ON THE DISTRIBUTION
OF POWERS OF REAL NUMBERS MODULO 1

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Dedicated to Professor Harald Niederreiter on the occasion of his 70th birthday

ABSTRACT. Given a strictly increasing sequence of positive real numbers tending to infinity $(q_n)_{n=1}^\infty$, and an arbitrary sequence of real numbers $(r_n)_{n=1}^\infty$. We study the set of $\alpha \in (1,\infty)$ for which $\lim_{n\to\infty} \|\alpha^{q_n} - r_n\| = 0$. In [3] Dubickas showed that whenever $\lim_{n\to\infty}(q_{n+1} - q_n) = \infty$, there always exists a transcendental α for which $\lim_{n\to\infty} \|\alpha^{q_n} - r_n\| = 0$. Adapting the approach of Bugeaud and Moshchevitin [2], we improve upon this result and show that whenever $\lim_{n\to\infty}(q_{n+1} - q_n) = \infty$, then for any interval $I \subset (1,\infty)$ the set of $\alpha \in I$ satisfying $\lim_{n\to\infty} \|\alpha^{q_n} - r_n\| = 0$ is of Hausdorff dimension 1.

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1. Introduction

It is a well known result of Koksma that for almost every $\alpha \in (1,\infty)$ the sequence $(\{\alpha^n\})_{n=1}^\infty$ is uniformly distributed modulo 1 [7]. Here and throughout almost every is meant with respect to the Lebesgue measure, and $\{\cdot\}$ denotes the fractional part of a real number. It is a long standing problem to determine the set of $\alpha \in (1,\infty)$ for which $\lim_{n\to\infty} \|\alpha^n\| = 0$, where $\|\cdot\|$ denotes the distance to the nearest integer. The only known examples of numbers satisfying this property are Pisot numbers, that is algebraic integers whose Galois conjugates all have modulus strictly less than 1. It was shown independently by Hardy [5] and Pisot [9], that if α is an algebraic number and $\lim_{n\to\infty} \|\alpha^n\| = 0$, then α is a Pisot number. Moreover, Pisot in [8] showed that there are only countably many $\alpha \in (1,\infty)$ satisfying $\lim_{n\to\infty} \|\alpha^n\| = 0$. This naturally leads to the question:

Is there a transcendental $\alpha \in (1,\infty)$ satisfying $\lim_{n\to\infty} \|\alpha^n\| = 0$?

This question is highly non trivial and currently out of our reach.

In this paper, instead of studying the distribution of the sequence $\{\alpha\}, \{\alpha^2\}, \{\alpha^3\}, \ldots$, we consider the distribution of the sequence $\{\alpha\}, \{\alpha^4\}, \{\alpha^9\}, \ldots$, or more generally $\{\alpha^{q_1}\}, \{\alpha^{q_2}\}, \{\alpha^{q_3}\}, \ldots$ where $(q_n)_{n=1}^\infty$ is a strictly increasing sequence of positive real numbers tending to infinity. We emphasise that the q_n are not necessarily natural numbers. We remark that if $\liminf_{n\to\infty}(q_{n+1}-q_n)>0$, then for almost every $\alpha \in (1,\infty)$ the sequence $(\{\alpha^{q_n}\})_{n=1}^\infty$ is uniformly distributed modulo 1. The proof of this statement is a simple adaptation of the proof of Theorem 1.10 in [1]. All of the sequences we will be considering shall satisfy $\liminf_{n\to\infty}(q_{n+1}-q_n) > 0$.

We are interested in the set of $\alpha \in (1,\infty)$ for which $(\{\alpha^{q_n}\})_{n=1}^\infty$ is not uniformly distributed modulo 1. In particular, the set of $\alpha \in (1,\infty)$ for which $\lim_{n\to\infty} \|\alpha^{q_n}\| = 0$, or more generally the set of $\alpha \in (1,\infty)$ which satisfy $\lim_{n\to\infty} \|\alpha^{q_n} - r_n\| = 0$, where $(r_n)_{n=1}^\infty$ is an arbitrary sequence of real numbers. Let

$$E(q_n, r_n) := \Big\{\alpha \in (1,\infty) : \lim_{n\to\infty} \|\alpha^{q_n} - r_n\| = 0\Big\}.$$

In [3], Dubickas showed that whenever $\lim_{n\to\infty}(q_{n+1} - q_n) = \infty$, then it is possible to construct a transcendental α contained in $E(q_n, r_n)$. Note that the α he constructs is always larger than 2. Our main result is the following.

THEOREM 1.1. *Let $(q_n)_{n=1}^\infty$ be a strictly increasing sequence of positive real numbers satisfying $\lim_{n\to\infty}(q_{n+1} - q_n) = \infty$, and let $(r_n)_{n=1}^\infty$ be an arbitrary sequence of real numbers. Then for any interval $I \subset (1,\infty)$ the set of $\alpha \in I$ satisfying $\lim_{n\to\infty} \|\alpha^{q_n} - r_n\| = 0$ is of Hausdorff dimension 1.*

The proof we give of Theorem 1.1 is based upon the approach of Bugeaud and Moshchevitin [2], which in turn is based upon the approach of Vijayaraghavan [10]. They show that for any $\epsilon > 0$ and $(r_n)_{n=1}^\infty$ a sequence of real numbers, there exists a set of Hausdorff dimension 1 for which $\|\alpha^n - r_n\| < \epsilon$ for all $n \geq 1$. The set of $\alpha \in (1,\infty)$ which satisfy $\|\alpha^n - r_n\| < \epsilon$ for all n sufficiently large is studied further in [6].

Given $(\{\alpha^{q_n}\})_{n=1}^\infty$ is uniformly distributed modulo 1 for almost every $\alpha \in (1,\infty)$, Theorem 1.1 is somewhat surprising in that it states that there exists a set, which in some sense is as large as we could hope for, which exhibits completely the opposite behaviour of uniform distribution. Indeed, taking $(r_n)_{n=1}^\infty$ to be the constant sequence $r_n = \kappa$ for some $\kappa \in (0,1)$, then for any interval $I \subset (1,\infty)$ the set of $\alpha \in I$ satisfying $\lim_{n\to\infty}\{\alpha^{q_n}\} = \kappa$ is of Hausdorff dimension 1.

2. Proof of Theorem 1.1

We prove Theorem 1.1 via a Cantor set construction. To help our exposition we briefly recall some of the theory from [4] on this type of construction. Let $E_1 \subset \mathbb{R}$ be an arbitrary closed interval and $E_1 \supset E_2 \supset E_3 \supset \cdots$ be a decreasing sequence of sets, where each E_n is a finite union of disjoint closed intervals, where each element of E_n contains at least two elements of E_{n+1}, and the maximum length of the intervals in E_n tends to 0 as $n \to \infty$. Then the set

$$E = \bigcap_{n=1}^{\infty} E_n \tag{1}$$

is the Cantor set associated to the sequence $(E_n)_{n=1}^{\infty}$. The following proposition appears at Example 4.6 in [4].

PROPOSITION 2.1. *Suppose in the construction of E above each interval in E_{n-1} contains at least m_n intervals of E_n which are separated by gaps of at least γ_n, where $0 < \gamma_{n+1} < \gamma_n$ for each n. Then*

$$\dim_H(E) \geq \liminf_{n \to \infty} \frac{\log m_1 \cdots m_{n-1}}{-\log m_n \gamma_n}.$$

We are now is a position to prove Theorem 1.1.

P r o o f o f T h e o r e m 1.1. We begin by fixing $\lambda \in (1, \infty)$, $\delta > 0$ some small positive constant, and let $(r_n)_{n=1}^{\infty}$ be our sequence of real numbers. Without loss of generality we may assume that $r_n \in [-1/2, 1/2)$ for all $n \in \mathbb{N}$. To prove our result it is sufficient to prove that $[\lambda, \lambda+\delta] \cap E(q_n, r_n)$ is of Hausdorff dimension 1.

(1) Replacing $(q_n)_{n=1}^{\infty}$ with $(\tilde{q}_n)_{n=1}^{\infty}$. Let $\epsilon > 0$ be some small positive constant. We now replace our sequence $(q_n)_{n=1}^{\infty}$ with $(\tilde{q}_n)_{n=1}^{\infty}$, and our sequence $(r_n)_{n=1}^{\infty}$ with $(\tilde{r}_n)_{n=1}^{\infty}$. We will pick our $(\tilde{q}_n)_{n=1}^{\infty}$ and $(\tilde{r}_n)_{n=1}^{\infty}$ in such a way that $E(\tilde{q}_n, \tilde{r}_n) \subset E(q_n, r_n)$. We then use Proposition 2.1 to determine a lower bound for $\dim_H(E(\tilde{q}_n, \tilde{r}_n) \cap [\lambda, \lambda + \delta])$, which in turn provides a lower bound for $\dim_H(E(q_n, r_n) \cap [\lambda, \lambda + \delta])$. The feature of the sequence $(\tilde{q}_n)_{n=1}^{\infty}$ that we will exploit in our proof, is that this new sequence does not grow too fast, yet importantly we still have $\lim_{n \to \infty}(\tilde{q}_{n+1} - \tilde{q}_n) = \infty$. The sequence $(\tilde{q}_n)_{n=1}^{\infty}$ and the rate at which we control the growth of $(\tilde{q}_n)_{n=1}^{\infty}$ shall depend on ϵ. For ease of exposition we drop the dependence of $(\tilde{q}_n)_{n=1}^{\infty}$ on ϵ from our notation.

We begin our construction by asking whether

$$q_{n+1} \leq (1 + \epsilon)q_n \tag{2}$$

is satisfied for all $n \in \mathbb{N}$. If it is, we set $(q_n)_{n=1}^{\infty} = (\tilde{q}_n)_{n=1}^{\infty}$, $(r_n)_{n=1}^{\infty} = (\tilde{r}_n)_{n=1}^{\infty}$ and stop. Suppose our sequence $(q_n)_{n=1}^{\infty}$ does not satisfy (2) for all $n \in \mathbb{N}$. Let $N \in \mathbb{N}$ be the first $n \in \mathbb{N}$ for which (2) fails. We now introduce additional terms in our sequence $(q_n)_{n=1}^{\infty}$, situated between q_N and q_{N+1} at

$$\tilde{q}_N^j := q_N + j\left(\frac{\epsilon q_N}{2}\right) \quad \text{for } j = 1, \ldots, m.$$

Here m is the smallest natural number for which $q_N + m(\frac{\epsilon q_N}{2}) \in [q_{N+1} - \epsilon q_N, q_{N+1}]$. To each $\tilde{q}_N^j$ we associate an arbitrary real number $\tilde{r}_N^j \in [-1/2, 1/2)$, these terms are then placed within the sequence $(r_n)_{n=1}^{\infty}$ between r_N and r_{N+1}. Importantly the elements q_N and q_{N+1} are still placed in the positions corresponding to r_N and r_{N+1}.

The following inequalities are straightforward consequences of our construction

$$\begin{aligned}
\tilde{q}_N^1 &\leq (1+\epsilon)q_n \\
\tilde{q}_N^{j+1} &\leq (1+\epsilon)\tilde{q}_N^j \quad \text{for } j = 1, \ldots, m-1, \\
q_{N+1} &\leq (1+\epsilon)\tilde{q}_N^m.
\end{aligned} \tag{3}$$

In other words, all of the new terms in our sequences satisfy (2). The new terms in our sequence also satisfy

$$\begin{aligned}
\tilde{q}_N^1 - q_N &= \frac{\epsilon q_N}{2} \\
\tilde{q}_N^{j+1} - \tilde{q}_N^j &= \frac{\epsilon q_N}{2} \quad \text{for } j = 1, \ldots, m-1, \\
q_{N+1} - \tilde{q}_N^m &\geq \frac{\epsilon q_N}{2}.
\end{aligned} \tag{4}$$

So if N was large the gaps between successive terms in our sequences would be large. This property is what allows us to ensure $\lim_{n \to \infty}(\tilde{q}_{n+1} - \tilde{q}_n) = \infty$.

We now take our new sequence and ask if it satisfies (2) for all $n \in \mathbb{N}$. If it does, then our construction is complete, and we set $(\tilde{q}_n)_{n=1}^{\infty}$ and $(\tilde{r}_n)_{n=1}^{\infty}$ to be our new sequences. If not, we find the smallest n for which it fails and repeat the above steps. Repeating this process indefinitely if necessary, we construct sequences $(\tilde{q}_n)_{n=1}^{\infty}$ and $(\tilde{r}_n)_{n=1}^{\infty}$ for which

$$\tilde{q}_{n+1} \leq (1+\epsilon)\tilde{q}_n, \tag{5}$$

holds for all $n \in \mathbb{N}$, we retain the property

$$\lim_{n \to \infty}(\tilde{q}_{n+1} - \tilde{q}_n) = \infty, \tag{6}$$

and

$$E(\tilde{q}_n, \tilde{r}_n) \subset E(q_n, r_n). \tag{7}$$

The fact that (6) holds is a consequence of (4). The final property (7) holds because the original terms in our sequence $(q_n)_{n=1}^{\infty}$ keep their corresponding r_n, and at each step in our construction we only ever introduced finitely many terms between a q_n and a q_{n+1}. So if α satisfies $\|\alpha^{\tilde{q}_n} - \tilde{r}_n\| \to 0$ then it also satisfies $\|\alpha^{q_n} - r_n\| \to 0$, i.e., (7) holds.

(2) Construction of our Cantor set. We now construct our Cantor set E. Our set E will be contained in $[\lambda, \lambda + \delta] \cap E(\tilde{q}_n, \tilde{r}_n)$ and we will be able to use Proposition 2.1 to obtain estimates on $\dim_H(E)$. We let

$$\epsilon_n := \frac{1}{2(\tilde{q}_{n+1} - \tilde{q}_n)},$$

by (6) we have $\epsilon_n \to 0$. Let us fix $\eta \in (0, 1)$ some parameter that we will eventually let tend to 1. Let N be sufficiently large that

$$2\epsilon_n \lambda^{\tilde{q}_{n+1} - \tilde{q}_n} = \frac{\lambda^{\tilde{q}_{n+1} - \tilde{q}_n}}{\tilde{q}_{n+1} - \tilde{q}_n} \geq \lceil \lambda^{\eta(\tilde{q}_{n+1} - \tilde{q}_n)} \rceil + 2 \quad \text{for all } n \geq N. \tag{8}$$

We may also assume that this N is sufficiently large that

$$(\lambda + \delta)^{\tilde{q}_N} - \lambda^{\tilde{q}_N} \geq 4 \quad \text{and} \quad \epsilon_n < 1/2 \quad \text{for all } n \geq N.$$

We let
$$a_n := \tilde{r}_n - \epsilon_n \quad \text{and} \quad b_n := \tilde{r}_n + \epsilon_n \quad \text{for all } n \in \mathbb{N}.$$

By our assumptions on $\tilde{r}_n$ and ϵ_n, we may assume that $a_n, b_n \in (-1, 1)$ for all $n \geq N$.

Since $(\lambda + \delta)^{\tilde{q}_N} - \lambda^{\tilde{q}_N} \geq 4$, there exists an integer j_N for which $j_N, j_N + 1, \ldots, j_N + m + 1$ are contained in $[\lambda^{\tilde{q}_N}, (\lambda + \delta)^{\tilde{q}_N}]$, where m is some natural number greater than or equal to 2. We ignore the first and the last of these integers and focus on $j_N + 1, \ldots j_N + m$. To each of these integers $h = j_N + 1, \ldots j_N + m$ we associate the interval

$$I_{N,h} := [(h + a_N)^{1/\tilde{q}_N}, (h + b_N)^{1/\tilde{q}_N}].$$

By our construction each $I_{N,h}$ is contained in $[\lambda, \lambda + \delta]$, and each $\alpha \in I_{N,h}$ satisfies $\|\alpha^{\tilde{q}_N} - \tilde{r}_N\| < \epsilon_N$. Let E_N be the set of all intervals $I_{N,h}$. For each h we have

$$(h + b_N)^{\tilde{q}_{N+1}/\tilde{q}_N} - (h + a_N)^{\tilde{q}_{N+1}/\tilde{q}_N} \geq \Big((h + b_N) - (h + a_N) \Big)(h + a_N)^{\frac{\tilde{q}_{N+1} - \tilde{q}_N}{\tilde{q}_N}}$$

$$\geq 2\epsilon_N \lambda^{\tilde{q}_{N+1} - \tilde{q}_N} \tag{9}$$

$$\geq \lceil \lambda^{\eta(\tilde{q}_{N+1} - \tilde{q}_N)} \rceil + 2.$$

Where the last inequality is by (8). Therefore there exists an integer j_{N+1} such that $j_{N+1}, j_{N+1} + 1, \ldots, j_{N+1} + \lceil \lambda^{\eta(\tilde{q}_{n+1} - \tilde{q}_n)} \rceil + 1$ are all contained in

$$[(h + a_N)^{\tilde{q}_{n+1}/\tilde{q}_n}, (h + b_N)^{\tilde{q}_{n+1}/\tilde{q}_n}].$$

To each $h = j_{N+1} + 1, \ldots, j_{N+1} + \lceil \lambda^{\eta(\tilde{q}_{n+1} - \tilde{q}_n)} \rceil$ we associate the interval

$$I_{N+1,h} = [(h + a_{N+1})^{1/\tilde{q}_{N+1}}, (h + b_{N+1})^{1/\tilde{q}_{N+1}}].$$

Importantly each interval $I_{N+1,h}$ is contained in an element of E_N, and this element contains precisely $m_N := \lceil \lambda^{\eta(\tilde{q}_{N+1} - \tilde{q}_N)} \rceil$ of these intervals. We let E_{N+1} denote the set of $I_{N+1,h}$. Any $\alpha \in I_{N+1,H}$ is contained in $[\lambda, \lambda + \delta]$ and satisfies

$$\|\alpha^{\tilde{q}_N} - \tilde{r}_N\| < \epsilon_N \quad \text{and} \quad \|\alpha^{\tilde{q}_{N+1}} - \tilde{r}_{N+1}\| < \epsilon_{N+1}.$$

We may show that (9) holds with $a_N, b_N, \tilde{q}_N, \tilde{q}_{N+1}$ replaced by $a_{N+1}, b_{N+1}, \tilde{q}_{N+1}, \tilde{q}_{N+2}$, and we may therefore repeat the above steps accordingly. Moreover, we may repeat the procedure described above for every subsequent n. To each $n \geq N$ we let E_n denote the set of interval $I_{n,h}$ produced in our construction. The following properties follow from our construction:

- Let $I_{n,h} \in E_n$, then for each $\alpha \in I_{n,h}$ we have

$$\|\alpha^{\tilde{q}_i} - \tilde{r}_i\| < \epsilon_n \quad \text{for} \ i = N, N+1, \ldots, n.$$

- $E_n \subset E_{n-1} \subset \cdots \subset E_N$.

- $E_n \subset [\lambda, \lambda + \delta]$.

If we let

$$E = \bigcap_{n=N}^{\infty} E_n,$$

it is clear that any $x \in E$ is contained in $[\lambda, \lambda + \delta]$, and satisfies $\|x^{\tilde{q}_n} - \tilde{r}_n\| < \epsilon_n$ for all $n \geq N$, so

$$E \subset [\lambda, \lambda + \delta] \cap E(\tilde{q}_n, \tilde{r}_n).$$

It is a consequence of our construction that each element of E_n contains exactly

$$m_n := \lceil \lambda^{\eta(\tilde{q}_{n+1} - \tilde{q}_n)} \rceil \tag{10}$$

elements of E_{n+1}. It may also be shown that the distance between any two intervals in E_n is always at least

$$\gamma_n := \frac{c}{\tilde{q}_n(\lambda + \delta)^{\tilde{q}_n - 1}}, \tag{11}$$

where c is some positive constant that is independent of n.

Applying Proposition 2.1, combined with (10) and (11), we obtain the following bounds on the Hausdorff dimension of E:

$$\dim_H(E) \geq \liminf_{n\to\infty} \frac{\log m \cdot m_N \cdots m_{n-1}}{-\log m_n \gamma_n}$$

$$\geq \liminf_{n\to\infty} \frac{\log 2 \cdot \lceil \lambda^{\eta(\tilde{q}_{N+1}-\tilde{q}_N)} \rceil \cdots \lceil \lambda^{\eta(\tilde{q}_n-\tilde{q}_{n-1})} \rceil}{-\log \frac{c\lceil \lambda^{\eta(\tilde{q}_{n+1}-\tilde{q}_n)} \rceil}{\tilde{q}_n(\lambda+\delta)^{\tilde{q}_n-1}}}$$

$$\geq \liminf_{n\to\infty} \frac{\eta(\tilde{q}_n - \tilde{q}_N)\log\lambda + \log 2}{-\log \frac{c\lceil \lambda^{\eta(\tilde{q}_{n+1}-\tilde{q}_n)} \rceil}{\tilde{q}_n(\lambda+\delta)^{\tilde{q}_n-1}}}$$

$$\geq \liminf_{n\to\infty} \frac{\eta(\tilde{q}_n - \tilde{q}_N)\log\lambda + \log 2}{-\log\lceil \lambda^{\eta(\tilde{q}_{n+1}-\tilde{q}_n)} \rceil - \log c + (\tilde{q}_n - 1)\log(\lambda+\delta) + \log\tilde{q}_n}$$

$$\geq \liminf_{n\to\infty} \frac{\eta(\tilde{q}_n - \tilde{q}_N)\log\lambda + \log 2}{-\eta(\tilde{q}_{n+1}-\tilde{q}_n)\log\lambda - \log c + (\tilde{q}_n - 1)\log(\lambda+\delta) + \log\tilde{q}_n}$$

$$\geq \frac{\eta\log\lambda}{\eta\epsilon\log\lambda + \log(\lambda+\delta)}.$$

Since η was arbitrary we may let it converge to 1 so

$$\dim_H\big([\lambda, \lambda+\delta] \cap E(\tilde{q}_n, \tilde{r}_n)\big) \geq \frac{\log\lambda}{\epsilon\log\lambda + \log(\lambda+\delta)}.$$

Therefore, by (7)

$$\dim_H\big([\lambda, \lambda+\delta] \cap E(q_n, r_n)\big) \geq \frac{\log\lambda}{\epsilon\log\lambda + \log(\lambda+\delta)},$$

but since ϵ is arbitrary we may conclude that

$$\dim_H\big([\lambda, \lambda+\delta] \cap E(q_n, r_n)\big) \geq \frac{\log\lambda}{\log(\lambda+\delta)}.$$

The argument we have presented also works for any $\delta' \in (0, \delta)$ this implies

$$\dim_H\big([\lambda, \lambda+\delta'] \cap E(q_n, r_n)\big) \geq \log\lambda/\log(\lambda+\delta').$$

Moreover

$$[\lambda, \lambda+\delta'] \cap E(q_n, r_n) \subset [\lambda, \lambda+\delta] \cap E(q_n, r_n),$$

so

$$\dim_H\big([\lambda, \lambda+\delta] \cap E(q_n, r_n)\big) \geq \dim_H\big([\lambda, \lambda+\delta'] \cap E(q_n, r_n)\big) \geq \frac{\log\lambda}{\log(\lambda+\delta')}.$$

Letting δ' tend to zero we deduce that $\dim_H\big([\lambda, \lambda+\delta] \cap E(q_n, r_n)\big) = 1$. $\qquad \square$

We conclude with a few remarks on our proof and the speed at which $\|\alpha^{q_n} - r_n\|$ converges to zero. In our proof of Theorem 1.1 we set

$$\epsilon_n = 1/2(\tilde{q}_{n+1} - \tilde{q}_n).$$

This choice of ϵ_n is somewhat arbitrary, our proof still works with any sequence ϵ_n which tends to zero, as long as for any $\eta \in (0,1)$ we have

$$2\epsilon_n \lambda^{\tilde{q}_{n+1}-\tilde{q}_n} \geq \lceil \lambda^{\eta(\tilde{q}_{n+1}-\tilde{q}_n)} \rceil + 2$$

for all n sufficiently large.

If $(q_n)_{n=1}^\infty$ satisfies $\lim_{n\to\infty} q_{n+1}/q_n = 1$, then it is not necessary to introduce the sequences $(\tilde{q}_n)_{n=1}^\infty$ and $(\tilde{r}_n)_{n=1}^\infty$ in the proof of Theorem 1.1. This means we can say something about the speed of convergence. If ϵ_n decays to zero sufficiently slowly that for any $\eta \in (0,1)$ and $\lambda \in (1,\infty)$, we have

$$2\epsilon_n \lambda^{q_{n+1}-q_n} \geq \lceil \lambda^{\eta(q_{n+1}-q_n)} \rceil + 2,$$

for all n sufficiently large. Then the argument given in the proof of Theorem 1.1 yields a set of Hausdorff dimension 1 within any interval satisfying

$$\|\alpha^{q_n} - r_n\| = O(\epsilon_n).$$

As an example, for any $k \in \mathbb{N}$ there exists a set of Hausdorff dimension 1 within any interval satisfying

$$\|\alpha^{n^2}\| = O(n^{-k}).$$

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