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uniform distribution theory

ON THE SEQUENCE $\alpha n!$

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ABSTRACT. We prove that there exists $\alpha \in \mathbb{R}$ such that for any N the discrepancy D_N of the sequence $\{\alpha n!\}$, $1 \leqslant n \leqslant N$ satisfies $D_N = O(\log N)$.

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1. Low discrepancy sequences

Consider a sequence $\xi_k, k = 1, 2, 3 \ldots$ of points from the interval $[0, 1)$. The discrepansy D_N of the first N points of the sequence is defined as

$$D_N = \sup_{\gamma \in [0,1)} |N\gamma - \#\{j : \ 1 \leqslant j \leqslant N, \xi_j < \gamma\}|.$$

According to the famous W. M. Schmidt's theorem for any infinite sequence one has

$$\liminf_{N \to \infty} \frac{D_N}{\log N} > 0.$$

This statement is sharp enough. Consider a real number α with bounded partial quotients in its continued fraction expansion. Then the discrepancy of the sequence

$$\{\alpha k\}, \quad k = 1, 2, 3, \ldots$$

(here $\{\cdot\}$ stands for the fractional part) satisfies the inequality

$$D_N \leqslant M \log N,$$

where M depends on the bound for the partial quotients of α, see [4, p. 125, Theorem 3.4].

Roughly speaking, this result was obtained long ago by Ostrowski [7] and Khintchine [2]. For further information concerning discrepancy bounds one can see wonderful books [1], [4] and [6].

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As for exponentially increasing sequences we would like to refer to Levin's paper [5]. Given integer $q \geqslant 2$ Levin proved the existence of real α such that the discrepancy D_N for the sequence

$$\{\alpha q^k\}, \quad k = 1, 2, 3, \dots \tag{1}$$

satisfies the inequality

$$D_N = O\big((\log N)^2\big). \tag{2}$$

Up to now it is not known if there exists α such that the order of the discrepancy for the sequence (1) is smaller that that for (2). However if the sequence n_k grows faster than exponentially, it is quite easy to construct α such that the discrepancy of the sequence $\{\alpha n_k\}$ is small. This is just the purpose of the present short communication.

Theorem 1. *Suppose that a sequence of positive numbers n_k, $k = 1, 2, 3, \dots$ satisfies the condition*

$$\inf_k \frac{n_{k+1}}{k n_k} > 0. \tag{3}$$

There exists $\alpha \in \mathbb{R}$ such that for any N the discrepancy D_N of the sequence $\{\alpha n_k\}$, $1 \leqslant k \leqslant N$ satisfies $D_N = O(\log N)$.

We immediately deduce

Corollary 2. *Then there exists $\alpha \in \mathbb{R}$ such that for any N the discrepancy D_N of the sequence $\{\alpha n!\}$, $1 \leqslant n \leqslant N$ satisfies $D_N = O(\log N)$.*

Here we should note that in [8] it is shown that for every sequence x_k there exists α such that $||\alpha k! - x_k|| \to 0, k \to \infty$ (here $||\cdot||$ stands for the distance to the nearest integer.

Korobov [3, Theorem 3 and Example 1], constructed the real numbers α for which the sequence $\{\alpha k!\}$ is uniformly distributed. However his construction does not give optimal bounds for the discrepancy.

2. Lemmas

In the sequel F_i stands for the ith Fibonacci number, so $F_0 = F_1 = 1, F_{i+1} = F_i + F_{i-1}$ and $\phi = \frac{1+\sqrt{5}}{2}$.

Lemma 3. *Any positive integer N can be represented in a form $N = \sum_{i=1}^{r} b_i F_i$, where $b_1 \in \{1, 2, 3\}, b_i \in \{1, 2\}, 2 \leqslant i \leqslant r$ and $r \leqslant 1 + \log_\phi N$.*

P r o o f. It is a well-known fact that any positive integer can be represented in a form $N = \sum_{i=1}^{t} a_i F_i$ with $a_i \in \{0, 1\}$, $t \leqslant 1 + \log_\phi N$ and in the sequence

$$a_1, a_1, \dots, a_t \tag{4}$$

there are two consecutive not ones. Now we give an algorithm how to construct from the sequence (4) a sequence $b_1, b_2, \dots, b_r$ with all positive b_i and $r \leqslant t$.

We shall use two procedures.

Procedure 1. If we have two consecutive zeros, that is, we have a pattern $a_i, 0, 0, a_{i+3}$, with $a_{i+3} = 1$, we can replace it by the pattern $a_i, 1, 1, 0$. The sum $N = \sum_{i=1}^{t} a_i F_i$ will remain the same as $F_{i+3} = F_{i+2} + F_{i+1}$. But the number of zeros in the sequence will decrease by one.

Procedure 1 enables one to get from the sequence (4) another sequence of ones and zeros without two consecutive zeros.

Procedure 2. If we have a pattern $a_i, 0, a_{i+2}$, we may replace it by the pattern $a_i + 1, 1, a_{i+2} - 1$. If $a_{i+2} = 1$, then there will be no zero in $(i+1)$th position but a zero will appear in $(i+2)$th position. The total number of zeros will not change.

The algorithm is as follows. Procedure 1 enables one to get a sequence

$$a'_1, a'_1, \dots, a'_{t'} \tag{5}$$

of ones and zeros, where there are no two consecutive zeros with $t' \leqslant t$ and $a_{t'} = 1$.

If $a'_1 = 0$, then $a'_2 = 1$ and we may replace the pattern a'_1, a'_2 by $2, 0$ as $F_2 = 2 = 2F_1$. So we may suppose that in (5) one has $a'_1 \in \{1, 2\}$. But it may happen that $a'_2 = a'_3 = 0$. By applying Procedures 1 we obtain a sequence

$$a''_1, a''_1, \dots, a''_{t''} \tag{6}$$

with $t'' \leqslant t'$, where $a'_1 \in \{1, 2\}$ and all other elements are equal to 1 or 0 with no two consecutive zeros. Now we take the zero in the smallest position and apply Procedure 2. The zero will turn into the next position. Then either there are two consecutive zeros (and we can reduce the number of zeros by Procedure 1) or we can move the zero in the next position again. Each "moving to the next position" increases the previous digit by 1. But all the time the previous digit is 1. The only exception is in the very beginning of the process, when $a''_1 = 2$. Then a''_1 must turn into 3.

In such a way we get the necessary representation for N. □

From Lemma 3 we immediately deduce

Corollary 4. *For any positive integer N the set of the first N positive integers can be partitioned into segments of consecutive integers in the following way:*

$$\{1,2,\dots,N\} = \mathcal{A} \sqcup \bigsqcup_{i=2}^{r} \bigsqcup_{j=1}^{b_i} \{R_{i,j}, R_{i,j}+1,\dots,R_{i,j}+F_i-1\}, \tag{7}$$

where

$$\mathcal{A} = \{1\} \quad or \quad \{1,2\} \quad or \quad \{1,2,3\}, b_i \in \{1,2\}$$

and

$$R_{i,j} \geqslant F_i, \tag{8}$$

$$r \leqslant 1 + \log_\phi N. \tag{9}$$

Let R, i be positive integers. We consider the sequence

$$\{\phi k\}, \quad R \leqslant k < R + F_i \tag{10}$$

The following statement is well-known. It is the main argument of the classical proofs of the logarithmic order of discrepansy of the sequence $\{\alpha k\}$ in the case when α has bounded partial quotients in its continued fraction expansion. It immediately follows from the inequality $||\phi F_i|| \leqslant 1/F_i$, where $||\cdot||$ stands for the distance to the nearest integer. It means that the the set (10) is close to the set

$$\frac{\nu}{F_i}, \ 1, \ 0 \leqslant \nu \leqslant F_i - 1,$$

and hence discrepancy of the sequence (10) is bounded.

Lemma 5. *There is a substitution $\sigma_1,\dots,\sigma_{F_i}$ of the sequence $1,\dots,F_i$ such that*

$$\left|\{\phi(R+k)\} - \frac{\sigma_k}{F_i}\right| \leqslant \frac{1}{F_i}, \ 0 \leqslant k \leqslant F_i - 1.$$

Lemma 6. *Consider an arbitrary sequence ξ_k, $k = 1,2,3,\dots$ from the interval $[0,1)$. Suppose that a sequence n_k satisfies (3). Then there exist $c > 0$ and $\alpha \in \mathbb{R}$ such that*

$$||\alpha n_k - \xi_k|| \leqslant \frac{c}{k}, \quad k = 1,2,3,\dots \tag{11}$$

P r o o f. From (3) we see that $\frac{n_{k+1}}{n_k} \geqslant \kappa k$ for some positive κ. Fix k. Then the set

$$\left\{\alpha \in \mathbb{R} : \ ||\alpha n_k - \xi_k|| \leqslant \frac{c}{k}\right\}$$

is a union of segments of the form $\left[\frac{\xi_k+z}{n_k} - \frac{c}{kn_k}, \frac{\xi_k+z}{n_k} + \frac{c}{kn_k}\right]$, $z \in \mathbb{Z}$. The length of each segment is equal to $\frac{2c}{kn_k}$. The distance between the centers of neighboring segments is equal to $\frac{1}{n_k}$. We see that $\frac{c}{kn_k} \geqslant \frac{c\kappa}{n_{k+1}}$. So if c is large enough, we can choose integers z_k to get a sequence of nested segments

$$\left[\frac{\xi_1+z_1}{n_1}-\frac{c}{n_1},\frac{\xi_1+z_1}{n_1}+\frac{c}{n_1}\right]\supset\cdots\supset\left[\frac{\xi_k+z_k}{n_k}-\frac{c}{kn_k},\frac{\xi_k+z_k}{n_k}+\frac{c}{kn_k}\right]\supset$$
$$\supset\left[\frac{\xi_{k+1}+z_{k+1}}{n_{k+1}}-\frac{c}{(k+1)n_{k+1}},\frac{\xi_{k+1}+z_{k+1}}{n_{k+1}}+\frac{c}{(k+1)n_{k+1}}\right]\supset\cdots$$

The common point of these segments satisfies (11). □

Proof of Theorem 1. We take the sequence $\xi_k=\{\phi k\}, k=1,2,3,\dots$ and apply Lemma 6. Then we get real α. This is just the number which we need. Take positive integer N. Then in the decomposition (7) for each segment

$$\{R_{i,j},R_{i,j}+1,\dots,R_{i,j}+F_i-1\}$$

its right endpoint is $\geqslant F_i$. So from the inequalities of Lemmas 5 and 6 we get

$$\left|\left|\alpha n_k-\frac{\sigma_k}{F_i}\right|\right|\leqslant\frac{1}{F_i}+\frac{c}{R_{i,j}}\leqslant\frac{1+c}{F_i},\quad\text{for all } i,j.$$

This means that each sequence

$$\{\alpha n_k\},\ \ R_{i,j}<k\leqslant R_{i,j}+F_i \tag{12}$$

has discrepancy $O(1)$. But the sequence $\{\alpha n_k\}$, $1\leqslant k\leqslant N$ is partitioned into $O(\log N)$ sequences of the form (12), as all b_i are bounded by 3 and we have estimate (9). So this sequence has discrepancy $O(\log N)$. □

Remark 7. Fibonacci numbers are not significant for the proof. For the decomposition of the type (7) one may use Ostrowski's unique expression

$$N=\sum_{i=1}^{r}b_iq_i, b_i\leqslant a_i,\quad\text{for any irrational}\quad\alpha_0=[a_0;a_1,\dots,a_n,\dots]$$

with bounded partial quotients (here q_i is the denominator of ith convergent to α_0).

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