Uniform Distribution Theory **10** (2015), no.1, 1–17

uniform distribution theory

ON THE DISCREPANCY AND EMPIRICAL DISTRIBUTION FUNCTION OF $\{n_k\alpha\}$

István Berkes[1] and Marko Raseta[2]

Dedicated to Professor Harald Niederreiter on the occasion of his 70th birthday

ABSTRACT. By a classical result of Philipp (1975), for any sequence $(n_k)_{k\ge1}$ of positive integers satisfying the Hadamard gap condition, the discrepancy of $(n_kx)_{1\le k\le N}$ mod 1 satisfies the law of the iterated logarithm. For sequences $(n_k)_{k\ge1}$ growing subexponentially this result becomes generally false and the asymptotic behavior of the discrepancy remains unknown. In this paper we show that for randomly sampled subsequences $(n_k)_{k\ge1}$ the discrepancy D_N of $(n_kx)_{1\le k\le N}$ mod 1 and its L^p version $D_N^{(p)}$ not only satisfy a sharp form of the law of the iterated logarithm, but we also describe the precise asymptotic behavior of the empirical process of the sequence $(n_kx)_{1\le k\le N}$, leading to substantially stronger consequences.

Communicated by W.G. Nowak

1. Introduction

An infinite sequence $(x_k)_{k\ge1}$ of real numbers is said to be uniformly distributed mod 1 if for every pair a, b of real numbers with $0 \le a < b \le 1$ we have

$$\lim_{N\to\infty}\frac{\sum_{k=1}^N I_{[a,b)}(x_k)}{N} = b-a.$$

Here $I_{[a,b)}$ denotes the indicator function of the interval $[a, b)$, extended with period 1. It is easily seen that $(x_k)_{k\ge1}$ is uniformly distributed mod 1 iff $D_N(x_k)\to0$ (or equivalently $D_N^*(x_k)\to0$) where $D_N(x_k)$ and $D_N^*(x_k)$ are the the *discrepancy*, resp. *star discrepancy* of the finite sequence $(x_k)_{1\le k\le N}$ defined by

2010 Mathematics Subject Classification: 11K38, 42A61, 46E22, 60F15, 62G30.
Keywords: Discrepancy, empirical distribution, random subsequences, reproducing kernel Hilbert space, law of the iterated logarithm.
[1]Research supported by FWF Grant P24302-N18 and OTKA Grant K 108615.
[2]Research supported by FWF Projekt W1230.

$$D_N(x_k) := \sup_{0\le a<b\le 1} \left| \frac{\sum_{k=1}^N I_{[a,b)}(x_k)}{N} - (b-a) \right|,$$
$$D_N^*(x_k) := \sup_{0<a\le 1} \left| \frac{\sum_{k=1}^N I_{[0,a)}(x_k)}{N} - a \right|,$$

respectively. Clearly, $D_N^* \le D_N \le 2D_N^*$. By a classical result of Weyl [23], for any increasing sequence $(n_k)_{k\ge1}$ of positive integers, $(n_k x)_{k\ge1}$ is uniformly distributed mod 1 for almost all x and consequently, $D_N(n_k x) \to 0$ a.e. Computing the precise order of magnitude of $D_N(n_k x)$ is a difficult problem and has been solved only in a few special cases. In the case $n_k = k$, Kesten [12] proved that

$$ND_N(kx) \sim \frac{2}{\pi^2} \log N \log\log N \qquad \text{in measure.} \tag{1.1}$$

For further related results, see Drmota and Tichy [6] and Kuipers and Niederreiter [13]. Philipp [15], [16] proved that if $(n_k)_{k\ge1}$ satisfies the Hadamard gap condition

$$n_{k+1}/n_k \ge q > 1 \qquad k = 1, 2, \dots, \tag{1.2}$$

then

$$\frac{1}{4\sqrt{2}} \le \limsup_{N\to\infty} \frac{ND_N(n_k x)}{\sqrt{2N\log\log N}} \le C_q \qquad \text{a.e.,} \tag{1.3}$$

where C_q is a constant depending only on q. Note that by the Chung–Smirnov law of the iterated logarithm (see e.g. [20], p. 504) we have

$$\limsup_{N\to\infty} \frac{ND_N(\xi_k)}{\sqrt{2N\log\log N}} = 1/2 \qquad \text{a.s.} \tag{1.4}$$

if $(\xi_k)_{k\ge1}$ is a sequence of independent random variables, uniformly distributed over $(0,1)$. A comparison of (1.3) and (1.4) shows that under (1.2) the sequence $(\{n_k x\})_{k\ge1}$ behaves like an i.i.d. sequence of random variables; here, and in the sequel, $\{t\}$ denotes the fractional part of t. However, the analogy is not complete: the value of the limsup in (1.3) can be different from 1/2 and can also be a nonconstant function of x. Fukuyama [8] showed that in the case $n_k = \theta^k$, $\theta > 1$ the limsup actually equals a constant Σ_θ for almost all x, where

$$\begin{aligned} \Sigma_\theta &= 1/2 \quad \text{if } \theta^r \text{ is irrational for all } r\in\mathbb{N}, \\ \Sigma_\theta &= \sqrt{42}/9, \quad \text{if } \theta = 2, \\ \Sigma_\theta &= \frac{\sqrt{(\theta+1)\theta(\theta-2)}}{2\sqrt{(\theta-1)^3}} \quad \text{if } \theta \ge 4 \text{ is an even integer,} \end{aligned} \tag{1.5}$$

$$\Sigma_\theta = \frac{\sqrt{\theta+1}}{2\sqrt{\theta-1}} \quad \text{if } \theta \geq 3 \text{ is an odd integer.}$$

Aistleitner [1] showed that the limsup equals 1/2 for a large class of integer sequences $(n_k)_{k\geq 1}$ satisfying certain Diophantine conditions. For examples for a nonconstant limsup function in (1.3) and its version for the star discrepancy D_N^*, see Aistleitner [2] and Fukuyama and Miyamoto [11].

Given an increasing sequence $(n_k)_{k\geq 1}$ of real numbers, let

$$F_N(t) = F_N(t, x) = \frac{1}{N}\sum_{k=1}^{N} I_{(-\infty,\, t]}(\{n_k x\})$$

denote the empirical distribution function of the sample $\{n_1 x\}, \ldots, \{n_N x\}$. Philipp [16] proved that under (1.2) the sequence

$$\alpha_N(t,x) = \sqrt{\frac{N}{2\log\log N}}(F_N(t,x) - t), \quad 0 \leq t \leq 1, \ N = 1, 2, \ldots \tag{1.6}$$

is relatively compact in the Skorohod space $D[0,1]$ for almost all x and under additional number theoretic assumptions on n_k he determined the class of its limit functions in the $D[0,1]$ metric. Since

$$D_N^*(n_k x) = \sup_t |F_N(t,x) - t|,$$

this leads immediately to a precise LIL for $D_N(n_k x)$, but a functional LIL yields far deeper information on the behavior of the sequence $(\{n_k x\})_{k\geq 1}$ than (1.3): it yields the precise asymptotics of various other functionals of the curve in (1.6) with constants obtained as extreme values of functionals on Hilbert space. Unlike the constants in (1.5) above, they can only be evaluated approximately.

For sequences $(n_k)_{k\geq 1}$ growing slower than exponentially, the LIL (1.3) becomes generally false (see Berkes and Philipp [3]), and the asymptotic behavior of $D_N(n_k x)$ remains open. It is then natural to ask about 'typical' behavior of the discrepancy, which requires to study $D_N(n_k x)$ for random sequences $(n_k)_{k\geq 1}$. A simple and natural model is when $(n_k)_{k\geq 1}$ is an increasing random walk, i.e. when $n_{k+1} - n_k$ are i.i.d. positive random variables. This model was investigated by Schatte [17], [19], Weber [22], Berkes and Weber [4]. For other randomization methods in the context of the LIL for the discrepancy of $(n_k x)$ mod 1 we refer to Fukuyama [9], [10]. The purpose of the present paper is to investigate the random walk model in more detail and to prove a functional LIL for the empirical process of $(\{n_k x\})_{1\leq k\leq N}$ in this model.

Let $X_1, X_2, \ldots$ be i.i.d. positive random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that X_1 is supported on a finite interval

$[a, b] \subset (0, \infty)$ and has a bounded density. Let U be a random variable uniformly distributed on $(0, 1)$, independent of the sequence $(X_k)_{k \geq 1}$. Clearly, the existence of such a U can be guaranteed by a suitable enlargement of the probability space. Put $S_n = \sum_{k=1}^n X_k$ and let

$$F_N^*(t, x) = \frac{1}{N} \sum_{k=1}^N I_{(-\infty, t]}(\{S_k x\})$$

denote the empirical distribution function of the sample $\{S_1 x\}, \ldots, \{S_N x\}$. We will prove the following result.

Theorem. *With $\mathbb{P}$-probability one the sequence of functions*

$$\alpha_N(t, x) = \sqrt{\frac{N}{2 \log \log N}} (F_N^*(t, x) - t), \quad 0 \leq t \leq 1, \ N = 1, 2, \ldots \tag{1.7}$$

is relatively compact in the Skorohod space $D[0, 1]$ for almost all $x > 0$ in the sense of Lebesgue measure and its class of limit functions is identical with the unit ball B_Γ of the reproducing kernel Hilbert space determined by the covariance function

$$\Gamma(s, s') = \mathbb{E} g_s(U) g_{s'}(U) + \sum_{k=1}^\infty \mathbb{E} g_s(U) g_{s'}(U + S_k x) + \sum_{k=1}^\infty \mathbb{E} g_{s'}(U) g_s(U + S_k x). \tag{1.8}$$

Here $g_s = I_{(0,s)} - s$ is the centered indicator function of the interval $(0, s)$, extended with period 1.

The absolute convergence of the series (1.8) will follow from the proof of the theorem. For background on reproducing kernel Hilbert spaces see e.g. Oodaira [14].

In the case when X_1 is uniformly distributed on $(0, 1)$, the $\{S_k x\}$ are easily seen to be independent, uniform r.v.'s and $\Gamma(s, s')$ reduces to the covariance function $s(1 - s')$ $(s < s')$ of the Brownian bridge. In this case the limit set in the theorem reduces to the set

$$K = \left\{ y(t) : y \text{ is absolutely continuous in } [0, 1], \; y(0) = y(1) = 0 \text{ and } \int_0^1 y'(t)^2 dt \leq 1 \right\}$$

obtained in the i.i.d. case by Finkelstein [7].

As noted above, $\sup_{0\le t\le 1}|F_N^*(t,x)-t|$ equals the star discrepancy of the sequence $\{S_k x\}_{1\le k\le N}$, while $(\int_0^1 |F_N^*(t,x)-t|^p\,dt)^{1/p}$ is the L_p discrepancy $D_N^{(p)}(S_k x)$ of the same sequence. Thus immediate consequences of our theorem are

$$\limsup_{N\to\infty}\sqrt{\frac{N}{2\log\log N}}D_N^*(S_k x)=\sup_{y\in B_\Gamma}\|y\|_\infty \tag{1.9}$$

and

$$\limsup_{N\to\infty}\sqrt{\frac{N}{2\log\log N}}D_N^{(p)}(S_k x)=\sup_{y\in B_\Gamma}\|y\|_p \qquad (p\ge 1) \tag{1.10}$$

$\mathbb{P}$-a.s. for almost all x. With a different representation of the limit, relation (1.9) was obtained earlier in Schatte [19]. However, the Theorem above provides much more precise information than (1.9), (1.10): for example, as a standard application of our theorem, one can characterize precisely how frequently $(N/(2\log\log N))^{1/2}D_N^*(S_k x)$ can get close to its limsup or get asymptotics for weighted versions of the discrepancy. We refer to Strassen [21] for applications of functional laws of the iterated logarithm.

2. Proof of the theorem

By Fubini's theorem, it suffices to show that for any fixed $x>0$ with $\mathbb{P}$-probability 1 the sequence (1.7) is relatively compact in $D[0,1]$ and the set of its limit functions is B_Γ. Since for any $x>0$ the sequence $(X_k x)_{k\ge 1}$ also satisfies the condition of our theorem, without loss of generality we can assume $x=1$.

Throughout this section, $f:\mathbb{R}\to\mathbb{R}$ will denote bounded measurable functions satisfying

$$f(t+1)=f(t), \qquad \int_0^1 f(t)dt=0, \tag{2.1}$$

and the L_2-Lipschitz condition

$$\left(\int_0^1 |f(t+h)-f(t)|^2dt\right)^{1/2}\le Kh \tag{2.2}$$

for some constant $K>0$. Note that if $f=I_{(a,b)}-(b-a)$ for some $0\le a<b\le 1$, (extended as usual with period 1) then f satisfies (2.1), (2.2) with $K=1$. Put

$$A^{(f)}=\|f\|^2+2\sum_{k=1}^{\infty}\mathbb{E}f(U)f(U+S_k), \tag{2.3}$$

where U is a uniform $(0,1)$ random variable, independent of $(X_j)_{j\geq 1}$. The absolute convergence of the series (2.3) will follow from the proof of Lemma 1.

Note that our main theorem deals with the sequence $\{S_1x\}, \{S_2x\}, \dots$, or, equivalently, with partial sums of random variables defined on the torus $\mathbf{T} = \mathbb{R}/\mathbb{Z}$ where summation is meant mod 1. In this interpretation (which we adopt in the sequel), f is an element of $L^2(\mathbf{T})$ and all fractional part signs can be dropped, leading to a substantial simplification of the formulas.

In the sequel, C and λ will denote positive constants, not always the same, which depend (at most) on the function f and the distribution of the random variable X_1.

Lemma 1. *Let f satisfy* (2.1), (2.2)*, let $\ell, b, b_1, b_2, \dots$ be positive integers and define a sequence of sets by*

$$I_1 := \{1, 2, \dots, b\}$$

$$I_2 := \{p_1, p_1+1, \dots, p_1+b_1\} \quad \text{where} \quad p_1 \geq b+\ell+2$$

$$\vdots$$

$$I_n := \{p_{n-1}, p_{n-1}+1, \dots, p_{n-1}+b_{n-1}\} \quad \text{where} \quad p_{n-1} \geq p_{n-2}+b_{n-2}+\ell+2$$

$$\vdots$$

Then there exists a sequence $\delta_1, \delta_2, \dots$ of random variables, not depending on f, satisfying the following properties:

(i) $|\delta_n| \leq Ce^{-\lambda\ell}$ *for all* $n \in \mathbb{N}$,

(ii) *The random variables*

$$\sum_{i\in I_1} f(S_i), \ \sum_{i\in I_2} f(S_i - \delta_1), \dots, \ \sum_{i\in I_n} f(S_i - \delta_{n-1}), \dots$$

are independent.

Proof. We will construct the sequence $(\delta_n)_{n\in\mathbb{N}}$ by induction. Define

$$\delta_1 := (S_{b+l} - S_b) - F_{S_{b+l}-S_b}(S_{b+l} - S_b),$$

where, for any random variable Y, F_Y denotes the distribution function of Y. By the assumptions of our theorem and Theorem 1 of Schatte [18] we have

$$\sup_t |F_{S_n}(t) - t| \leq Ce^{-\lambda n} \qquad n \in \mathbb{N}. \tag{2.4}$$

Since $S_{b+\ell} - S_b \stackrel{d}{=} S_\ell$ for all b and all ℓ, it follows easily that $|\delta_1| \leq Ce^{-\lambda\ell}$. Furthermore we have

$$S_{p_1} - \delta_1 = S_{p_1} - (S_{b+\ell} - S_b) + F_{S_{b+\ell}-S_b}(S_{b+\ell} - S_b)$$

$$= (X_1 + \cdots + X_b) + (X_{b+\ell+1} + \cdots + X_{p_1}) + F_{S_{b+\ell}-S_b}(S_{b+\ell} - S_b) .$$

Similarly,

$$\begin{aligned} S_{p_1+1} - \delta_1 &= (X_1 + \cdots + X_b) + (X_{b+\ell+1} + \cdots + X_{p_1+1}) \\ &\quad + F_{S_{b+l}-S_b}(S_{b+\ell} - S_b) \\ &\vdots \\ S_{p_1+b_1} - \delta_1 &= (X_1 + \cdots + X_b) + (X_{b+\ell+1} + \cdots + X_{p_1+1}) \\ &\quad + F_{S_{b+\ell}-S_b}(S_{b+\ell} - S_b) . \end{aligned}$$

Thus applying Lemma 1 of [17] with

$$\begin{aligned} X &= (X_1, X_2, \ldots, X_b) \\ U &= F_{S_{b+\ell}-S_b}(S_{b+\ell} - S_b) \\ (W_1, \ldots, W_{p_1+b_1}) &= ((X_{b+\ell+1} + \ldots X_{p_1}), \ldots, (X_{b+\ell+1} + \cdots + X_{p_1+b_1})) \\ W &= X_1 + \cdots + X_b, \end{aligned}$$

it follows that

$$\sum_{j\in I_1} f(S_j) \text{ is independent of } \sum_{j\in I_2} f(S_j - \delta_1).$$

Now suppose $\delta_1, \ldots, \delta_{n-1}$ have been constructed and define

$$Y_n = S_{p_{n-1}+b_{n-1}+\ell} - S_{p_{n-1}+b_{n-1}}, \qquad \delta_n = Y_n - F_{Y_n}(Y_n).$$

As before, it follows easily that $|\delta_n| \le Ce^{-\lambda\ell}$. We let

$$\begin{aligned} &X = \left(X_1, \ldots, X_{p_{n-1}+b_{n-1}}, \delta_1, \ldots, \delta_{n-1}\right) \\ &U = F_{Y_n}(Y_n) \\ &W = X_1 + \cdots + X_{p_{n-1}+b_{n-1}} \\ &(W_1, \ldots, W_{p_n+b_n}) \\ &= \left(X_{p_{n-1}+b_{n-1}+\ell+1} + \cdots + X_{p_n}, \ldots, X_{p_{n-1}+b_{n-1}+\ell+1} + \cdots + X_{p_n+b_n}\right) . \end{aligned}$$

Then, again by Lemma 1 of [17] it follows that

$$\sum_{i\in I_{n+1}} f(S_i - \delta_n) \text{ is independent of } \left(\sum_{i\in I_1} f(S_i), \ldots, \sum_{i\in I_n} f(S_i - \delta_{n-1})\right),$$

which completes the induction step and the proof of the lemma. □

Put $\widetilde{m}_k = \sum_{j=1}^{k} \lfloor j^{1/2} \rfloor$, $\widehat{m}_k = \sum_{j=1}^{k} \lfloor j^{1/4} \rfloor$ and let $m_k = \widetilde{m}_k + \widehat{m}_k$. Using Lemma 1 we can construct sequences $(\Delta_k)_{k\in\mathbb{N}}$, $(\Pi_k)_{k\in\mathbb{N}}$ of random variables such that $\Delta_0 = 0$, $\Pi_0 = 0$,

$$|\Delta_k| \le Ce^{-\lambda \sqrt[4]{k}}, \qquad |\Pi_k| \le Ce^{-\lambda\sqrt{k}} \tag{2.5}$$

and

$$T_k^{(f)} := \sum_{j=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor} \left(f(S_j - \Delta_{k-1}) - \mathbb{E}f(S_j - \Delta_{k-1})\right)$$

$$T_k^{*(f)} := \sum_{j=m_{k-1}+\lfloor\sqrt{k}\rfloor+1}^{m_k} \left(f(S_j - \Pi_{k-1}) - \mathbb{E}f(S_j - \Pi_{k-1}\right)$$

are sequences of independent random variables.

Lemma 2. *Under the conditions of Lemma 1 we have*

$$\sum_{k=1}^{n} \mathbb{V}\mathrm{ar}(T_k^{(f)}) \sim A^{(f)}\widetilde{m}_n, \qquad \sum_{k=1}^{n} \mathbb{V}\mathrm{ar}(T_k^{*(f)}) \sim A^{(f)}\widehat{m}_n,$$

where $A^{(f)}$ *is defined by* (2.3).

Proof. Since f does not change in the proof, we will drop the upper index f from $T_k^{(f)}$, $T_k^{*(f)}$ and $A^{(f)}$. Clearly

$$\mathbb{V}\mathrm{ar}(T_k) = \sum_{j=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor} \mathbb{E}f^2(S_j - \Delta_{k-1})$$
$$+ 2 \sum_{\varrho=1}^{\lfloor\sqrt{k}\rfloor-1} \sum_{\ell=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor-\varrho} \mathbb{E}f(S_\ell - \Delta_{k-1})f(S_{\ell+\varrho} - \Delta_{k-1}) - L^{(k)}$$

where

$$L^{(k)} := \left(\sum_{j=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor} \mathbb{E}f(S_j - \Delta_{k-1}) \right)^2 .$$

By (2.1), (2.2), (2.4) and (2.5) we have

$$\|f(S_j - \Delta_{k-1}) - f(S_j)\| \le Ce^{-\lambda \sqrt[4]{k-1}}$$

and

$$\left|\mathbb{E}f(S_j)\right| = \left|\mathbb{E}f\big(S_j\big) - \mathbb{E}f\left(F_{S_j}\big(S_j\big)\right)\right| \le Ce^{-\lambda j}$$

since $F_{S_j}(S_j)$ is a uniformly distributed random variable and thus the last expectation in the previous displayed formula equals 0 by (2.1). Thus

$$L^{(k)} \leq Cke^{-\lambda\sqrt[4]{k-1}}.$$

Let now

$$\Lambda^{(k)} := \sum_{j=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor} \gamma_{j,k} \qquad O^{(k)} := \sum_{j=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor} \varepsilon_j,$$

where

$$\begin{aligned}\gamma_{j,k} &= \mathbb{E}f^2(S_j-\Delta_{k-1}) - \mathbb{E}f^2(S_j)\\ \varepsilon_j &= \mathbb{E}f^2(S_j) - \mathbb{E}f^2\left(F_{S_j}(S_j)\right).\end{aligned}$$

Repeating the argument above for the function $f^2 - \|f\|^2$, we get

$$\sum_{j=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor} \mathbb{E}f^2(S_j-\Delta_{k-1}) = \|f\|^2\lfloor\sqrt{k}\rfloor + \Lambda^{(k)} + O^{(k)}$$

and

$$|\Lambda^{(k)}| \leq C\sqrt{k}e^{-\lambda(k-1)^{1/4}}, \qquad |O^{(k)}| \leq Ce^{-\lambda(m_{k-1}+1)}.$$

We now turn to the cross terms. Define $T_{\ell,\varrho} = X_{\ell+1}+\cdots+X_{\ell+\varrho}$ and split the product expectation $\mathbb{E}f(S_\ell-\Delta_{k-1})f(S_{\ell+\varrho}-\Delta_{k-1})$ into the sum of terms

$$\begin{aligned}e_{\ell,\varrho,k} &:= \mathbb{E}f(S_\ell-\Delta_{k-1})f(S_{\ell+\varrho}-\Delta_{k-1}) - \mathbb{E}f(S_\ell)f(S_{\ell+\varrho}-\Delta_{k-1})\\ g_{\ell,\varrho,k} &:= \mathbb{E}f(S_\ell)f(S_{\ell+\varrho}-\Delta_{k-1}) - \mathbb{E}f(S_\ell)f(S_{\ell+\varrho})\\ h_{\ell,\varrho} &:= \mathbb{E}f(S_\ell)f(S_{\ell+\varrho}) - \mathbb{E}f\big(F_{S_\ell}(S_\ell)\big)f(S_{\ell+\varrho})\\ i_{\ell,\varrho} &:= \mathbb{E}f\big(F_{S_\ell}(S_\ell)\big)f(S_\ell+T_{\ell,\varrho}) - \mathbb{E}f\big(F_{S_\ell}(S_\ell)\big)f\big(F_{S_\ell}(S_\ell)+T_{\ell,\varrho}\big)\\ C_{\ell,\varrho} &:= \mathbb{E}f\big(F_{S_\ell}(S_\ell)\big)f\big(F_{S_\ell}(S_\ell)+T_{\ell,\varrho}\big).\end{aligned}$$

Here $F_{S_\ell}(S_\ell)$ is a uniformly distributed variable independent of $T_{\ell,\varrho}$ and thus letting U denote a uniform random variable independent of $(X_j)_{j\geq 1}$,

$$C_{\ell,\varrho} = C_\varrho = \mathbb{E}f(U)f(U+S_\varrho)$$

does not depend on ℓ. Exactly as before,

$$|e_{\ell,\varrho,k}| \leq Ce^{-\lambda(k-1)^{1/4}} \quad |g_{\ell,\varrho,k}| \leq Ce^{-\lambda(k-1)^{1/4}} \quad |h_{\ell,\varrho}| \leq Ce^{-\lambda\ell} \quad |i_{\ell,\varrho}| \leq Ce^{-\lambda\ell}.$$

Thus letting

$$E^{(k)} = 2\sum_{\varrho=1}^{\lfloor\sqrt{k}\rfloor-1}\sum_{\ell=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor-\varrho} e_{\ell,\varrho,k} \qquad G^{(k)} = 2\sum_{\varrho=1}^{\lfloor\sqrt{k}\rfloor-1}\sum_{\ell=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor-\varrho} g_{\ell,\varrho,k}$$

$$H^{(k)} = 2 \sum_{\varrho=1}^{\lfloor\sqrt{k}\rfloor-1} \sum_{\ell=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor-\varrho} h_{\ell,\varrho} \qquad I^{(k)} = 2 \sum_{\varrho=1}^{\lfloor\sqrt{k}\rfloor-1} \sum_{\ell=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor-\varrho} i_{\ell,\varrho}$$

we have

$$|E^{(k)}| \le Cke^{-\lambda(k-1)^{1/4}}, \qquad |G^{(k)}| \le Cke^{-\lambda(k-1)^{1/4}}$$
$$|H^{(k)}| \le C\sqrt{k}\, e^{-\lambda(m_{k-1}+1)}, \qquad |I^{(k)}| \le C\sqrt{k}\, e^{-\lambda(m_{k-1}+1)}.$$

Furthermore,

$$\sum_{\varrho=1}^{\lfloor\sqrt{k}\rfloor-1} \sum_{\ell=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor-\varrho} C_{\ell,\varrho} = \sum_{\varrho=1}^{\lfloor\sqrt{k}\rfloor-1} \sum_{\ell=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor-\varrho} C_{\varrho} =$$
$$= \lfloor\sqrt{k}\rfloor \sum_{\varrho=1}^{\infty} C_\varrho - \lfloor\sqrt{k}\rfloor \sum_{\varrho=\lfloor\sqrt{k}\rfloor}^{\infty} C_\varrho - \sum_{\varrho=1}^{\lfloor\sqrt{k}\rfloor-1} \varrho C_\varrho.$$

Thus using the independence of the T_k we get

$$\mathbb{V}\mathrm{ar}\left(\sum_{k=1}^{n} T_k\right) = \sum_{k=1}^{n} \mathbb{V}\mathrm{ar}(T_k) = O(1) + \sum_{k=4}^{n} \mathbb{V}\mathrm{ar}(T_k)$$
$$= O(1) + \sum_{k=4}^{n}\Bigg(\lfloor\sqrt{k}\rfloor + \Lambda^{(k)} + O^{(k)} + 2E^{(k)} + 2G^{(k)} + 2H^{(k)} + 2I^{(k)}$$
$$+ \lfloor\sqrt{k}\rfloor \cdot 2\sum_{\varrho=1}^{\infty} C_\varrho - 2\lfloor\sqrt{k}\rfloor \sum_{\varrho=\lfloor\sqrt{k}\rfloor}^{\infty} C_\varrho - 2\sum_{\varrho=1}^{\lfloor\sqrt{k}\rfloor-1} \varrho C_\varrho - L^{(k)}\Bigg).$$

Using the same techniques as before, we get $|C_\varrho| \le Ce^{-\lambda\varrho}$. Hence the previously established inequalities yield

$$\mathbb{V}\mathrm{ar}\left(\sum_{k=1}^{n} T_k\right) \sim A\widetilde{m}_n \sim Am_n.$$

Similarly,

$$\mathbb{V}\mathrm{ar}\left(\sum_{k=1}^{n} T_k^*\right) \sim A\widehat{m}_n,$$

completing the proof of Lemma 2. □

Since

$$\mathrm{Cov}(T_k^{(f)}, T_k^{(g)}) = \frac{1}{4}\left(\mathrm{Var}(T_k^{(f+g)}) - \mathrm{Var}(T_k^{(f-g)})\right),$$

Lemma 2 implies

Corollary. *We have*

$$\sum_{k=1}^{n} Cov(T_k^{(f)}, T_k^{(g)}) \sim \frac{1}{4}\left(A^{(f+g)} - A^{(f-g)}\right)\widetilde{m}_n$$

and

$$\sum_{k=1}^{n} Cov(T_k^{*(f)}, T_k^{*(g)}) \sim \frac{1}{4}\left(A^{(f+g)} - A^{(f-g)}\right)\widehat{m}_n.$$

From (2.3) it follows that

$$A^{(f+g)} - A^{(f-g)} = 4\langle f, g\rangle + 4\sum_{k=1}^{\infty}\mathbb{E}f(U)g(U+S_k) + 4\sum_{k=1}^{\infty}\mathbb{E}g(U)f(U+S_k). \quad (2.6)$$

Let $0 < t_1 < \ldots < t_r \le 1$ and put

$$\mathbf{Y}_k = (f_{(0,t_1)}(S_k), f_{(0,t_2)}(S_k), \ldots, f_{(0,t_r)}(S_k))$$

where $f_{(a,b)} = I_{(a,b)} - (b-a)$, with the indicator $I_{(a,b)}$ extended with period 1, as before.

Lemma 3. *With $\mathbb{P}$-probability 1, the class of limit points of the sequence*

$$\left\{(2N\log\log N)^{-1/2}\sum_{k=1}^{N}\mathbf{Y}_k, \qquad N = 1, 2, \ldots\right\} \quad (2.7)$$

in $\mathbb{R}^r$ is the ellipsoid

$$\left\{(x_1, \ldots, x_r) : \sum_{i,j=1}^{r}\Gamma(t_i, t_j)x_i x_j \le 1\right\} \quad (2.8)$$

where Γ is the function defined in equation (1.8).

Proof. Let

$$\mathbf{T}_k = \left(T_k^{(f_{(0,t_1)})}, \ldots, T_k^{(f_{(0,t_r)})}\right), \qquad \mathbf{T}_k^* = \left(T_k^{*(f_{(0,t_1)})}, \ldots, T_k^{*(f_{(0,t_r)})}\right).$$

and let $\mathbf{\Sigma_k}$ denote the covariance matrix of the vector $\mathbf{T}_k$. From the Corollary and (2.6) it follows that

$$m_n^{-1}(\mathbf{\Sigma_1} + \ldots + \mathbf{\Sigma_n}) \longrightarrow \mathbf{\Sigma}$$

where

$$\mathbf{\Sigma} = (\Gamma(t_i, t_j))_{1 \le i,j \le r}.$$

Clearly

$$|\mathbf{T}_k| \le C_r k^{1/2} = o(m_k \log\log m_k)^{1/2}$$

where C_r is a constant depending on r, showing that the sequences $(\mathbf{T}_k)_{k\ge 1}$ of independent random variables satisfies Kolmogorov's condition of the LIL. Thus Theorem 1 in Berning [5] implies that the set of limit points of

$$\left\{ (2m_n \log\log m_n)^{-1/2} \sum_{k=1}^{n} \mathbf{T}_k \right\} \tag{2.9}$$

is the ellipsoid (2.8). A similar statement holds for the sequence $(\mathbf{T}_k^*)_{k\ge 1}$, implying that

$$\left| \sum_{k=1}^{n} \mathbf{T}_k^* \right| = O(\widehat{m}_n \log\log \widehat{m}_n)^{1/2} = o(m_n \log\log m_n)^{1/2} \qquad \text{a.s.}$$

Also, for any $f = f_{(0,t_i)}$, $1 \le i \le r$ we have

$$\max_{m_k+1 \le \ell \le m_{k+1}} \left| \sum_{j=m_k+1}^{\ell} (f(S_j) - \mathbb{E} f(S_j)) \right|$$
$$= O(m_{k+1} - m_k) = O(k^{1/2}) = o(m_k \log\log m_k)^{1/2}.$$

From these relations Lemma 3 follows immediately. □

Lemma 4. *Let f satisfy* (2.1), (2.2). *Then we have*

$$\mathbb{E}\left(\sum_{k=M+1}^{M+N} f(S_k) \right)^2 \le C\|f\| N. \tag{2.10}$$

where C is a constant depending only on the constant K in (2.2) *and the distribution of X_1.*

Note that in the previous lemmas and their proofs the constants C and λ depend on f and the distribution of X_1. In Lemma 4 (as well as in Lemma 5 below) the dependence of the constant C on f is more specific: it is only through the constant K in (2.2). As we pointed out before, if $f = I_{(a,b)} - (b-a)$ for some $0 \le a < b \le 1$, then f satisfies (2.2) with $K = 1$ and thus Lemma 4 (as well as Lemma 5 below) hold uniformly for all centered indicators f.

P r o o f. In what follows, C and λ will denote positive constants, possibly different at different places, depending on the distribution of X_1 and the constant K in (2.2). We first show

$$|\mathbb{E} f(S_k) f(S_\ell)| \le C e^{-\lambda(\ell-k)} \|f\| \qquad (k < \ell). \tag{2.11}$$

Indeed, by Lemma 4.3 and relation (4.8) of Berkes and Weber [4] there exists a r.v. Δ with $|\Delta| \le Ce^{-\lambda(\ell-k)}$ such that $S_\ell - \Delta$ is a uniform r.v. independent of S_k. Hence

$$\mathbb{E}f(S_\ell - \Delta) = \int_0^1 f(t)dt = 0$$

and thus

$$\mathbb{E}f(S_k)f(S_\ell - \Delta) = \mathbb{E}f(S_k)\mathbb{E}f(S_\ell - \Delta) = 0. \tag{2.12}$$

On the other hand,

$$\begin{aligned} &|\mathbb{E}f(S_k)f(S_\ell) - \mathbb{E}f(S_k)f(S_\ell - \Delta)| \\ &\le \mathbb{E}\big(|f(S_k)|\,|f(S_\ell) - f(S_\ell - \Delta)|\big) \le \\ &\big(\mathbb{E}f^2(S_k)\big)^{1/2}\big(\mathbb{E}|f(S_\ell) - f(S_\ell - \Delta)|^2\big)^{1/2}. \end{aligned} \tag{2.13}$$

Since X_1 has a bounded density, by Theorem 1 of Schatte [18] the density φ_n of S_n exists for all $n \ge 1$ and satisfies $\varphi_n \to 1$ uniformly on $[0,1]$. Thus

$$P(S_n \in I) \le C|I| \qquad (n \ge 1) \tag{2.14}$$

whence we get

$$\mathbb{E}f^2(S_k) \le C\int_0^1 f^2(t)dt = C\|f\|^2. \tag{2.15}$$

On the other hand, $|\Delta| \le Ce^{-\lambda(\ell-k)}$ implies

$$\mathbb{E}|f(S_\ell) - f(S_\ell - \Delta)|^2 \le Ce^{-\lambda(\ell-k)} \tag{2.16}$$

which, together with (2.13)–(2.16), gives

$$|\mathbb{E}f(S_k)f(S_\ell) - \mathbb{E}f(S_k)f(S_\ell - \Delta)| \le Ce^{-\lambda(\ell-k)}.$$

Thus using (2.12) we get (2.11). Now by (2.11)

$$\Big|\sum_{M+1\le k<\ell\le M+N} \mathbb{E}f(S_k)f(S_\ell)\Big| \le CN\|f\|\sum_{\ell\ge1} e^{-\lambda\ell} \le CN\|f\|$$

which, together with (2.15), completes the proof of Lemma 4. □

Lemma 5. *Let f satisfy* (2.1), (2.2). *Then for any $M \ge 0$, $N \ge 1$, real $t \ge 1$ and $\|f\| \ge N^{-1/4}$ we have*

$$P\Big\{\Big|\sum_{k=M+1}^{M+N} f(S_k)\Big| \ge t\|f\|^{1/4}(N\log\log N)^{1/2}\Big\} \\ \le \exp\Big(-Ct\|f\|^{-1/2}\log\log N\Big) + t^{-2}N^{-1} \tag{2.17}$$

where C is a constant depending only on the constant K in (2.2) *and the distribution of X_1.*

P r o o f. Put

$$\psi(n) = \sup_{0\le x\le 1} |P(S_n \le x) - x|. \tag{2.18}$$

By Theorem 1 of [18] we have

$$\psi(n) \le Ce^{-\lambda n} \qquad (n \ge 1). \tag{2.19}$$

Divide the interval $[M+1, M+N]$ into subintervals $I_1, \dots, I_L$, with $L \sim N^{19/20}$, where each interval I_ν contains $\sim N^{1/20}$ terms. We set

$$\sum_{k=M+1}^{M+N} f(S_k) = \eta_1 + \dots + \eta_L$$

where

$$\eta_\nu = \sum_{k\in I_\nu} f(S_k).$$

We deal with the sums $\sum \eta_{2j}$ and $\sum \eta_{2j+1}$ separately. Since there is a separation $\sim N^{1/20}$ between the even block sums η_{2j}, we can apply Lemma 4.3 of [4] to get

$$\eta_{2j} = \eta_{2j}^* + \eta_{2j}^{**}$$

where

$$\eta_{2j}^* = \sum_{k\in I_{2j}} f(S_k - \Delta_j)$$

$$\eta_{2j}^{**} = \sum_{k\in I_{2j}} (f(S_k) - f(S_k - \Delta_j)) \tag{2.20}$$

where the Δ_j are r.v.'s with $|\Delta_j| \le \psi(N^{1/20}) \le N^{-10}$ and the r.v.'s η_{2j}^* $j = 1, 2, \dots$ are independent. Relation (2.16) in the proof of Lemma 4 shows that the L_2 norm of each summand in η_{2j}^{**} is $\le C\psi(N^{1/20}) \le CN^{-10}$ and thus for $\|f\| \ge N^{-1/4}$ we have

$$\|\eta_{2j}^{**}\| \le CN^{-9} \le C\|f\|N^{-8}. \tag{2.21}$$

Thus

$$\Big\|\sum \eta_{2j}^{**}\Big\| \le C\|f\|N^{-7}$$

and therefore by the Markov inequality

$$P\left(\Big|\sum \eta_{2j}^{**}\Big| \ge t\|f\|^{1/4}(N\log\log N)^{1/2}\right)$$
$$\le Ct^{-2}\|f\|^{-1/2}(N\log\log N)^{-1}\|f\|^2 N^{-14} \le t^{-2}N^{-1}. \tag{2.22}$$

Let now $|\lambda| = \mathcal{O}(N^{-1/16})$, then $|\lambda\eta^*_{2j}| \le C|\lambda|N^{1/20} \le 1/2$ for $N \ge N_0$ and thus using $e^x \le 1 + x + x^2$ for $|x| \le 1/2$ we get, using $E\eta^*_{2j} = 0$,

$$E\left(\exp\lambda\left(\sum_j \eta^*_{2j}\right)\right) = \prod_j E\left(e^{\lambda\eta^*_{2j}}\right) \le \prod_j E(1 + \lambda\eta^*_{2j} + \lambda^2\eta^{*2}_{2j})$$
$$= \prod_j (1 + \lambda^2 E\eta^{*2}_{2j}) \le \exp\left(\lambda^2 \sum_j E\eta^{*2}_{2j}\right). \tag{2.23}$$

By Lemma 4
$$\|\eta_{2j}\| \le C\|f\|^{1/2}N^{1/40},$$
which, together with (2.21) and the Minkowski inequality, implies
$$\|\eta^*_{2j}\| \le C\|f\|^{1/2}N^{1/40}.$$
Thus the last expression in (2.23) cannot exceed
$$\exp\left(\lambda^2 C\|f\| \sum_j N^{1/20}\right) \le \exp(\lambda^2 C\|f\|N).$$
Hence choosing
$$\lambda = (\log\log N/N)^{1/2}\|f\|^{-3/4}$$
(note that by $\|f\| \ge N^{-1/4}$ we have $|\lambda| = \mathcal{O}(N^{-1/6})$) and using the Markov inequality, we get

$$P\left\{\left|\sum_j \eta^*_{2j}\right| \ge t\|f\|^{1/4}(N\log\log N)^{1/2}\right\}$$
$$\le \exp\left\{-\lambda t\|f\|^{1/4}(N\log\log N)^{1/2} + \lambda^2 C\|f\|N\right\} \tag{2.24}$$
$$= \exp(-\|f\|^{-1/2}t\log\log N + C\|f\|^{-1/2}\log\log N)$$
$$\le \exp(-C'\|f\|^{-1/2}t\log\log N)$$

completing the proof of Lemma 5. □

With Lemmas 1–5 at hand, the proof of the Theorem can be completed easily. Given any $0 < t_1 < \ldots < t_r = 1$, let $B_\Gamma(t_1, \ldots, t_r)$ denote the set of points of the form $(g(t_1), \ldots, g(t_r))$, where $g(\cdot) \in B_\Gamma$. By the standard method of proving functional laws of the iterated logarithm developed in Strassen [21], Finkelstein [7], it suffices to prove that with probability 1 the sequence $\alpha_N(\cdot)$ is relatively compact in the $D[0,1]$ topology and for any $0 < t_1 < \ldots < t_r = 1$, $r = 1, 2, \ldots$ the set of limit points of the vector $(\alpha_N(t_1), \ldots, \alpha_N(t_r))$ is identical with the set $B_\Gamma(t_1, \ldots, t_r)$. Since $B_\Gamma(t_1, \ldots, t_r)$ coincides with the ellipsoid (2.8), the second statement follows from Lemma 3. On the other hand, the equicontinuity statement can be proved by a dyadic chaining argument, similar to the proof of Proposition 3.3.2 in Philipp [16]. (The uniformity of the statements of Lemmas 4 and 5 over all centered indicator functions is needed in this chaining argument.) Since the necessary modifications are routine, we omit the details.

Acknowledgement. We would like to thank the referee for his careful reading of the paper and for his remarks leading to a substantial improvement of the presentation.

REFERENCES

[1] C. Aistleitner, On the law of the iterated logarithm for the discrepancy of lacunary sequences. *Trans. Amer. Math. Soc.* **362** (2010), 5967–5982.

[2] C. Aistleitner, Irregular discrepancy behavior of lacunary series II. *Monatsh. Math.* **161** (2010), 255–270.

[3] I. Berkes and W. Philipp, The size of trigonometric and Walsh series and uniform distribution mod 1. J. London Math. Soc. **50** (1994), 454–464.

[4] I. Berkes and M. Weber, On the convergence of $\sum c_k f(n_k x)$. *Mem. Amer. Math. Soc.* **201** (2009), no. 943, viii+72 pp.

[5] J. Berning, On the multivariate law of the iterated logarithm. *Ann. Probability* **7** (1979), 980–988.

[6] M. Drmota and R. F. Tichy, Sequences, Discrepancies and Applications, Lecture Notes Math. **1651**, Springer, Berlin, Heidelberg, New York, 1997.

[7] H. Finkelstein, The law of the iterated logarithm for empirical distributions. *Ann. Math. Statist.* **42** (1971) 607–615.

[8] K. Fukuyama, The law of the iterated logarithm for discrepancies of $\{\theta^n x\}$. *Acta Math. Hungar.* **118** (2008), 155–170.

[9] K. Fukuyama, A metric discrepancy result for a lacunary sequence with small gaps. Monatsh. Math. **162** (2011), 277–288.

[10] K. Fukuyama, A central limit theorem and a metric discrepancy result for sequences with bounded gaps. In: *Dependence in probability, analysis and number theory*, I. Berkes, R. Bradley, H. Dehling, R. Tichy, eds., pp. 233–246. Kendrick Press, Heber City, UT, 2010.

[11] K. Fukuyama and S. Miyamoto, Metric discrepancy results for Erdős-Fortet sequence. *Studia Sci. Math. Hung.* **49** (2012), 52–78.

[12] H. Kesten, The discrepancy of random sequences $\{kx\}$, Acta Arith. **10** (1964/65), 183-213.

[13] L. Kuipers and H. Niederreiter, Uniform Distribution of Sequences, Wiley, New York, 1974.

[14] H. Oodaira, The law of the iterated logarithm for Gaussian processes. *Ann. Probability* **1** (1973), 954–967.

[15] W. Philipp, Limit theorems for lacunary series and uniform distribution mod 1. *Acta Arith.* **26** (1975), 241–251.

[16] W. Philipp, A functional law of the iterated logarithm for empirical distribution functions of weakly dependent random variables, *Annals of Prob.* **5** (1977), 319–350.

[17] P. Schatte, On a law of the iterated logarithm for sums mod 1 with applications to Benford's law, *Prob. Th. Rel. Fields* **77** (1988), 167–178.

[18] P. Schatte, On the asymptotic uniform distribution of sums reduced mod 1, *Math. Nachr.* **115** (1984), 275–281.

[19] P. Schatte, On a uniform law of the iterated logarithm for sums mod 1 and Benford's law. Lithuanian Math. J. **31** (1991), 133–142.

[20] G. Shorack and J. Wellner, Empirical Processes with Applications in Statistics. *Wiley*, 1986.

[21] V. Strassen, An invariance principle for the law of the iterated logarithm. *Z. Wahrsch. Verw. Gebiete* **3**, 211–226 (1964).

[22] M. Weber, Discrepancy of randomly sampled sequences of integers, *Math. Nachr.* **271** (2004), 105–110.

[23] H. Weyl, Über die Gleichverteilung von Zahlen mod. Eins, Math. Ann. **77** (1916), 313-352.

Received January 24, 2014
Accepted May 19, 2014

I. Berkes
Institute of Statistics
Graz University of Technology
Kopernikusgasse 24
8010 Graz, Austria
E-mail: berkes@tugraz.at

M. Raseta
Department of Mathematics
University of York
York YO10 5DD
Great Britain
E-mail: marko.raseta@york.ac.uk