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HILBERT SPACE WITH REPRODUCING KERNEL AND UNIFORM DISTRIBUTION PRESERVING MAPS, II

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ABSTRACT. For Hilbert space H with reproducing kernel $K(\mathbf{x},\mathbf{y})$, we express the mean square worst-case error

$$\int_{[0,1]^s} \sup_{\substack{f\in H\\ ||f||\le 1}} \left| \frac{1}{N}\sum_{n=0}^{N-1} f\big(\Phi(\{\mathbf{x}_n+\boldsymbol{\sigma}\})\big) - \int_{[0,1]^s} f(\mathbf{x})\mathrm{d}\mathbf{x}\right|^2 \mathrm{d}\boldsymbol{\sigma}$$

as

$$\frac{1}{N^2}\sum_{n,m=0}^{N-1}\int_{[0,1]^s} K\big(\Phi(\mathbf{x}),\Phi(\mathbf{y})\big)\mathrm{d}_{\mathbf{x}}\mathrm{d}_{\mathbf{y}} g_{m,n}(\mathbf{x},\mathbf{y}) - \int_{[0,1]^{2s}} K(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y},$$

where $\Phi(\mathbf{x})$ is a uniform distribution preserving map, $\mathbf{x}_0,\dots,\mathbf{x}_{N-1}\in[0,1)^s$, and $g_{m,n}(\mathbf{x},\mathbf{y})$ are copulas associated with points $\mathbf{x}_m$ and $\mathbf{x}_n$. Applying this, for dimension $s=1$, we find that the minimum of the mean square worst-case error is attained in the sequence $x_n=\frac{n}{N}$, for the kernel $K(x,y)=1-\max(x,y)$ and $\Phi(x)=x$.

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1. Introduction

Let $\mathbf{x}\in[0,1]^s$. Let H be a Hilbert space of some functions $f(\mathbf{x})$ from $L^2([0,1]^s)$, with a reproducing kernel $K(\mathbf{x},\mathbf{y})$ from $L^1([0,1]^s)$, and let

$$\mathbf{x}_0,\dots,\mathbf{x}_{N-1}\in[0,1)^s$$

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be an N-terms sequence. I.H. Sloan and H. Woźniakowski [11] proved that

$$\begin{aligned}
&\sup_{\substack{f\in H\\ ||f||\le 1}} \left| \frac{1}{N}\sum_{n=0}^{N-1} f(\mathbf{x}_n) - \int_{[0,1]^s} f(\mathbf{x})\mathrm{d}\mathbf{x} \right|^2 \\
&= \int_{[0,1]^s}\int_{[0,1]^s} K(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} - \frac{2}{N}\sum_{n=0}^{N-1}\int_{[0,1]^s} K(\mathbf{x}_n,\mathbf{y})\mathrm{d}\mathbf{y} \\
&+ \frac{1}{N^2}\sum_{n,m=0}^{N-1} K(\mathbf{x}_m,\mathbf{x}_n), \qquad (1)
\end{aligned}$$

where the left hand side of (1) is said to be the worst-case square error. In the quasi-Monte Carlo (QMC) integration the following mean square worst--case error

$$\int_{[0,1]^s} \sup_{\substack{f\in H\\ ||f||\le 1}} \left| \frac{1}{N}\sum_{n=0}^{N-1} f(\mathbf{x}_n\oplus\boldsymbol{\sigma}) - \int_{[0,1]^s} f(\mathbf{x})\mathrm{d}\mathbf{x} \right|^2 \mathrm{d}\boldsymbol{\sigma} \qquad (2)$$

is studied for digital shift $\mathbf{x}\oplus\boldsymbol{\sigma}$. For a special map $\Phi(\mathbf{x})$ called the tent map and for some sequence $\mathbf{x}_n$ the following mean square worst-case error

$$\int_{[0,1]^s} \sup_{\substack{f\in H\\ ||f||\le 1}} \left| \frac{1}{N}\sum_{n=0}^{N-1} f\big(\Phi(\mathbf{x}_n\oplus\boldsymbol{\sigma})\big) - \int_{[0,1]^s} f(\mathbf{x})\mathrm{d}\mathbf{x} \right|^2 \mathrm{d}\boldsymbol{\sigma} \qquad (3)$$

is better than (2), cf. [2]. In the previous paper [1] there is noted that the tent map $\Phi(\mathbf{x})$ and shift $\mathbf{x}\oplus\boldsymbol{\sigma}$ belong to the so called uniform distribution preserving maps. Here a map $\Phi : [0,1]^s \to [0,1]^s$ is called uniform distribution preserving (u.d.p.) map if for every uniformly distributed (u.d.) sequence $\mathbf{x}_n$, $n = 1, 2, \dots$, the image $\Phi(\mathbf{x}_n)$ is again a u.d. sequence. By Weyl limit [10, p. 1–62] we have the following main criterion of u.d.p. [10, 2.5.1.].

Theorem 1. *A map $\Phi(\mathbf{x})$ is u.d.p. if and only if $\Phi(\mathbf{x})$ is Riemann integrable and for every Riemann integrable $f : [0,1]^s \to \mathbb{R}$ we have*

$$\int_{[0,1]^s} f\big(\Phi(\mathbf{x})\big)\mathrm{d}\mathbf{x} = \int_{[0,1]^s} f(\mathbf{x})\mathrm{d}\mathbf{x}. \qquad (4)$$

In [1] the sequence $\mathbf{x}_0, \dots, \mathbf{x}_{N-1}$ is replaced by $\Psi(\mathbf{x}_0,\boldsymbol{\sigma}), \dots, \Psi(\mathbf{x}_{N-1},\boldsymbol{\sigma})$, where $\Psi(\mathbf{x},\boldsymbol{\sigma})$ is a u.d.p. map with respect to $\mathbf{x}$ and $\boldsymbol{\sigma}$, simultaneously.

Applying (4) the mean square worst-case error (1) is expressed as

$$\int_{[0,1]^s} \sup_{\substack{f\in H\\ ||f||\le 1}} \left| \frac{1}{N}\sum_{n=0}^{N-1} f\big(\Psi(\mathbf{x}_n,\boldsymbol{\sigma})\big) - \int_{[0,1]^s} f(\mathbf{x})\mathrm{d}\mathbf{x} \right|^2 \mathrm{d}\boldsymbol{\sigma} =$$

$$\frac{1}{N^2}\sum_{n,m=0}^{N-1} \int_{[0,1]^s} K\big(\Psi(\mathbf{x}_m,\boldsymbol{\sigma}),\Psi(\mathbf{x}_n,\boldsymbol{\sigma})\big)\mathrm{d}\boldsymbol{\sigma} - \int_{[0,1]^{2s}} K(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \quad (5)$$

because of (4), u.d.p. of $\Psi(\mathbf{x},\boldsymbol{\sigma})$ implies

$$\int_{[0,1]^s} K\big(\Psi(\mathbf{x}_n,\boldsymbol{\sigma}),\mathbf{y}\big)\mathrm{d}\boldsymbol{\sigma} = \int_{[0,1]^s} K(\boldsymbol{\sigma},\mathbf{y})\mathrm{d}\boldsymbol{\sigma}$$

and thus

$$\frac{2}{N}\sum_{n=0}^{N-1}\int_{[0,1]^s}\int_{[0,1]^s} K(\Psi(\mathbf{x}_n,\boldsymbol{\sigma}),\mathbf{y})\mathrm{d}\mathbf{y}\mathrm{d}\boldsymbol{\sigma} = 2\int_{[0,1]^{2s}} K(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}. \quad (6)$$

By using Fourier-Walsh expansion of the kernel $K(\mathbf{x},\mathbf{y})$

$$K(\mathbf{x},\mathbf{y}) = \sum_{\mathbf{k},\mathbf{k}'\in\mathbb{N}_0^s} \widehat{K}(\mathbf{k},\mathbf{k}')\overline{\mathrm{wal}_{\mathbf{k}}(\mathbf{x})}\mathrm{wal}_{\mathbf{k}'}(\mathbf{y}),$$

where

$$\widehat{K}(\mathbf{k},\mathbf{k}') = \int_{[0,1]^{2s}} K(\mathbf{x},\mathbf{y})\mathrm{wal}_{\mathbf{k}}(\mathbf{x})\overline{\mathrm{wal}_{\mathbf{k}'}(\mathbf{y})}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \quad (7)$$

in [1, Thm. 2] the following theorem is proved. [1]

Theorem 2. *For every sequence $\mathbf{x}_0,\dots,\mathbf{x}_{N-1}$ in the unit cube $[0,1)^s$ and every u.d.p. map $\Phi(\mathbf{x})$ and an arbitrary kernel $K(\mathbf{x},\mathbf{y})$ with Fourier-Walsh expansion* (7) *we have*

$$\int_{[0,1]^s} \sup_{\substack{f\in H\\ ||f||\le 1}} \left| \frac{1}{N}\sum_{n=0}^{N-1} f\big(\Phi(\mathbf{x}_n\oplus\boldsymbol{\sigma})\big) - \int_{[0,1]^s} f(\mathbf{x})\mathrm{d}\mathbf{x} \right|^2 \mathrm{d}\boldsymbol{\sigma} =$$

$$\sum_{\substack{\mathbf{k}\in\mathbb{N}_0^s\\ \mathbf{k}\ne\mathbf{0}}} \widehat{K}_1(\mathbf{k},\mathbf{k})\left|\frac{1}{N}\sum_{n=0}^{N-1}\mathrm{wal}_{\mathbf{k}}(\mathbf{x}_n)\right|^2, \quad (8)$$

where $\widehat{K}_1(\mathbf{k},\mathbf{k}) = \int_{[0,1]^{2s}} K\big(\Phi(\mathbf{x}),\Phi(\mathbf{y})\big)\mathrm{wal}_{\mathbf{k}}(\mathbf{x})\overline{\mathrm{wal}_{\mathbf{k}}(\mathbf{y})}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}$.

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[1]Theorem 2 extend [2, Thm. 4] or [3, Thm. 12.7], which was proved for a Sobolev weighted space.

Theorem 2 separates the map $\Phi(\mathbf{x})$ and the sequence $\mathbf{x}_n$ in the mean square worst-case error (8).

The aim of this paper is to present a new expression of the mean square worst-case error by applying the theory of distribution functions (d.f.s) of sequences and the Riemann-Stieltjes integration. In Part 2, there is proved another type of separation as in (8), denoted by (9). Then in Part 3 we apply (9) in one-dimensional case for the special kernel $K(x,y) = 1 - \max(x,y)$ and for some extension $\Phi(x)$ of the tent map $\Phi(x) = 1-|2x-1|$. In Part 4 we also find the mean square worst-case error for $\Phi(x) = bx \bmod 1$. Finally, in Part 6, for the identity map $\Phi(x) = x$ we find that the minimum of the mean square worst-case error (9) over arbitrary points $x_0, x_1, \dots, x_{N-1}$ in $[0,1)$ is attained at $x_i - x_{i-1} = \frac{1}{N}$.

2. Distribution functions method for dimension s

Following [1] we define:

• $\Psi(\mathbf{x},\boldsymbol{\sigma})$ is a u.d.p. map of the form $\Phi(\mathbf{x}\oplus\boldsymbol{\sigma})$ or $\Phi(\{\mathbf{x}+\boldsymbol{\sigma}\})$, where $\Phi(\mathbf{x})$ is an arbitrary u.d.p. map.

• $x = \frac{x_0}{b} + \frac{x_1}{b^2} + \dots$ is a b-adic representation of $x \in [0,1)$, and

• $\sigma = \frac{\sigma_0}{b} + \frac{\sigma_1}{b^2} + \dots$, then

• $x \oplus \sigma = \frac{x_0+\sigma_0 \pmod b}{b} + \frac{x_1+\sigma_1 \pmod b}{b^2} + \dots$,

• $\mathbf{x}\oplus\boldsymbol{\sigma} = (x_1\oplus\sigma_1, x_2\oplus\sigma_2, \dots, x_s\oplus\sigma_s)$.

• $\{\mathbf{x}+\boldsymbol{\sigma}\} = (\{x_1+\sigma_1\}, \{x_2+\sigma_2\}, \dots, \{x_s+\sigma_s\})$,

• $\boldsymbol{\sigma}_i$, $i = 0,1,\dots$, is a u.d. sequence in $[0,1)^s$,

• $g_{m,n}(\mathbf{x},\mathbf{y})$ is the asymptotic distribution function (a.d.f.) of the sequence $(\mathbf{x}_m\oplus\boldsymbol{\sigma}_i, \mathbf{x}_n\oplus\boldsymbol{\sigma}_i)$, $i = 0,1,2,\dots$,

• also the same notation $g_{m,n}(\mathbf{x},\mathbf{y})$ is used for a.d.f. of the sequence $(\{\mathbf{x}_m+\boldsymbol{\sigma}_i\}, \{\mathbf{x}_n+\boldsymbol{\sigma}_i\})$, $i = 0,1,2,\dots$

• We distinguish $g_{m,n}(\mathbf{x},\mathbf{y})$ depending on whether $\Psi(\mathbf{x},\boldsymbol{\sigma}) = \Phi(\mathbf{x}\oplus\boldsymbol{\sigma})$ or $\Psi(\mathbf{x},\boldsymbol{\sigma}) = \Phi(\{\mathbf{x}+\boldsymbol{\sigma}\})$.

• Note that $g_{m,n}(\mathbf{x},\mathbf{1}) = \mathbf{x}$ and $g_{m,n}(\mathbf{1},\mathbf{y}) = \mathbf{y}$, and thus $g_{m,n}(\mathbf{x},\mathbf{y})$ is a copula, see definition in [12, 2.3.].

• $\Phi(x) = 1-|2x-1|$, $x\in[0,1]$, is u.d.p. map called the tent map or baker's map, see [5].

• Here, as usual, $\{x\}$ is the fractional part of x.

The following result holds for an arbitrary dimension s.

Theorem 3. *For every sequence $\mathbf{x}_0,\dots,\mathbf{x}_{N-1}$ in the unit cube $[0,1)^s$ and every u.d.p. map $\Phi(\mathbf{x})$ and an arbitrary continuous kernel $K(\mathbf{x},\mathbf{y})$ we have*

$$\int_{[0,1]^s}\sup_{\substack{f\in H\\ ||f||\le 1}}\left|\frac{1}{N}\sum_{n=0}^{N-1}f\big(\Psi(\mathbf{x}_n,\boldsymbol{\sigma})\big)-\int_{[0,1]^s}f(\mathbf{x})\mathrm{d}\mathbf{x}\right|^2\mathrm{d}\boldsymbol{\sigma}=$$

$$\frac{1}{N^2}\sum_{n,m=0}^{N-1}\int_{[0,1]^s}K\big(\Phi(\mathbf{x}),\Phi(\mathbf{y})\big)\mathrm{d}_{\mathbf{x}}\mathrm{d}_{\mathbf{y}}g_{m,n}(\mathbf{x},\mathbf{y})-\int_{[0,1]^{2s}}K(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}.\quad(9)$$

P r o o f. The integral (5) can be computed as limits

$$\frac{1}{M}\sum_{i=0}^{M-1}K\big(\Psi(\mathbf{x}_m,\boldsymbol{\sigma}_i),\Psi(\mathbf{x}_n,\boldsymbol{\sigma}_i)\big)\to\int_{[0,1]^s}K\big(\Psi(\mathbf{x}_m,\boldsymbol{\sigma}),\Psi(\mathbf{x}_n,\boldsymbol{\sigma})\big)\mathrm{d}\boldsymbol{\sigma},$$

$$\to\int_{[0,1]^s}K\big(\Phi(\mathbf{x}),\Phi(\mathbf{y})\big)\mathrm{d}_{\mathbf{x}}\mathrm{d}_{\mathbf{y}}g_{m,n}(\mathbf{x},\mathbf{y}),$$

where $M\to\infty$. Now, from (5), we have (9). □

Remark 4. Theorem 3 separates the map $\Phi(\mathbf{x})$ and the sequence $\mathbf{x}_n$ again by (9). Further an explicit formula of d.f. $g_{m,n}(\mathbf{x},\mathbf{y})$ is given only for one-dimensional case in Part 5. Where it is used that

- $g_{m,n}(x,y)=|h_m^{-1}([0,x))\cap h_n^{-1}([0,y))|$, where
- $h_n(\sigma)=x_n\oplus\sigma$ or $h_n(\sigma)=\{x_n+\sigma\}$ and

the Riemann-Stieltjes integral (9) is computed using integration by parts.

3. The case $s=1$

Theorem 5. *Let $\Psi(x,\sigma)$ be $\Phi(x\oplus\sigma)$ or $\Phi(\{x+\sigma\})$, where $\Phi(x)$ is an arbitrary u.d.p. map. Let $x_0,\dots,x_{N-1}$ be a sequence in $[0,1)$ and $K(x,y)$ be a continuous kernel. Then*

$$\int_0^1\sup_{\substack{f\in H\\ ||f||\le 1}}\left|\frac{1}{N}\sum_{n=0}^{N-1}f\big(\Psi(x_n,\sigma)\big)-\int_0^1 f(x)\mathrm{d}x\right|^2\mathrm{d}\sigma=$$

$$\int_0^1\int_0^1\left(\frac{1}{N^2}\sum_{m,n=0}^{N-1}g_{m,n}(x,y)-xy\right)\mathrm{d}_x\mathrm{d}_yK\big(\Phi(x),\Phi(y)\big).\quad(10)$$

P r o o f. To compute $\int_0^1\int_0^1 K\big(\Phi(x),\Phi(y)\big)\mathrm{d}_x\mathrm{d}_y g_{m,n}(x,y)$, we use integration by parts

$$\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y) = F(1,1) - \int_0^1 g(1,y)\mathrm{d}_y F(1,y) - \int_0^1 g(x,1)\mathrm{d}_x F(x,1) + \int_0^1\int_0^1 g(x,y)\mathrm{d}_x\mathrm{d}_y F(x,y) \quad (11)$$

which holds for an arbitrary continuous $F(x,y)$ and every d.f. $g(x,y)$.

Remark 6. For every m,n we have

$$g_{m,n}(x,1) = x \quad \text{and} \quad g_{m,n}(1,y) = y,$$

i.e. $g_{m,n}(x,y)$ is a two-dimensional copula. Every copula is continuous [7]. Furthermore by Fréchet-Hoeffiding bounds

$$\max(x+y-1,0) \le g_{m,n}(x,y) \le \min(x,y) \quad \text{for} \quad (x,y)\in[0,1]^2$$

and thus

$$\max(x+y-1,0) - xy \le \frac{1}{N^2}\sum_{m,n=0}^{N-1} g_{m,n}(x,y) - xy \le \min(x,y) - xy. \quad (12)$$

P r o o f o f (11). Integration by parts gives

$$\begin{aligned}
&\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y)\\
&= \left[\int_0^1 F(x,y)\mathrm{d}_y g(x,y)\right]_{x=0}^{x=1} - \int_0^1\int_0^1 \mathrm{d}_y g(x,y)\mathrm{d}_x F(x,y)\\
&= \int_0^1 F(1,y)\mathrm{d}_y g(1,y) - \int_0^1\int_0^1 \mathrm{d}_y g(x,y)\mathrm{d}_x F(x,y)\\
&= [F(1,y)g(1,y)]_{y=0}^{y=1} - \int_0^1 g(1,y)\mathrm{d}_y F(1,y)\\
&\quad - \left[\int_0^1 g(x,y)\mathrm{d}_x F(x,y)\right]_{y=0}^{y=1}\\
&\quad + \int_0^1\int_0^1 g(x,y)\mathrm{d}_y\mathrm{d}_x F(x,y).
\end{aligned}$$

Using $g(0,y) = g(x,0) = 0$ for every $x,y\in[0,1]$ we have (11). □

Now, applying (11) to (9) we find

$$\begin{aligned}
&\int_0^1 \sup_{\substack{f\in H\\ ||f||\leq 1}} \left| \frac{1}{N}\sum_{n=0}^{N-1} f\big(\Psi(x_n,\sigma)\big) - \int_0^1 f(x)\mathrm{d}x \right|^2 \mathrm{d}\sigma \\
&= K\big(\Phi(1),\Phi(1)\big) - \int_0^1 y\mathrm{d}_y K\big(\Phi(1),\Phi(y)\big) \\
&\quad - \int_0^1 x\mathrm{d}_x K\big(\Phi(x),\Phi(1)\big) \\
&\quad + \int_0^1\int_0^1 \left(\frac{1}{N^2}\sum_{m,n=0}^{N-1} g_{m,n}(x,y)\right)\mathrm{d}_x\mathrm{d}_y K\big(\Phi(x),\Phi(y)\big) \\
&\quad - \int_0^1\int_0^1 K(x,y)\mathrm{d}x\mathrm{d}y. \qquad (13)
\end{aligned}$$

From u.d.p. of $\Phi(x)$ and (11) follows that

$$\begin{aligned}
\int_0^1\int_0^1 K(x,y)\mathrm{d}x\mathrm{d}y &= \int_0^1\int_0^1 K\big(\Phi(x),\Phi(y)\big)\mathrm{d}x\mathrm{d}y \\
&= K\big(\Phi(1),\Phi(1)\big) - \int_0^1 y\mathrm{d}_y K\big(\Phi(1),\Phi(y)\big) \\
&\quad - \int_0^1 x\mathrm{d}_x K\big(\Phi(x),\Phi(1)\big) \\
&\quad + \int_0^1\int_0^1 xy\mathrm{d}_x\mathrm{d}_y K\big(\Phi(x),\Phi(y)\big). \qquad (14)
\end{aligned}$$

Adding (13) and (14) we derive (10). □

In the following theorem we apply Theorem 5, or precisely (13), to the new u.d.p. map called the tent map. The simple tent map is defined as

$$\Phi_0(x) = 1 - |2x-1|.$$

We extend this map to the map $\Phi(x)$ with the graph in Fig. 1 and define by the following: putting

$$I_i = \left[\frac{i-1}{b}, \frac{i}{b}\right), \quad i = 1,2,\dots,b,$$

we define

$$\Phi(x) = \big((-1)^{i-1}(bx-i-1)\big) + \frac{1}{2}(1+(-1)^i) \text{ if } x\in I_i\,. \qquad (15)$$

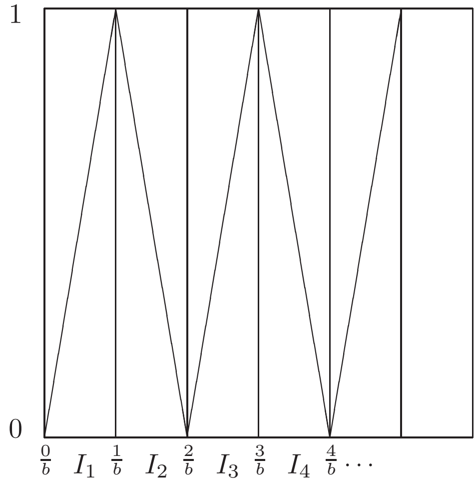

FIGURE 1. The graph of the tent map $\Phi(x)$.

THEOREM 7. *Let $K(x,y)=1-\max(x,y)$, $\Phi(x)$ be the tent u.d.p. map and $2|b$. Then we have*

$$\int_0^1 \sup_{\substack{f\in H\\ ||f||\le 1}} \left|\frac{1}{N}\sum_{n=0}^{N-1} f\big(\Phi(x_n\oplus\sigma)\big)-\int_0^1 f(x)\mathrm{d}x\right|^2 \mathrm{d}\sigma$$
$$=-\frac{1}{3}+\frac{1}{N^2}\sum_{m,n=0}^{N-1}\left[\sum_{\substack{i,j=1\\ 2|i-j}}^{b} b\int_{\frac{i-1}{b}}^{\frac{i}{b}} g_{m,n}\left(x,x+\frac{j-i}{b}\right)\mathrm{d}x\right.$$
$$\left.-\sum_{\substack{i,j=1\\ 2\nmid i-j}}^{b} b\int_{\frac{i-1}{b}}^{\frac{i}{b}} g_{m,n}\left(x,-x+\frac{i+j-1}{b}\right)\mathrm{d}x\right]. \tag{16}$$

The same holds also for $\Phi(\{x_n+\sigma\})$ instead of $\Phi(x_n\oplus\sigma)$. The base b of the tent u.d.p. map $\Phi(x)$ is independent of the base of he shift $x_n\oplus\sigma$.

P r o o f. The Riemann-Stieltjes integral $\int_0^1\int_0^1 g(x,y)\mathrm{d}_x\mathrm{d}_yF(x,y)$ is defined as the limit

$$\sum_{k=1}^{m}\sum_{l=1}^{n} g(\alpha_k,\beta_l)\big(F(x_k,y_l)+F(x_{k+1},y_{l+1})-F(x_k,y_{l+1})-F(x_{k+1},y_l)\big)\to$$
$$\int_0^1\int_0^1 g(x,y)\mathrm{d}_x\mathrm{d}_yF(x,y) \tag{17}$$

if the maximal diameter of the rectangles $[x_k, x_{k+1}] \times [y_l, y_{l+1}]$ tends to zero. This integral exists for a continuous $g(x, y)$ and for $F(x, y)$ with a bounded variation. In the following we assume that these conditions are valid. For partitions $0 = x_0 < x_1 < \cdots < x_m = 1$ of x-axis and $0 = y_0 < y_1 < \cdots < y_n = 1$ of y-axis we assume that they are the same and contain $\frac{i}{b}$, $i = 1, 2, \ldots, b$. Denote the differential $\mathrm{d}_x\mathrm{d}_y F(x, y)$ for the rectangle $\Box = [x_k, x_{k+1}] \times [y_l, y_{l+1}]$ by

$$\Box F(x, y) = F(x_k, y_l) + F(x_{k+1}, y_{l+1}) - F(x_k, y_{l+1}) - F(x_{k+1}, y_l). \tag{18}$$

If the diameter of $\Box$ tends to zero, $\Box \to 0$, then we find the differential $\mathrm{d}_x\mathrm{d}_y F(x, y)$ as

$$\mathrm{d}_x\mathrm{d}_y F(x, y) = F(x, y) + F(x + \mathrm{d}x, y + \mathrm{d}y) - F(x, y + \mathrm{d}y) - F(x + \mathrm{d}x, y).$$

By the definition of Riemann-Stieltjes integral

$$\int_0^1 \int_0^1 g(x, y)\mathrm{d}_x\mathrm{d}_y F(x, y) = \sum_{i,j=1}^{b} \iint_{I_i \times I_j} g(x, y)\mathrm{d}_x\mathrm{d}_y F(x, y).$$

In the following we put $F(x, y) = \max\bigl(\Phi(x), \Phi(y)\bigr)$ and distinguish four cases: 1^0. i, j-are even, $x \in I_i$ and $y \in I_j$, see Fig. 2.

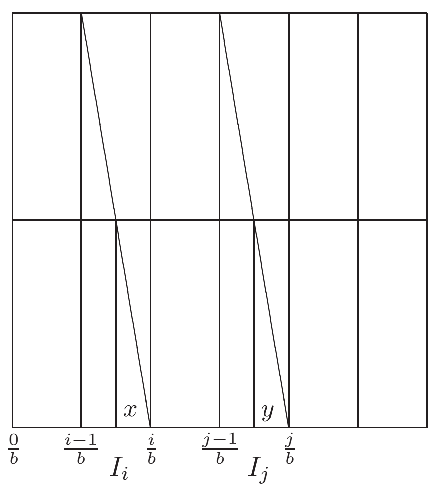

FIGURE 2. The graph of the tent map $\Phi(x)$ in the case 1^0.

Then we have

$$\Phi(x) = \Phi(y) \Leftrightarrow b\left(\frac{i}{b} - x\right) = b\left(\frac{j}{b} - y\right) \Leftrightarrow y = x + \frac{j - i}{b}$$

and

$$\Phi(x) = \Phi(y) > \Phi(y + \mathrm{d}y).$$

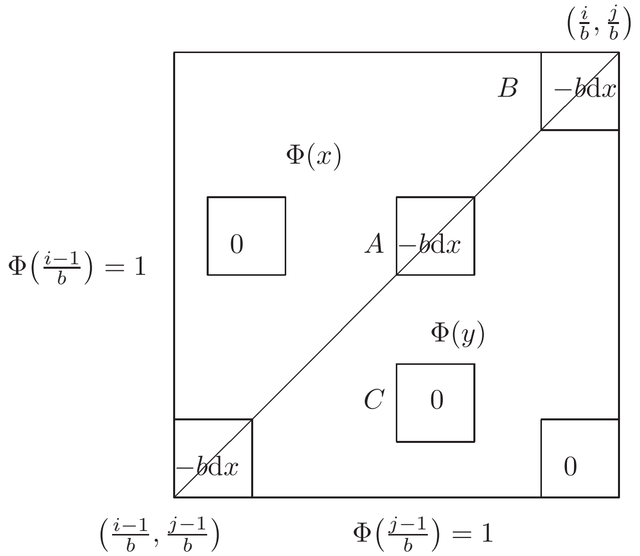

Figure 3.

In Fig. 3 we have the differential $F(x,y)$ in $I_i \times I_j$ in the case 1^0, where the differential $\square F(x,y)$ for squares $\square = A, B, C$, respectively, is computed by means of definition (18):

$$
\begin{aligned}
AF(x,y) &= \big(\Phi(x_k) = \Phi(y_l)\big) + \big(\Phi(x_{k+1}) = \Phi(y_{l+1})\big) - \Phi(x_k) - \Phi(y_l)\\
&= \Phi(x_{k+1}) - \Phi(x_k) = -b(x_{k+1} - x_k),\\
BF(x,y) &= \big(\Phi(x_k) = \Phi(y_l)\big) + \big(\Phi(x_{k+1}) = \Phi(y_{l+1}) = 0\big) - \Phi(x_k) - \Phi(y_l)\\
&= \Phi(x_{k+1}) - \Phi(x_k) = -b(x_{k+1} - x_k),\\
CF(x,y) &= \Phi(y_l) + \Phi(y_{l+1}) - \Phi(y_l) - \Phi(y_{l+1}) = 0.
\end{aligned}
$$

From this

$$
\iint_{I_i \times I_j} g(x,y)\mathrm{d}_x\mathrm{d}_y F(x,y) = \int_{\frac{i-1}{b}}^{\frac{i}{b}} g\left(x, x + \frac{j-i}{b}\right)(-b)\mathrm{d}x. \tag{19}
$$

Similarly,

2^0. i, j-are odd, $x \in I_i$ and $y \in I_j$, see Fig. 4.

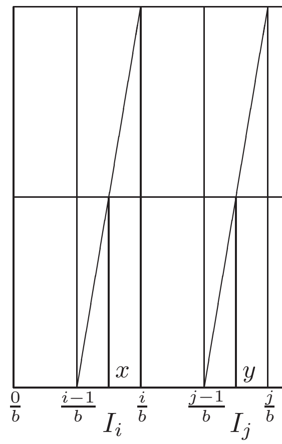

FIGURE 4. The graph of $\Phi(x)$ in the case 2^0.

$$\Phi(x) = \Phi(y) \Leftrightarrow b\left(x - \frac{i-1}{b}\right) = b\left(y - \frac{j-1}{b}\right) \Leftrightarrow y = x + \frac{j-i}{b}$$

and

$$\Phi(x) = \Phi(y) < \Phi(y + \mathrm{d}y).$$

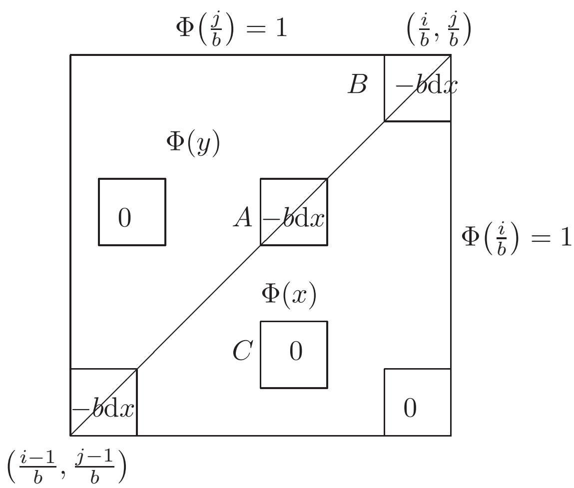

FIGURE 5.

In Fig. 5 we have the differential $F(x, y)$ in $I_i \times I_j$ in the case 2^0, where the differential $\square F(x, y)$ for squares $\square = A, B, C$, respectively, is computed by means of definition (18):

$$\begin{aligned}AF(x,y) &= \big(\Phi(x_k)=\Phi(y_l)\big)+\big(\Phi(x_{k+1})=\Phi(y_{l+1})\big)-\Phi(y_{l+1})-\Phi(x_{k+1})\\ &= \Phi(x_k)-\Phi(x_{k+1})=-b(x_{k+1}-x_k),\end{aligned}$$

$$BF(x,y)=\big(\Phi(x_k)=\Phi(y_l)\big)+1-1-1=\Phi(x_k)-\Phi(1)=-b(1-x_k),$$

$$CF(x,y)=\Phi(x_k)+\Phi(x_{k+1})-\Phi(x_k)-\Phi(x_{k+1})=0.$$

Similarly, for other $I_i\times I_j$. Then for Riemann-Stieltjes integral

$$\iint_{I_i\times I_j} g(x,y)\mathrm{d}_x\mathrm{d}_y F(x,y)=\int_{\frac{i-1}{b}}^{\frac{i}{b}} g\left(x,x+\frac{j-i}{b}\right)(-b)\mathrm{d}x. \tag{20}$$

3^0. i-even and j-odd, and $x\in I_i$ and $y\in I_j$, see Fig. 6.

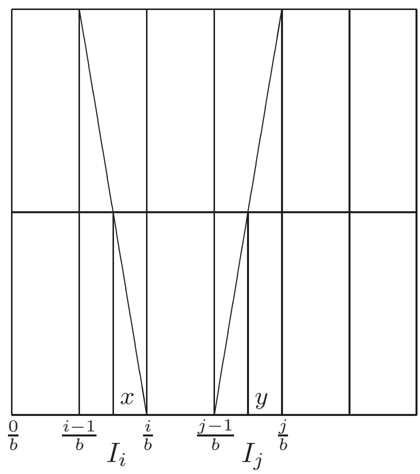

FIGURE 6. The graph of $\Phi(x)$ in the case 3^0.

$$\Phi(x)=\Phi(y)\Leftrightarrow b\left(\frac{i}{b}-x\right)=b\left(y-\frac{j-1}{b}\right)\Leftrightarrow y=-x+\frac{i+j-1}{b}$$

and

$$\Phi(x)=\Phi(y)<\Phi(y+\mathrm{d}y).$$

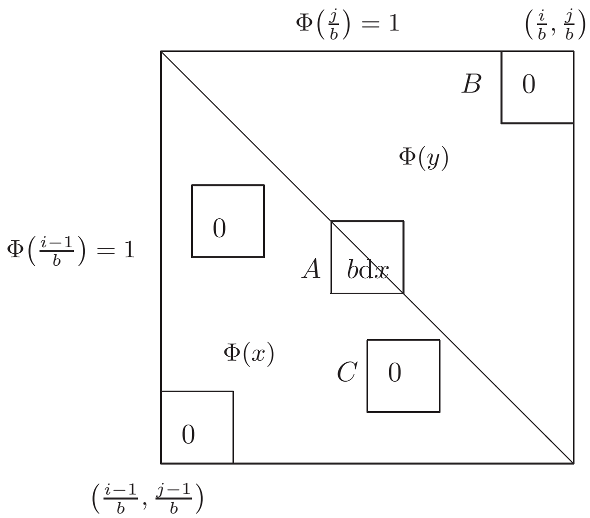

FIGURE 7.

In Fig. 7 we have the differential $F(x,y)$ in $I_i \times I_j$ in the case 3^0, where the differential $\square F(x,y)$ for squares $\square = A, B, C$, respectively, is computed by means of definition (18):

$$\begin{aligned} AF(x,y) &= \Phi(x_k) + \Phi(y_{l+1}) - \big(\Phi(x_k) = \Phi(y_{l+1})\big) - \big(\Phi(x_{k+1}) = \Phi(y_l)\big) \\ &= \Phi(x_k) - \Phi(x_{k+1}) = b(x_{k+1} - x_k), \\ BF(x,y) &= \Phi(y_l) + 1 - 1 - \Phi(y_l) = 0, \\ CF(x,y) &= \Phi(x_k) + \Phi(x_{k+1}) - \Phi(x_k) - \Phi(x_{k+1}) = 0. \end{aligned}$$

From this

$$\iint_{I_i \times I_j} g(x,y)\mathrm{d}_x\mathrm{d}_y F(x,y) = \int_{\frac{i-1}{b}}^{\frac{i}{b}} g\left(x, -x + \frac{i+j-1}{b}\right) b\,\mathrm{d}x. \tag{21}$$

Similarly,

4^0. i-odd and j-even.

$$\iint_{I_i \times I_j} g(x,y)\mathrm{d}_x\mathrm{d}_y F(x,y) = \int_{\frac{i-1}{b}}^{\frac{i}{b}} g\left(x, -x + \frac{i+j-1}{b}\right) b\,\mathrm{d}x. \tag{22}$$

For an application of (13) we note that

$$\Phi(1) = 0, \qquad K\big(\Phi(1), \Phi(1)\big) = 1, \qquad K\big(\Phi(x), \Phi(1)\big) = 1 - \Phi(x),$$

$$\begin{aligned} -\int_0^1 x\mathrm{d}_x K\big(\Phi(x), \Phi(1)\big) &= \int_0^1 x\mathrm{d}_x\Phi(x) \\ &= \int_0^{\frac{1}{b}} xb\mathrm{d}x + \int_{\frac{1}{b}}^{\frac{2}{b}} x(-b)\mathrm{d}x + \int_{\frac{2}{b}}^{\frac{3}{b}} xb\mathrm{d}x + \cdots = -\frac{1}{2}, \end{aligned}$$

$$\int_0^1\int_0^1 K(x,y)\mathrm{d}x\mathrm{d}y = \frac{1}{3}, \qquad g(x,y) = \frac{1}{N^2}\sum_{m,n=0}^{N-1} g_{m,n}(x,y),$$

$$K\big(\Phi(x), \Phi(y)\big) = 1 - F(x,y).$$

Then we find (16). □

4. Further generalization of tent map

To study u.d.p. mat $\Phi(x) = bx \bmod 1$ we use the further generalization of tent u.d.p. map defined in (15). We extended it to a two-parametric u.d.p. tent map $\Phi_{b_1,b_2}(x)$ defined by the graph in Fig. 8 that is also u.d.p. map.

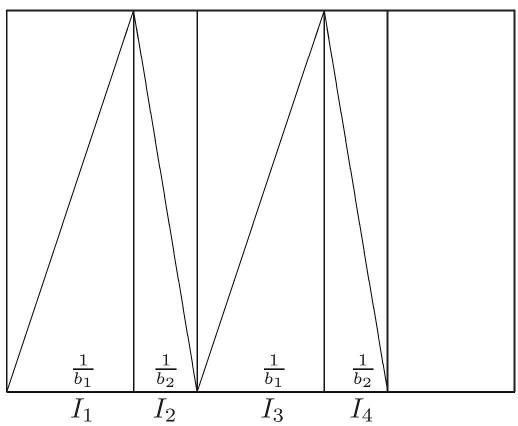

FIGURE 8. $\Phi_{b_1,b_2}(x)$-tent function.

Here note that

$$I_i = \begin{cases} \left[\frac{i}{k} - \frac{1}{b_2}, \frac{i}{k}\right] & \text{if } i \text{ is even,} \\ \left[\frac{i-1}{k}, \frac{i-1}{k} + \frac{1}{b_1}\right] & \text{if } i \text{ is odd.} \end{cases}$$

Then, similarly to (16), the mean square worst-case error can be expressed as

$$\begin{aligned}
&\int_0^1 \sup_{\substack{f\in H\\ ||f||\le 1}} \left| \frac{1}{N}\sum_{n=0}^{N-1} f\big(\Phi_{b_1,b_2}(x_n\oplus\sigma)\big) - \int_0^1 f(x)\mathrm{d}x\right|^2 \mathrm{d}\sigma \\
&= -\frac{1}{3} + \frac{1}{N^2}\sum_{m,n=0}^{N-1}\Bigg[\sum_{\substack{i,j=1\\ i,j-\text{even}}}^{k} b_2\int_{\frac{i}{k}-\frac{1}{b_2}}^{\frac{i}{k}} g_{m,n}\left(x, x+\frac{j-i}{k}\right)\mathrm{d}x \\
&\quad + \sum_{\substack{i,j=1\\ i,j-\text{odd}}}^{k} b_1\int_{\frac{i-1}{k}}^{\frac{i-1}{k}+\frac{1}{b_1}} g_{m,n}\left(x, x+\frac{j-i}{k}\right)\mathrm{d}x \\
&\quad - \sum_{\substack{i,j=1\\ i-\text{even},j-\text{odd}}}^{k} b_2\int_{\frac{i}{k}-\frac{1}{b_2}}^{\frac{i}{k}} g_{m,n}\left(x, \frac{b_2}{b_1}\left(\frac{i}{k}-x\right)+\frac{j-1}{k}\right)\mathrm{d}x \\
&\quad - \sum_{\substack{i,j=1\\ i-\text{odd},j-\text{even}}}^{k} b_1\int_{\frac{i-1}{k}}^{\frac{i-1}{k}+\frac{1}{b_1}} g_{m,n}\left(x, \frac{b_1}{b_2}\left(\frac{i-1}{k}-x\right)+\frac{j}{k}\right)\mathrm{d}x\Bigg], \qquad (23)
\end{aligned}$$

where $K(x,y) = 1-\max(x,y)$, $\Phi_{b_1,b_2}(x)$ is a two-parameters tent u.d.p. map, $\frac{2}{k} = \frac{1}{b_1} + \frac{1}{b_2}$ and $2|k$. This also holds for $\Phi(\{x_n+\sigma\})$ instead of $\Phi(x_n\oplus\sigma)$. The bases b_1, b_2 for the u.d.p. map $\Phi_{b_1,b_2}(x)$ are independent of the base of the shift $x_n\oplus\sigma$. The proof of (23) is similar to the proof of Theorem 7.

Let $\Phi(x) = bx \bmod 1$ be the u.d.p. map and $k = 2b$. Since $K\big(\Phi(x),\Phi(y)\big)$ for $\Phi(x) = bx \bmod 1$ is discontinuous, we cannot use (11). But we can use limits

$$\lim_{b_2\to\infty}\Phi_{b_1,b_2}(x_m\oplus\sigma) = \Phi(x_m\oplus\sigma) \text{ on } [0,1],$$

or

$$\lim_{b_2\to\infty}\Phi_{b_1,b_2}(\{x_m+\sigma\}) = \Phi(\{x_m+\sigma\}),$$

as $b_2\to\infty$, k =constant, $b_1\to b=\frac{k}{2}$. From (23), the following limits hold:

$$b_2\int_{\frac{i}{k}-\frac{1}{b_2}}^{\frac{i}{k}} g_{m,n}\left(x, x+\frac{j-i}{k}\right)\mathrm{d}x \to g_{m,n}\left(\frac{i}{k},\frac{j}{k}\right),$$

$$b_1\int_{\frac{i-1}{k}}^{\frac{i-1}{k}+\frac{1}{b_1}} g_{m,n}\left(x, x+\frac{j-i}{k}\right)\mathrm{d}x \to \frac{k}{2}\int_{\frac{i-1}{k}}^{\frac{i}{k}+\frac{2}{k}} g_{m,n}\left(x, x+\frac{j-i}{k}\right)\mathrm{d}x,$$

$$b_2\int_{\frac{i}{k}-\frac{1}{b_2}}^{\frac{i}{k}} g_{m,n}\left(x,\frac{b_2}{b_1}\left(\frac{i}{k}-x\right)+\frac{j-1}{k}\right)\mathrm{d}x =$$

$$b_1\int_{\frac{j-1}{k}}^{\frac{j-1}{k}+\frac{1}{b_1}} g_{m,n}\left(\frac{i}{k}+\frac{b_1}{b_2}\left(\frac{j-1}{k}-y\right),y\right)\mathrm{d}y \to \frac{k}{2}\int_{\frac{j-1}{k}}^{\frac{j-1}{k}+\frac{2}{k}} g_{m,n}\left(\frac{i}{k},x\right)\mathrm{d}x,$$

$$b_1\int_{\frac{i-1}{k}}^{\frac{i-1}{k}+\frac{1}{b_1}} g_{m,n}\left(x,\frac{b_1}{b_2}\left(\frac{i-1}{k}-x\right)+\frac{j}{k}\right)\mathrm{d}x \to \frac{k}{2}\int_{\frac{j-1}{k}}^{\frac{j-1}{k}+\frac{2}{k}} g_{m,n}\left(x,\frac{j}{k}\right)\mathrm{d}x.$$

Finally, the limit

$$\int_0^1 \sup_{\substack{f\in H\\ ||f||\le 1}} \left|\frac{1}{N}\sum_{n=0}^{N-1} f\big(\Phi_{b_1,b_2}(x_n\oplus\sigma\big) - \int_0^1 f(x)\mathrm{d}x\right|^2 \mathrm{d}\sigma \to$$

$$\int_0^1 \sup_{\substack{f\in H\\ ||f||\le 1}} \left|\frac{1}{N}\sum_{n=0}^{N-1} f\big(\Phi(x_n\oplus\sigma)\big) - \int_0^1 f(x)\mathrm{d}x\right|^2 \mathrm{d}\sigma$$

follows from the expression (5) and from the limit

$$\int_0^1 K\big(\Phi_{b_1,b_2}(x_m\oplus\sigma),\Phi_{b_1,b_2}(x_n\oplus\sigma)\big)\mathrm{d}\sigma \to \int_0^1 K\big(\Phi(x_m\oplus\sigma),\Phi(x_n\oplus\sigma)\big)\mathrm{d}\sigma$$

which follows from the Lebesgue theorem on the dominated convergence. Then we have:

Theorem 8. *Let $K(x,y)=1-\max(x,y)$ be the kernel, let $\Phi(x)=bx \bmod 1$ be the u.d.p. map and $k=2b$. Then we have*

$$\int_0^1 \sup_{\substack{f\in H\\ ||f||\le 1}} \left|\frac{1}{N}\sum_{n=0}^{N-1} f\big(\Phi(x_n\oplus\sigma)\big) - \int_0^1 f(x)\mathrm{d}x\right|^2 \mathrm{d}\sigma$$

$$= -\frac{1}{3}+\frac{1}{N^2}\sum_{m,n=0}^{N-1}\Bigg[\sum_{\substack{i,j=1\\ i,j-even}}^{k} g_{m,n}\left(\frac{i}{k},\frac{j}{k}\right)+\sum_{\substack{i,j=1\\ i,j-odd}}^{k}\frac{k}{2}\int_{\frac{i-1}{k}}^{\frac{i}{k}+\frac{2}{k}} g_{m,n}\left(x,x+\frac{j-i}{k}\right)\mathrm{d}x$$

$$-\sum_{\substack{i,j=1\\ i-even,j-odd}}^{k}\frac{k}{2}\int_{\frac{j-1}{k}}^{\frac{j-1}{k}+\frac{2}{k}} g_{m,n}\left(\frac{i}{k},x\right)\mathrm{d}x-\sum_{\substack{i,j=1\\ i-odd,j-even}}^{k}\frac{k}{2}\int_{\frac{i-1}{k}}^{\frac{i-1}{k}+\frac{2}{k}} g_{m,n}\left(x,\frac{j}{k}\right)\mathrm{d}x\Bigg]. \tag{24}$$

The same holds also for $\Phi(\{x_n+\sigma\})$ instead of $\Phi(x_n\oplus\sigma)$. The base b for this u.d.p. map $\Phi(x)$ is independent of the base of the shift $x_n\oplus\sigma$.

5. Explicit formula for copulas $g_{m,n}(x,y)$

To apply Theorems 5 and 7 we need an expression of $g_{m,n}(x,y)$. For the sequence $(\{x_m+\sigma_i\},\{x_n+\sigma_i\})$, $i=1,2,\dots$ Let $0\le u\le v\le 1$ be fixed and $x,y\in[0,1]$ be variables and define:

$$h_u(x)=\{x+u\},\qquad h_v(y)=\{y+v\},$$

$$H(u,v,x,y)=|h_u^{-1}([0,x))\cap h_v^{-1}([0,y))|.$$

Then we have

$$H(x_m,x_n,x,y)=g_{m,n}(x,y).$$

Using graphs of $h_u(x)$ and $h_v(y)$,

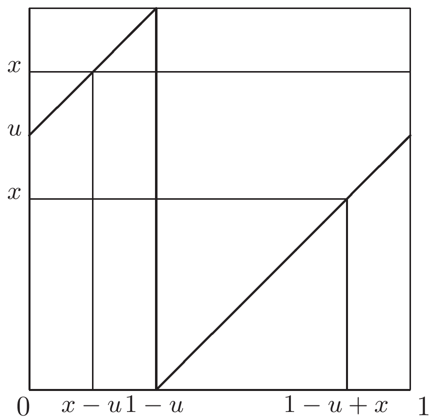

FIGURE 9. The graph of $h_u(x)$.

then we see

$$h_u^{-1}([0,x])=\begin{cases}[1-u,1-u+x] & \text{if } x\le u,\\ [0,x-u]\cup[1-u,1] & \text{if } u\le x.\end{cases}\tag{25}$$

Hence

$$H(u,v,x,y)=$$

$$\begin{cases}\big|[1-u,1-u+x]\cap[1-v,1-v+y]\big| & \text{if } x\le u, y\le v,\\ \big|[1-u,1-u+x]\cap([0,y-v]\cup[1-v,1])\big| & \text{if } x\le u, y>v,\\ \big|([0,x-u]\cup[1-u,1])\cap[1-v,1-v+y]\big| & \text{if } x>u, y\le v,\\ \big|([0,x-u]\cup[1-u,1])\cap([0,y-v]\cup[1-v,1])\big| & \text{if } x>u, y>v.\end{cases}\tag{26}$$

Now we are using minimum and maximum formulae for the length of intersection of two intervals $[\alpha,\beta]$ and $[\gamma,\delta]$

$$\big|[\alpha,\beta]\cap[\gamma,\delta]\big| = \max\big(\min(\beta,\delta)-\max(\alpha,\gamma),0\big). \tag{27}$$

Insert (27) into (26), and we see that

$$H(u,v,x,y) =$$
$$\begin{cases} \max\big(\min(y,x-u+v),0\big) & \text{if } x\le u,\ y\le v,\\ \max\big(\min(x,y-v-1+u),0\big)+\max(v-u+x,0) & \text{if } x\le u,\ y>v,\\ y & \text{if } x>u,\ y\le v,\\ \min(x-u+v,y)+\max(y-v-1+u,0) & \text{if } x>u,\ y>v. \end{cases} \tag{28}$$

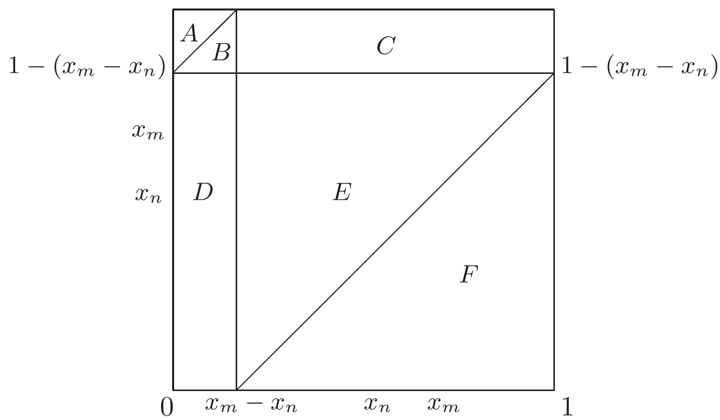

FIGURE 10. Division of $[0,1]^2$.

Using division in Fig. 10 we have

$$H(u,v,x,y) = \begin{cases} x & \text{if } (x,y)\in A,\\ y-\big(1-(u-v)\big) & \text{if } (x,y)\in B,\\ x+y-1 & \text{if } (x,y)\in C,\\ 0 & \text{if } (x,y)\in D,\\ x-(u-v) & \text{if } (x,y)\in E,\\ y & \text{if } (x,y)\in F. \end{cases} \tag{29}$$

Since for a u.d. sequence $u_n \in [0,1)$, $n = 1,2,\dots$, the two-dimensional sequence (u_n, u_n) has an a.d.f. $g(x,y) = \min(x,y)$, then $H(u,v,x,y) = \min(x,y)$ for $u = v$. It can also be seen from (29).

Taken together we get $g_{m,m}(x,y) = \min(x,y)$ and for $x_m \neq x_n$ we have

$$g_{m,n}(x,y) = \begin{cases} x & \text{if } (x,y) \in A, \\ y - (1 - |x_m - x_n|) & \text{if } (x,y) \in B, \\ x + y - 1 & \text{if } (x,y) \in C, \\ 0 & \text{if } (x,y) \in D, \\ x - |x_m - x_n| & \text{if } (x,y) \in E, \\ y & \text{if } (x,y) \in F. \end{cases} \tag{30}$$

6. Application of Theorem 3

In this part we find global minimum of the mean square worst-case error for dimension $s = 1$, kernel $K(x,y) = 1 - \max(x,y)$ and $\Phi(x) = x$.

Theorem 9. *Let $K(x,y) = 1 - \max(x,y)$, $\Phi(\{x+\sigma\}) = \{x+\sigma\}$, $0 \le x_0 < x_1 < \dots < x_{N-2} < x_{N-1} \le 1$ and put $t_i = x_{i+1} - x_i$, $i = 0,1,\dots,N-2$. Then the minimum of mean square worst-case error* (9) *is attained at $t_i = \frac{1}{N}$, $i = 0,1,\dots,N-2$ and this minimum is*

$$\int_0^1 \sup_{\substack{f\in H\\ ||f||\le 1}} \left| \frac{1}{N}\sum_{n=0}^{N-1} f(\{x_n+\sigma\}) - \int_0^1 f(x)\mathrm{d}x \right|^2 \mathrm{d}\sigma = \frac{1}{6N^2}. \tag{31}$$

Proof. We start with the formula (9). The following proof is divided into the parts 1–5 according to the following.

In the part 1 we shall express the integral (9) for the case $s = 1$ as

$$\begin{aligned} &\int_0^1 \int_0^1 K\big(\Phi(x),\Phi(y)\big)\mathrm{d}_x\mathrm{d}_y g_{m,n}(x,y) \\ &= \int_0^{|x_m-x_n|} K(\Phi(x),\Phi(x+1-|x_m-x_n|)\mathrm{d}x \\ &\quad + \int_{|x_m-x_n|}^1 K(\Phi(x),\Phi(x-|x_m-x_n|)\mathrm{d}x. \end{aligned} \tag{32}$$

In the part 2, for $\Phi(x) = x$, and $T_{m,n} = |x_m - x_n| = \sum_{k=m}^{n-1} t_k$, $m < n$, we shall prove

$$\frac{\partial}{\partial t_i}\left(\int_0^{T_{m,n}} K(x, x+1-T_{m,n})\mathrm{d}x + \int_{T_{m,n}}^1 K(x, x-T_{m,n})\mathrm{d}x\right) = 2T_{m,n} - 1 \quad (33)$$

assuming that $T_{m,n}$ contains term t_i.

In the part 3 we shall see that the zero partial derivative

$$\frac{\partial}{\partial t_i}\left(\sum_{\substack{m<n\\ m,n=0}}^{N-1} \int_0^1 \int_0^1 K(x,y)\mathrm{d}_x\mathrm{d}_y g_{m,n}(x,y)\right) = \sum_{\substack{m<n, m,n=0\\ T_{m,n} \text{ contains } t_i}}^{N-1} \Big(2T_{m,n} - 1\Big) = 0 \quad (34)$$

is equivalent to the following linear equation

$$\begin{aligned} t_0(N-2-i+1) + t_1 2(N-2-i+1) + \dots \\ \dots + t_k(k+1)(N-2-i+1) + \dots \\ \dots + t_i(i+1)(N-2-i+1) + t_{i+1}(i+1)(N-2-(i+1)+1) + \dots \\ \dots + t_s(i+1)(N-2-s+1) + \dots \\ \dots + t_{N-2}(i+1) = \frac{1}{2}(i+1)(N-2-i+1). \end{aligned} \quad (35)$$

In the part 4 we shall prove that the system (35), $i = 0, 1, \dots, N-2$, is regular and has the unique solution $t_i = \frac{1}{N}$ for $i = 0, 1, \dots, N-2$.

In the final part 5 we shall compute the minimum of the mean square worst--case error for $t_i = \frac{1}{N}$.

P r o o f o f 1. In this case we calculate (9) directly by Riemann-Stieltjes integration. As in (18) we denote the differential $\mathrm{d}_x\mathrm{d}_y g_{m,n}(x,y)$ for the rectangle $\square = [x_k, x_{k+1}] \times [y_l, y_{l+1}]$ as

$$\begin{aligned} \square g_{m,n}(x,y) = g_{m,n}(x_k, y_l) + g_{m,n}(x_{k+1}, y_{l+1}) \\ - g_{m,n}(x_k, y_{l+1}) - g_{m,n}(x_{k+1}, y_l). \end{aligned} \quad (36)$$

If diameter $\square \to 0$, then we find the differential $\mathrm{d}_x\mathrm{d}_y g_{m,n}(x,y)$ as

$$\begin{aligned} \mathrm{d}_x\mathrm{d}_y g_{m,n}(x,y) = g_{m,n}(x,y) + g_{m,n}(x+\mathrm{d}x, y+\mathrm{d}y) \\ - g_{m,n}(x, y+\mathrm{d}y) - g_{m,n}(x+\mathrm{d}x, y). \end{aligned}$$

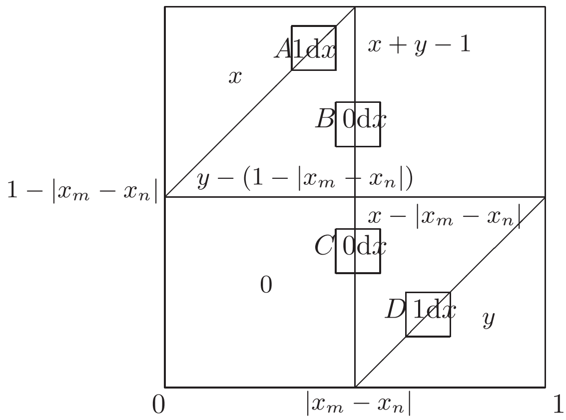

FIGURE 11. Differential of $g_{m,n}(x,y)$.

From Fig. 11 we obtain:

$$\begin{aligned}
Ag_{m,n}(x,y) &= \big(x_k = y_l - (1-|x_m-x_n|)\big) + \big(x_{k+1} = y_{l+1} - (1-|x_m-x_n|)\big)\\
&\qquad - x_k - \big(y_l - (1-|x_m-x_n|)\big)\\
&= y_l - \big(1-|x_m-x_n|\big) + x_{k+1} - x_k - \big(y_l - (1-|x_m-x_n|)\big)\\
&= (x_{k+1}-x_k) \to 1.\mathrm{d}x,
\end{aligned}$$

$$\begin{aligned}
Bg_{m,n}(x,y) &= \big(y_l - (1-|x_m-x_n|)\big) + (x_{k+1}+y_{l+1}-1)\\
&\qquad - \big(y_{l+1} - \big(1-|x_m-x_n|\big)\big) - (x_{k+1}+y_l-1) = 0,
\end{aligned}$$

$$Cg_{m,n}(x,y) = 0 + (x_{k+1}-|x_m-x_n|) - 0 - (x_{k+1}-|x_m-x_n|) = 0,$$

$$\begin{aligned}
Dg_{m,n}(x,y) &= (x_k - |x_m-x_n| = y_l) + (x_{k+1} - |x_m-x_n| = y_{l+1})\\
&\qquad - (x_k - |x_m-x_n|) - y_l\\
&= y_l + (x_{k+1}-|x_m-x_n|) - (x_k-|x_m-x_n|) - y_l\\
&= (x_{k+1}-x_k) \to 1.\mathrm{d}x.
\end{aligned}$$

Thus the differential $\mathrm{d}_x\mathrm{d}_y g_{m,n}(x,y)$ is nonzero only on two lines:

(i) $y = x + (1-|x_m-x_n|)$ if $x\in[0,|x_m-x_n|]$, and

(ii) $y = x - |x_m-x_n|$ if $x\in[|x_m-x_n|,1]$.

Then (i) and (ii) imply (32). This holds for an arbitrary kernel satisfying (1) and an arbitrary u.d.p. function $\Phi(x)$. □

P r o o f o f 2. Here we assume that $K(x,y) = 1 - \max(x,y)$ and $\Phi(x) = x$. We have, for $t+T \in [0,1]$, t is a variable and T is a constant

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{t+T} K(x,x+1-t)\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_{t+T}^{t+T+\mathrm{d}t} K(x,x+1-(t+T)\mathrm{d}x + \int_0^{t+T+\mathrm{d}t} \frac{K\big(x,x+1-(t+T+\mathrm{d}t)\big) - K\big(x,x+1-(t+T)\big)}{\mathrm{d}t}\mathrm{d}x,$$

$$K(t+T,1) + \int_0^{t+T}\left[\frac{\partial K(x,y)}{\partial y}\right]_{y=x+1-(t+T)}\frac{\mathrm{d}}{\mathrm{d}t}\big(x+1-(t+T)\big)\mathrm{d}x = 0+t+T. \quad (37)$$

Similarly,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{t+T}^{1} K\big(x,x-(t+T)\big)\mathrm{d}x\\ &= -K(t+T,0) + \int_{t+T}^{1}\left[\frac{\partial K(x,y)}{\partial y}\right]_{y=x-(t+T)}\frac{\mathrm{d}}{\mathrm{d}t}\big(x-(t+T)\big)\mathrm{d}x\\ &= -1+t+T+0\,. \end{aligned} \quad (38)$$

Now, (37) and (38) imply (33). □

The (34) implies (35) bearing in mind the following number of

$$T_{m,n}, 0 \le m < n \le N-1;$$

$$\begin{aligned}
&\#\{T_{m,n} \text{ contains } t_0, t_i\} = 1(N-2-i+1)\,,\\
&\#\{T_{m,n} \text{ contains } t_1, t_i\} = 2(N-2-i+1)\,,\\
&\#\{T_{m,n} \text{ contains } t_k, t_i\} = (k+1)(N-2-i+1),\ \ k<i\,,\\
&\#\{T_{m,n} \text{ contains } t_i\} = (i+1)(N-2-i+1)\,,\\
&\#\{T_{m,n} \text{ contains } t_{i+1}, t_i\} = (i+1)\big(N-2-(i+1)+1\big)\,,\\
&\#\{T_{m,n} \text{ contains } t_s, t_i\} = (i+1)(N-2-s+1)\,,\ s>i\,,\\
&\#\{T_{m,n} \text{ contains } t_{N-2}, t_i\} = (i+1)\,,\\
&\#\{T_{m,n}\} = (N-2+1)+(N-2)+\cdots+1 = \tfrac{(N-1)N}{2}\,.
\end{aligned}$$

□

Proof of 4. Put in (35)

$$t_0 = t_1 = \dots = t_k = \dots = t_i = \dots = t_s = \dots = t_{N-2} = 1.$$

Then we have

$$\begin{aligned}
&(N-2-i+1) + 2(N-2-i+1) + \dots + (k+1)(N-2-i+1) + \dots \\
&\quad + (i+1)(N-2-i+1) \\
&\quad + (i+1)\bigl(N-2-(i+1)+1\bigr) + \dots + (i+1)(N-2-s+1) + \dots + (i+1) \\
&= \frac{(i+1)(i+2)}{2}(N-2-i+1) + (i+1)\frac{(N-2-i)(N-2-i+1)}{2} \\
&= \frac{(i+1)}{2}(N-2-i+1)(i+2+N-2-i).
\end{aligned}$$

Thus

$$t_0 = t_1 = \dots = t_k = \dots = t_i = \dots = t_s = \dots = t_{N-2} = \frac{1}{N}$$

solve (35) for every $i = 0, 1, \dots, N-2$. The regularity of (35) can be proved by induction. For $N = 4$ the system of linear equations (35) has the form

$$\begin{aligned}
t_0 \cdot 3 + t_1.2 + t_2 \cdot 1 &= \frac{1}{2} \cdot 3, \\
t_0 \cdot 2 + t_1.4 + t_2 \cdot 2 &= \frac{1}{2} \cdot 4, \\
t_0 \cdot 1 + t_1.2 + t_2 \cdot 3 &= \frac{1}{2} \cdot 3,
\end{aligned}$$

and it is regular, having the solution $t_0 = t_1 = t_2 = \frac{1}{4}$. □

Proof of 5. To compute minimum of the mean square worst-case error (31) we start with (5) in one-dimensional form

$$\begin{aligned}
&\frac{1}{N^2}\sum_{m,n=0}^{N-1}\int_0^1\int_0^1 K(x,y)\mathrm{d}_x\mathrm{d}_y g_{m,n}(x,y) - \int_0^1\int_0^1 K(x,y)\mathrm{d}x\mathrm{d}y \\
&\quad = \frac{1}{N^2}\sum_{m=0}^{N-1}\int_0^1\int_0^1 K(x,y)\mathrm{d}_x\mathrm{d}_y \min(x,y) \\
&\qquad + \frac{2}{N^2}\sum_{\substack{m<n \\ m=n=0}}^{N-1}\int_0^1\int_0^1 K(x,y)\mathrm{d}_x\mathrm{d}_y g_{m,n}(x,y) - \int_0^1\int_0^1 K(x,y)\mathrm{d}x\mathrm{d}y. \qquad (39)
\end{aligned}$$

We have
$$\int_0^1\int_0^1 K(x,y)\mathrm{d}_x\mathrm{d}_y \min(x,y) = \int_0^1 K(x,x)\mathrm{d}x = \frac{1}{2},$$
$$\int_0^1\int_0^1 K(x,y)\mathrm{d}x\mathrm{d}y = \frac{1}{3}, \tag{40}$$
and
$$\begin{aligned}&\int_0^1\int_0^1 K(x,y)\mathrm{d}_x\mathrm{d}_y g_{m,n}(x,y)\\ &= \int_0^{T_{m,n}} K(x,x+1-T_{m,n})\mathrm{d}x + \int_{T_{m,n}}^1 K(x,x-T_{m,n})\mathrm{d}x\\ &= \frac{T_{m,n}^2}{2} + \frac{(1-T_{m,n})^2}{2}.\end{aligned} \tag{41}$$
Summing (41) for $T_{m,n} = \frac{n-m}{N}$, we have
$$\begin{aligned}&\sum_{\substack{m<n\\ m=n=0}}^{N-1}\left(\left(\frac{n-m}{N}\right)^2 + \left(1-\frac{n-m}{N}\right)^2\right)\\ &= (N-1)\left(\left(\frac{1}{N}\right)^2 + \left(1-\frac{1}{N}\right)^2\right) + (N-2)\left(\left(\frac{2}{N}\right)^2 + \left(1-\frac{2}{N}\right)^2\right)\\ &\quad + (N-3)\left(\left(\frac{3}{N}\right)^2 + \left(1-\frac{3}{N}\right)^2\right) + \cdots + 1\cdot\left(\left(\frac{N-1}{N}\right)^2 + \left(1-\frac{N-1}{N}\right)^2\right)\\ &= \frac{1}{N}\frac{(N-1)N(2N-1)}{6}.\end{aligned} \tag{42}$$
Input (40) and (42) into (39) we find the minimum (31). □

Thus the proof of Theorem 9 is finished.

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