Uniform Distribution Theory **10** (2015), no.2, 77–91

uniform distribution theory

# STABILITY OF BALANCING SEQUENCE MODULO $p$

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*Dedicated to Professor Harald Niederreiter on the occasion of his 70th birthday*

ABSTRACT. Stability is an important aspect of a number sequence. It is known that Fibonacci sequence is stable modulo 2 and 5. The objective of the paper is to study the stability of the balancing sequence modulo primes.

*Communicated by András Sárközy*

## 1. Introduction

As introduced by Behera and Panda [1], balancing numbers $x$ and balancers $r$ are solutions of the Diophantine equation

$$1 + 2 + 3 + \cdots + (x - 1) = (x + 1) + (x + 2) + \cdots + (x + r). \qquad (1.1)$$

As a consequence of (1.1), if $x$ is a balancing number then $x^2 = \frac{(x+r)(x+r+1)}{2}$ is a triangular number or equivalently, $8x^2 + 1$ is a perfect square and $\sqrt{8x^2 + 1}$ is called a Lucas-balancing number [8]. Writing $8x^2 + 1 = y^2$, we are lead to the Pell's equation $y^2 - 8x^2 = 1$ satisfied by the Lucas-balancing and balancing numbers. The $n^{th}$ balancing and Lucas-balancing number are denoted by $B_n$ and $C_n$ respectively. The balancing numbers satisfy the recurrence relation $B_0 = 0, B_1 = 1$ and $B_{n+1} = 6B_n - B_{n-1}$ for $n \geq 2$. The sequence of balancing numbers modulo $m$ is periodic and the period modulo $m$ is denoted by $\pi(m)$ [9]. By definition, $\pi(m)$ is the smallest natural number to satisfy $B_{\pi(m)} \equiv 0, B_{\pi(m)+1} \equiv 1 \pmod{m})$. The computation of $\pi(m)$ depends on the factorization of $m$, but for arbitrary primes $p$, there is no exact formula for $\pi(p)$, though certain divisibility relations for $\pi(p)$ are known [9]. The rank of apparition or simply rank of balancing sequence modulo $m$ is the least positive

2010 Mathematics Subject Classification: 11A06, 11A07.
Keywords: balancing number, uniform distribution modulo m, stability, prime.

integer $r$ such that $B_r \equiv 0 \pmod{m}$ and let it be denoted by $\alpha(m)$. Thus, $\alpha(m)$ is the index of the first non-zero balancing number $B_n$ which is divisible by $m$. Niven [7] introduced the notion of uniform distribution of a sequence of integer as follows. A sequence of integers $\mathcal{A} = \{a_n : n = 0, 1, \ldots\}$ is called uniformly distributed modulo $m \geq 2$ if

$$\lim_{N\to\infty} \frac{1}{N} \#\{n < N : a_n \equiv b \pmod{m}\} = \frac{1}{m}$$

for any $b \in \{0, 1, \ldots, m-1\}$.

For a fixed modulus $m$ and feasible residue $r$, we denote the number of occurrences of $r$ in a period of the balancing sequence by $\nu_B(m, r)$. This function is the frequency distribution function of balancing sequence modulo $m$. In the early 1970's, interest in the distribution functions of binary recurrence sequences centered on the characterization of those sequences that have constant frequency distribution function, i.e., sequences that are uniformly distributed. Denoting by

$$\Omega_B(m) := \{\nu_B(m, r) : r \in \{0, 1, \ldots, m-1\}\} \setminus \{0\}, \tag{1.2}$$

the set of all frequencies of the feasible residues modulo $m$ in a period, the balancing sequence is uniformly distributed whenever $\#\{\Omega_B(m)\} = 1$. Stability of the balancing sequence comes into picture when $\#\{\Omega_B(m)\}$ is not constant and this generalize the concept of uniform distribution and also the notion of $f$-uniform distribution modulo prime powers [11]. The concrete and precise definition of stability was due to Carlip and Jacobson [3]. We prefer to state this definition for the balancing sequence, though it can be stated for any arbitrary sequence.

**Definition 1.1.** The balancing sequence is said to be stable modulo a prime $p$ if there is a positive integer $N$ such that $\Omega_B(p^k) = \Omega_B(p^N)$ for all $k \geq N$.

For better understanding of the above concept, the following examples will be helpful.

**Example 1.2.** Consider the balancing sequence modulo $3, 9, 27$. It is easy to see that

$$\nu_B(3, 0) = 2,\ \nu_B(3, 1) = 1 = \nu_B(3, 2),$$

$$\nu_B(9, 0) = \nu_B(9, 3) = \nu_B(9, 6) = 2,\ \nu_B(9, 1) = \nu_B(9, 8) = 3,$$

$$\nu_B(27, 1) = \nu_B(27, 26) = 6, \nu_B(27, 8) = \nu_B(27, 19) = 3,$$

$$\nu_B(27, i) = 2 \text{ for } i \equiv 0, \pm 3, \pm 6, \pm 9, \pm 12 \pmod{27}.$$

Therefore

$$\Omega_B(3) = \{1, 2\},\ \ \Omega_B(3^2) = \{2, 3\},\ \ \Omega_B(3^3) = \{2, 3, 6\}. \tag{1.3}$$

From (1.3), we observe that the elements in the set $\Omega_B(3^k)$ increases as $k$ increases.

**Example 1.3.**

$$\nu_B(5,0) = 2 = \nu_B(5,1) = \nu_B(5,4),$$

$$\nu_B(25,i) = 2 \text{ for } i \equiv 0, \pm1, \pm4, \pm5, \pm6, \pm10, \pm14, \pm16 \pmod{25}.$$

Hence

$$\Omega_B(5) = \{2\} = \Omega_B(5^2). \tag{1.4}$$

Equation (1.3) is an indication that the balancing sequence may not be stable modulo 3, while (1.4) shows the possible stability of the sequence modulo 5.

Bundschuh [2] studied the stability of Lucas sequence modulo 2 and 5 and found that the sequence is not stable for these two primes. Somer and Carlip [10] demonstrated several classes of binary recurrences which are not $p$-stable and established sufficient criteria for such recurrences to be $p$-stable. We will show that the balancing sequence is stable for two particular classes of primes. In this paper, we will completely describe the function $\nu_B(p^k,\cdot)$. We will show that $\nu_B(2^k,\cdot) = \{1\}$ and hence $\Omega_B(2^k) = \{1\}, \nu_B(p^k,\cdot) = \{1\}$ when $p \equiv -1 \pmod 8$ and $\nu_B(p^k,\cdot) = \{2\}$ when $p \equiv -3 \pmod 8$. These results would confirm the stability of the balancing sequence modulo $p$ when $p \equiv -1$, $-3 \pmod 8$. Finally we have shown that balancing sequence is not stable modulo primes $p \equiv 3 \pmod 8$. However, for some primes $p \equiv 1 \pmod 8$ the balancing sequence is stable.

## 2. Preliminaries

In this section, we present some results which will be needed in the sequel. Throughout the remaining part of this paper, $p$ represents an odd prime. For any non-zero integer $a$, $\mathrm{ord}_p a = m$ if $p^m \mid a$ but $p^{m+1} \nmid a$. Important properties of $\mathrm{ord}_p$ are $\mathrm{ord}_p(ab) = \mathrm{ord}_p(a) + \mathrm{ord}_p(b), \mathrm{ord}_p(a+b) \geq \min(\mathrm{ord}_p(a), \mathrm{ord}_p(b))$ [5]. Thus $a \equiv b \pmod{p^k}$ is equivalent to $\mathrm{ord}_p(a-b) \geq k$.

We also need the following results relating to periods of balancing numbers.

**Theorem 2.1.** ([9]) *For any natural number $n > 1$, $\pi(n) = n$ if and only if $n = 2^k$ for any $k \in \mathbb{N}$.*

**Theorem 2.2.** ([9]) *For any odd prime $p$, $\pi(p)$ divides $p-1$ if $p \equiv \pm1 \pmod 8$ and $\pi(p)$ divides $p+1$ if $p \equiv \pm3 \pmod 8$. Thus, if $p$ is an odd prime, then $\pi(p)$ divides $p^2-1$.*

**Lemma 2.3.** *If the integers $m$ and $n$ are of the same parity, then*

$$B_m - B_n = 2B_{\frac{m-n}{2}}C_{\frac{m+n}{2}}, \tag{2.1}$$

$$C_m - C_n = 16B_{\frac{m-n}{2}}B_{\frac{m+n}{2}}. \tag{2.2}$$

Proof. It is well known that $B_{x\pm y} = B_xC_y \pm C_xB_y$ [8]. Thus $B_{x+y} - B_{x-y} = 2B_yC_x$; and taking $x+y=m, x-y=n$, we get $B_m - B_n = 2B_{\frac{m-n}{2}}C_{\frac{m+n}{2}}$. Similarly by virtue of the formula $C_{x\pm y} = C_xC_y \pm 8B_xB_y$, [8] $C_m - C_n = 16B_{\frac{m-n}{2}}B_{\frac{m+n}{2}}$ follows. □

**Lemma 2.4.** *If $B_n \equiv 0 \pmod{p}$, then $B_{2n} \equiv 0 \pmod{p}$ and $B_{2n+1} \equiv 1 \pmod{p}$.*

Proof. Since $B_n \equiv 0 \pmod{p}, B_{2n} = 2B_nC_n \equiv 0 \pmod{p}$ and $B_{2n+1} = B_n \cdot B_{n+2} - B_{n-1} \cdot B_{n+1} \equiv -(B_n^2 - 1) \equiv 1 \pmod{p}$ since $B_{n+1}B_{n-1} = B_n^2 - 1$, (see [8]). □

**Lemma 2.5.** *For any prime $p$, $\pi(p) = \alpha(p)$ or $\pi(p) = 2\alpha(p)$.*

Proof. Since $B_{\alpha(p)} \equiv 0 \pmod{p}$ by Lemma 2.4 we get $B_{2\alpha(p)} \equiv 0 \pmod{p}$ and $B_{2\alpha(p)+1} \equiv 1 \pmod{p}$. Thus, $\pi(p)|2\alpha(p)$ and hence

$$\pi(p) = \alpha(p) \quad \text{or} \quad \pi(p) = 2\alpha(p). \qquad \square$$

**Lemma 2.6.** *If $\alpha(p^2) \neq \alpha(p)$ then $\alpha(p^l) = p^{l-1}\alpha(p)$. Further, if $k$ is the largest integer such that $\alpha(p^k) = \alpha(p)$ and $l > k$, then $\alpha(p^l) = p^{l-k}\alpha(p)$*

Proof. The congruence $B_{\alpha(p^l)} \equiv 0 \pmod{p^l}$ gives $B_{\alpha(p^l)} = kp^l$ for some natural number $k$. By De-Moivre's Theorem for balancing numbers ([8])

$$C_{p\alpha(p^l)} + \sqrt{8}B_{p\alpha(p^l)} = \left(C_{\alpha(p^l)} + \sqrt{8}B_{\alpha(p^l)}\right)^p.$$

Hence, for $l > 1$

$$\begin{aligned} B_{p\alpha(p^l)} &= k\binom{p}{1}C^{p-1}_{\alpha(p^l)}p^l + 8k^3\binom{p}{3}C^{p-3}_{\alpha(p^l)}p^{3l} + \cdots + 8^{\frac{p-1}{2}}k^pp^{pl} \\ &\equiv 0 \pmod{p^{l+1}}. \end{aligned} \tag{2.3}$$

It is clear from above equation that $\alpha(p^{l+1})$ divides $p\alpha(p^l)$. Since $\alpha(p^l)$ divides $\alpha(p^{l+1})$, it follows that $\alpha(p^{l+1}) = \alpha(p^l)$ or $\alpha(p^{l+1}) = p\alpha(p^l)$. For $l = 1$, the conclusion is that $\alpha(p^2) = \alpha(p)$ or $\alpha(p^2) = p\alpha(p)$; so if $\alpha(p^2) \neq \alpha(p)$, then $\alpha(p^2) = p\alpha(p)$. Continuing in this process we will arrive at $\alpha(p^l) = p^{l-1}\alpha(p)$. Further, if $k$ is the largest integer such that $\alpha(p^k) = \alpha(p)$, then $\alpha(p^{k+t}) = p\alpha(p^{k+t-1}) = \cdots = p^t\alpha(p^k) = p^t\alpha(p)$ for each natural number $t$. □

The following lemma, which relates the order of $B_n$ with order of $n$, will play a crucial role.

**Lemma 2.7.** *If $n \in \mathbb{N}$ and $p$ is any arbitrary prime then $\alpha(p) \mid n$ if and only if $p \mid B_n$. Furthermore, if $\alpha(p) \mid n$, then*

$$\mathrm{ord}_p B_n \geq 1 + \mathrm{ord}_p n. \tag{2.4}$$

Proof. The proof of first part follows directly from the definition of $\alpha(p)$. Let $\mathrm{ord}_p B_n = t$ and $\mathrm{ord}_p n = s$. Then $n = kp^s$ where $\gcd(k,p) = 1$, and $\alpha(p) \mid n$ implies $\alpha(p) \mid kp^s$. Since $\alpha(p) \mid p^2 - 1$, by Theorem 2.2, $\gcd(\alpha(p), p) = 1$. Thus $\alpha(p)|k$ which gives $k = a\alpha(p)$ for some integer $a$. Putting the value of $k$ in $n$, we get $n = a\alpha(p)p^s$. By definition, $p^t \| B_n$ if and only if $\alpha(p^t) \| n$. If $\alpha(p) \neq \alpha(p^2)$ then by Lemma 2.6, $p^{t-1}\alpha(p) \| n$. Putting the value of $n$, we get $p^{t-1}\alpha(p) \| a\alpha(p)p^s$ which implies $p^{t-1} \| a \cdot p^s$. Since $\gcd(k,p) = 1$ and $k = a\alpha(p)$, we have $\gcd(a,p) = 1$. Therefore $t - 1 = s$. If $m > 1$ is the largest integer such that $\alpha(p^m) = \alpha(p)$, then $p^{t-m}\alpha(p) \| n$ and proceeding as above we will reach at $t - m = s$. Hence combining both the cases we conclude that $t \geq 1 + s$. □

Similar results also hold for the Lucas balancing sequence. The proof of the following lemma is similar to that of Lemma 2.7 and is omitted.

**Lemma 2.8.** *Let $n \in \mathbb{N}$ and for any prime $p \equiv 3 \pmod 8$ define*

$$\beta(p) = \min\{r : C_r \equiv 0 \pmod p\}.$$

*Then*

$$\beta(p) \mid n \Leftrightarrow p \mid C_n \quad \text{and} \quad \beta(p) \mid n \Rightarrow \mathrm{ord}_p C_n = 1 + \mathrm{ord}_p n\,. \tag{2.5}$$

## 3. Stability of balancing sequence modulo 2

The Fibonacci sequence is stable modulo 2 and 5 [4]. In this section, we will show that the balancing sequence is also stable modulo 2.

**Theorem 3.1.** *$\nu_B(2^k, b) = 1$ for every residue $b$ modulo $2^k$ and for any $k \in \mathbb{N}$.*

Proof. By virtue of Theorem 2.1, for $n > 1$, $\pi(n) = n$ if and only if $n = 2^k$ for any $k \in \mathbb{N}$. Using this result, we will show that each residue $b \in \{0, 1, \cdots, 2^k - 1\}$ occurs only once in a period modulo $2^k$. Since $B_n$ is even or odd according as $n$ is even or odd, it follows that the least residue of $B_n, 0 \leq n \leq 2^k - 1$ modulo $2^k$ is also even or odd according as $n$ is even or odd. To complete the proof, we have to show that no two least residue of $B_n, 0 \leq n \leq 2^k - 1$ are congruent modulo $2^k$. Since $B_{2m+1}$ and $B_{2n}$ are incongruent modulo $2^k$, it is sufficient to show that

$$B_{2m+1} \not\equiv B_{2n+1} \pmod{2^k} \text{ for } 0 < 2m + 1 < 2n + 1 < 2^k, \tag{3.1}$$

and

$$B_{2i} \not\equiv B_{2j} \pmod{2^k} \text{ for } 0 \leq 2i < 2j \leq 2^k. \tag{3.2}$$

Since $\pi(2^k) = 2^k$, it follows that $2^k \mid B_n$ if and only if $2^k \mid n$. Let us assume the contrary of (3.1), i. e.,

$$B_{2m+1} \equiv B_{2n+1} \pmod{2^k} \quad \text{for } 0 < 2m+1 < 2n+1 < 2^k,$$

hence $2^k$ divides $B_{2n+1} - B_{2m+1}$. Using (2.1), $2^k \mid 2B_{n-m}C_{m+n+1}$ implies $2^{k-1} \mid B_{n-m}$ as $\gcd(2, C_x) = 1$ for any natural number $x$. It easily follows from induction on $k$ that if $2^{k-1} \mid B_{n-m}$, then $2^{k-1} \mid n-m$ which is a contradiction since $n - m < 2^{k-1}$. Thus (3.1) holds. In a similar fashion, (3.2) can be proved. □

From equation (1.2), we have $\Omega_B(2^k) = \{1\}$. The following corollary, which ascertains the stability of balancing sequence modulo 2, is a consequence of the above theorem.

**Corollary 3.2.** *The balancing sequence is stable modulo* 2.

## 4. Stability of balancing sequence modulo primes $p \equiv -1, -3 \pmod 8$

In this section, we will establish the stability of the balancing sequences modulo primes $p$ congruent to $-1, -3$ modulo 8.

The following lemmas dealing with some periodicity results will prove their usefulness while proving main results of this section.

**Lemma 4.1.** *If $A = \{a_1, a_2, \dots, a_r\}$ are distinct residues modulo $p$, then $A + mp$ for $m = 0, 1, \dots, p^{k-1} - 1$ are also distinct residues modulo $p^k$.*

Proof. Suppose that for some integers $1 \le l, m \le r$ and $0 \le i, j \le p^{k-1} - 1$,

$$a_l + ip \equiv a_m + jp \pmod{p^k}. \tag{4.1}$$

This implies that $p^k \mid (i-j)p - (a_l - a_m)$ and hence, $p$ must divide $a_l - a_m$. In other words, $a_l \equiv a_m \pmod p$, which is a contradiction since $a_i$'s are distinct residues modulo $p$ for $1 \le i \le r$. □

**Lemma 4.2.** *If $p \equiv -1 \pmod 8$, then $\pi(p) \mid \frac{p-1}{2}$. Furthermore, $\pi(p)$ is odd.*

Proof. If $p \equiv -1 \pmod 8$, then $p = 8x - 1$ for some integer $x$. By Theorem 2.2, $\pi(p) | p - 1 = 8x - 2$. Thus,

$$B_{8x-2} \equiv 0 \pmod p, \quad B_{8x-3} \equiv -B_1, \quad B_{8x-4} \equiv -B_2 \pmod p$$

and so on. In other words, $B_r + B_{8x-2-r} \equiv 0 \pmod p$ for $r = 1, 2, \dots, 4x - 2$. In particular, $B_{4x-2} + B_{4x} \equiv 0 \pmod p$ which implies that $6B_{4x-1} \equiv 0 \pmod p$.

Hence $B_{4x-1} = B_{\frac{p-1}{2}} \equiv 0 \pmod p$ as $\gcd(6,p) = 1$. We claim that $B_{\frac{p+1}{2}} \equiv 1 \pmod p$. Observe that

$$\mathrm{ord}_p(B_{\frac{p+1}{2}} - B_1) = \mathrm{ord}_p(2 \cdot B_{\frac{p-1}{2}} \cdot C_{\frac{p+3}{2}}) \geq 0 + 1 + \mathrm{ord}_p\left(\frac{p-1}{2}\right) + \mathrm{ord}_p(C_{\frac{p+3}{2}}) \geq 1$$

which shows that $B_{\frac{p+1}{2}} \equiv 1 \pmod p$ and combining with $B_{\frac{p-1}{2}} \equiv 0 \pmod p$, we conclude that $\pi(p) \mid \frac{p-1}{2} = 4x - 1$, which implies that $\pi(p)$ is odd. □

**Lemma 4.3.** *If $p \equiv -1 \pmod 8$, then $\pi(p) = \alpha(p)$.*

Proof. By Lemma 4.2, $\pi(p)$ is odd. Thus, $B_1 + B_{\pi(p)-1} \equiv 0, B_2 + B_{\pi(p)-2} \equiv 0 \pmod p$, and in general $B_r + B_{\pi(p)-r} \equiv 0 \pmod p$ for $r = 1, 2, \ldots, \frac{\pi(p)-1}{2}$. By virtue of Theorem 2.8 of [8], $B_n | B_{kn}$ and hence if $B_n \equiv 0 \pmod m$, then $B_{kn} \equiv 0 \pmod m$. Since $B_n \equiv 0 \pmod p$ implies that $\alpha(p) | n$, it follows that $\alpha(p) | \pi(p)$ and thus $\alpha(p) \leq \pi(p)$. If $\alpha(p) < \pi(p)$, then $B_{\alpha(p)} \equiv 0 \pmod p$ implies $B_{\pi(p)-\alpha(p)} \equiv 0 \pmod p$. Hence, at least two $B_n$'s out of $B_1, B_2, \ldots, B_{\pi(p)-1}$ are congruent to zero. If $t$ be the index of the second one, then $t = 2\alpha(p)$, which shows that $2\alpha(p) < \pi(p)$ – a contradiction to $\pi(p) | 2\alpha(p)$. Therefore $\pi(p) = \alpha(p)$. □

**Lemma 4.4.** *If $p \equiv -3 \pmod 8$, then $B_{\frac{p+1}{2}} \equiv 0 \pmod p$.*

Proof. If $p = 8x - 3$, then by Theorem 2.2,

$$B_p \equiv -1 \pmod p, \quad B_{p+1} \equiv 0 \pmod p. \tag{4.2}$$

Using the recurrence relation $B_{n+1} = 6B_n - B_{n-1}$ and (4.2) it is easy to see that

$$B_{\frac{p-1}{2}} + B_{\frac{p+3}{2}} \equiv 0 \pmod p. \tag{4.3}$$

Hence $6B_{\frac{p+1}{2}} \equiv 0 \pmod p$ and $(6,p) = 1$ implies $B_{\frac{p+1}{2}} \equiv 0 \pmod p$. □

**Lemma 4.5.** *If $p \equiv -3 \pmod 8$, then for every $x$ such that $0 \leq x \leq \frac{\pi(p)}{2}$,*

$$B_x \equiv B_{\frac{\pi(p)}{2}-x} \pmod p \quad \textit{and} \quad B_x \equiv -B_{\frac{\pi(p)}{2}+x} \pmod p.$$

*Furthermore, $\pi(p) = 2\alpha(p)$.*

Proof. If $B_x \equiv B_y \pmod p$ for some $0 \leq y < x \leq \frac{\pi(p)}{2}$, then $C_x \equiv \pm C_y \pmod p$ since $C_n = \sqrt{8B_n^2 + 1}$. Therefore, $B_{x \pm y} = B_x C_y \pm B_y C_x \equiv 0 \pmod p$. By Lemma 4.4, for $0 \leq x \leq \frac{\pi(p)}{2}, B_x = 0$ if and only if $x = 0, \frac{\pi(p)}{2}$. Hence $x \pm y = 0$ or $\frac{\pi(p)}{2}$. We observe that $x - y = 0$ gives trivial solution $x = y$ which is not possible since $x > y$. Again, $x + y = 0$ gives $x = -y$ which is also not possible since both $x$ and $y$ are non-negative and $x > y$. $x - y = \frac{\pi(p)}{2}$ gives $x = \frac{\pi(p)}{2} + y$. This is absurd since $0 \leq x \leq \frac{\pi(p)}{2}$. Thus, we are left with one option $x+y=\frac{\pi(p)}{2}$ or equivalently, $x=\frac{\pi(p)}{2}-y$. Hence, $B_y \equiv B_{\frac{\pi(p)}{2}-y} \pmod p$. From 4.3 we have $B_{\frac{\pi(p)}{2}-k} \equiv -B_{\frac{\pi(p)}{2}+k} \pmod p$. Thus, $B_x \equiv -B_{\frac{\pi(p)}{2}+x} \pmod p$ and the proof is complete. □

The following two theorems, assuring the stability of the balancing sequence modulo $p$ for $p \equiv -1,-3 \pmod 8$, are important results of this section.

**Theorem 4.6.** *If $p \equiv -3 \pmod 8$ and $k \in \mathbb{N}$, then $\nu_B(p^k,b)=2$ for each feasible residue $b$ modulo $p^k$. Hence the balancing sequence is stable modulo $p$ for $p \equiv -3 \pmod 8$.*

Proof. Firstly, we will show that the number of occurrences of each feasible residue modulo $p$ in a period is 2. In Lemma 4.5, we have shown that each feasible residue of the balancing numbers $B_x$, $x \in \{0,1,\dots,\frac{\pi(p)}{2}\}$ modulo $p$ occurs twice. Since $B_x \equiv -B_{\frac{\pi}{2}+x} \pmod p$, it follows that $\#\{x : B_x \equiv b \pmod p), 0 \le x \le \pi(p)-1\} = 2$, i.e., $\nu_B(p,b)=2$ holds for each feasible residue $b$ modulo $p$. Using Lemma (4.1) we get $\nu_B(p^k,b)=2$ for each feasible residue $b$ modulo $p^k$. □

From equation (1.2), $\Omega_B(p^k)=\{2\}$.

**Theorem 4.7.** *If $p \equiv -1 \pmod 8$ and $k \in \mathbb{N}$, then $\nu_B(p^k,b)=1$ for each feasible residue $b$ modulo $p^k$. Hence the balancing sequence is stable modulo $p$ for $p \equiv -1 \pmod 8$.*

Proof. Since $p \equiv -1 \pmod 8$, by Lemma 4.3 $\pi(p)=\alpha(p)$. Therefore, each feasible residue $b$ occurs only once such that $B_r \equiv b \pmod p$ for $0 \le r < \pi(p)$; otherwise $\alpha(p) < \pi(p)$. Now Lemma 4.1 confirms that $\nu_B(p^k,b)=1$ for each feasible residue $b$ of the balancing sequence modulo $p^k$. □

Therefore from equation (1.2), $\Omega_B(p^k)=\{1\}$.

## 5. Stability of balancing sequence modulo primes $p \equiv 1,3 \pmod 8$

Modulo 8, there are four classes of primes $p \equiv \pm1,\pm3 \pmod 8$. In the last section, we have proved that the balancing sequence is stable modulo primes $p \equiv -1,-3 \pmod 8$. But unfortunately, the sequence is not stable modulo in general for primes $p \equiv 1,3 \pmod 8$. However, for certain primes of this class, the balancing sequence is indeed stable.

The following lemmas, relating to the structure of the period and behaviour of balancing numbers occurring in a period, will play crucial roles while proving the main results of this section.

**Lemma 5.1.** *If* $p \equiv 3 \pmod 8$, *then* $4 \mid \pi(p)$.

P r o o f. Firstly, we will prove

$$B_{\frac{p-1}{2}} \equiv 1 \pmod p. \tag{5.1}$$

Observe that

$$\operatorname{ord}_p\Big(B_{\frac{p-1}{2}} - B_1\Big) = \operatorname{ord}_p\Big(2 \cdot B_{\frac{p-3}{4}} C_{\frac{p+1}{4}}\Big) = \operatorname{ord}_p\Big(B_{\frac{p-3}{4}}\Big) + \operatorname{ord}_p\Big(C_{\frac{p+1}{4}}\Big). \tag{5.2}$$

In view of Theorem 2.2, $\pi(p) \mid p+1$. Using this result in (5.2) we get

$$\operatorname{ord}_p\Big(B_{\frac{p-1}{2}} - B_1\Big) \geq 0 + 1 + \operatorname{ord}_p\Big(\frac{p+1}{4}\Big) \geq 1. \tag{5.3}$$

Proceeding as in Lemma 4.4 and using (4.2), it is easy to see that

$$B_{\frac{p+1}{2}} \equiv 0 \pmod p.$$

Using the recurrence $B_n = 6B_{n-1} - B_{n-2}$ and (5.1), we get $B_{\frac{p+3}{2}} \equiv 1 \pmod p$, which confirms that $\pi(p) \nmid (p+1)/2 = 4x+2$; but $\pi(p) \mid p+1 = 8x+4$ which implies that $4 \mid \pi(p)$. □

**Lemma 5.2.** *If* $p \equiv 3 \pmod 8$ *and* $x \in \mathbb{N}$, *then* $B_{p^x \frac{\pi(p)}{4}} \not\equiv B_{3 \cdot p^x \frac{\pi(p)}{4}} \pmod p$.

P r o o f. Firstly, we will show that for $x \in \mathbb{N}$

$$B_{p^x \frac{\pi(p)}{4}} \equiv (-1)^x B_{\frac{\pi(p)}{4}} \pmod p. \tag{5.4}$$

If $x$ is even, then

$$\operatorname{ord}_p\Big(B_{p^x \frac{\pi(p)}{4}} - B_{\frac{\pi(p)}{4}}\Big) = \operatorname{ord}_p\Big(2B_{\frac{\pi(p)}{4}\frac{p^x-1}{2}} \cdot C_{\frac{\pi(p)}{4}\frac{p^x+1}{2}}\Big). \tag{5.5}$$

and $\alpha(p) \mid \frac{\pi(p)}{4} \cdot \frac{p^x-1}{2}$. Therefore, using Lemma 2.7, we get

$$\begin{aligned}&\operatorname{ord}_p\Big(B_{p^x \frac{\pi(p)}{4}} - B_{\frac{\pi(p)}{4}}\Big)\\ &\geq \operatorname{ord}_p 2 + 1 + \operatorname{ord}_p\Big(\frac{\pi(p)}{4}\Big(\frac{p^x-1}{2}\Big)\Big) + \operatorname{ord}_p\Big(C_{\frac{\pi(p)}{4}(\frac{p^x+1}{2})}\Big)\\ &\geq 1.\end{aligned}$$

Now, let $x$ be odd. Since $B_{-n} = -B_n$, it can be easily proved that

$$B_{p^x \frac{\pi(p)}{4}} \equiv -B_{\frac{\pi(p)}{4}} \pmod p. \tag{5.6}$$

A similar argument as above will lead to

$$B_{3 \cdot p^x \frac{\pi(p)}{4}} \equiv (-1)^x B_{3 \cdot \frac{\pi(p)}{4}} \pmod p. \tag{5.7}$$

To complete the proof, it remains to show that $B_{\frac{\pi(p)}{4}} \not\equiv B_{3\cdot\frac{\pi(p)}{4}} \pmod p$. It is obvious since $B_{\frac{\pi(p)}{4}} \equiv -B_{3\cdot\frac{\pi(p)}{4}} \pmod p$. □

**Lemma 5.3.** *If $p \equiv 3 \pmod 8$ and $k \in \mathbb{N}$, then there are two distinct feasible residues of $B_n$ with $0 \le n < \pi(p)p^{k-1}$ occurring at least $p^{[k/2]}$ times in a period modulo $p^k$.*

Proof. Let $n$ be a non-negative integer, $j \in \{0,1\}$ and $\pi\big(p^{[(k-1)/2]+1}\big)|n$. We claim that

$$B_{n+\frac{\pi(p)}{4}(1+2j)p^{[(k-1)/2]}} \equiv B_{\frac{\pi(p)}{4}(1+2j)p^{[(k-1)/2]}} \pmod{p^k}. \tag{5.8}$$

If $p \equiv 3 \pmod 8$, then by virtue of Lemma 5.1, $4 \mid \pi(p)$ and $\pi(p) \mid n$ implies $4 \mid n$. Thus,

$$\begin{aligned}
&\mathrm{ord}_p\Big(B_{n+\frac{\pi(p)}{4}(1+2j)p^{[(k-1)/2]}} - B_{\frac{\pi(p)}{4}(1+2j)p^{[(k-1)/2]}}\Big)\\
&= \mathrm{ord}_p\Big(2B_{\frac{n}{2}}C_{\frac{n}{2}+\frac{\pi(p)}{4}(1+2j)p^{[(k-1)/2]}}\Big)\\
&= \mathrm{ord}_p(2) + \mathrm{ord}_p\Big(B_{\frac{n}{2}}\Big) + \mathrm{ord}_p\Big(C_{\frac{n}{2}+\frac{\pi(p)}{4}(1+2j)p^{[(k-1)/2]}}\Big)\\
&\ge 0 + 1 + \mathrm{ord}_p\Big(\frac{n}{2}\Big) + 1 + \mathrm{ord}_p\Big(\frac{n}{2} + \frac{\pi(p)}{4}(1+2j)p^{[(k-1)/2]}\Big)\\
&\ge 2(1+[(k-1)/2]) > 2(1+(k-1)/2-1) > k-1,
\end{aligned}$$

which proves (5.8). Since $\pi\big(p^{[(k-1)/2]+1}\big) \mid n$ by assumption, $\pi(p)p^{[(k-1)/2]}|n$. Therefore,

$$n = \pi(p)p^{[(k-1)/2]}i \quad \text{with some} \quad i < p^{[k/2]}. \tag{5.9}$$

Thus,

$$\begin{aligned}
0 \le n + \frac{\pi(p)}{4}(1+2j)p^{[(k-1)/2]} &= \Big(\pi(p)\cdot i + \frac{\pi(p)}{4}(1+2j)\Big)p^{[(k-1)/2]}\\
&\le \Big(\pi(p)p^{[k/2]} - \pi(p) + \frac{3\pi(p)}{4}\Big)p^{[(k-1)/2]}\\
&= \pi(p)\cdot p^{k-1} - \frac{\pi(p)}{4}p^{[(k-1)/2]} < \pi(p)\cdot p^{k-1}.
\end{aligned}$$

Now, it remains to show that $B_{\frac{\pi(p)}{4}p^{[(k-1)/2]}}$ and $B_{\frac{3\pi(p)}{4}p^{[(k-1)/2]}}$ are incongruent modulo $p^k$; it is enough to show that they are incongruent modulo $p$, which is established in Lemma 5.2. □

**Lemma 5.4.** *If $p \equiv 3 \pmod 8$ and $k \in \mathbb{N}$, then for every integer $x$ with $1 \le x \le [(k-1)/2]$ there exist $(p-1)p^{k-2x-1}$ distinct feasible residue $b$ of $B_n$ with $0 \le n < \pi(p)p^{k-1}$ occurring at least $2p^x$ times in a period modulo $p^k$.*

P r o o f. Let $n$ be a non-negative integer and $p^{x-1}\|n$. Then

$$\begin{aligned}&\operatorname{ord}_p(B_{n+\pi(p)p^{k-x-1}}-B_n)\\&=\operatorname{ord}_p\Big(2B_{\frac{\pi(p)}{2}p^{k-x-1}}C_{n+\frac{\pi(p)}{2}p^{k-x-1}}\Big)\\&\geq 1+\operatorname{ord}_p\Big(\frac{\pi(p)}{2}p^{k-x-1}\Big)+1+\operatorname{ord}_p\Big(n+\frac{\pi(p)}{2}p^{k-x-1}\Big)\\&=1+(k-x-1)+1+x-1=k.\end{aligned}$$

(Here we are using the inequality $\operatorname{ord}_p(a+b)\geq\min\big(\operatorname{ord}_p(a),\operatorname{ord}_p(b)\big)$ as $0\leq x-1\leq\frac{k-3}{2}$ and $\frac{k-1}{2}\leq k-x-1\leq k-2$.) Therefore

$$B_{n+\pi(p)p^{k-x-1}}\equiv B_n \pmod{p^k} \tag{5.10}$$

for all $n$ such that $p^{x-1}\|n$. We need to count the number of integers $n$ with $0\leq n<\pi(p)p^{k-1}$ and $p^{x-1}\|n$ for which a given $b$ occurs as a residue of $B_n$ modulo $p^k$. This is equivalent to counting the number of integers $n$ with $0\leq n<\pi(p)\cdot p^{k-x-1}$ and $p^{x-1}\|n$ for which a given $b$ occurs as a residue of $B_n$ modulo $p^k$ and then to multiply this number by $p^x$. Hence we have to check the distribution of the $2(p-1)p^{k-2x-1}$ numbers

$$B_j \pmod{p^k}: 1\leq j<\pi(p)p^{k-x-1},\ \ 2\nmid j,\ \ \frac{\pi(p)}{4}\mid j,\ \ p^{x-1}\|j. \tag{5.11}$$

We claim that half of them, i. e., $(p-1)p^{k-2x-1}$ of the $B_n$'s are pairwise incongruent modulo $p^k$ and other half are congruent to the first half in some way; more specifically,

$$B_{\frac{\pi(p)}{2}p^{k-x-1}-\frac{\pi(p)}{4}j}\equiv B_{\frac{\pi(p)}{4}j} \pmod{p^k},\qquad \text{for } 1\leq j<p^{k-x-1},\ \ 2\nmid j,\ \ p^{x-1}\|j \tag{5.12}$$

and

$$B_{\pi(p)p^{k-x-1}-\frac{\pi(p)}{4}j}\equiv B_{\frac{\pi(p)}{2}p^{k-x-1}+\frac{\pi(p)}{4}j} \pmod{p^k},\qquad \text{for } 1\leq j<p^{k-x-1},\ \ 2\nmid j,\ \ p^{x-1}\|j. \tag{5.13}$$

Observe that

$$\begin{aligned}&\operatorname{ord}_p\Big(B_{\frac{\pi(p)}{2}p^{k-x-1}-\frac{\pi(p)}{4}j}-B_{\frac{\pi(p)}{4}j}\Big)\\&=\operatorname{ord}_p\Big(2B_{\frac{\pi(p)}{4}(p^{k-x-1}-j)}C_{\frac{\pi(p)}{4}p^{k-x-1}}\Big)\\&=\operatorname{ord}_p(2)+\operatorname{ord}_p\Big(B_{\frac{\pi(p)}{4}(p^{k-x-1}-j)}\Big)+\operatorname{ord}_p\Big(C_{\frac{\pi(p)}{4}p^{k-x-1}}\Big)\\&\geq 1+\operatorname{ord}_p\Big(\frac{\pi(p)}{4}(p^{k-x-1}-j)\Big)+1+\operatorname{ord}_p\Big(\frac{\pi(p)}{4}p^{k-x-1}\Big)\\&=2+x-1+k-x-1=k\end{aligned}$$

and

$$\begin{aligned}&\mathrm{ord}_p\Big(B_{\pi(p)\cdot p^{k-x-1}-\frac{\pi(p)}{4}j}-B_{\frac{\pi(p)}{2}p^{k-x-1}+\frac{\pi(p)}{4}j}\Big)\\&=\mathrm{ord}_p\Big(2B_{\frac{\pi(p)}{4}p^{k-x-1}-\frac{\pi(p)}{4}j}C_{\frac{3\pi(p)}{4}p^{k-x-1}}\Big)\\&\geq 1+\mathrm{ord}_p\Big(\frac{\pi(p)}{4}(p^{k-x-1}-j)\Big)+1+\mathrm{ord}_p\Big(\frac{3\pi(p)}{4}p^{k-x-1}\Big)\\&\geq 2+x-1+k-x-1=k.\end{aligned}$$

Hence, it only remains to show that

$$B_{\pi(p)\cdot p^{k-x-1}-\frac{\pi(p)}{4}j}\not\equiv B_{\frac{\pi(p)}{4}j}\ (\mathrm{mod}\ p^k),\ 1\leq j<p^{k-x-1},\ \ 2\nmid j,\ \ p^{x-1}\|j.\quad(5.14)$$

Since

$$\mathrm{ord}_p\Big(B_{\pi(p)p^{k-x-1}-\frac{\pi(p)}{4}j}-B_{-\frac{\pi(p)}{4}j}\Big)=\mathrm{ord}_p\Big(2B_{\frac{\pi(p)}{2}p^{k-x-1}}C_{\frac{\pi(p)}{2}p^{k-x-1}-\frac{\pi(p)}{4}j}\Big)\geq k,$$

we have

$$B_{\pi(p)p^{k-x-1}-\frac{\pi(p)}{4}j}\equiv B_{-\frac{\pi(p)}{4}j}\equiv -B_{\frac{\pi(p)}{4}j}\pmod{p^k}$$
$$\text{for }\ 1\leq j<p^{k-x-1},\ \ 2\nmid j,\ \ p^{x-1}\|j,$$

from which (5.14) follows and the proof is complete. □

**Lemma 5.5.** *If $p\equiv 3\,(\mathrm{mod}\ 8)$, then there exist $\frac{\pi(p)}{2}-1$ distinct feasible residue $b$ of $B_n$ with $0\leq n<\pi(p)$ occurring exactly twice.*

Proof. In view of Lemma 5.3 with $k=1$, there exists two distinct feasible residue $b$ of $B_n$ modulo $p$ for $n=0,1,\dots,\pi(p)-1$ occurring only once. Hence we need to check the distribution of the remaining $\pi(p)-2$ residues, namely,

$$B_r\pmod p,\ \text{for } 0\leq r<\pi(p),\quad r\notin\Big\{\frac{\pi(p)}{4},\frac{3\pi(p)}{4}\Big\}.$$

We claim that half of them, i. e, $\frac{\pi(p)}{2}-1$ of $B_n$'s are pairwise incongruent modulo $p$ and the other half are congruent to the first half in some manner, i. e.,

$$B_i\equiv B_{\frac{\pi(p)}{2}-i}\ (\mathrm{mod}\ p)\ \text{ and }\ B_{\frac{\pi(p)}{2}+i}\equiv B_{\pi(p)-i}\ (\mathrm{mod}\ p)\ \text{ for }\ 0\leq i<\frac{\pi(p)}{4}.\quad(5.15)$$

But

$$\begin{aligned}\mathrm{ord}_p(B_{\frac{\pi(p)}{2}-i}-B_i)&=\mathrm{ord}_p\Big(2B_{\frac{\pi(p)}{4}-i}C_{\frac{\pi(p)}{4}}\Big)\\&\geq\mathrm{ord}_p(2)+1+\mathrm{ord}_p\Big(B_{\frac{\pi(p)}{4}-i}\Big)+\mathrm{ord}_p\Big(C_{\frac{\pi(p)}{4}}\Big)\geq 1\end{aligned}$$

shows that $B_i\equiv B_{\frac{\pi(p)}{2}-i}\,(\mathrm{mod}\ p)$. Similarly it can be easily seen that $B_{\frac{\pi(p)}{2}+i}\equiv B_{\pi(p)-i}\,(\mathrm{mod}\ p)$. To complete the proof, it remains to show

$$B_i\not\equiv B_{\frac{\pi(p)}{2}+i}\pmod p.$$

Since, $B_i \equiv -B_{\frac{\pi(p)}{2}+i} \pmod p$ and the case $B_i \equiv 0 \equiv B_{\frac{\pi(p)}{2}+i} \pmod p$ contradicts the definition of period, it follows that $B_i \not\equiv B_{\frac{\pi(p)}{2}+i} \pmod p$. □

**Remark 5.6.** If $p \equiv 3 \pmod 8$ and $k \in \mathbb{N}$, then there exist $p^{k-1}(\frac{\pi(p)}{2}-1)$ distinct feasible residues $b$ of $B_n$ modulo $p^k$ with $0 \leq n < \pi(p)p^{k-1}$ occurring exactly twice.

We are now in a position to prove an important theorem of this section.

**Theorem 5.7.** *If $p \equiv 3 \pmod 8$ and for $i \in \{0,1\}$, then*

$$\nu_B(p^k, b) = \begin{cases} p^{[k/2]} & \text{if } b \equiv B_{\left(\frac{\pi(p)}{4}+i\frac{\pi(p)}{2}\right)p^{[(k-1)/2]}} \pmod{p^k}, \\ 2.p^x & \text{if } b \equiv B_{\frac{\pi(p)}{4}j+i\frac{\pi(p)}{2}p^{k-x-1}}, \text{ and} \\ & \qquad p^{x-1} \| j,\ 2 \nmid j,\ 1 \leq j < p^{k-x-1} \\ & \qquad \text{for } x \in \{1,2,\ldots,[(k-1)/2]\}, \\ 2 & \text{otherwise.} \end{cases}$$

Proof. In view of Lemma 5.3, 5.4 and Remark 5.6 we have the following results:

$$\nu_B(p^k, b) \geq p^{[k/2]},\ \nu_B(p^k, b) \geq 2 \cdot p^x \text{ and } \nu_B(p^k, b) = p^{k-1}\left(\frac{\pi(p)}{2}-1\right). \quad (5.16)$$

Hence,

$$\begin{aligned} \sum_{b=0}^{p^k-1} \nu_B(p^k) &\geq 2p^{[k/2]} + \sum_{x=1}^{[(k-1)/2]} (p-1)p^{k-2x-1}(2p^x) + p^{k-1}\left(\frac{\pi(p)}{2}-1\right) \\ &= \pi(p)p^{k-1}. \end{aligned} \quad (5.17)$$

In view of [9, Theorem 3.5], the left hand side of (5.17) equals $\pi(p) \cdot p^{k-1}$. Thus, equality holds in (5.16) for every feasible residue $b$ modulo $p^k$. □

**Remark 5.8.** In the above theorem, the second case occurs if $k \geq 3$ and in this case, there are exactly $(p-1)p^{k-2x-1}$ distinct feasible residues $b$ occur modulo $p^k$. In the third case, for each $k \in \mathbb{N}$, exactly $p^{k-1}\left(\frac{\pi(p)}{2}-1\right)$ distinct feasible residues $b$ modulo $p^k$ occur.

Using (1.2), we get $\Omega_B(p^k) = \{2, 2p, 2p^2, \ldots, 2p^{[(k-1)/2]}, p^{[k/2]}\}$. Thus, the following corollary is a direct consequence of Theorem 5.7.

**Corollary 5.9.** *If $p \equiv 3 \pmod 8$, then balancing sequence is not stable modulo $p$.*

We next search for primes $p \equiv 1 \pmod 8$ for which the balancing sequence is stable. In the following theorem, we limit the search for such primes in the class of associated Pell numbers.

**Theorem 5.10.** *If the prime $p \equiv 1 \pmod 8$ is an odd indexed associated Pell number, then balancing sequence is stable modulo $p$.*

Proof. Since $p$ is an odd indexed associated Pell number, $\pi(p)$ is odd [9, Theorem 4.3]. Using the arguments given in the proof of Lemma 4.3, it is easy to see that $\pi(p) = \alpha(p)$. Now, proceeding like the proof of Theorem 4.7, one can easily verify that the balancing sequences is stable modulo such a prime. □

For some members in the class of primes $p \equiv 1 \pmod 8$, $\pi(p)$ is a multiple of 4. For example 17 is one such prime with $\pi(17) = 8$. The following theorem confirms that the balancing sequence is not stable modulo any such prime.

**Theorem 5.11.** *Let $p$ be a prime such that $p \equiv 1 \pmod 8$ and $4 \mid \pi(p)$. If $i \in \{0, 1\}$, then*

$$\nu_B(p^k, b) = \begin{cases} p^{[k/2]} & \text{if } b \equiv B_{\left(\frac{\pi(p)}{4} + i\cdot\frac{\pi(p)}{2}\right)p^{[(k-1)/2]}} \pmod{p^k}, \\ 2p^x & \text{if } b \equiv B_{\frac{\pi(p)}{4}j + i\cdot\frac{\pi(p)}{2}p^{k-x-1}}, \text{ and} \\ & \qquad p^{x-1} \| j,\ 2 \nmid j,\ 1 \le j < p^{k-x-1} \\ & \qquad \text{for } x \in \{1, 2, \dots, [(k-1)/2]\} \\ 2 & \text{otherwise} \end{cases}$$

Proof. The proof is similar to the proof of Theorem 5.7, hence it is omitted. □

There are some primes $p \equiv 1 \pmod 8$ for which $4 \nmid \pi(p)$. Such type of primes are excluded from Theorem 5.11. For example, if $p = 137, \pi(p) = 34$ and one can check that the balancing sequence is stable modulo 137. It is an open problem to identify some more subclass of primes for which the balancing sequence is stable.

**Acknowledgements.** It is a pleasure to thank the unanimous referee for his valuable comments and suggestions which improved the presentation of the paper to a great extent.

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Received January 12, 2015
Accepted April 24, 2015

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