied the set $\Lambda(x_n)$ of all such points. Inspired by I.J. Schoenberg [34], P. Kostyrko, M. Mačaj, T. Šalát and O. Strauch [21] was found:

Theorem 13. *The set $\Lambda(x_n)$, for $x_n \in [0,1)$ $n = 1, 2, \dots$, coincides with the set of all discontinuity points of d.f.s $g(x) \in G(x_n)$.*

From it follows:

(i) Let x_n be a sequence of real numbers. If for every $k = 1, 2, \dots$ the difference sequence $x_{n+k} - x_n$, $n = 1, 2, \dots$ has $\Lambda(x_{n+k} - x_n) = \emptyset$, then $\Lambda(x_n) = \emptyset$.

(ii) For u.d. sequence x_n we have $\Lambda(x_n \bmod 1) = \emptyset$.

(iii) $\omega_h = \limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} \right|^2$ for $h = 1, 2, \dots$ and assume that $\lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} \omega_h = 0$. The every $g(x) \in G(x_n)$ is a continuous, thus $\Lambda(x_n) = \emptyset$.

Example 5. By Example 1 every d.f. $g(x) \in G(\log n \bmod 1)$ is continuous, thus we have $\Lambda(\log n \bmod 1) = \emptyset$. More generally, for $x_n = c\log n \bmod 1$, $c \neq 0$, we have

$$\omega_h = \frac{1}{4\pi^2 h^2 c^2 + 1}$$

which implies $\lim_{H\to\infty} \frac{1}{H}\sum_{h=1}^{H} \omega_h = 0$ and thus by (iii) $\Lambda(c\log n \bmod 1) = \emptyset$, again.

Example 6. By [36] starting with $\log\log n \bmod 1$ all the sequences of iterate logarithm $\log\log\ldots\log n \bmod 1$ have

$$G(\log\log\ldots\log n \bmod 1) = \{c_\alpha(x); \alpha \in [0,1]\} \cup \{h_\alpha(x); \alpha \in [0,1]\}.$$

Here $c_\alpha(x)$ is one-step d.f. for which

$$c_\alpha(x) = 0 \quad \text{for} \quad x \in [0,\alpha],\ c_\alpha(x) = 1 \quad \text{for} \quad x \in (\alpha, 1] \quad \text{and} \quad h_\alpha : [0,1] \to [0,1]$$

is a constant distribution function, where $h_\alpha(0) = 0$, $h_\alpha(1) = 1$, and $h_\alpha(x) = \alpha$ if $x \in (0,1)$. Thus we have $\Lambda(\log\log\ldots\log n \bmod 1) = [0,1]$.

Example 7. Let $\alpha = \frac{p}{q}\pi$, where p and q are positive integers and g.c.d. $(p,q)=1$. By D. Berend, M. D. Boshernitzan, and G. Kolesnik [6] the sequence

$$x_n = n\cos(n\cos n\alpha) \bmod 1, \quad n = 1, 2, \ldots$$

has $G(x_n) = \{g(x)\}$, where

$$g(x) = \begin{cases} x & \text{if } q \text{ is odd}, \\ \left(1 - \frac{1}{q}\right)x + \frac{1}{q}c_0(x) & \text{if } q \text{ is even}, \end{cases}$$

and $c_0(x)$ is the one-jump d.f. which the jump in 0. This implies

$$\Lambda(x_n) = \begin{cases} \emptyset & \text{if } q \text{ is odd}, \\ \{0\} & \text{if } q \text{ is even}. \end{cases}$$

2.5. Uniformly maldistributed sequences

G. Myerson [24]:

Definition 4. The sequence $x_n \in [0,1)$ $n = 1, 2, \ldots$, is said to be *uniformly maldistributed* (u.m.) if for every nonempty proper subinterval $I \subset [0,1]$ we have both

$$\liminf_{N\to\infty} \frac{1}{N}\#\{n \le N; x_n \in I\} = 0 \text{ and } \limsup_{N\to\infty} \frac{1}{N}\#\{n \le N; x_n \in I\} = 1.$$

He mentioned that the first condition is superfluous, and showed that

Example 8. The sequence $x_n = \{\log\log n\}$ of fractional parts of the iterated logarithm is u.m.

In [36] is proved: Let $c_\alpha(x)$ is one-step d.f. for which $c_\alpha(x) = 0$ for $x \in [0, \alpha]$ and $c_\alpha(x) = 1$ for $x \in (\alpha, 1]$.

Theorem 14. *The sequence x_n is u.m. if and only if*

$$\{c_\alpha(x); \alpha \in [0,1]\} \subset G(x_n).$$

Example 9. By Example 6, starting with $x_n = \{\log\log n\}$, all the sequences $x_n = \{\log\log\dots\log n\}$, $n = n_0, n_0+1, \dots$ are u.m.

Thus, in the theory of uniform maldistribution we need not consider d.f.s other than one-jump d.f. $c_\alpha(x)$ which has a jump of size 1 at α. This suggests the definition (see [36]):

Definition 5. The sequence x_n is said to be *uniformly maldistributed in the strict sense* (u.m.s.) if $G(x_n) = \{c_\alpha(x); \alpha \in [0,1]\}$.

Theorem 15. *For every sequence $x_n \in [0,1)$ we have*

$$G(x_n) \subset \{c_\alpha(x); \alpha \in [0,1]\} \iff \lim_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^{N} |x_m - x_n| = 0.$$

Moreover, if $G(x_n) \subset \{c_\alpha(x); \alpha \in [0,1]\}$, then $G(x_n) = \{c_\alpha(x); \alpha \in I\}$, where I is a closed subinterval of $[0,1]$ which can be found as

$$I = \left[\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n,\ \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n\right],$$

and the length $|I|$ of I can also be found as

$$|I| = \limsup_{M,N\to\infty} \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} |x_m - x_n|.$$

The following theorem is immediately evident from the preceding,

Theorem 16. *The sequence $x_n \in [0,1)$ is u.m.s. if and only if*

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^{N} |x_m - x_n| = 0 \text{ and } \limsup_{M,N\to\infty} \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} |x_m - x_n| = 1,$$

or alternatively $\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n - \liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n = 1$.

Example 10. Let x_n, $n = 1, 2, \dots$ be defined as

$$x_n = \left\{ 1 + (-1)^{\left[\sqrt{[\sqrt{\log_2 n}]}\right]} \left\{ \sqrt{[\sqrt{\log_2 n}]} \right\} \right\},$$

were $[x]$ denotes the integral part and $\{x\}$ the fractional part of x. Then

$$G(x_n) = \{c_\alpha(x); \alpha \in [0, 1]\}.$$

2.6. Benford's law

The first digit problem:

2.6.1. Historical notes

An infinite sequence $x_n \geq 1$, $n = 1, 2, \dots,$ of real numbers satisfies *Benford's law*, if the frequency (the asymptotic density) of occurrences of a given first digit a, when x_n is expressed in the decimal form is given by $\log_{10}\left(1 + \frac{1}{a}\right)$ for every $a = 1, 2, \dots, 9$ (0 as a possible first digit is not admitted).

It was S. Newcomb (1881), who firstly noted "*That the ten digits do not occur with equal frequency must be evident to anyone making use of logarithm tables*". F. Benford (1938) compared the empirical frequency of occurrences of a with $\log_{10}((a+1)/a)$ in twenty different tables. Since x_n has the first digit a if and only if

$$\log_{10} x_n \bmod 1 \in [\log_{10} a, \log_{10}(a+1)),$$

Benford's law for x_n follows from the uniform distribution of $\log_{10} x_n \bmod 1$. For the asymptotic density of the second-place digit b he found

$$\sum_{a=1}^{9} \log_{10}\left(1 + \frac{1}{10a + b}\right).$$

F. Benford rediscovered Newcomb's observation from (1881). P. Diaconis [8] suggested the following generalization:

2.6.2. Generalization of Benford's law

Let $b \geq 2$ be an integer considered as a base for the development of a real number $x > 0$ and $M_b(x)$ be the mantissa of x defined by $x = M_b(x) \times b^{n(x)}$ such that $1 \leq M_b(x) < b$ holds, where $n(x)$ is a uniquely determined integer. Let $K = k_1 k_2 \cdots k_r$ be a positive integer expressed in the base b , that is

$$K = k_1 \times b^{r-1} + k_2 \times b^{r-2} + \cdots + k_{r-1} \times b + k_r,$$

where $k_1 \neq 0$ and at the same time $K = k_1 k_2 \cdots k_r$ is considered as an $r-$consecutive block of digits in the base b. Note that for x of the type $x = 0.00\cdots$

$\cdots 0k_1k_2\cdots k_r\cdots$, $k_1 > 0$, we have $M_b(x) = k_1.k_2\cdots k_r\cdots$ and the first zero digits are omitted. Thus arbitrary $x > 0$ has the first r-digits, starting a non-zero digit, equal to $k_1k_2\cdots k_r$ if and only if [7]

$$k_1.k_2\cdots k_r \le M_b(x) < k_1.k_2\cdots(k_r+1). \tag{21}$$

Since $\log_b M_b(x) = \log_b x \bmod 1$ the inequality (21) is equivalent to

$$\log_b\left(\frac{K}{b^{r-1}}\right) \le \log_b x \bmod 1 < \log_b\left(\frac{K+1}{b^{r-1}}\right). \tag{22}$$

Here we use the shorthand notation $\frac{K}{b^{r-1}} = k_1.k_2\cdots k_r$.

Definition 6. A sequence x_n, $n = 1, 2, \ldots$, of positive real numbers satisfies *Benford's law* (abbreviated to B.L.) [8] in base b, if for every $r = 1, 2, \ldots$ and every r-digits integer $K = k_1k_2\cdots k_r$ we have the density

$$\lim_{N\to\infty} \frac{\#\{n \le N;\ \text{leading block of } r \text{ digits (beginning with} \ne 0) \text{ of } x_n = K\}}{N}$$
$$= \log_b\left(\frac{K+1}{b^{r-1}}\right) - \log_b\left(\frac{K}{b^{r-1}}\right). \tag{23}$$

From (22) and from definition (23) it follows immediately:

Theorem 17. *A sequence x_n, $n = 1, 2, \ldots$, of positive real numbers satisfies B.L. in base b if and only if the sequence* $\log_b x_n \bmod 1$ *is u.d. in* $[0, 1)$.

Example 11. The sequence of Fibonacci numbers F_n, factorials $n!$, and n^n, and n^{n^2} satisfy Benford's law, but the sequence n, and all primes p_n does not, see [44, 2.12.26], [3].

P. Diaconis [8] and A. I. Pavlov [29] have been the first, who applied uniform distribution theory to B.L. For instance:

(i) By the criterion of P.B Kennedy [44, p. 2–13, 2.2.9], P. Diaconis proved: *If a sequence $x_n > 0$, $n = 1, 2, \ldots$, satisfies B.L. in the base b, then*

$$\limsup_{n\to\infty} n\left|\log\frac{x_{n+1}}{x_n}\right| = \infty. \tag{24}$$

(ii) Applying van der Corput difference theorem [22, p. 26, Th.3.1] A.I. Pavlov proved: *Assume $x_n > 0$, $n = 1, 2, \ldots$. If for every $k = 1, 2, \ldots$ the ratio sequence $\frac{x_{n+k}}{x_n}$, $n = 1, 2, \ldots$, satisfies B.L. in the base b, then the original sequence x_n, $n = 1, 2, \ldots$ also satisfies B.L. in the base b.*

---

[7] If $k_1 = k_2 = \cdots = k_r = b - 1$ then we have $k_1.k_2\cdots(k_r+1) = b$.

[8] precisely known as generalized or strong

(iii) In [3] we have: *The positive sequences* x_n *and* $\frac{1}{x_n}$, $n = 1, 2, \dots$ *satisfy B.L. in the base* b *simultaneously.* Proof: Both two sequences u_n and $-u_n$ are u.d. mod1 simultaneously, since their Weyl's sums are complex conjugate each other.

(iv) *The positive sequences* x_n *and* nx_n, $n = 1, 2, \dots$ *satisfy B.L. in the base* b *simultaneously.* Proof: Both two sequences u_n and $u_n + \log n$ are u.d. mod1 simultaneously, see [44, p. 2–27, 2.3.6.].

(v) *Assume that a sequence* $0 < x_1 \le x_2 \le \dots$ *satisfies B.L. in the integer base* $b > 1$. *Then*

$$\lim_{n\to\infty} \frac{\log x_n}{\log n} = \infty. \tag{25}$$

Proof: It follows from the theorem of H. Niederreiter [44, p. 2–12, 2.2.8] that every monotone u.d. sequence u_n mod 1 must satisfy $\lim_{n\to\infty} \frac{|u_n|}{\log n} = \infty$. Here, it suffices to put $u_n = \log x_n$, instead of $u_n = \log_b x_n$.

(vi) *For a sequence* $x_n > 0$, $n = 1, 2, \dots,$ *assume that*
  (i) $\lim_{n\to\infty} x_n = \infty$ *monotonically,*
  (ii) $\lim_{n\to\infty} \log \frac{x_{n+1}}{x_n} = 0$ *monotonically.*

Then the sequence x_n *satisfies B.L. in every base* b *if and only if*

$$\lim_{n\to\infty} n \log \frac{x_{n+1}}{x_n} = \infty. \tag{26}$$

Proof: It follows from Fejér's difference theorem in the form in [44, p. 2–13, 2.2.11].

(vii) [44, p. 2–14, 2.2.12] implies: *Let* $x_n > 0$ *be a sequence, which satisfies* $\lim_{n\to\infty} \log_b \frac{x_{n+1}}{x_n} = \theta$ *with* θ *is irrational. Then* x_n *satisfies B.L. in the base* b.

V. Baláž, K. Nagasaka and O. Strauch [3] study d.f.s of a sequence $x_n \in (0, 1)$ which satisfy B.L. Using the Fig. 1

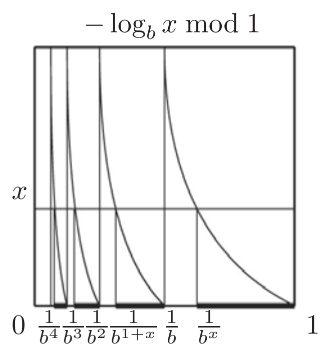


Figure 1: Intervals $f_i^{-1}([0, x))$, $i = 0, 1, 2, \dots$

and that $f_i^{-1}([0,x)) = \left(\frac{1}{b^{i+x}}, \frac{1}{b^i}\right]$ they proved:

Theorem 18. *Let x_n, $n = 1,2,\dots$, be a sequence in $(0,1)$ and $G(x_n)$ be the set of all d.f.s of x_n. Assume that every d.f. $g(x) \in G(x_n)$ is continuous at $x = 0$. Then the sequence x_n satisfies B.L. in the base b if and only if for every $g(x) \in G(x_n)$ we have*

$$x = \sum_{i=0}^{\infty} \left(g\left(\frac{1}{b^i}\right) - g\left(\frac{1}{b^{i+x}}\right)\right) \text{ for } x \in [0,1]. \tag{27}$$

Example 12. We present the following solutions of (27):

$$g(x) = \begin{cases} x & \text{if } x \in \left[0, \frac{1}{b}\right], \\ 1 + \frac{\log x}{\log b} + (1-x)\frac{1}{b-1} & \text{if } x \in \left[\frac{1}{b}, 1\right]. \end{cases}$$

$$g^*(x) = \begin{cases} 0 & \text{if } x \in \left[0, \frac{1}{b^2}\right], \\ 2 + \frac{\log x}{\log b} & \text{if } x \in \left[\frac{1}{b^2}, \frac{1}{b}\right], \\ 1 & \text{if } x \in \left[\frac{1}{b}, 1\right] \end{cases}$$

$$g^{**}(x) = \begin{cases} 0 & \text{if } x \in \left[0, \frac{1}{b^3}\right], \\ 3 + \frac{\log x}{\log b} & \text{if } x \in \left[\frac{1}{b^3}, \frac{1}{b^2}\right], \\ 1 & \text{if } x \in \left[\frac{1}{b^2}, 1\right] \end{cases}$$

If H is the set of all $t_1 g(x) + t_2 g^*(x) + t_3 g^{**}(x)$, $t_1 + t_2 + t_3 = 1$, $t_1, t_2, t_3 \geq 0$, then there exists a sequence x_n such that $G(x_n) = H$ (see 2.1), and this x_n satisfies B.L.

By the following Example 13, there exist an integer sequences $x_n \in \mathbb{N}$ which satisfies B.L. for an arbitrary base b. By the following Theorem 19 no such real sequence $x_n \in [0,1)$ exists.

Example 13. By [44, p. 2–117, 2.12.14], the sequence

$$\alpha n \log^\tau n \bmod 1, \quad \alpha \neq 0,\ 0 < \tau \leq 1,$$

is u.d. From this follows that $x_n = n^n$ satisfies B.L. for an arbitrary integer base b, because $\log_b n^n = n \log n \frac{1}{\log b}$.

In [3] there is proved:

Theorem 19. *For a sequence $x_n \in (0,1)$, $n = 1,2,\dots$, assume that every d.f. $g(x) \in G(x_n)$ is continuous at $x = 0$. Then there exist only finitely many different integer bases b for which the sequence x_n satisfies B.L. simultaneously. Moreover, if the sequence x_n satisfies B.L. in base b, and for some $k = 1,2,\dots$ there exists kth integer root $\sqrt[k]{b}$, then x_n satisfies B.L. also in base $\sqrt[k]{b}$.*

As it is well known that the increasing sequence of all positive integers $1, 2, 3, \dots$ does not satisfy B.L. in every base $b \geq 2$. It follows from the fact that $\log_b n \bmod 1$ is not u.d. For a density of n for which r initial digits are $K = k_1 k_2 \dots k_r$, A.I. Pavlov [29] proved that

$$\liminf_{N\to\infty} \frac{\#\{n \leq N; n \text{ has the first } r \text{ digits } = K\}}{N} = \frac{1}{K(b-1)}, \tag{28}$$

$$\limsup_{N\to\infty} \frac{\#\{n \leq N; n \text{ has the first } r \text{ digits } = K\}}{N} = \frac{b}{(K+1)(b-1)}. \tag{29}$$

Using the theory of d.f.s boundaries (28) and (29)are extended in [3] to the following (30): By G. Pólya and G. Szegö [32] the d.f.s of $\log_b n \bmod 1$ is of the form (see also Theorem 21)

$$g_u(x) = \frac{b^{\min(x,u)} - 1}{b^u} + \frac{1}{b^u} \frac{b^x - 1}{b - 1},$$

where the parameter u runs $[0, 1]$ and by [15], for increasing sequence N_i, $i = 1, 2, \dots$, we have

$$\log_b N_i \bmod 1 \to u \Longrightarrow F_{N_i}(x) \to g_u(x).$$

Thus

$$\frac{\#\{n \leq N_i; n \text{ has the first } r \text{ digits } = K\}}{N_i} \to \big(g_u(x_2) - g_u(x_1)\big) \tag{30}$$

as $i \to \infty$, $x_1 = \log_b(k_1.\, k_2 k_3 \dots k_r)$, $x_2 = \log_b\big(k_1.\, k_2 k_3 \dots (k_r + 1)\big)$, and the minimum is appeared in $u = x_1$ and the maximum in $u = x_2$.

2.6.3. General scheme of solution of the First Digit Problem

Many authors think that if the sequence x_n does not satisfy B.L., then the relative density of indices n for which the b-expansion of x_n start with leading digits $K = k_1 k_2 \dots k_r$

$$\frac{1}{N} \# \left\{ n \leq N; \log_b \left(\frac{K}{b^{r-1}} \right) \leq \{\log_b x_n\} < \log_b \left(\frac{K+1}{b^{r-1}} \right) \right\}.$$

do not follow any distribution in the sense of natural density, see S. Eliahou, B. Massé and D. Schneider (2013). These authors as an alternate result shown that the sequence $\log_{10} n^r \bmod 1$, $n = 1, 2, \dots$, and the sequence $\log_{10} p_n^r \bmod 1$ $n = 1, 2, \dots$, p_n are all prime numbers, have the discrepancy $O(r^{-1})$. Thus, for $r \to \infty$, these sequences tends to uniform distribution and thus n^r and p_n^r tends to B.L. Using theory of d.f.'s we can give the following general solution of the first digit problem (see [28]):

Theorem 20. *Let $g(x) \in G(\log_b x_n \bmod 1)$ and $\lim_{i\to\infty} F_{N_i}(x) = g(x)$. Then*

$$\lim_{N_i\to\infty} \frac{\#\{n \le N_i;\ \text{first } r \text{ digits (starting a non-zero digit) of } x_n = K\}}{N_i}$$
$$= g\left(\log_b\left(\frac{K+1}{b^{r-1}}\right)\right) - g\left(\log_b\left(\frac{K}{b^{r-1}}\right)\right).$$

2.6.4. Distribution functions of sequences involving logarithm

Distribution functions of $\log_b x_n \bmod 1$ we need in generalized B.L., Theorem 20. It can be computed by following theorem: [9]

Assumptions. Let the real-valued function $f(x)$ be strictly increasing for $x \ge 1$ and let

$f^{-1}(x)$ be its inverse function and

$F_N(x) = \frac{\#\{n\le N; f(n) \bmod 1 \in [0,x)\}}{N}$ for $x \in [0,1]$.

Assume that

(i) $\lim_{x\to\infty} f'(x) = 0$,

(ii) $\lim_{k\to\infty} f^{-1}(k+1) - f^{-1}(k) = \infty$,

(iii) $\lim_{k\to\infty} \frac{f^{-1}(k+w(k))}{f^{-1}(k)} = \psi(w)$ for every sequence $w(k) \in [0,1]$ for which $\lim_{k\to\infty} w(k) = w$, where this limit defines the function
$$\psi : [0,1] \to [1, \psi(1)],$$

(iv) $\psi(1) > 1$.

Theorem 21 ([43]). *Then* (i)–(iv) *imply*

$$G(f(n) \bmod 1) = \left\{g_w(x) = \frac{1}{\psi(w)}\frac{\psi(x)-1}{\psi(1)-1} + \frac{\min(\psi(x),\psi(w))-1}{\psi(w)}; w \in [0,1]\right\}.$$

Now, if $w(i) = \{f(N_i)\} \to w$, then $F_{N_i}(x) \to g_w(x)$ for every $x \in [0,1]$.

Theorem 22 (Y. Ohkubo [27]). *Then* (i)–(iv) *imply*

$$G(f(p_n) \bmod 1) = \left\{g_w(x) = \frac{1}{\psi(w)}\frac{\psi(x)-1}{\psi(1)-1} + \frac{\min(\psi(x),\psi(w))-1}{\psi(w)}; w \in [0,1]\right\}.$$

Now, if $w(i) = \{f(p_{N_i})\} \to w$, then $F_{N_i}(x) \to g_w(x)$ for every $x \in [0,1]$.

Example 14 (Example of natural numbers). For the sequence

$f(n) = \log_b n^s$, $n = 1,2,\dots$, $f^{-1}(x) = b^{\frac{x}{s}}$,

$\lim_{k\to\infty} \frac{f^{-1}(k+w)}{f^{-1}(k)} = \frac{b^{\frac{k+w}{s}}}{b^{\frac{k}{s}}} = b^{\frac{w}{s}} = \psi(w)$, and by Theorem 21

[9] Theorem 21 is a simple version of Theorems 34, 35 and 36.

$G(\log_b n^s \bmod 1) = \big\{g_w(x) = \frac{1}{b^{\frac{w}{s}}}\frac{b^{\frac{x}{s}}-1}{b^{\frac{1}{s}}-1} + \frac{\min(b^{\frac{x}{s}},b^{\frac{w}{s}})-1}{b^{\frac{w}{s}}}; w\in[0,1]\big\}.$

If $\lim_{i\to\infty}\{f(N_i)\} = \lim_{i\to\infty}\{\log_b(N_i^s)\} = w$, then

$$\lim_{i\to\infty}\frac{\#\{n\le N_i;\ \text{first } r \text{ digits of } n^s \text{ are } k_1k_2\dots k_r\}}{N_i}$$
$$= g_w\big(\log_b k_1.k_2k_3\dots(k_r+1)\big) - g_w\big(\log_b k_1.k_2k_3\dots k_r\big)$$
$$= \frac{b^{(\log_b k_1.k_2\dots(k_r+1))/s}-1}{b^{1/s}-1} - \frac{b^{(\log_b k_1.k_2\dots k_r)/s}-1}{b^{1/s}-1}$$
$$= \frac{(k_1.k_2k_3\dots(k_r+1))^(1/s) - (k_1.k_2k_3\dots k_r))^(1/s)}{b^{1/s}-1},$$

where we assume $N_i = b^i$ which gives $\lim_{i\to\infty}\{\log_b(b^{is}\} = 0 = w$.

Example 15 (Example of primes)**.** $f(p_n) = \log_b p_n^s$, $n = 1,2,\dots$, p_n is the nth prime.

$$f(x) = \log_b x^s,$$
$$G(\log_b p_n^s \bmod 1) = \Big\{g_w(x) = \frac{1}{b^{\frac{w}{s}}}\frac{b^{\frac{x}{s}}-1}{b^{\frac{1}{s}}-1} + \frac{\min(b^{\frac{x}{s}},b^{\frac{w}{s}})-1}{b^{\frac{w}{s}}}; w\in[0,1]\Big\}.$$

If $N_i = \pi(b^{\frac{i}{s}})+1$, then $\lim_{\imath\to\infty}\{f(p_{N_i})\} = 0$ (cf. [28]) and $g_0(x) = \frac{b^{x/s}-1}{b^{1/s}-1}$. Thus

$$\lim_{i\to\infty}\frac{\#\{n\le N_i;\ \text{first } r \text{ digits of } p_n^s = k_1k_2\dots k_r\}}{N_i}$$
$$= g_0\Big(\log_b\big(k_1.k_2k_3\dots(k_r+1)\big)\Big) - g_0\big(\log_b(k_1.k_2k_3\dots k_r)\big)$$
$$= \frac{b^{(\log_b(k_1.k_2\dots(k_r+1)))/s}-1}{b^{1/s}-1} - \frac{b^{(\log_b(k_1.k_2\dots k_r))/s}-1}{b^{1/s}-1}$$
$$= \frac{(k_1.k_2k_3\dots(k_r+1))^{(1/s)} - (k_1.k_2k_3\dots k_r))^{(1/s)}}{b^{1/s}-1}.$$

2.6.5. Two-dimensional Benford's law

Unsolved Problems [42, 1.38, p. 186]:

F. Luca and P. Stanica [23] proved

Theorem 23. *There exists infinite many n such that Fibonacci number F_n starts with digits K_1 and $\varphi(F_n)$ starts with digits K_2 in the base b representation. Here K_1 and K_2 are arbitrary and $\varphi(x)$ is the Euler function.*

In the following we see that this claim is equivalent that the sequence

$$\big(\log_b F_n, \log_b \varphi(F_n)\big) \bmod 1, \quad n = 1, 2, \dots,$$

is everywhere dense in $[0,1]^2$, but the authors use the following method:

(i) By the first author $\varphi(F_n)/F_n$ is dense in $[0,1]$. Thus, for an interval I with arbitrary small length and containing K_2/K_1, there exists $\varphi(F_a)/F_a \in I$.

(ii) Then $\varphi(F_{ap})/F_{ap} \in I$ for all sufficiently large prime p.

(iii) There exists infinitely many primes p such that F_{ap} starts with K_1.

(iv) Finally, multiplying I by F_{ap} they find $\varphi(F_{ap})$ which starts with K_2.

Now we shall extend Theorem 20 to the two-dimensional case.

Let $x_n > 0$, $y_n > 0$, $n = 1, 2, \dots;$

$F_N(x,y) = \frac{\#\{n \le N; \{\log_b x_n\} < x \text{ and } \{\log_b y_n\} < y\}}{N}$,

$K_1 = k_1^{(1)} k_2^{(1)} \dots k_{r_1}^{(1)}$,

$K_2 = k_1^{(2)} k_2^{(2)} \dots k_{r_2}^{(2)}$,

$u_1 = \log_b \left(\frac{K_1}{b^{r_1-1}}\right)$,

$u_2 = \log_b \left(\frac{K_1+1}{b^{r_1-1}}\right)$,

$v_1 = \log_b \left(\frac{K_2}{b^{r_2-1}}\right)$,

$v_2 = \log_b \left(\frac{K_2+1}{b^{r_2-1}}\right)$,

x_n has the first r_1 digits $= K_1 \iff \{\log_b x_n\} \in [u_1, u_2)$;

y_n has the first r_2 digits $= K_2 \iff \{\log_b y_n\} \in [v_1, v_2)$;

Theorem 24. *Let $g(x,y) \in G(\{\log_b x_n\}, \{\log_b y_n\})$ and $\lim_{k\to\infty} F_{N_k}(x,y) = g(x,y)$ for $(x,y) \in [0,1]^2$. Then*

$$\lim_{k\to\infty} \frac{\#\{n \le N_k; x_n \textit{ has the first } r_1 \textit{ digits } = K_1 \textit{ and } y_n \textit{ has the first } r_2 \textit{ digits } = K_2\}}{N_k}$$
$$= g(u_2, v_2) + g(u_1, v_1) - g(u_2, v_1) - g(u_1, v_2). \tag{31}$$

Example 16.

$$G(\{\log_b n\}, \{\log_b (n+1)\})$$
$$= \left\{ g_u(x,y) = \frac{b^{\min(x,y)} - 1}{b-1} \frac{1}{b^u} + \frac{b^{\min(x,y,u)} - 1}{b^u}; u \in [0,1] \right\}.$$

$g_u(x,y) = \min(g_u(x), g_u(y))$ by Sklar theorem, where $g_u(x) = \frac{b^x - 1}{b-1} \cdot \frac{1}{b^u} + \frac{b^{\min(x,u)} - 1}{b^u}$. Thus

$$\lim_{k\to\infty} \frac{\#\{n \le N_k; n \text{ has the first } r_1 \text{ digits } = K_1 \text{ and } (n+1) \text{ has the first } r_2 \text{digits} = K_2\}}{N_k}$$
$$= g_u(u_2, v_2) + g_u(u_1, v_1) - g_u(u_2, v_1) - g_u(u_1, v_2)$$

If $K_1 = K_2$ then $= g_u(u_2) - g_u(u_1)$. It can be found directly.

Example 17. Let $x_n \in [0,1)$, $n = 1, 2, \dots$, be a u.d. sequence. Then

(I)
$$x_n \quad \text{and} \quad \log_b n \bmod 1$$
are statistically independent (Theorem 8);

(II)
$$x_n \quad \text{and} \quad \log_b(n \log n) \bmod 1$$
are statistically independent (Y. Ohkubo (2011) [27]); see Theorem 9.

(III)
$$x_n \quad \text{and} \quad \log_b p_n \bmod 1$$
are statistically independent (Y. Ohkubo (2011) [27]); see Theorem 10.

$$G(x_n, \{\log_b p_n\}) = \{x.g_u(y); u \in [0,1]\},$$

where $g_u(x) = \frac{b^x-1}{b-1} \cdot \frac{1}{b^u} + \frac{b^{\min(x,u)}-1}{b^u}$ and $F_{N_k}(x,y) \to x.g_u(y)$ if $\{\log_b N_k\} \to u$.

Example 18. Let $x_n \in [0,1)$, $n = 1, 2, \dots$, be u.d. sequence. Then x_n and $\log_b n \bmod 1$ are statistically independent, i.e.,

$$G(x_n, \{\log_b n\}) = \{g_u(x,y) = x.g_u(y); u \in [0,1]\},$$

where $g_u(x) = \frac{b^x-1}{b-1} \cdot \frac{1}{b^u} + \frac{b^{\min(x,u)}-1}{b^u}$. This was proved by G. Rauzy (1973), see [44, p. 2–27, 2.3.6.]

Example 19. By Theorem 10 we have

$$G(\{\log_b F_n\}, \{\log_b p_n\}) = \{x.g_u(y); u \in [0,1]\}$$

and let

$$\{\log_b p_{N_k}\} \to u.$$

Then

$$\lim_{k\to\infty} \frac{\#\{n \le N_k; F_n \text{ has the first } r_1 \text{ digits} = K_1 \text{ and } p_n \text{ has the first } r_2 \text{ digits} = K_2\}}{N_k}$$
$$= u_2 g_u(v_2) + u_1 g_u(v_1) - u_2 g_u(v_1) - u_1 g_u(v_2), \tag{32}$$

where

$$u_1 = \log_b\left(\frac{K_1}{b^{r_1-1}}\right), \quad u_2 = \log_b\left(\frac{K_1+1}{b^{r_1-1}}\right),$$

$$v_1 = \log_b\left(\frac{K_2}{b^{r_2-1}}\right), \quad v_2 = \log_b\left(\frac{K_2+1}{b^{r_2-1}}\right),$$

and

$$g_u(x) = \frac{b^x-1}{b-1} \cdot \frac{1}{b^u} + \frac{b^{\min(x,u)}-1}{b^u}.$$

2.7. Two-dimensional copulas

[10] If distribution function $c(x,y)$ satisfies

$$c(x,1)=x \quad \text{and} \quad c(1,y)=y,$$

then $c(x,y)$ is called *copula*. These d.f.'s introduced by M. Sklar (1959) and all their basic properties can be found in R.B. Nelsen (1999).

Let $G_{2,1}$ be the set of all two-dimensional copulas. There are some basic properties $G_{2,1}$:

(I) $G_{2,1}$ is closed under pointwise limit and convex linear combinations.

(II) For every $g(x,y) \in G_{2,1}$ and every $(x_1,y_1),(x_2,y_2) \in [0,1]^2$ we have $|g(x_2,y_2)-g(x_1,y_1)| \le |x_2-x_1|+|y_2-y_1|$, [26, p. 9]. Also [26, p. 9, Coroll. 2.2.6]: The horizontal, vertical and diagonal sections of copula are all nondecreasing and uniformly continuous on $[0,1]$.

(III) For every $g(x,y) \in G_{2,1}$ we have $g_3(x,y)=\max(x+y-1,0) \le g(x,y) \le \min(x,y) = g_2(x,y)$, where $g_3(x,y)$ and $g_2(x,y)$ are copulas (Fréchet--Hoeffding bounds, see R.B. Nelsen (1999) [p. 9][26]).

(IV) M. Sklar (1959) proved (cf. (cf. Nelsen[p. 15, Th. 2.3.3])):

Theorem 25. *For every d.f. $g(x,y)$ on $[0,1]^2$ there exists $c(x,y) \in G_{2,1}$ such that*

$$g(x,y)=c(g(x,1),g(1,y)) \quad \textit{for every} \quad (x,y) \in [0,1]^2.$$

If $g(x,1)$ and $g(1,y)$ are continuous, then the copula $c(x,y)$ is unique .

• For d.f. $g(x,y)$ denote the marginal $g_1(x)=g(x,1)$ and $g_2(y)=g(1,y)$ and by Sklar $g(x,y)=c(g_1(x),g_2(y))$. Then for every continuous $F(x,y)$ we have

$$\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x,y) = \int_0^1\int_0^1 F(g_1^{(-1)}(x),g_2^{(-1)}(y))\mathrm{d}c(x,y), \tag{33}$$

see M. Hofer and M.R. Iacò [18].

(VI) Examples:

$g_\theta(x,y)=(\min(x,y))^\theta(xy)^{1-\theta}$, where $\theta \in [0,1]$ (Cuadras-Augé family, cf. Nelsen [1999, p. 12, Ex. 2.5]),

$g_4(x,y)=\frac{xy}{x+y-xy}$ (see Nelsen [1999, p. 19, 2.3.4]),

$\tilde{g}(x,y)=x+y-1+g(1-x,1-y)$ for every $g(x,y) \in G_{2,1}$ (Survival copula, see Nelsen [1999, p. 28, 2.6.1]).

---

[10] We denote by $G_{s,k}$ the set of all d.f.s $g(\mathbf{x})$ on $[0,1]^s$ for which all k-dimensional marginal (i.e., face) d.f.s satisfy $g(1,\dots,1,x_{i_1},1,\dots,1,x_{i_2},1,\dots,1,x_{i_k},1,\dots,1)=x_{i_1}x_{i_2}\dots x_{i_k}$. For $k=1$, these d.f.'s are called *copulas*

Possible a new types of copulas can be produced by: If a sequences (x_n, y_n) have both x_n and y_n u.d. then all d.f.'s $g(x,y)$ of the sequence (x_n, y_n) has marginals

$$g(x,1) = x, \qquad g(1,y) = y.$$

Theorem 26. *Let X be a non-empty, closed and connected set of copulas. Then there exists a two-dimensional sequence (x_n, y_n), $n = 1, 2, \dots$, in $[0,1)^2$ such that $G(x_n, y_n) = X$.*

Example 20. For the sequence $(u + z_n, v + z_n) \bmod 1$, z_n, $n = 1, 2, \dots$, z_n is u.d. we denote a.d.f. as $g_{u,v}(x,y)$. Then by Weyl's limit relation

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} K(\{u + z_n, v + z_n\}) &= \int_0^1 K(\{u+z\}, \{v+z\}) \mathrm{d}z \\ &= \int_0^1 \int_0^1 K(x,y) \mathrm{d}_x \mathrm{d}_y g_{u,v}(x,y). \end{aligned} \tag{34}$$

In the following, for $g_{u,v}(x,y)$, we prove (39).

Let $0 \le u \le v \le 1$ be fixed and $x, y \in [0,1]$ be variables and define

$$h_u(x) = x + u \bmod 1, \quad h_v(y) = y + v \bmod 1.$$

Then

$$g_{u,v}(x,y) = |h_u^{-1}([0,x)) \cap h_v^{-1}([0,y))|.$$

Using the following graphs of $h_u(x)$

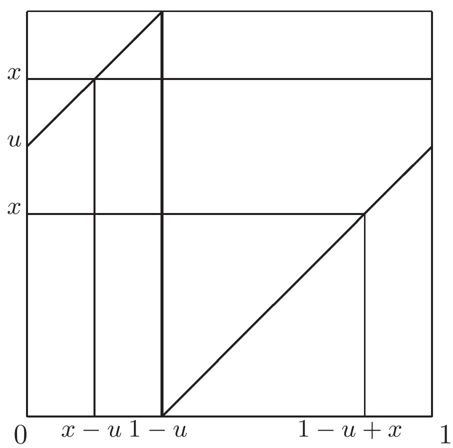


Figure 1: The graph of $h_u(x)$.

then we see

$$h_u^{-1}([0,x]) = \begin{cases} [1-u, 1-u+x] & \text{if } x \le u, \\ [0, x-u] \cup [1-u, 1] & \text{if } u \le x. \end{cases} \tag{35}$$

Hence

$$\begin{aligned} & g_{u,v}(x,y) \\ & = \begin{cases} \big|[1-u, 1-u+x] \cap [1-v, 1-v+y]\big| & \text{if } x \le u, y \le v, \\ \big|[1-u, 1-u+x] \cap ([0, y-v] \cup [1-v, 1])\big| & \text{if } x \le u, y > v, \\ \big|([0, x-u] \cup [1-u, 1]) \cap [1-v, 1-v+y]\big| & \text{if } x > u, y \le v, \\ \big|([0, x-u] \cup [1-u, 1]) \cap ([0, y-v] \cup [1-v, 1])\big| & \text{if } x > u, y > v. \end{cases} \end{aligned} \tag{36}$$

Now we used minimum and maximum formula for the length of intersection of two intervals $[\alpha, \beta]$ and $[\gamma, \delta]$

$$\big|[\alpha, \beta] \cap [\gamma, \delta]\big| = \max\big(\min(\beta, \delta) - \max(\alpha, \gamma), 0\big). \tag{37}$$

Insert (37) into (36) we see

$$\begin{aligned} & g_{u,v}(x,y) \\ & = \begin{cases} \max\big(\min(y, x-u+v), 0\big) & \text{if } x \le u, y \le v, \\ \max\big(\min(x, y-v-1+u), 0\big) + \max(v-u+x, 0) & \text{if } x \le u, y > v, \\ y & \text{if } x > u, y \le v, \\ \min(x-u+v, y) + \max(y-v-1+u, 0) & \text{if } x > u, y > v, \end{cases} \end{aligned} \tag{38}$$

which implies

$$g_{u,v}(x,y) = \begin{cases} x & \text{if } (x,y) \in A, \\ y - (1 - |u-v|) & \text{if } (x,y) \in B, \\ x + y - 1 & \text{if } (x,y) \in C, \\ 0 & \text{if } (x,y) \in D, \\ x - |u-v| & \text{if } (x,y) \in E, \\ y & \text{if } (x,y) \in F, \end{cases} \tag{39}$$

where

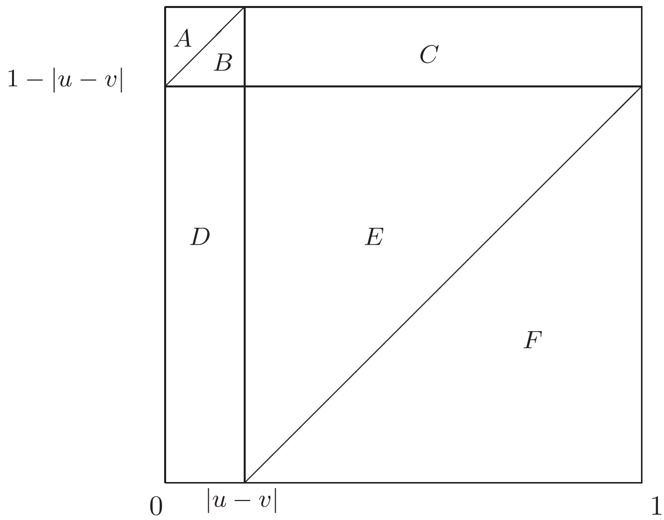


Figure 2: Division $[0,1]^2$ by (39).

2.7.1. Applications

Let $F(x,y)$ be a Riemann integrable function defined on $[0,1]^2$ and x_n, y_n, $n = 1, 2, \dots$, be two u.d. sequences in $[0,1)$. A problem is to find limit points of the sequence

$$\frac{1}{N}\sum_{n=1}^{N} F(x_n, y_n), \quad N = 1, 2, \dots \tag{40}$$

For $F(x,y) = |x-y|$, this problem was formulated by F. Pillichshammer and S. Steinerberger [30]. They proved:

Theorem 27. *Let x_n and y_n be two uniformly distributed sequences in $[0,1)$. Then*

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} |x_n - y_n| \le \frac{1}{2}$$

and in particular

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} |x_{n+1} - x_n| \le \frac{1}{2}$$

and this result is best possible.

They also found:

• $\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}|x_{n+1}-x_n| = \frac{2(b-1)}{b^2}$ for van der Corput sequence x_n in the base b and

• $\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}|x_{n+1}-x_n| = 2\{\alpha\}(1-\{\alpha\})$ for $x_n = n\alpha \bmod 1$, where α is irrational.

Applying Helly theorems we see that limit points of (40) form the set

$$\left\{\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y); g(x,y)\in G((x_n,y_n))\right\}, \tag{41}$$

where $G(x_n,y_n)$ is the set of all d.f.s of the two-dimensional sequence (x_n,y_n), $n=1,2,\dots$ In this case, two-dimensional sequence (x_n,y_n) does not need to be u.d. but every d.f. $g(x,y)\in G((x_n,y_n))$ satisfies:

(i) $g(x,1)=x$ for $x\in[0,1]$ and

(ii) $g(1,y)=y$ for $y\in[0,1]$.

Thus d.f. $g(x,y)$ is a copula.

2.8. Extremes of $\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y)$

The problem of to find extremes of (40) is equivalent to find extreme values of $\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y)$ over copulas $g(x,y)$. In [11] is proved:

Theorem 28. *Let $F(x,y)$ be a Riemann integrable function defined on $[0,1]^2$. For differential of $F(x,y)$ let us assume that $\mathrm{d}_x\mathrm{d}_y F(x,y) > 0$ for every $(x,y)\in(0,1)^2$. Then*

$$\max_{g(x,y)\text{-}copula}\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y) = \int_0^1 F(x,x)\mathrm{d}x,$$

$$\min_{g(x,y)\text{-}copula}\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y) = \int_0^1 F(x,1-x)\mathrm{d}x, \tag{42}$$

where, precisely, max *is attained in $g(x,y)=\min(x,y)$ and* min *in $g(x,y) = \max(x+y-1,0)$, uniquely.*

P r o o f. The integration by parts gives

$$\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y) = F(1,1) - \int_0^1 g(1,y)\mathrm{d}_y F(1,y) - \int_0^1 g(x,1)\mathrm{d}_x F(x,1) + \int_0^1\int_0^1 g(x,y)\mathrm{d}_x\mathrm{d}_y F(x,y) \tag{43}$$

which holds for every Riemann integrable $F(x,y)$ and d.f. $g(x,y)$ which does not have any common discontinuity points. Then Fréchet-Hoeffding bounds (see Nelsen [26, p. 9])

$$\max(x+y-1,0) \le g(x,y) \le \min(x,y)$$

and the assumption $\mathrm{d}_x\mathrm{d}_y F(x,y) > 0$ implies

$$\begin{aligned}\int_0^1\int_0^1 \max(x+y-1,0)\mathrm{d}_x\mathrm{d}_y F(x,y) &\le \int_0^1\int_0^1 g(x,y)\mathrm{d}_x\mathrm{d}_y F(x,y)\\ &\le \int_0^1\int_0^1 \min(x,y)\mathrm{d}_x\mathrm{d}_y F(x,y).\end{aligned}$$

Since every copula is continuous, then the left inequality is attained if and only if $g(x,y) = \max(x+y-1,0)$ and the right if and only if $g(x,y) = \min(x,y)$.

Directly by definition of a.d.f., for every u.d. sequence $x_n \in [0,1)$, it can be proved that

a) the sequence (x_n, x_n), $n = 1,2,\dots$, has the a.d.f. $g(x,y) = \min(x,y)$ and
b) the sequence $(x_n, 1-x_n)$, $n = 1,2,\dots$, has the a.d.f. $g(x,y) = \max(x+y-1,0)$. From it

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} F(x_n,x_n) = \int_0^1 F(x,x)\mathrm{d}x = \int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y \min(x,y), \tag{44}$$

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} F(x_n,1-x_n) &= \int_0^1 F(x,1-x)\mathrm{d}x\\ &= \int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y \max(x+y-1,0). \qquad (45)\end{aligned}$$

If $\mathrm{d}_x\mathrm{d}_y F(x,y) < 0$, the right hand sides of (42) are exchanged.

□

Using copulas we gives in the following, an alternative proof of F. Pillichshammer and S. Steinerberger [30] Theorem 27:

Example 21. Putting $F(x,y) = |x-y|$, we have $F(1,1) = 0$, $F(1,x) = 1-x$, $F(y,1) = 1-y$, and computing, for $y > x$,

$$\mathrm{d}_x\mathrm{d}_y|y-x| = (y+\mathrm{d}y-(x+\mathrm{d}x)) = (y-x)-(y-(x+\mathrm{d}x))-(y+\mathrm{d}y-x) = 0,$$

and for $y = x$, $\mathrm{d}y = \mathrm{d}x$,

$$\mathrm{d}_x\mathrm{d}_y|y-x| = |x+\mathrm{d}x-(x+\mathrm{d}x)|+|x-x|-|(x+\mathrm{d}x)-x|-|x-(x+\mathrm{d}x)| = -2\mathrm{d}x$$

then we have

$$\int_0^1\int_0^1 |x-y|\mathrm{d}_x\mathrm{d}_y g(x,y) = \int_0^1 g(x,1)\mathrm{d}x + \int_0^1 g(1,y)\mathrm{d}y - 2\int_0^1 g(x,x)\mathrm{d}x. \quad (46)$$

Thus for a copula $g(x,y)$, $g(x,1)=x$, $g(1,y)=y$ we have

$$\int_0^1\int_0^1 |x-y|\mathrm{d}_x\mathrm{d}_y g(x,y) = 1 - 2\int_0^1 g(x,x)\mathrm{d}x. \quad (47)$$

We shall compute (47) for van der Corput sequence $\gamma_q(n)$, $n=0,1,\dots$, in base q. We have that every point $(\gamma_q(n),\gamma_q(n+1))$, $n=0,1,2,\dots$, lies on the diagonals of intervals

$$\left[0, 1-\frac{1}{q}\right] \times \left[\frac{1}{q}, 1\right] \quad (48)$$

$$\left[1-\frac{1}{q^i}, 1-\frac{1}{q^{i+1}}\right] \times \left[\frac{1}{q^{i+1}}, \frac{1}{q^i}\right], \qquad i = 1,2,\dots \quad (49)$$

By Fig. 1 we find the so-called von Neumann-Kakutani transformation $T : [0,1] \to [0,1]$, see Fig. 1. Because $\gamma_q(n)$ is u.d., the sequence $\big(\gamma_q(n),\gamma_q(n+1)\big)$ has a.d.f. $g(x,y)$ which is copula of the form

$$\begin{aligned} g(x,y) =& |\mathrm{Project}_X(([0,x]\times[0,y])\cap \mathrm{graph}\ T)| \\ =& \min\big(\big|[0,x]\cap I_X\big|, \big|[0,y]\cap I_Y\big|\big) \\ &+ \sum_{i=1}^{\infty} \min\big(\big|[0,x]\cap I_X^{(i)}\big|, \big|[0,y]\cap I_Y^{(i)}\big|\big), \end{aligned} \quad (50)$$

where $\mathrm{Project}_X$ is the projection of a two dimensional set to the X-axis. [11]

[11] Copula $g(x,y)$ of the type (50) is called *shuffle of M* (see [26, p. 69]). It is a copula whose support is a collection of line segments with slope $+1$ or -1.

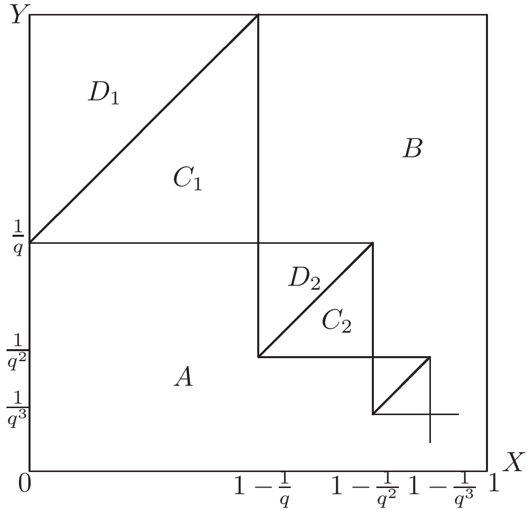


Figure 1: Line segments containing $(\gamma_q(n), \gamma_q(n+1))$, $n = 1, 2, \dots$
The graph of the von Neumann-Kakutani transformation T.

The sum (50) implies

$$g(x,y) = \begin{cases} 0 & \text{if } (x,y) \in A, \\ 1-(1-y)-(1-x) = x+y-1 & \text{if } (x,y) \in B, \\ y - \frac{1}{q^i} & \text{if } (x,y) \in C_i, \\ x - 1 + \frac{1}{q^{i-1}} & \text{if } (x,y) \in D_i, \end{cases} \tag{51}$$

$i = 1, 2, \dots$ From (51) it follows

$$g(x,x) = \begin{cases} 0, & \text{if } x \in \left[0, \frac{1}{q}\right], \\ x - \frac{1}{q}, & \text{if } x \in \left[\frac{1}{q}, 1 - \frac{1}{q}\right], \\ 2x-1, & \text{if } x \in \left[1 - \frac{1}{q}, 1\right], \end{cases} \tag{52}$$

(for $q = 2$, the mean equality misses) and by (47)

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |\gamma_q(n) - \gamma_q(n+1)| = 1 - 2\int_0^1 g(x,x)\mathrm{d}x = \frac{2(q-1)}{q^2}.$$

As we see in part 2.7.1 in uniform distribution theory the problem of optimizing the integral

$$\int_0^1 \int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y) \tag{53}$$

over copulas $g(x,y)$ is motivated by computing optimal limit points of the sequence $\frac{1}{N}\sum_{n=1}^{N} F(x_n, y_n)$, $N = 1,2,\dots$ over uniform distribution sequences x_n and y_n, $n = 1,2,\dots$ But problem of optimizing (53) belongs to the well-known mass transportation problems, or the Monge-Kantorovich transportation problem, see e.g., L. Ambrosio and N. Gigli (2013) [1]. In Theorem 28 we have seen that the solution of the problem in uniform distribution theory depends on the sign of partial derivatives $\frac{\partial^2 F(x,y)}{\partial x \partial y}$, see Fig. 2.

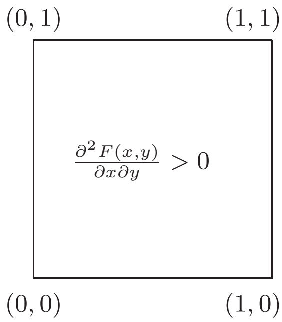


Figure 2.

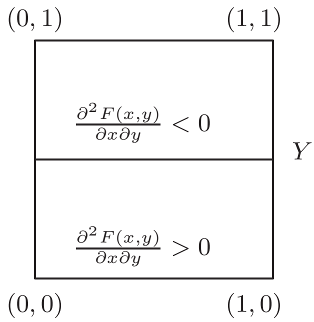


Figure 3.

A criterion for Fig. 3 follows from [11, Th. 7]:

Theorem 29. *Let us assume that a copula $g(x,y)$ maximizes the integral*

$$\int_0^1 \int_0^1 F(x,y) \mathrm{d}_x \mathrm{d}_y g(x,y).$$

Let $[X_1, X_2] \times [Y_1, Y_2]$ be an interval in $[0,1]^2$ such that the differential

$$g(X_2, Y_2) + g(X_1, Y_1) - g(X_2, Y_1) - g(X_1, Y_2) > 0.$$

Assume that for every interior point (x,y) of $[X_1, X_2] \times [Y_1, Y_2]$ the differential $\mathrm{d}_x \mathrm{d}_y F(x,y)$ has constant signum. Then we have:

(i) *if $\mathrm{d}_x \mathrm{d}_y F(x,y) > 0$, then*

$$g(x,y) = \min\big(g(x,Y_2) + g(X_1,y) - g(X_1,Y_2),\\ g(x,Y_1) + g(X_2,y) - g(X_2,Y_1)\big) \quad (54)$$

(ii) *if $\mathrm{d}_x \mathrm{d}_y F(x,y) < 0$, then*

$$g(x,y) = \max\big(g(x,Y_2) + g(X_2,y) - g(X_2,Y_2),\\ g(x,Y_1) + g(X_1,y) - g(X_1,Y_1)\big) \quad (55)$$

for every $(x,y) \in [X_1, X_2] \times [Y_1, Y_2]$.

Theorem 29 can also be used for Fig. 4. Such a method is described in R.F. Tichy, S. Thonhauser, O. Strauch, M.R. Iacó and V. Baláž [49]:

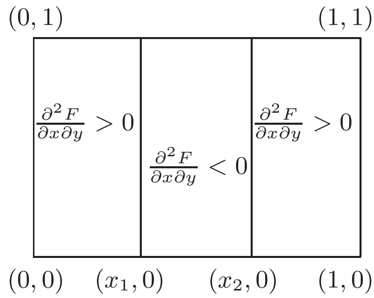


Figure 4.

Let $g(x, y)$ be a copula which maximize $\int_0^1 \int_0^1 F(x, y) \mathrm{d}_x \mathrm{d}_y g(x, y)$ and $F(x, y)$ satisfies Fig. 4. Then $g(x, y)$ satisfies (54) if $x \in (0, x_1) \cup (x_2, 1)$ and (55) if $x \in (x_1, x_2)$. Denote

$$g(x_1, y) = h_1(y), \quad \text{and} \quad g(x_2, y) = h_2(y).$$

Thus:
If $x \in (0, x_1)$, then

$$\begin{aligned} g(x, y) &= \min(g(0, y) + g(x, 1) - g(0, 1), g(x, 0) + g(x_1, y) - g(x_1, 0) \\ &= \min(0 + x - 0, 0 + h_1(y) - 0) \\ &= \min(x, h_1(y)). \end{aligned}$$

If $x \in (x_1, x_2)$, then

$$\begin{aligned} g(x, y) &= \max(g(x, 1) + g(x_2, y) - g(x_2, 1), g(x_1, y) + g(x, 0) - g(x_1, 0) \\ &= \max(x + h_2(y) - x_2, h_1(y) + 0 - 0) \\ &= \max(x + h_2(y) - x_2, h_1(y)). \end{aligned}$$

If $x \in (x_2, 1)$, then

$$\begin{aligned} g(x, y) &= \min(g(x_2, y) + g(x, 1) - g(x_2, 1), g(x, 0) + g(1, y) - g(1, 0) \\ &= \min(h_2(y) + x - x_2, 0 + y - 0) \\ &= \min(x - x_2 + h_2(y), y). \end{aligned}$$

Summary

$$g(x,y)=\begin{cases}\min\bigl(x,h_1(y)\bigr) & \text{if } x\in[0,x_1],\\ \max\bigl(x+h_2(y)-x_2,h_1(y)\bigr) & \text{if } x\in[x_1,x_2],\\ \min\bigl(x-x_2+h_2(y),y\bigr) & \text{if } x\in[x_2,1]\end{cases}\tag{56}$$

where $y\in[0,1]$ and $h_1(y)$ and $h_2(y)$ we must calculated.
The differential $\mathrm{d}_x\mathrm{d}_y g(x,y)$ is nonzero only for points (x,y) on the curves

$$\begin{aligned}&x=h_1(y),y\in[0,1],\\ &x=x_2-h_2(y)+h_1(y),y\in[0,1],\\ &x=x_2-h_2(y)+y,y\in[0,1],\end{aligned}\tag{57}$$

and we have

$$\mathrm{d}_x\mathrm{d}_y g(x,y)=\begin{cases}h_1'(y)\mathrm{d}y & \text{if } x\in[0,x_1],x=h_1(y),\\ (h_2'(y)-h_1'(y))\mathrm{d}y & \text{if } x\in[x_1,x_2],x=x_2-h_2(y)+h_1(y),\\ (1-h_2'(y))\mathrm{d}y & \text{if } x\in[x_2,1],x=x_2-h_2(y)+y.\end{cases}\tag{58}$$

Let

$$F(x,y)=\begin{cases}F_1(x,y) & \text{if } x\in(0,x_1),\frac{\partial^2F_1(x,y)}{\partial x\partial y}>0\\ F_2(x,y) & \text{if } x\in(x_1,x_2),\frac{\partial^2F_2(x,y)}{\partial x\partial y}<0\\ F_3(x,y) & \text{if } x\in(x_2,1),\frac{\partial^2F_3(x,y)}{\partial x\partial y}>0.\end{cases}$$

Then (56), (58) and (57) give

$$\begin{aligned}\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y)&=\int_0^1 F_1\bigl(h_1(y),y\bigr)h_1'(y)\mathrm{d}y\\ &+\int_0^1 F_2\bigl(x_2-h_2(y)+h_1(y),y\bigr)\bigl(h_2'(y)-h_1'(y)\bigr)\mathrm{d}y\\ &+\int_0^1 F_3\bigl(x_2-h_2(y)+y,y\bigr)\bigl(1-h_2'(y)\bigr)\mathrm{d}y.\end{aligned}\tag{59}$$

Denote

$$\begin{aligned}&G(y,h_1,h_2,h_1',h_2')\\ &=F_1\bigl(h_1(y),y\bigr)h_1'(y)+F_2\bigl(x_2-h_2(y)+h_1(y),y\bigr)\bigl(h_2'(y)-h_1'(y)\bigr)\\ &+F_3\bigl(x_2-h_2(y)+y,y\bigr)(1-h_2'(y)).\end{aligned}$$

Then

$$\max_{g(x,y)-\text{ copula}} \int_0^1 \int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y) = \max_{h_1,h_2} \int_0^1 G(y,h_1,h_2,h_1',h_2')\mathrm{d}y, \quad (60)$$

where h_1, h_2 give a copula in (56). To do this we use the following criterion:

Theorem 30. *The function $g(x,y)$ defined by* (56) *is a copula if and only if*

(i) $h_1(y)$ *and* $h_2(y)$ *are increasing;*

(ii) $h_1(0) = 0$, $h_2(0) = 0$*;*

(iii) $h_1(1) = x_1$, $h_2(1) = x_2$*;*

(iv) $0 \le h_1(y) \le h_2(y) \le y$*;*

(v) $0 \le h_1'(y) \le h_2'(y) \le 1$*;*

If the (h_1, h_2) is maximum of $\int_0^1 G(y,h_1,h_2,h_1',h_2')\mathrm{d}y$ but not satisfy assumptions of Theorem 30, then we have only

$$\max_{g(x,y)-\text{copula}} \int_0^1 \int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y) \le \int_0^1 G(y,h_1,h_2,h_1',h_2')\mathrm{d}y. \quad (61)$$

For solution of (60) we can using the calculus of variations: If (h_1, h_2) extremize the integral $\int_0^1 G(y,h_1,h_2,h_1',h_2')\mathrm{d}y$ then (h_1, h_2) must be satisfied the following system of Euler-Lagrange differential equations

$$\frac{\partial G}{\partial h_1} - \frac{\mathrm{d}}{\mathrm{d}y}\frac{\partial G}{\partial h_1'} = 0,$$
$$\frac{\partial G}{\partial h_2} - \frac{\mathrm{d}}{\mathrm{d}y}\frac{\partial G}{\partial h_2'} = 0.$$

The solution (h_1, h_2) maximize $\int_0^1 G(y,h_1,h_2,h_1',h_2')\mathrm{d}y$ if

$$\frac{\partial^2 G}{\partial h_1' \partial h_1'} \le 0, \quad \begin{vmatrix} \frac{\partial^2 G}{\partial h_1' \partial h_1'} & \frac{\partial^2 G}{\partial h_1' \partial h_2'} \\ \frac{\partial^2 G}{\partial h_2' \partial h_1'} & \frac{\partial^2 G}{\partial h_2' \partial h_2'} \end{vmatrix} \le 0.$$

To compare (60)we give L. Uckelmann's (1997) [48] the mass transportation problems: Let

$F(x,y) = \Phi(x+y)$ for $(x,y) \in [0,1]^2$;

For $0 < k_1 < k_2 < 2$ let $\Phi(x)$ be a twice differentiable function such that $\Phi(x)$ is strictly convex on $[0,k_1] \cup [k_2,2]$ and concave on $[k_1,k_2]$, i.e.,

$\Phi''(x) > 0$ for $x \in [0,k_1) \cup (k_2,2]$',

$\Phi''(x) > 0$ for $x \in [0,k_1) \cup (k_2,2]$.

If α and β are the solutions of

$$\Phi(2\alpha) - \Phi(\alpha+\beta) + (\beta-\alpha)\Phi'(\alpha+\beta) = 0,$$
$$\Phi(2\beta) - \Phi(\alpha+\beta) + (\alpha-\beta)\Phi'(\alpha+\beta) = 0$$

such that $0 < \alpha < \beta < 1$, then the optimal copula $C(x, y)$ is the shuffle of M

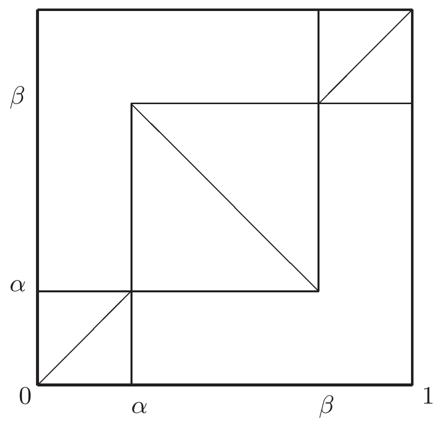


Figure 5.

with the support

$$\Gamma(x) = \begin{cases} x & \text{for } x \in [0,\alpha] \cup [\beta,1], \\ \alpha+\beta-x & \text{for } x \in (\alpha,\beta). \end{cases}$$

Then

$$\max \int_0^1 \int_0^1 F(x,y)dC(x,y) = \int_0^\alpha \Phi(2x)dx + (\beta-\alpha)\Phi(\alpha+\beta) + \int_\beta^1 \Phi(2x)dx.$$

2.9. Example of three-dimensional copula

See [12]: In this part we apply the Weyl's limit relation to calculate the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} F\big(\gamma_q(n), \gamma_q(n+1), \gamma_q(n+2)\big)$$
$$= \int_0^1 \int_0^1 \int_0^1 F(x,y,z)\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z g(x,y,z),$$

where $\gamma_q(n)$ is the van der Corput sequence in base q, $g(x, y, z)$ is the asymptotic distribution function of $\big(\gamma_q(n), \gamma_q(n+1), \gamma_q(n+2)\big)$, and $F(x, y, z) = \max(x, y, z)$.

Let $q \geq 3$ be an integer.

We start with that every point $\big(\gamma_q(n), \gamma_q(n+1)\big)$, $n = 0, 1, 2, \dots$, lies on the diagonals of intervals (48) and (49). Then all terms of the sequence $\big(\gamma_q(n), \gamma_q(n+2)\big), n = 0, 1, 2, \dots$, lie in the diagonals of the following intervals

$$\left[0, 1 - \frac{2}{q}\right] \times \left[\frac{2}{q}, 1\right], \tag{62}$$

$$\left[1 - \frac{1}{q^i}, 1 - \frac{1}{q^{i+1}}\right] \times \left[\frac{1}{q} + \frac{1}{q^{i+1}}, \frac{1}{q} + \frac{1}{q^i}\right], i = 1, 2, \dots, \tag{63}$$

$$\left[1 - \frac{1}{q} - \frac{1}{q^k}, 1 - \frac{1}{q} - \frac{1}{q^{k+1}}\right] \times \left[\frac{1}{q^{k+1}}, \frac{1}{q^k}\right], k = 1, 2, \dots, \tag{64}$$

Every maximal 3-dimensional interval I containing points

$$\big(\gamma_q(n),\ \gamma_q(n+1),\ \gamma_q(n+2)\big)$$

will be written as $I = I_X \times I_Y \times I_Z$, where I_X, I_Y, I_Z are projections of I to the X, Y, Z, axes, respectively. Moreover if

$$\gamma_q(n) \in I_X, \quad \text{then} \quad \gamma_q(n+1) \in I_Y \quad \text{and} \quad \gamma_q(n+2) \in I_Z.$$

From u.d. of $\gamma_q(n)$ follows that the lengths $|I_X| = |I_Y| = |I_Z|$. Combining intervals (48), (62), (63), (64), (49) of equal lengths by following Figure 3, then we find that every point $\big(\gamma_q(n), \gamma_q(n+1), \gamma_q(n+2)\big)$ is contained in diagonals of the intervals

$$I = \left[0, 1 - \frac{2}{q}\right] \times \left[\frac{1}{q}, 1 - \frac{1}{q}\right] \times \left[\frac{2}{q}, 1\right], \tag{65}$$

$$I^{(i)} = \left[1 - \frac{1}{q^i}, 1 - \frac{1}{q^{i+1}}\right] \times \left[\frac{1}{q^{i+1}}, \frac{1}{q^i}\right] \times \left[\frac{1}{q} + \frac{1}{q^{i+1}}, \frac{1}{q} + \frac{1}{q^i}\right], i = 1, 2, \dots, \tag{66}$$

$$J^{(k)} = \left[1 - \frac{1}{q} - \frac{1}{q^k}, 1 - \frac{1}{q} - \frac{1}{q^{k+1}}\right] \times \left[1 - \frac{1}{q^k}, 1 - \frac{1}{q^{k+1}}\right] \times \left[\frac{1}{q^{k+1}}, \frac{1}{q^k}\right],$$
$$k = 1, 2, \dots, \tag{67}$$

where $|I| = 0$ if $q = 2$. These intervals are maximal with respect to inclusion, see Fig. 1:

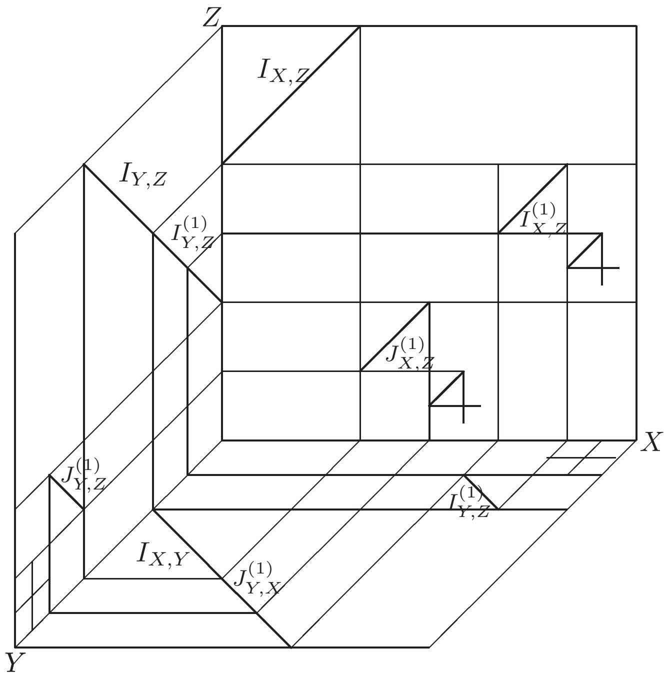


Figure 1: Mapping between intervals with equal lengths.

Now, let T be the union of diagonals of (66), (67) and (65). Again, as in (50), the a.d.f. $g(x, y, z)$ has the form [12]

$$g(x, y, z) = |\text{Project}_X([0, x] \times [0, y] \times [0, z] \cap T)| \tag{68}$$

and it can be rewritten as

$$\begin{aligned} g(x, y, z) = \min \big(|[0, x] \cap I_X|, |[0, y] \cap I_Y|, |[0, z] \cap I_Z|\big) \\ + \sum_{i=1}^{\infty} \min \left(\big|[0, x] \cap I_X^{(i)}\big|, \big|[0, y] \cap I_Y^{(i)}\big|, \big|[0, z] \cap I_Z^{(i)}\big|\right) \\ + \sum_{k=1}^{\infty} \min \left(\big|[0, x] \cap J_X^{(k)}\big|, \big|[0, y] \cap J_Y^{(k)}\big|, \big|[0, z] \cap J_Z^{(k)}\big|\right). \end{aligned} \tag{69}$$

To calculate minimums in (69) we can use the following Fig. 2 (here $q = 3$):

[12] Since $g(x, y, z)$ is continuous, we use in the calculation closed intervals.

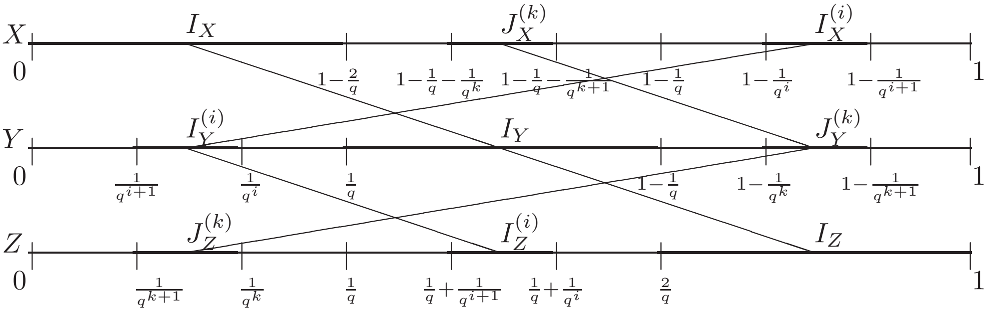


Figure 2: Projections of intervals $I, I^{(i)}, J^{(k)}$ on axes X, Y, Z.

As an example of application of (69) and Fig. 2 we compute $g(x,x,x)$ for $q \geq 3$ without the knowledge of $g(x,y,z)$, [13]

$$g(x,x,x) = \begin{cases} 0 & \text{if } x \in \left[0, \frac{2}{q}\right], \\ x - \frac{2}{q} & \text{if } x \in \left[\frac{2}{q}, 1 - \frac{1}{q}\right], \\ 3x - 2 & \text{if } x \in \left[1 - \frac{1}{q}, 1\right]. \end{cases} \tag{70}$$

P r o o f.

1. Let $x \in \left[0, \frac{1}{q}\right]$.
Then $|[0,x] \cap I_Z| = 0$, $\left|[0,x] \cap I_Z^{(i)}\right| = 0$, $\left|[0,x] \cap J_Y^{(k)}\right| = 0$, consequently $g(x,x,x) = 0$.

2. Let $x \in \left[\frac{1}{q}, \frac{2}{q}\right]$.
Then $|[0,x] \cap I_Z| = 0$, $\left|[0,x] \cap J_Y^{(k)}\right| = 0$, $\left|[0,x] \cap I_X^{(i)}\right| = 0$, consequently $g(x,x,x) = 0$.

3. Let $x \in \left[\frac{2}{q}, 1 - \frac{1}{q}\right]$.
Then $\left|[0,x] \cap I_X^{(i)}\right| = 0$, $\left|[0,x] \cap J_Y^{(k)}\right| = 0$, consequently
$g(x,x,x) = \min\left(1 - \frac{2}{q}, x - \frac{1}{q}, x - \frac{2}{q}\right) = x - \frac{2}{q}$.

4. Let $x \in \left[1 - \frac{1}{q}, 1\right]$.
Specify $x \in I_X^{(k_1)}$, $x \in J_Y^{(k_1)}$. Then $\left|[0,x] \cap I_X^{(k)}\right| = 0$, $\left|[0,x] \cap J_Y^{(k)}\right| = 0$ for $k > k_1$. Thus (69) implies

[13]For $q = 3$ the middle member in (70) is omitted.

$$
\begin{aligned}
g(x,x,x) = &\min\left(1-\frac{2}{q}, 1-\frac{1}{q}-\frac{1}{q}, x-\frac{2}{q}\right)\\
&+\sum_{i=1}^{k_1}\min\left(\left|[0,x]\cap I_X^{(i)}\right|, \left|[0,y]\cap I_Y^{(i)}\right|, \left|[0,z]\cap I_Z^{(i)}\right|\right)\\
&+\sum_{k=1}^{k_1}\min\left(\left|[0,x]\cap J_X^{(k)}\right|, \left|[0,y]\cap J_Y^{(k)}\right|, \left|[0,z]\cap J_Z^{(k)}\right|\right)\\
&= x-\frac{2}{q}+\sum_{i=1}^{k_1-1}\left(\frac{1}{q^i}-\frac{1}{q^{i+1}}\right)+x-1+\frac{1}{q^{k_1}}\\
&+\sum_{k=1}^{k_1-1}\left(\frac{1}{q^k}-\frac{1}{q^{k+1}}\right)+x-1+\frac{1}{q^{k_1}} = 3x-2.
\end{aligned}
$$

□

For $q = 2$ we have

$$
g(x,x,x) = \begin{cases} 0 & \text{if } x \in \left[0, \frac{1}{2}\right], \\ x-\frac{1}{2} & \text{if } x \in \left[\frac{1}{2}, \frac{3}{4}\right], \\ 3x-2 & \text{if } x \in \left[\frac{3}{4}, 1\right]. \end{cases} \tag{71}
$$

The knowledge [14] of the a.d.f. $g(x,y,z)$ of the sequence

$$
\big(\gamma_q(n), \gamma_q(n+1), \gamma_q(n+2)\big), n = 1, 2, \dots
$$

allows us to compute the following limit by the Weyl limit relation (4) in dimension $s = 3$.

$$
\begin{aligned}
&\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}F\big(\gamma_q(n), \gamma_q(n+1), \gamma_q(n+2)\big)\\
&= \int_0^1\int_0^1\int_0^1 F(x,y,z)\mathrm{d}_x\mathrm{d}_y\mathrm{d}_x g(x,y,z),
\end{aligned} \tag{72}
$$

where $F(x,y,z)$ is an arbitrary continuous function defined in $[0,1]^3$. To calculate the Riemann-Stieltjes integral (72) we will use the integration by parts.

Lemma 31. *Assume that $F(x,y,z)$ is a continuous in $[0,1]^3$ and $g(x,y,z)$ is a d.f.*

---

[14] $g(x,y,z)$ is explicitly given in [12] by using 27 possibilities.

Then

$$\begin{aligned}
&\int_0^1\int_0^1\int_0^1 F(x,y,z)\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z g(x,y,z)\\
&= F(1,1,1,) - \int_0^1 g(1,1,z)\mathrm{d}_z F(1,1,z) - \int_0^1 g(1,y,1)\mathrm{d}_y F(1,y,1)\\
&\quad - \int_0^1 g(x,1,1)\mathrm{d}_x F(x,1,1) + \int_0^1\int_0^1 g(1,y,z)\mathrm{d}_y\mathrm{d}_z F(1,y,z)\\
&\quad + \int_0^1\int_0^1 g(x,1,z)\mathrm{d}_x\mathrm{d}_z F(x,1,z) + \int_0^1\int_0^1 g(x,y,1)\mathrm{d}_x\mathrm{d}_y F(x,y,1)\\
&\quad - \int_0^1\int_0^1\int_0^1 g(x,y,z)\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z F(x,y,z). \qquad (73)
\end{aligned}$$

Here

$$\begin{aligned}
\mathrm{d}_x\mathrm{d}_y F(x,y) &= F(x+\mathrm{d}x, y+\mathrm{d}y) + F(x,y) - F(x+\mathrm{d}x,y) - F(x,y+\mathrm{d}y),\\
\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z F(x,y,z) &= F(x+\mathrm{d}x,y+\mathrm{d}y,z+\mathrm{d}z) - F(x,y,z)\\
&\quad + F(x+\mathrm{d}x,y,z) + F(x,y+\mathrm{d}y,z) + F(x,y,z+\mathrm{d}z)\\
&\quad - F(x+\mathrm{d}x,y+\mathrm{d}y,z) - F(x,y+\mathrm{d}y,z+\mathrm{d}z)\\
&\quad - F(x+\mathrm{d}x,y,z+\mathrm{d}z). \qquad (74)
\end{aligned}$$

Note that

$$\mathrm{d}_x\mathrm{d}_y F(x,y) = \frac{\partial^2 F(x,y)}{\partial x\partial y}\mathrm{d}x\mathrm{d}y,$$

$$\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z F(x,y,z) = \frac{\partial^3 F(x,y,z)}{\partial x\partial y\partial z}\mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

if the partial derivatives exist. Put

$$F(x,y,z) = \max(x,y,z).$$

We have

$$\mathrm{d}_x F(x,1,1) = \mathrm{d}_y F(1,y,1) = \mathrm{d}_z F(1,1,z) = 0,$$

$$\mathrm{d}_x\mathrm{d}_y F(x,y,1) = \mathrm{d}_x\mathrm{d}_z F(x,1,z) = \mathrm{d}_y\mathrm{d}_z F(1,y,z) = 0,$$

The differential $\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z F(x,y,z)$ is non-zero if and only if $x = y = z$ and in this case $\mathrm{d}_x\mathrm{d}_y\mathrm{d}_z F(x,y,z) = \mathrm{d}x$.

P r o o f. For every interval

$$J = \left[x_1^{(1)}, x_2^{(1)}\right] \times \left[x_1^{(2)}, x_2^{(2)}\right] \times \cdots \times \left[x_1^{(s)}, x_2^{(s)}\right] \subset [0,1]^s$$

and every continuous $F(x_1, x_2, \dots, x_s)$ the differential $\Delta(F, J)$ is defined as

$$\Delta(F, J) = \sum_{\varepsilon_1=1}^{2} \cdots \sum_{\varepsilon_s=1}^{2} (-1)^{\varepsilon_1 + \cdots + \varepsilon_s} F(x_{\varepsilon_1}^{(1)}, \dots, x_{\varepsilon_s}^{(s)}). \tag{75}$$

Putting $F(x_1, x_2, \dots, x_s) = \max(x_1, x_2, \dots, x_s)$, $x_1^{(i)} = x$, $x_2^{(i)} = x + \mathrm{d}x$ we have

$$\begin{aligned}
\Delta(F, J) &= (-1)^{1+1+\cdots+1} x + \sum_{\varepsilon_1=1}^{2} \cdots \sum_{\varepsilon_s=1}^{2} (-1)^{\varepsilon_1+\cdots+\varepsilon_s} (x + \mathrm{d}x) \\
&= \sum_{\varepsilon_1=1}^{2} \cdots \sum_{\varepsilon_s=1}^{2} (-1)^{\varepsilon_1+\cdots+\varepsilon_s} (x + \mathrm{d}x) - (-1)^{1+1+\cdots+1} \mathrm{d}x \\
&= (-1)^{s+1} \mathrm{d}x.
\end{aligned}$$

□

Then by (73)

$$\begin{aligned}
&\int_0^1 \int_0^1 \int_0^1 F(x, y, z) \mathrm{d}_x \mathrm{d}_y \mathrm{d}_z g(x, y, z) \\
&= 1 - \int_0^1 \int_0^1 \int_0^1 g(x, y, z) \mathrm{d}_x \mathrm{d}_y \mathrm{d}_z F(x, y, z) \\
&= 1 - \int_0^1 g(x, x, x) \mathrm{d}x.
\end{aligned} \tag{76}$$

For $q \geq 3$ and by (70) we have

$$\int_0^1 g(x, x, x) \mathrm{d}x = \int_{\frac{2}{q}}^{1-\frac{1}{q}} \left(x - \frac{2}{q}\right) \mathrm{d}x + \int_{1-\frac{1}{q}}^{1} (3x - 2) \mathrm{d}x = \frac{1}{2} - \frac{2}{q} + \frac{3}{q^2}.$$

For $q = 2$ and by (71) we have

$$\int_0^1 g(x, x, x) \mathrm{d}x = \int_{\frac{1}{2}}^{\frac{3}{4}} \left(x - \frac{1}{2}\right) \mathrm{d}x + \int_{\frac{3}{4}}^{1} (3x - 2) \mathrm{d}x = \frac{3}{16}.$$

Therefore for $q \geq 3$, by (72) and by (76) we have

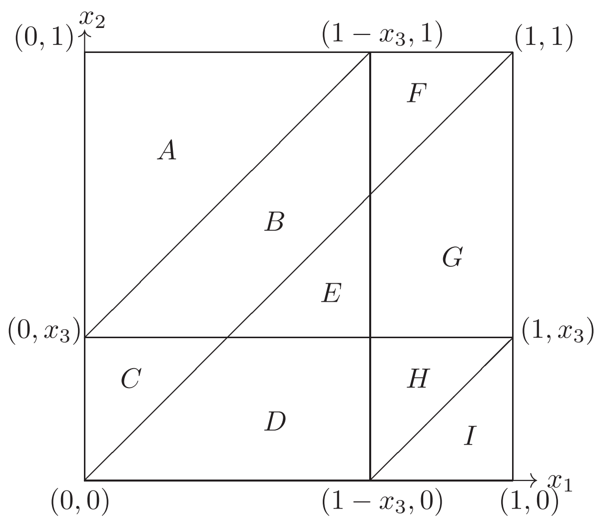
$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \max\bigl(\gamma_q(n), \gamma_q(n+1), \gamma_q(n+2)\bigr) = \frac{1}{2} + \frac{2}{q} - \frac{3}{q^2}. \tag{77}$$

Example 22. Example of a copula in $G_{3,2}$ [44, 3-5,3.2.8].

Let u_n and v_n be two u.d. and statistically independent sequences in $[0,1)$. Then the sequence

$$\mathbf{x}_n = (u_n, v_n, \{u_n - v_n\}), \quad n = 1, 2, \dots,$$

has the a.d.f. $g(\mathbf{x})$ which can be described as follows:
Divide the unit square $[0,1]^2$ into regions $A, B, C, D, E, F, G, H, I$ as shown on the following Figure 2



Then

$$g(x_1,x_2,x_3) = \begin{cases} x_1x_3, & \text{if } (x_1,x_2) \in A, \\ -\frac{1}{2}(x_1^2+x_2^2+x_3^2)+x_1x_2+x_2x_3, & \text{if } (x_1,x_2) \in B, \\ -\frac{1}{2}x_1^2+x_1x_2, & \text{if } (x_1,x_2) \in C, \\ \frac{1}{2}x_2^2, & \text{if } (x_1,x_2) \in D, \\ -\frac{1}{2}x_3^2+x_2x_3, & \text{if } (x_1,x_2) \in E, \\ -\frac{1}{2}x_2^2+x_1x_2+x_1x_3+x_2x_3-x_1-x_3+\frac{1}{2}, & \text{if } (x_1,x_2) \in F, \\ \frac{1}{2}x_1^2+x_1x_3+x_2x_3-x_1-x_3+\frac{1}{2}, & \text{if } (x_1,x_2) \in G, \\ \frac{1}{2}(x_1^2+x_2^2+x_3^2)+x_1x_3-x_1-x_3+\frac{1}{2}, & \text{if } (x_1,x_2) \in H, \\ x_1x_2+x_2x_3-x_2 & \text{if } (x_1,x_2) \in I. \end{cases}$$

The Weyl criterion implies that the two–dimensional sequence $(u_n, \{u_n - v_n\})$ is u.d., thus the face d.f.s are

$$g(1,x_2,x_3) = x_2x_3, \quad g(x_1,1,x_3) = x_1x_3, \quad g(x_1,x_2,1) = x_1x_2.$$

Another d.f. having these three properties (distinct from the u.d.) is

$$g(x_1, x_2, x_3) = \min(x_1x_2, x_1x_3, x_2x_3).$$

2.10. Ratio sequences

Let $x_1 < x_2 < \dots$ be an increasing sequence of positive integers and consider the sequence of blocks X_n, $n = 1, 2, \dots$, with blocks

$$X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \cdots, \frac{x_n}{x_n}\right)$$

and denote by $F(X_n, x)$ the step d.f.

$$F(X_n, x) = \frac{\#\{i \le n; \frac{x_i}{x_n} < x\}}{n},$$

for $x \in [0, 1)$ and $F(X_n, 1) = 1$. A d.f. g is a d.f. of the sequence of single blocks X_n, if there exists an increasing sequence of positive integers $n_1, n_2, \dots$ such that $\lim_{k\to\infty} F(X_{n_k}, x) = g(x)$ a.e. on $[0, 1]$. Denote by $G(X_n)$ the set of all d.f.s of the sequence of single blocks X_n. $G(X_n)$ has the following properties:

- (iii) Assume that all d.f.s in $G(X_n)$ are continuous at 1. Then all d.f.s in $G(X_n)$ are continuous on $(0, 1]$, i.e., only possible discontinuity is in 0.
- (iv) If $\underline{d}(x_n) > 0$, then for every $g(x) \in G(X_n)$ we have [45, Th. 6.2(iii)] $\frac{\underline{d}(x_n)}{\overline{d}(x_n)} x \le g(x) \le \frac{\overline{d}(x_n)}{\underline{d}(x_n)} x$ for every $x \in [0, 1]$. Thus $\underline{d}(x_n) = \overline{d}(x_n) > 0$ implies u.d. of the block sequence X_n, $n = 1, 2, \dots$.
- (v) If $\underline{d}(x_n) > 0$, then every $g(x) \in G(X_n)$ is continuous on $[0, 1]$.
- (vi) If $\underline{d}(x_n) > 0$, then there exists $g(x) \in G(X_n)$ such that $g(x) \ge x$ for every $x \in [0, 1]$, [45, Th. 6.2(ii)]. By [2, Th. 6)] every $G(X_n)$ contains $g(x) \ge x$.
- (vii) If $\overline{d}(x_n) > 0$, then there exists $g(x) \in G(X_n)$ such that $g(x) \le x$ for every $x \in [0, 1]$.
- (viii) Assume that $G(X_n)$ is singleton, i.e., $G(X_n) = \{g(x)\}$. Then either $g(x) = c_0(x)$ for $x \in [0, 1]$; or $g(x) = x^\lambda$ for some $0 < \lambda \le 1$ and $x \in [0, 1]$. Moreover, if $\overline{d}(x_n) > 0$, then $g(x) = x$.
- (ix) $\max_{g\in G(X_n)} \int_0^1 g(x)\mathrm{d}x \ge \frac{1}{2}$.
- (x) Assume that every d.f. $g(x) \in G(X_n)$ has a constant value on the fixed interval $(u, v) \subset [0, 1]$ (maybe different). If $\underline{d}(x_n) > 0$ then all d.f.'s in $G(X_n)$ has infinitely many intervals with constant values.

(xi) There exists an increasing sequence x_n, $n = 1, 2, \dots$, of positive integers such that $G(X_n) = \{h_\alpha(x); \alpha \in [0,1]\}$, where $h_\alpha(x) = \alpha$, $x \in (0,1)$ is the constant d.f.

(xii) There exists an increasing sequence x_n, $n = 1, 2, \dots$, of positive integers such that $c_1(x) \in G(X_n)$ but $c_0(x) \notin G(X_n)$, where $c_0(x)$ and $c_1(x)$ are one–jump d.f.'s with the jump of height 1 at $x = 0$ and $x = 1$, respectively.

(xiii) There exists an increasing sequence x_n, $n = 1, 2, \dots$, of positive integers such that $G(X_n)$ is non-connected.

(xiv) $G(X_n) = \{x^\lambda\}$ if and only if $\lim_{n\to\infty}(x_{k.n}/x_n) = k^{1/\lambda}$ for every $k = 1, 2, \dots$. Here as in (viii) we have $0 < \lambda \le 1$.

(xv) If $\underline{d}(x_n) > 0$, then all d.f.s $g(x) \in G(X_n)$ are continuous, nonsingular and bounded by $h_1(x) \le g(x) \le h_2(x)$, where

$$h_1(x) = \begin{cases} x\frac{\underline{d}}{\overline{d}} & \text{if } x \in \left[0, \frac{1-\overline{d}}{1-\underline{d}}\right], \\ \frac{\underline{d}}{\frac{1}{x}-(1-\underline{d})} & \text{otherwise,} \end{cases} \qquad h_2(x) = \min\left(x\frac{\overline{d}}{\underline{d}}, 1\right).$$

Furthermore, there exists x_n, $n = 1, 2, \dots$, such that $h_2(x) \in G(X_n)$ and for every x_n we have $h_1(x) \notin G(X_n)$, [2, Th. 7] and moreover

(xvi) for a given fixed $g(x) \in G(X_n)$ we have $h_{1,g}(x) \le g(x) \le h_{2,g}(x)$, where

$$h_{1,g}(x) = \begin{cases} x\frac{\underline{d}}{d_g} & \text{if } x < y_0 = \frac{1-d_g}{1-\underline{d}}, \\ x\frac{1}{d_g} + 1 - \frac{1}{d_g} & \text{if } y_0 \le x \le 1, \end{cases} \tag{78}$$

$$h_{2,g}(x) = \min\left(x\frac{\overline{d}}{d_g}, 1\right), \tag{79}$$

where if $\lim_k \to \infty F(X_{n_k}, x) = g(x)$, then $d_g = \lim_{k\to\infty} \frac{n_k}{x_{n_k}}$ if exists.

Applying (xv) it is proved [2]:

Theorem 32. *For every increasing sequence $x_1 < x_2 < \dots$ of positive integers with the lower and upper asymptotic densities $0 < \underline{d} \le \overline{d}$ we have*

$$\frac{1}{2}\frac{\underline{d}}{\overline{d}} \le \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{x_n}, \tag{80}$$

and

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{x_n} \le \frac{1}{2} + \frac{1}{2}\left(\frac{1-\min(\sqrt{\underline{d}}, \overline{d})}{1-\underline{d}}\right)\left(1 - \frac{\underline{d}}{\min(\sqrt{\underline{d}, \overline{d}})}\right). \tag{81}$$

Here the equations in (80) *and* (81) *can be attained.*

Example 23. O. Strauch and J.T. Tóth (2001): Put $x_n = p_n$, the nth prime and denote
$$X_n = \left(\frac{2}{p_n}, \frac{3}{p_n}, \dots, \frac{p_{n-1}}{p_n}, \frac{p_n}{p_n}\right).$$
The sequence of blocks X_n is u.d. and therefore the ratio sequence p_m/p_n, $m = 1, 2, \dots, n$, $n = 1, 2, \dots$ is u.d. in $[0, 1]$. This generalizes a result of A. Schinzel (cf. W. Sierpiński (1964, p. 155)). Note that from u.d. of X_n applying the L^2 discrepancy of X_n we get the following interesting limit
$$\lim_{n\to\infty} \frac{1}{n^2 p_n} \sum_{i,j=1}^{n} |p_i - p_j| = \frac{1}{3}.$$

The set $G(X_n)$ can be complicated, see F. Filip, L. Mišík and J.T. Tóth [13]:

Example 24. Let a_k, n_k, $k = 1, 2, \dots$, and x_n, $n = 1, 2, \dots$ be three increasing integer sequences and $h_1 < h_2$ be two positive integers. Assume that

(i) $\frac{n_k}{n_{k+1}} \to 0$ for $k \to \infty$;

(ii) $\frac{a_k}{n_{k+1}} \to 0$ for $k \to \infty$;

(iii) for odd k we have
$a_k^{h_2} \le x_{n_k} = (a_{k-1} + n_k - n_{k-1})^{h_1} \le (a_k + 1)^{h_2}$ and
$x_i = (a_k + i - n_k)^{h_2}$ for $n_k < i \le n_{k+1}$;

(iv) for even k we have
$a_k^{h_1} \le x_{n_k} = (a_{k-1} + n_k - n_{k-1})^{h_2} \le (a_k + 1)^{h_1}$ and
$x_i = (a_k + i - n_k)^{h_1}$ for $n_k < i \le n_{k+1}$.

Then $\frac{x_n}{x_{n+1}} \to 1$ and the set $G(X_n)$ of all distribution functions of the sequence of blocks X_n is $G(X_n) = G_1 \cup G_2 \cup G_3 \cup G_4$, where

$G_1 = \{x^{\frac{1}{h_2}} \cdot t; t \in [0, 1]\}$,

$G_2 = \{x^{\frac{1}{h_2}}(1 - t) + t; t \in [0, 1]\}$,

$G_3 = \{\max(0, x^{\frac{1}{h_1}} - (1 - x^{\frac{1}{h_1}})u); u \in [0, \infty)\}$ and

$G_4 = \{\min(1, x^{\frac{1}{h_1}} \cdot v); v \in [1, \infty)\}$.

In Unsolved Problems [42, 1.9] there are given many open questions on $G(X_n)$, one from them: Characterize a nonempty set H of d.f.s for which there exists an increasing sequence of positive integers x_n such that $G(X_n) = H$. In [2] is given the following partial result:

Theorem 33. *Let H be a nonempty set of d.f.s defined on $[0, 1]$. Then there exists a positive integer sequence $x_1 < x_2 < \dots$ such that $H \subset G(X_n)$.*

3. Calculation methods of $G(x_n)$

3.1. Calculation of d.f.s by definition

We give a proof of Example 6 via definition of d.f.s.

Example 25. Starting with $x_n = \{\log\log n\}$ all the sequences $\{\log\log\dots\log n\}$ have

$$G(x_n) = \{c_\alpha(x); \alpha \in [0,1]\} \cup \{h_\alpha(x); \alpha \in [0,1]\}.$$

Proof. For the first iterated logarithm we chose an index-sequence N_k as

$$N_k = [\exp\exp(k+\alpha)]$$

Then we have $\lim_{k\to\infty} F_{N_k}(x) = c_\alpha(x)$. For $N_k = [\exp\exp(k+\varepsilon_k)]$, where $\varepsilon_k \to 0$ such that $\big(\exp\exp(k+\varepsilon_k)\big)/(\exp\exp k) \to \beta$, we have $\lim_{k\to\infty} F_{N_k}(x) = h_\alpha(x)$, where $\alpha = (\beta-1)/\beta$.

On the other hand, let $\lim_{n\to\infty} F_{N_n}(x) = g(x)$. Then $N_n = \exp\exp(k_n+\varepsilon_n)$, where $k_n = [\log\log N_n]$, $\varepsilon_n = \{\log\log N_n\}$, and the sequence $(\varepsilon_n)_{n=1}^\infty$ cannot have different limit points. □

Similarly, but complicated, it can be found:

Example 26. The set of all d.f.s of the two-dimensional sequence

$$(\log n, \log\log n) \bmod 1$$

has the form, see [43]:

$$\begin{aligned} G((\log n, \log\log n) \bmod 1) = {} & \{g_{u,v}(x,y); u\in[0,1], v\in[0,1]\} \\ & \cup \{g_{u,0,j,\alpha}(x,y); u\in[0,1], \alpha\in A, j=1,2,\dots\} \\ & \cup \{g_{u,0,0,\alpha}(x,y); u\in[\alpha,1], \alpha\in A\}, \end{aligned} \tag{82}$$

where A is the set of all limit points of the sequence $e^n \bmod 1$, $n=1,2,\dots$, and, [15] for $(x,y)\in[0,1)^2$,

$$\begin{aligned} g_{u,v}(x,y) &= g_u(x)\cdot c_v(y), \\ g_{u,0,j,\alpha}(x,y) &= g_{u,0,j,\alpha}(x)\cdot c_0(y), \\ g_{u,0,0,\alpha}(x,y) &= g_{u,0,0,\alpha}(x)\cdot c_0(y), \end{aligned}$$

where

$$g_u(x) = \frac{e^{\min(x,u)}-1}{e^u} + \frac{1}{e^u}\frac{e^x-1}{e-1},$$

[15]The exact form of A is a well-known open problem.

$$c_v(y) = \begin{cases} 0 & \text{if } 0 \le y \le v, \\ 1 & \text{if } v < y \le 1, \end{cases} \qquad \text{and} \qquad c_v(0) = 0, c_v(1) = 1,$$

$$\begin{aligned} g_{u,0,j,\alpha}(x) &= \frac{e^{\max(\alpha,x)} - e^{\alpha}}{e^{j+u}} + \frac{e^{\min(x,u)} - 1}{e^u} + \frac{1}{e^u}\frac{e^x - 1}{e - 1}\left(1 - \frac{1}{e^{j-1}}\right), \\ g_{u,0,0,\alpha}(x) &= \frac{e^{\max(\min(x,u),\alpha)} - e^{\alpha}}{e^u}. \end{aligned} \tag{83}$$

A proof via d.f.s. consider limits of

$$\begin{aligned} F_N(x,y) &= \frac{\#\{3 \le n \le N; (\{\log n\}, \{\log\log n\}) \in [0,x) \times [0,y)\}}{N} \\ &= \frac{\sum_{j=0}^{J} \#\{[e^{e^j}, e^{e^{j+y}}) \cap (\cup_{k=1}^{K}[e^k, e^{k+x})) \cap \mathbf{N}\}}{N}, \end{aligned}$$

where

$$K = [\log N], \qquad J = [\log\log N], \qquad \mathbf{N} = \{1, 2, \dots, N\}.$$

O. Strauch and O. Blažeková in [43] and R. Giuliano Antonini and O. Strauch in [15] generalized Example 26 and Koksma [19], [20, Kap. 8] (cf. [22, p. 58, Th. 7.7]) to the following Theorems 34, 35 and 36. Assume:

(I) $f(x)$ be a real-valued function defined for $x \ge 1$ such that $f(x)$ is strictly increasing with inverse function $f^{-1}(x)$.

(II) $\lim_{k\to\infty} \frac{f^{-1}(k+x) - f^{-1}(k)}{f^{-1}(k+1) - f^{-1}(k)} = \tilde{g}(x)$ for each $x \in [0,1]$, point of continuity of $\tilde{g}(x)$;

(III) $\lim_{k\to\infty} \frac{f^{-1}(k+u)}{f^{-1}(k)} = \psi(u)$ for each $u \in [0,1]$, point of continuity of $\psi(u)$, or $\psi(u) = \infty$ for $u > 0$;

(IV) $\lim_{k\to\infty} f^{-1}(k+1) - f^{-1}(k) = \infty$.

For computing $G(f(n) \bmod 1)$ we have the following three theorems.

Theorem 34. *If $1 < \psi(1) < \infty$ and $f'(x) \to 0$ as $x \to \infty$, then*

$$G(f(n) \bmod 1) = \left\{ g_u(x) = \frac{\min(\psi(x), \psi(u)) - 1}{\psi(u)} + \frac{1}{\psi(u)}\tilde{g}(x); u \in [0,1] \right\}, \tag{84}$$

where

$$\tilde{g}(x) = \frac{\psi(x) - 1}{\psi(1) - 1} \quad \textit{and} \quad F_{N_i}(x) \to g_u(x) \quad \textit{as} \quad i \to \infty$$

if and only if $f(N_i) \bmod 1 \to u$.

The lower d.f. $\underline{g}(x)$ *and the upper d.f.* $\overline{g}(x)$ *of* $f(n) \bmod 1$ *are*

$$\underline{g}(x) = \tilde{g}(x), \quad \overline{g}(x) = 1 - \frac{1}{\psi(x)}\big(1 - \tilde{g}(x)\big).$$

Furthermore $\underline{g}(x) = g_0(x) = g_1(x)$ *belongs to* $G(f(n) \bmod 1)$ *but* $\overline{g}(x) = g_x(x)$ *does not.*

Theorem 35. *If* $\psi(1) = 1$, *then the sequence* $f(n) \bmod 1$, $n = 1, 2, \dots$ *has a.d.f.* $\tilde{g}(x)$, *i.e.,*

$$G(f(n) \bmod 1) = \{\tilde{g}(x)\}. \tag{85}$$

Theorem 36. *Let* $\psi(u) = \infty$, *for every* $u > 0$ *and for* $u = 0$ *the limit* $\psi(u)$ *is not defined in the way that for every* $t \in [0, \infty)$ *there exists a sequence* $u(k) \to 0$ *such that* (i) $\lim_{k\to\infty} \frac{f^{-1}(k+u(k))}{f^{-1}(k)} = t$. *Then we have*

$$G(f(n) \bmod 1) = \{c_u(x); u \in [0,1]\} \cup \{h_\beta(x); \beta \in [0,1]\}, \tag{86}$$

where $F_{N_i} \to c_u(x)$ *if and only if* $f(N_i) \bmod 1 \to u > 0$ *and* $F_{N_i} \to h_\beta(x)$ *if and only if* $f(N_i) \bmod 1 \to 0$ *and* $\frac{f^{-1}([f(N_i)])}{N_i} \to 1 - \beta$.

In proofs of Theorems 34, 35 and 36 there are studied limits of step d.f. $F_N(x)$ expressed as

$$F_N(x) = \frac{\sum_{k=0}^{K-1} A_N\big([k, k+x)\big)}{N} + \frac{A_N\big([K, K+x) \cap [K, K+w)\big)}{N} + \frac{O\left(A_N\big([1, f^{-1}(0))\big)\right)}{N},$$

where

$$K = [f(N)], \qquad w = \{f(N)\}, \qquad A_N\big([x,y)\big) = \#\{n \le N; f(n) \in [x,y)\}.$$

Example 27. Applying Theorem 34 to the function $f(x) = \log(x \log^{(i)} x)$, where $\log^{(i)} x$ is the ith iterated logarithm $\log \dots \log x$, we find

$$G\big(\log(n \log^{(i)} n) \bmod 1\big) = G(\log n \bmod 1).$$

3.2. Connectivity of $G(x_n)$

The connectivity of $G(x_n)$ implies the following simple theorem: Define

- For the d.f. g the Graph(g) be the continuous curve formed by all the points $(x, g(x))$ for $x \in [0,1]$, and the all line segments connecting the points of discontinuity $(x, \liminf_{x'\to x} g(x'))$ and $(x, \limsup_{x'\to x} g(x'))$.
- Denote $\underline{g}_H(x) = \inf_{g\in H} g(x)$ and $\overline{g}_H(x) = \sup_{g\in H} g(x)$.

Theorem 37 ([37]). *Let H be a non–empty, closed, and connected set of d.f.s. Assume that for every $g \in H$ there exists a point $(x, y) \in \mathrm{Graph(g)}$ such that $(x, y) \notin \mathrm{Graph}(\widetilde{\mathrm{g}})$ for any $\widetilde{g} \in H$ with $\widetilde{g} \neq g$. If*

(i) *$\underline{g} = \underline{g}_H$ and $\overline{g} = \overline{g}_H$ for the lower d.f. $\underline{g}$ and the upper d.f. $\overline{g}$ of the sequence $x_n \in [0, 1)$, and*

(ii) *$G(x_n) \subset H$,*

then $G(x_n) = H$.

To prove $G(x_n) \subset H$ it can be used:

Theorem 38. *Let $F(x, y)$ be a real continuous function defined on $[0, 1]^2$ and $G(F)$ is the set of all d.f. $g(x)$ which satisfy $\int_0^1 \int_0^1 F(x, y)\mathrm{d}g(x)\mathrm{d}g(y) = 0$. Then for every sequence $x_n \in [0, 1)$ we have*

$$G(x_n) \subset G(F) \Longleftrightarrow \lim_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^{N} F(x_m, x_n) = 0. \tag{87}$$

P r o o f. Using the definition of the Riemann-Stieltjes integral, we have

$$\int_0^1 \int_0^1 F(x, y)\mathrm{d}F_N(x)\mathrm{d}F_N(y) = \frac{1}{N^2} \sum_{m,n=1}^{N} F(x_m, x_n).$$

Suppose that $\lim_{k\to\infty} F_{N_k}(x) = g(x)$ for all continuity points x of g. Then, applying the Helly-Bray lemma, we find

$$\int_0^1 \int_0^1 F(x, y)\mathrm{d}F_{N_k}(x)\mathrm{d}F_{N_k}(y) = \int_0^1 \int_0^1 F(x, y)\mathrm{d}g(x)\mathrm{d}g(y),$$

and the implication $\Longleftarrow$ in (87) follows immediately.

In order to show the implication $\Longrightarrow$, assume

$$\lim_{k\to\infty} \frac{1}{{N_k}^2} \sum_{m,n=1}^{N_k} F(x_m, x_n) = \beta > 0.$$

By the Helly selection principle there exists a subsequence N'_k of N_k such that

$$\lim_{k\to\infty} F_{N'_k}(x) = g(x) \in G(x_n).$$

Again, by the Helly-Bray theorem we find $\int_0^1 \int_0^1 F(x, y)\mathrm{d}g(x)\mathrm{d}g(y) = \beta$. We conclude $g \notin G(F)$. □

Theorem 37 implies:

- Let $g_1 \neq g_2$ be two d.f.s. Denote

$$F_{g_2}(x,y) = \int_0^x g_2(t)\mathrm{d}t + \int_0^y g_2(t)\mathrm{d}t - \max(x,y) + \int_0^1 (1-g_2(t))^2\mathrm{d}t,$$

$$F_{g_1,g_2}(x) = \frac{\int_0^x \big(g_2(t)-g_1(t)\big)\mathrm{d}t - \int_0^1 \big(1-g_2(t)\big)\big(g_2(t)-g_1(t)\big)\mathrm{d}t}{\int_0^1 \big(g_2(t)-g_1(t)\big)^2\mathrm{d}t},$$

$$F_{g_1,g_2}(x,y) = F_{g_2}(x,y) - F_{g_1,g_2}(x)F_{g_1,g_2}(y). \int_0^1 \big(g_2(t)-g_1(t)\big)^2\mathrm{d}t.$$

Theorem 39. *For given sequence $x_n \in [0,1)$ we have*

$$G(x_n) = \{tg_1(x) + (1-t)g_2(x); t \in [0,1]\}$$

if and only if

(i) $\lim_{N\to\infty} \frac{1}{N^2}\sum_{m,n=1}^N F_{g_1,g_2}(x_m,x_n) = 0,$

(ii) $\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^N F_{g_1,g_2}(x_n) = 0,$

(iii) $\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^N F_{g_1,g_2}(x_n) = 1.$

Example 28. Put

$$F_1(x,y) = 1 - \max(x,y) - \frac{3}{4}(1-x^2)(1-y^2),$$

$$F_2(x,y) = \frac{x+y}{2} - \max(x,y) + \frac{1}{4} - 3(x-x^2)(y-y^2),$$

$$F_3(x,y) = 1 - \max(x,y),$$

$$F_4(x,y) = \frac{x+y}{2} - \max(x,y) + \frac{1}{4},$$

and

$H_1 = \{tx + (1-t)c_1(x); t \in [0,1]\}$, and $H_2 = \{tx + (1-t)h_{1/2}(x); t \in [0,1]\}$.

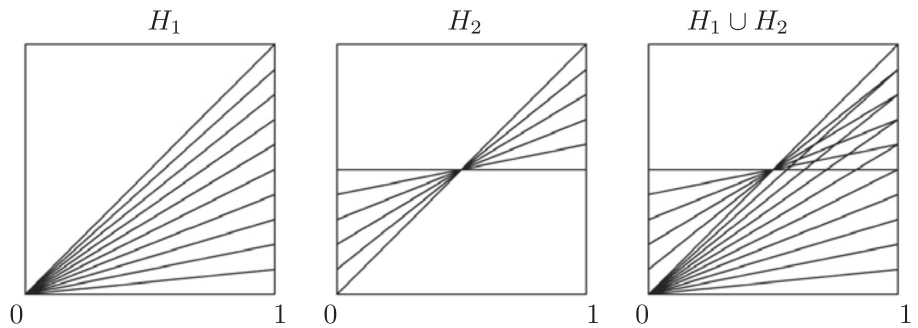


Figure 1: D.f.s H_1, H_2 and $H_1 \cup H_2$.

Then $G(x_n) = H_1 \cup H_2$ for a sequence x_n in $[0,1)$ if and only if

(i) $\lim_{N\to\infty} \frac{1}{N^4} \sum_{m,n,k,l=1}^{N} F_1(x_m, x_n) F_2(x_k, x_l) = 0$,

(ii) $\liminf_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^{N} F_3(x_m, x_n) = 0$.

(iii) $\liminf_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^{N} F_4(x_m, x_n) = 0$.

Here $c_1(x)$ is the one–jump d.f. with jump at $x = 1$ and $h_{1/2}(x)$ is the d.f. taking constant value 1/2.

3.2.1. The moment problem $\int_0^1 \int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y) = 0$

These investigation is motivated by Theorem 37. Again, for a given continuous $F : [0,1]^2 \to \mathbb{R}$, let $G(F)$ denote the set of all distribution functions g which solve the following moment problem $\int_0^1 \int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y) = 0$.

Theorem 40 ([37]). *Let $F : [0,1]^2 \to \mathbb{R}$ be a continuous and symmetric function. For every distribution functions $g(x)$, $\tilde{g}(x)$ we have*

$$\int_0^1 \int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y) = 0 \Longleftrightarrow \int_0^1 \int_0^1 F(x,y)\mathrm{d}\tilde{g}(x)\mathrm{d}\tilde{g}(y)$$
$$= \int_0^1 \big(g(x) - \tilde{g}(x)\big)\Big(2\mathrm{d}_x F(x,1) - \int_0^1 \big(g(y) + \tilde{g}(y)\big)\mathrm{d}_y \mathrm{d}_x F(x,y)\Big) \quad (88)$$

Example 29. Putting $\tilde{g}(x) = c_0(x)$ in (88), we have

$$\int_0^1 \int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y) = 0 \Longleftrightarrow$$
$$F(0,0) = \int_0^1 \big(g(x) - 1\big)\Big(2\mathrm{d}_x F(x,1) - \int_0^1 \big(g(y) + 1\big)\mathrm{d}_y \mathrm{d}_x F(x,y)\Big). \quad (89)$$

For $F(x,y) = F_0(x,y)$ we have

$$g(x) = x \Longleftrightarrow \frac{1}{3} = \int_0^1 g(x)\big(2x - g(x)\big)\mathrm{d}x.$$

Here

$$F_0(x,y) = \frac{1}{3} + \frac{x^2 + y^2}{2} - \max(x,y)$$
$$= \frac{1}{3} + \frac{x^2 + y^2}{2} - \frac{x + y}{2} - \frac{|x - y|}{2}$$

This function $F_0(x,y)$ is inspired by the classical L^2 discrepancy

$$\int_0^1 \big(F_N(x)-x\big)^2 \mathrm{d}x$$

since

$$\int_0^1 \big(F_N(x)-x\big)^2 \mathrm{d}x = \frac{1}{N^2}\sum_{m,n=1}^{N} F_0(x_m,x_n).$$

3.3. Computation $G\big(h(x_n,y_n)\big)$ by $g(x,y)\in G\big((x_n,y_n)\big)$

We describe a result for $h(x,y)=x+y \bmod 1$.

Theorem 41. *Let x_n and y_n be two sequences in $[0,1)$ and $G((x_n,y_n))$ denote the set of all d.f.s of the two-dimensional sequence (x_n,y_n). If*

$$z_n = x_n + y_n \bmod 1,$$

then the set $G(z_n)$ of all d.f.s of z_n has the form

$$G(z_n) = \left\{g(t)=\int_{0\le x+y<t} 1.\mathrm{d}g(x,y)+\int_{1\le x+y<1+t} 1.\mathrm{d}g(x,y);\ \ g(x,y)\in G\big((x_n,y_n)\big)\right\}$$

assuming that all the used Riemann-Stieltjes integrals exist.

Example 30. Applying Theorem 41 to the $G\big((\log n,\log\log n) \bmod 1\big)$ in Example 26 it can be found, for

$$G\big(\log(n\log n) \bmod 1\big) = \{g_{u,v}(x); u\in[0,1], v\in[0,1]\},$$

that

$$g_{u,v}(x)=\begin{cases} g_u(1+x-v)-g_u(1-v) & \text{if } 0\le x\le v,\\ g_u(x-v)+1-g_u(1-v) & \text{if } v<x\le 1.\end{cases}$$

Directly by means of computation we see that

$$g_{u,v}(x)=g_w(x), \quad \text{for} \quad w=u+v \bmod 1,$$

defined in Example 1 and thus $G\big(\log(n\log n) \bmod 1\big) = G\big(\log n \bmod 1\big)$. The same holds by Example 27.

3.4. Solution of $(X_1,X_2,X_3)=\left(\int_0^1 g(x)\mathrm{d}x, \int_0^1 xg(x)\mathrm{d}x, \int_0^1 g^2(x)\mathrm{d}x\right)$

See [39], [44, 2.2.21]:

It is motivated by L^2 discrepancy criterion for g-distributed sequences: A sequence x_n in $[0,1]$ has a.d.f. $g(x)$ if and only if

$$\lim_{N\to\infty}\Bigg(1+\int_0^1 g^2(x)\mathrm{d}x-2\int_0^1 g(x)\mathrm{d}x+\frac{2}{N}\sum_{n=1}^N\int_0^{x_n} g(x)\mathrm{d}x - \frac{1}{N}\sum_{n=1}^N x_n-\frac{1}{2N^2}\sum_{m,n=1}^N|x_m-x_n|\Bigg)=0,$$

or equivalently, if and only if

(i) $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N x_n=\int_0^1 x\mathrm{d}g(x)$,

(ii) $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N\int_0^{x_n} g(x)\mathrm{d}x=\int_0^1\left(\int_0^x g(t)\mathrm{d}t\right)\mathrm{d}g(x)$,

(iii) $\lim_{N\to\infty}\frac{1}{N^2}\sum_{m,n=1}^N|x_m-x_n|=\int_0^1\int_0^1|x-y|\mathrm{d}g(x)\mathrm{d}g(y)$.

Since the left-hand side of (iii) contains $g(x)$ we shall instead it by the second moment and we solve

$$(s_1,s_2,s_3)=\left(\int_0^1 x\mathrm{d}g(x),\int_0^1 x^2\mathrm{d}g(x),\int_0^1\int_0^1|x-y|\mathrm{d}g(x)\mathrm{d}g(y)\right),\qquad(90)$$

where

$s_1=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N x_n$,

$s_2=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N x_n^2$ and

$s_3=\lim_{N\to\infty}\frac{1}{N^2}\sum_{m,n=1}^N|x_m-x_n|$.

Using $(s_1,s_2,s_3)=(1-X_1,1-2X_2,2(X_1-X_3))$ then (90) can be transform to

$$(X_1,X_2,X_3)=\left(\int_0^1 g(x)\mathrm{d}x,\int_0^1 xg(x)\mathrm{d}x,\int_0^1 g^2(x)\mathrm{d}x\right).$$

Define, for every nondecreasing $g:[0,1]\to[0,1]$, the operator

$$\mathbf{F}(g)=\left(\int_0^1 g(x)\mathrm{d}x,\int_0^1 xg(x)\mathrm{d}x,\int_0^1 g^2(x)\mathrm{d}x\right).$$

For $\mathbf{F}$, we introduce the body

$$\mathbf{\Omega}=\big\{\mathbf{F}(g);g\,[0,1]\to[0,1],g\text{ is nondecreasing}\big\},$$

and $\partial\mathbf{\Omega}$ denote the *boundary* of $\mathbf{\Omega}$.

Let $g(u_1,v_1,u_2,v_2)$ denote the distribution function $h(x)$ defined by

$$h(x)=\begin{cases}0 & \text{for }\ 0\le x\le v_1\,,\\ \frac{u_2-u_1}{v_2-v_1}x+u_1-v_1\frac{u_2-u_1}{v_2-v_1} & \text{for }\ v_1<x\le v_2\,,\\ 1 & \text{for }\ v_2<x\le 1\end{cases}$$

and put $\mathbf{X}=(X_1,X_2,X_3)$. O. Strauch [39] proved

Theorem 42. *For the moment problem $\mathbf{X} = \mathbf{F}(g)$, to have only a finite number of solutions in distribution functions g it is necessary and sufficient that $\mathbf{X} \in \partial\mathbf{\Omega}$. We express the boundary $\partial\mathbf{\Omega}$ as*

$$\partial\mathbf{\Omega} = \bigcup_{1\le i\le 7} \Pi_i.$$

In addition, for $\mathbf{X} \in \Pi_i$, $i = 1, 2, \ldots, 6$, the moment problem $\mathbf{X} = \mathbf{F}(g)$ is uniquely solvable as $g = g^{(i)}$, and for $\mathbf{X} \in \Pi_7$ has precisely two solutions of types $g^{(7)}$ and $g^{(7^*)}$.

Theorem 43. *Let x_n, $n = 1, 2, \ldots$ be a sequence in $[0, 1]$ with the limits*

$X_1 = 1 - \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n$,

$X_2 = \frac{1}{2} - \frac{1}{2} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n^2$,

$X_3 = 1 - \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n - \frac{1}{2} \lim_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^{N} |x_m - x_n|$.

If

$$\mathbf{X} = (X_1, X_2, X_3) \in \bigcup_{1\le i\le 7} \Pi_i,$$

then the sequence x_n has an a.d.f. Precisely, if

$$\mathbf{X} \in \Pi_i, \ \ i = 1, \ldots, 6,$$

then x_n has a.d.f. $g^{(i)}$, and if

$$\mathbf{X} \in \Pi_7,$$

then x_n has a.d.f. either $g^{(7)}$ or $g^{(7^*)}$, depending on whether

$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int_0^{x_n} g^{(7)}(t)\mathrm{d}t = X_1 - X_3$ *or*

$\lim_{n\to\infty} \frac{1}{N} \sum_{0}^{N} \int_0^{x_n} g^{(7^*)}(t)\mathrm{d}t = X_1 - X_3$.

Here

$$\Pi_1 = \Big\{(X_1, X_2, X_3);\quad X_2 = \frac{1}{2} - \frac{1}{2}(1 - X_1)^2 - \frac{3}{2}(X_1 - X_3)^2,$$
$$\max\Big(\frac{4}{3}X_1 - \frac{1}{3}, \ \frac{2}{3}X_1\Big) \le X_3 \le X_1, \ 0 \le X_1 \le 1\Big\},$$

$$\Pi_2 = \Big\{(X_1, X_2, X_3);\quad X_2 = \frac{1}{2}X_1 + \frac{1}{2}\sqrt{\frac{1}{3}(X_3 - X_1^2)},$$
$$X_1^2 \le X_3 \le \min\Big(\frac{4}{3}X_1^2, \ \frac{4}{3}X_1^2 - \frac{2}{3}X_1 + \frac{1}{3}\Big), \ 0 \le X_1 \le 1\Big\},$$

$$\Pi_3 = \left\{(X_1, X_2, X_3);\quad X_2 = \frac{1}{2} - \frac{4}{9}\frac{(1-X_1)^3}{(1+X_3-2X_1)}, \right.$$
$$\left. \frac{4}{3}X_1^2 - \frac{2}{3}X_1 + \frac{1}{3} \le X_3 \le \frac{4}{3}X_1 - \frac{1}{3},\ \frac{1}{2} \le X_1 \le 1\right\},$$

$$\Pi_4 = \left\{(X_1, X_2, X_3);\quad X_2 = X_1 - \frac{4}{9}\frac{X_1^3}{X_3}, \frac{4}{3}X_1^2 \le X_3 \le \frac{2}{3}X_1,\ 0 \le X_1 \le \frac{1}{2}\right\},$$

$$\Pi_5 = \left\{(X_1, X_2, X_3);\quad X_2 = \frac{1}{2} - \frac{1}{2}\frac{(1-X_1)^3}{(1+X_3-2X_1)}, \right.$$
$$\left. X_1^2 \le X_3 \le X_1,\ 0 \le X_1 < \frac{1}{2}\right\},$$

$$\Pi_6 = \left\{(X_1, X_2, X_3);\quad X_2 = X_1 - \frac{1}{2}\frac{X_1^3}{X_3},\ X_1^2 \le X_3 \le X_1,\ \frac{1}{2} < X_1 \le 1\right\},$$

$$\Pi_7 = \left\{\left(\frac{1}{2}, \frac{1}{2} - \frac{1}{16X_3}, X_3\right);\quad \frac{1}{4} < X_3 < \frac{1}{2}\right\},$$

and

$$g^{(1)} = g(0, (1-X_1) - 3(X_1 - X_3), 1, (1-X_1) + 3(X_1 - X_3)),$$
$$g^{(2)} = g\left(X_1 - \sqrt{3(X_3 - X_1^2)}, 0, X_1 + \sqrt{3(X_3 - X_1^2)}, 1\right),$$
$$g^{(3)} = g\left(1 - \frac{3}{2}\frac{1+X_3-2X_1}{1-X_1}, 0, 1, \frac{4}{3}\frac{(1-X_1)^2}{(1+X_3-2X_1)}\right),$$
$$g^{(4)} = g\left(0, 1 - \frac{4X_1^2}{3X_3}, \frac{3X_3}{2X_1}, 1\right),$$
$$g^{(5)} = g\left(\frac{X_1 - X_3}{1-X_1}, 0, \frac{X_1 - X_3}{1-X_1}, \frac{(1-X_1)^2}{1+X_3-2X_1}\right),$$
$$g^{(6)} = g\left(\frac{X_3}{X_1}, 1 - \frac{X_1^2}{X_3}, \frac{X_3}{X_1}, 1\right),$$
$$g^{(7)} = g\left(1 - 2X_3, 0, 1 - 2X_3, \frac{1}{4X_3}\right),$$
$$g^{(7^*)} = g\left(2X_3, 1 - \frac{1}{4X_3}, 2X_3, 1\right).$$

Example 31. Since the straight line $X_2 - X_3 = \frac{1}{12}$ is touched to the projection of $\mathbf{\Omega}$ in $X_2 \times X_3$, then

$$\max_{g(x)-\text{ d.f.}} \int_0^1 \big(x - g(x)\big) g(x) \mathrm{d}x = \frac{1}{12}$$

and it is attained in $g(x) = \frac{x}{2}$.

Example 32. The points[16]

(X_1, X_2, X_3) for $g(x) = c_\alpha(x)$ and (X_1, X_2, X_3) for $g(x) = h_\alpha(x)$, $\alpha \in [0,1]$, lie on the contour of $\mathbf{\Omega}$ and for such (X_1, X_2, X_3) the d.f. $g(x)$ is given uniquely. This implies

$$\int_0^1 x^2 \mathrm{d}g(x) = \left(\int_0^1 x \mathrm{d}g(x)\right)^2 \iff \exists_{\alpha\in[0,1]} g(x) = c_\alpha(x),$$

$$\int_0^1 x^2 \mathrm{d}g(x) = \int_0^1 x \mathrm{d}g(x) \iff \exists_{\alpha\in[0,1]} g(x) = h_\alpha(x),$$

$$\left(1 - \int_0^1 x \mathrm{d}g(x)\right)\left(\int_0^1 x \mathrm{d}g(x)\right) = \frac{1}{2}\int_0^1 \int_0^1 |x-y| \mathrm{d}g(x)\mathrm{d}g(y) \iff \exists_{\alpha\in[0,1]} g(x) = h_\alpha(x)$$

and we also have

$$G(x_n) \subset \{c_\alpha(x); \alpha \in [0,1]\} \iff \lim_{N\to\infty} \left(\left(\frac{1}{N}\sum_{n=1}^N x_n\right)^2 - \frac{1}{N}\sum_{n=1}^N x_n^2 \right) = 0,$$

$$G(x_n) \subset \{h_\alpha(x); \alpha \in [0,1]\} \iff \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N (x_n - x_n^2) = 0,$$

$$G(x_n) \subset \{h_\alpha(x); \alpha \in [0,1]\} \iff \lim_{N\to\infty} \left(\frac{1}{N}\sum_{n=1}^N x_n - \left(\frac{1}{N}\sum_{n=1}^N x_n\right)^2 - \frac{1}{N^2}\sum_{m,n=1}^N |x_m - x_n| \right) = 0.$$

3.5. Mapping x_n to $f(x_n)$

Let $f : [0,1] \to [0,1]$ be a function such that, for all $x \in [0,1]$, $f^{-1}([0,x))$ can be expressed as a sum of finitely many pairwise disjoint subintervals $I_i(x)$ of $[0,1]$ with endpoints $\alpha_i(x) \le \beta_i(x)$. For any d.f. $g(x)$ we put

$$g_f(x) = \sum_i g\big(\beta_i(x)\big) - g\big(\alpha_i(x)\big) = \int_{f^{-1}\left([0,x)\right)} 1.\mathrm{d}g(x).$$

[16] $h_\alpha(x) = \alpha$ for $x \in (0,1)$.

The mapping $g \to g_f$ can be used for the studying $G(x_n)$ by the following statement:

Theorem 44. *Let* $x_n \bmod 1$ *be a sequence having* $g(x)$ *as a d.f. associated with the sequence of indices* $N_1, N_2, \dots$ *Suppose that any term* $x_n \bmod 1$ *is repeated only finitely many times. Then the sequence* $f(\{x_n\})$ *has the d.f.s* $g_f(x)$ *for the same* $N_1, N_2, \dots,$ *and vice-versa any distribution function of* $f(\{x_n\})$ *has this form.*

For example, Theorem 44 can be used to study the sequence $\xi(3/2)^n \bmod 1$, $n = 1, 2, \dots$ Consider

$$f(x) = 2x \bmod 1, \text{ and } h(x) = 3x \bmod 1.$$

In this case, for every $x \in [0, 1]$, we have

$$g_f(x) = g\big(f_1^{-1}(x)\big) + g\big(f_2^{-1}(x)\big) - g(1/2),$$

$$g_h(x) = g\big(h_1^{-1}(x)\big) + g\big(h_2^{-1}(x)\big) + g\big(h_3^{-1}(x)\big) - g(1/3) - g(2/3),$$

with inverse functions

$$f_1^{-1}(x) = x/2, \quad f_2^{-1}(x) = (x+1)/2,$$

and

$$h_1^{-1}(x) = x/3, \quad h_2^{-1}(x) = (x+1)/3, \quad h_3^{-1}(x) = (x+2)/3.$$

Pjateckiĭ-Šapiro [31], by means of the ergodic theory, proved that a necessary and sufficient condition that the sequence $\xi q^n \bmod 1$ with integer$q > 1$ has a distribution function $g(x)$ is that $g_\varphi(x) = g(x)$ for all $x \in [0, 1]$, where $\varphi(x) = qx \bmod 1$. For $\xi(3/2)^n \bmod 1$ we have the following similar necessity.

Theorem 45. *Any distribution function* $g(x)$ *of* $\xi(3/2)^n \bmod 1$ *satisfies the functional equation* $g_f(x) = g_h(x)$ *for all* $x \in [0, 1]$.

The above theorem yields to the following sets of uniqueness for distribution functions of $\xi(3/2)^n \bmod 1$.

Theorem 46. *Let* g_1, g_2 *be any two distribution functions satisfying* $g_{i_f}(x) = g_{i_h}(x)$ *for* $i = 1, 2$ *and* $x \in [0, 1]$. *Denote*

$$I_1 = [0, 1/3], \quad I_2 = [1/3, 2/3], \quad I_3 = [2/3, 1].$$

If $g_1(x) = g_2(x)$ *for* $x \in I_i \cup I_j$, $1 \le i \ne j \le 3$, *then* $g_1(x) = g_2(x)$ *for all* $x \in [0, 1]$.

Next we have an integral formula for testing $g_f = g_h$. Denote

$$F(x, y) = |\{2x\} - \{3y\}| + |\{2y\} - \{3x\}| - |\{2x\} - \{2y\}| - |\{3x\} - \{3y\}|.$$

Theorem 47. *The continuous distribution function g satisfies $g_f = g_h$ on $[0,1]$ if and only if $\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y) = 0$.*

The following theorem (see O. Strauch [40]) can be used for generating solutions of $g_f = g_h$.

Theorem 48. *Let g_1, g_2 be two absolutely continuous distribution functions satisfying $g_{1_h}(x) = g_{2_f}(x)$ for $x \in [0,1]$. Then the absolutely continuous distribution function $g(x)$ satisfies $g_f(x) = g_1(x)$ and $g_h(x) = g_2(x)$ for $x \in [0,1]$ if and only if $g(x)$ has the form*

$$g(x) = \begin{cases} \Psi(x), & \text{for } x \in [0,1/6], \\ \Psi(1/6) + \Phi(x-1/6), & \text{for } x \in [1/6,2/6], \\ \Psi(1/6) + \Phi(1/6) + g_1(1/3) - \Psi(x-2/6) & \\ +\Phi(x-2/6) - g_1(2x-1/3) + g_2(3x-1), & \text{for } x \in [2/6,3/6], \\ 2\Phi(1/6) + g_1(1/3) - g_1(2/3) + g_2(1/2) & \\ -\Psi(x-3/6) + g_1(2x-1), & \text{for } x \in [3/6,4/6], \\ -\Psi(1/6) + 2\Phi(1/6) + g_1(1/3) - g_1(2/3) + g_2(1/2) & \\ -\Phi(x-4/6) + g_1(2x-1), & \text{for } x \in [4/6,5/6], \\ -\Psi(1/6) + \Phi(1/6) + g_1(1/3) + \Psi(x-5/6) & \\ -\Phi(x-5/6) - g_1(2x-5/3) + g_2(3x-2), & \text{for } x \in [5/6,1], \end{cases}$$

where

$$\Psi(x) = \int_0^x \psi(t)\mathrm{d}t, \quad \Phi(x) = \int_0^x \phi(t)\mathrm{d}t, \ \text{for } x \in [0,1/6],$$

and $\psi(t)$, $\phi(t)$ are Lebesgue integrable functions on $[0,1/6]$ satisfying

(i) $0 \le \psi(t) \le 2g_1'(2t)$,

(ii) $0 \le \phi(t) \le 2g_1'(2t+1/3)$,

(iii) $2g_1'(2t) - 3g_2'(3t+1/2) \le \psi(t) - \phi(t) \le -2g_1'(2t+1/3) + 3g_2'(3t)$,

for almost all $t \in [0,1/6]$.

Example 33. The functions $c_0(x)$, $c_1(x)$, a x solve $g_f(x) = g_h(x)$ for all $x \in [0,1]$. Putting $g_1(x) = g_2(x) = x$ in Theorem 48, the following solution

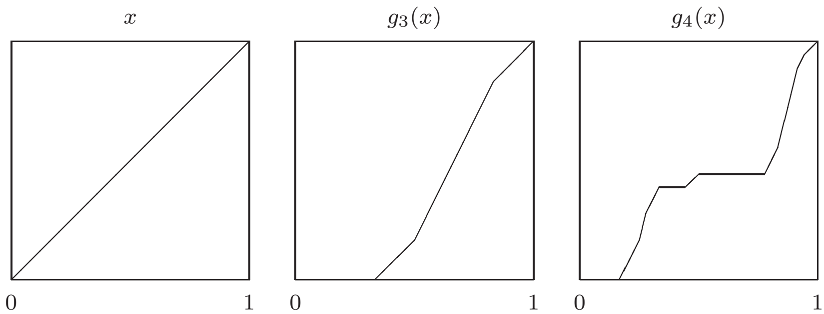
$g_3(x)$ of $g_f = g_h$ can be found:

$$g_3(x) = \begin{cases} 0 & \text{for } x \in [0, 2/6], \\ x - 1/3 & \text{for } x \in [2/6, 3/6], \\ 2x - 5/6 & \text{for } x \in [3/6, 5/6], \\ x & \text{for } x \in [5/6, 1]. \end{cases}$$

Taking $g_1(x) = g_2(x) = g_3(x)$, this $g_3(x)$ can be used as a starting point in Theorem 48 and we find

$$g_4(x) = \begin{cases} 0 & \text{for } x \in [0, 1/6], \\ 2x - 1/3 & \text{for } x \in [1/6, 3/12], \\ 4x - 5/6 & \text{for } x \in [3/12, 5/18], \\ 2x - 5/18 & \text{for } x \in [5/18, 2/6], \\ 7/18 & \text{for } x \in [2/6, 8/18], \\ x - 1/18 & \text{for } x \in [8/18, 3/6], \\ 8/18 & \text{for } x \in [3/6, 7/9], \\ 2x - 20/18 & \text{for } x \in [7/9, 5/6], \\ 4x - 50/18 & \text{for } x \in [5/6, 11/12], \\ 2x - 17/18 & \text{for } x \in [11/12, 17/18], \\ x & \text{for } x \in [17/18, 1], \end{cases}$$

see the following pictures



The study of d.f.s in $G(\{\xi(3/2)^n\})$ is also motivated by K. Mahler's (1968) conjecture: There is no $0 \neq \xi$ such that $0 \leq \{\xi(3/2)^n\} < 1/2$ for $n = 0, 1, 2, \dots$ Mahler's conjecture follows the conjecture: Let $g(x) \in G(\{\xi(3/2)^n\})$ and $I \subset [0, 1]$. If $g(x) = \mathit{constant}$ for all $x \in I$, then the length $|I| < 1/2$.

REFERENCES

[1] AMBROSIO, L.—GIGLI, N.: *A User's Guide to Optimal Transport. Modelling and Optimisation of Flows an Networks*, Lecture Notes in Mathematics. Vol. 2062, Springer-Verlag, Berlin, Heidelberg, 2013 (MR3050280).

[2] BALÁŽ, V.—MIŠÍK, L.—STRAUCH, O.—TÓTH, J. T.: *Distribution functions of ratio sequences, III*, Publ. Math. Debrecen **4770** (2013), 1-19.

[3] BALÁŽ, V.—NAGASAKA, K.—STRAUCH, O.: *Benford's law and distribution functions of sequences in* $(0,1)$, Mathematical Notes **88** (2010), no. 4, 485–501.

[4] BARONE, H .G.: *Limit points of sequences and their transforms by methods of summability*, Duke Math. J. **5** (1939), 740–752 (Zbl. 0022.21902).

[5] BENFORD, F.: *The law of anomalous numbers*, Proc. Amer. Phil. Soc. **78** (1938), 551–572 (Zbl 18, 265; JFM 64.0555.03).

[6] BEREND, D.—BOSHERNITZAN, M. D.—KOLESNIK, G.: *Distribution modulo* 1 *of some oscillating sequences, II*, Israel J. Math. **92** (1995), 125–147.

[7] COQUET, J.—LIARDET P.: *A metric study involving independent sequences*, J. Analyse Math. **49** (1987), 15–53 (MR 89e:11043).

[8] DIACONIS, P.: *The distribution of leading digits and uniform distribution mod 1*, The Annals of Probability **5** (1977), no. 1, 72–81.

[9] DRMOTA, M.— TICHY, R. F.: *Sequences, Discrepancies and Applications*, Lecture Notes in Mathematics Vol. 1651, Springer-Verlag, Berlin, Heidelberg, 1997.

[10] FAST, H.: *Sur la convergence statistique*, Colloq. Math. **2** (1951), 241–244.

[11] FIALOVÁ, J.—STRAUCH, O.: *On two-dimensional sequences composd by one--dimensional uniformly distributed seqiuences*, Uniform Distribution Theory 6 (2011), no. 1, 101–125.

[12] FIALOVÁ, J.—MIŠÍK, L.— O. STRAUCH, O.: *An asymptotic distribution function of the three-dimensional shifted van der Corput sequence*, Applied Mathematics **5** (2014), 2334–2359, http://dx.doi.org/10.4236/am.2014.515227

[13] FILIP F.—MIŠÍK, L.—TÓTH J. T.: *On distribution functions of certain block sequences*, Uniform Distribution Theory **2** (2007), no. 1, 115–126.

[14] FRIDY, J. A.: *Statistical limit points*, Proc. Amer. Math. Soc. **118** (1993), 1187–1192.

[15] GIULIANO ANTONINI, R.—STRAUCH, O.: *On weighted distribution functions of sequences*, Uniform distribution Theory **3** (2008), no. 1, 1–18.

[16] GRABNER, P. J.—TICHY, R. F.: *Remarks on statistical independence of sequences*, Math. Slovaca **44** (1994), 91–94 (MR 95k:11098).

[17] HLAWKA, E.: *The Theory of Uniform Distribution*, A B Academic Publishers, Berkhamsted, 1984 (the translation of the original German edition Hlawka 1979) (MR 85f:11056).

[18] HOFER, M.—IACÓ, M. R.: *Optimal bounds for integrals with respect to copulas and applications*, J. Optim. Theory Appl. **161** (2014), no. 3, 999–1011 (DOI 10.1007/s10957-013-0454-x).

[19] KOKSMA, J. F.: *Asymptotische verdeling van getallen modulo 1. I, II, II,* Mathematica (Leiden) **1** (1933), 245–248; **2** (1933), 1–6, 107–114 (Zbl 0007.33901).

[20] KOKSMA, J. F.: *Diophantische Approximationen*. In: Ergebnisse der Mathematik und Ihrer Grenzgebiete, Vol. IV, Springer, Berlin, 1936 (Zbl 12, 39602).

[21] KOSTYRKO, P.—MAČAJ, M.—ŠALÁT, T.—STRAUCH, O.: *On statistical limit points*, Proc. Amer. Math. Soc. **129** (2001), no. 9, 2647–2654, (MR 2002b:40003).

[22] KUIPERS, L.—NIEDERREITER, H.: *Uniform Distribution of Sequences*, John Wiley & Sons, New York, 1974; Reprint: Dover Publications, Inc. Mineola, New York, 2006.

[23] LUCA, F.—STĂNICĂ, P.: *On the first digits of the Fibonacci numbers and their Euler function*, Uniform Distribution Theory **9** (2014), no. 1, 21–25.

[24] MYERSON, G.: *A sampler of recent developments in the distribution of sequences*, Number theory with an emphasis on the Markoff spectrum (Provo, UT, 1991), Lecture Notes in Pure and App. Math. **147**, Marcel Dekker, New York, 1993, 163–190.

[25] NIEDERREITER, H.: *Random Number Generators and Quasi–Monte Carlo Methods.* In: CBMS-NSF Regional Conference Series in Applied Mathematics **63**, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1992.

[26] NELSEN, R. B.: *An Introduction to Copulas. Properties and Applications*, Lecture Notes in Statistics **139** Springer-Verlag, New York, 1999.

[27] OHKUBO, Y.: *On sequences involving primes* Uniform Distribution Theory, **6** (2011), no. 2, 221–238.

[28] OHKUBO, Y.—STRAUCH, O.: *Distribution of leading digits of numbers*, Uniform Distribution Theory (accepted).

[29] PAVLOV, A.I.: *On distribution of fractional parts and Benford's law* Izv. Aka. Nauk SSSR Ser. Mat. **45** (1981), no. 4, 760–774. (Russian)

[30] PILLICHSHAMMER, F.—STEINERBERGER, S.: *Average distance between consecutive points of uniformly distributed sequences*, Uniform Distribution Theory **4** (2009), no. 1, 51–67.

[31] PJATECKIĬ-ŠAPIRO, I. I.: *On the laws of distribution of the fractional parts of an exponential function*, Izv. Akad. Nauk SSSR, Ser. Mat. **15** (1951), 47–52 (Russian) (MR **13**, 213d).

[32] PÓLYA, G.—SZEGŐ, G.: *Aufgaben und Lehrsätze aus der Analysis* Vol.1,2; 3rd corr. ed. Grundlehren d. math. Wiss., Band 19, Springer Verlag, Berlin, Göttingen, Heidelberg, New York, 1964.

[33] RAUZY, G.: *Propriétés statistiques de suites arithmétiques*, *Le Mathmaticien* Vol **15**, Collection SUP, Presses Universitaires de France, Paris, 1976, 133 pp. (MR **53**#13152).

[34] SCHOENBERG, I. J.: *The integrability of certain functions and related summability methods*, Amer. Math. Monthly **66** (1959), 361–375.

[35] SKLAR, M.: *Fonctions de répartition à n dimensions et leurs marges*, Publ. Inst. Statis. Univ. Paris **8** (1959), 229–231 (MR **23**# A2899).

[36] STRAUCH, O.: *Uniformly maldistributed sequences in a strict sense*, Monatsh. Math. **120** (1995), 153–164 (MR 96g:11095).

[37] STRAUCH, O.: *On set of distribution functions of a sequence.* In: *Proc. Conf. Analytic and Elementary Number Theory*, in Honor of E. Hlawka's 80th Birthday, Vienna, July 18-20, 1996, Universität Wien and Universität für Bodenkultur (W. G. Nowak and J. Schoißengeier, eds.) Vienna, 1997, 214–229 (Zbl 886.11044).

[38] STRAUCH, O.: *L^2 discrepancy*, Math. Slovaca **44** (1994), 601–632 (MR 96c:11085).

[39] STRAUCH, O.: *A new moment problem of distribution functions in the unit interval*, Math. Slovaca **44** (1994), no. 2, 171–211.

[40] STRAUCH, O.: *On distribution functions of* $\xi(3/2)^n$ mod 1, Acta Arith. **LXXXI** (1997), no. 1, 25–35 (MR 98c:11075).

[41] STRAUCH, O.: *Moment problem of the type* $\int_0^1 \int_0^1 F(x,y)\mathrm{d}g(x)\mathrm{d}g(y) = 0$. In: Proceedings of the International Conference on Algebraic Number Theory and Diophantine Analysis held in Graz, August 30 to September 5, 1998 (F. Halter-Koch, R.F. Tichy eds.), Walter de Gruyter, Berlin, New York, 2000, 423–443 (MR 2001d:11079).

[42] STRAUCH, O.: *Unsolved Problems*, in: Unsolved Problems Section on the homepage of Uniform Distribution Theory, `www.boku.ac.at/MATH/udt/unsolvedproblems.pdf`; Tatra Mt. Math. Publ. 56 (2013), 109–229 (DOI: 10.2478/tmmp-2013-0029).

[43] STRAUCH, O.—BLAŽEKOVÁ, O.: *Distribution of the sequence* $p_n/n \bmod 1$, Uniform Distribution Theory **1** (2006), 45–63.

[44] STRAUCH, O.—PORUBSKÝ, Š.: *Distribution of Sequences: A Sampler*, Peter Lang, Frankfurt am Main, 2005; Electronic revised version December 11, 2013, `https//math.boku.ac.at/udt`

[45] STRAUCH, O.—TÓTH, J. T.: *Distribution functions of ratio sequences*, Publ. Math. Debrecen **58** (2001), no. 4, 751–778.

[46] VAN DER CORPUT, J. G.: *Verteilungsfunktionen I–VIII*, Proc. Akad. Amsterdam **38** (1935), 813–821, 1058–1066, **39** (1936), 10–19, 19–26, 149–153, 339–344, 489–494, 579–590.

[47] WINTNER, A.: *On the cyclical distribution of the logarithms of the prime numbers*, Quart. J. Math. Oxford **6** (1935), no. 1, 65–68 (Zbl 11, 149).

[48] UCKELMANN, L.: *Optimal couplings between one-dimensional distributions*, Distribution with given Marginals and Moment Pproblems, (V. Beneš and J. Štěpán eds.) (Prague, 1996) 275–281. Kluwer Acad. Publ., Dordrecht, 1997, MR1614681 (99a:60012).

[49] TICHY, R. F.—THONHAUSER, S.—STRAUCH, O.—IACÓ, M. R.—BALÁŽ, V.: *Extremes of* $\int_0^1 \int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y)$ (submitted).

[50] WINKLER, R.: *On the distribution behaviour of sequences*, Math. Nachr. **186** (1997), 303–312.

Received February 2, 2015
Accepted November 2, 2015

Oto Strauch
Mathematical Institute
Slovak Academy of Sciences
Štefánikova 49
814 73 Bratislava
SLOVAK REPUBLIC
E-mail: oto.strauch@mat.savba.sk