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DISTRIBUTION OF THE VALUES OF THE DERIVATIVE OF THE RIEMANN ZETA FUNCTION ON ITS a-POINTS

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ABSTRACT. In this paper, we study the value distribution of the derivative $\zeta'(s)$ of the Riemann zeta function at the a-points $\rho_a = \beta_a + i\gamma_a$ of $\zeta(s)$. Actually, we give an asymptotic formula and an upper bound for the sum

$$\sum_{\rho_a;\ 0<\gamma_a\leq T} \zeta'(\rho_a)\, X^{\rho_a} \quad \text{as} \quad T \longrightarrow \infty,$$

where X is a fixed positive number. This work continues the investigations of Fujii [1, 2, 3] and Garunkštis & Steuding [5].

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Let $\zeta(s)$ be the Riemann zeta function, $s = \sigma + it$ and a be a nonzero complex number. The zeros of $\zeta(s) - a$, which will be denoted by $\rho_a = \beta_a + i\gamma_a$, are called the a-points of $\zeta(s)$. First, we note that there is an a-points near any trivial zero $s = -2n$ for sufficiently large n and apart from these a-points there are only finitely many other a-points in the half-plane $\sigma \leq 0$. The a-points with $\beta_a \leq 0$ are said to be trivial. All other zeros lie in a strip $0 < \sigma < A$, where A depends on a and are called the nontrivial a-points. Their number satisfies a Riemann-von Mangoldt type formula (for some proof, we refer to Levinson's paper [6] or Selberg's paper [7]), namely

$$N_a(T) \quad = \sum_{0<\gamma_a<T;\ \beta_a>0} 1 \quad = \frac{T}{2\pi}\log\frac{T}{2\pi c_a e} + \mathcal{O}(\log T), \tag{1}$$

where

$$c_a = \begin{cases} 2 & \text{if} \quad a = 1, \\ 1 & \text{otherwise.} \end{cases}$$

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We observe that these asymptotics are essentially independent of a and that

$$N_a(T) \sim N(T), \quad T \to \infty,$$

where $N(T) = N_0(T)$ denotes the number of nontrivial zeros $\rho = \beta + i\gamma$ satisfying $0 < \gamma < T$. Levinson [6] showed that all but $\mathcal{O}\left(\frac{N(T)}{\log\log T}\right)$ of the a-points with imaginary part in $T < t < 2T$ lie in $\left|Re(s) - \frac{1}{2}\right| < \frac{(\log\log T)^2}{\log T}$. So the a-points are clustered around the line $Re(s) = \frac{1}{2}$.

In this paper, we are interested in the sum

$$\sum_{\rho_a=\beta_a+i\gamma_a;\ 0<\gamma_a\leq T} \zeta'\left(\rho_a\right) X^{\rho_a},$$

where X is a fixed positive constant. Our method is based on the formula stated by Garunkštis and Steuding in [5, §6, Remark ii)] with the choice of the function $f(s) = \zeta'(s)X^s$. There are several reasons why the last sum is of interest. The first one is, the estimation of this sum can be used to study the normal distribution of the values of $\log|\zeta'(\rho_a)|$, the second one is to study the vertical distribution of a-points of $\zeta(s)$. Recall that in the case of $a = 0$, recently Hiary and Odlyzko [4] studied the tail part of the conjectured normal distribution of the values of $\log\left|\zeta'\left(\frac{1}{2} + i\gamma_n\right)\right|$, and to do so, they have been concerned with the sum

$$\sum_{T\leq\gamma_n\leq T+H} \zeta'\left(\frac{1}{2} + i\gamma_n\right) e^{2\pi i n x}$$

as a function of x, where $\rho = \beta + i\gamma$ denotes the nontrivial zeros of $\zeta(s)$ and γ_n denotes the nth positive imaginary part of the zeros of $\zeta(s)$. They have approximated the later sum by

$$\sum_{T\leq\gamma_n\leq T+H} \zeta'\left(\frac{1}{2} + i\gamma_n\right) e^{2\pi i\tilde{\gamma_n}x},$$

where $\tilde{\gamma_n} = \frac{1}{2\pi}\gamma_n \log\frac{T}{2\pi}$. Similar sums were studied also by Steuding in [8], he proved that for x a positive real number $\neq 1$ and as $T \to \infty$,

$$\sum_{\rho_a=\beta_a+i\gamma_a;\ 0<\gamma_a<T} x^{\rho_a} = \big(\alpha(x) - x\Lambda(1/x)\big)\frac{T}{2\pi} + O\left(T^{\frac{1}{2}+\epsilon}\right),$$

where $\alpha(x)$ and $\Lambda(x)$ equal the Dirichlet series coefficients in $\zeta'(s)/(\zeta(s) - a) = \sum_{n\geq 2}\alpha(n)n^{-s}$ and $\zeta'(s)/\zeta(s) = \sum_{n\geq 2}\Lambda(n)n^{-s}$, respectively, if $x = n$ or $x = 1/n$ for some integer $n \geq 2$, and zero otherwise. Furthermore, he deduced that for any complex a, the ordinates γ_a of the a-points are uniformly distributed modulo one. Garunkštis and Steuding have shown in [5] [1], that if $T \longrightarrow \infty$,

$$\sum_{\rho_a;\ 0<\gamma_a\leq T} \zeta'(\rho_a) = \left(\tfrac{1}{2}-a\right)\tfrac{T}{2\pi}\log^2\tfrac{T}{2\pi} + (C_0 - 1 + 2a)\tfrac{T}{2\pi}\log\tfrac{T}{2\pi} + (1 - C_0 - C_0^2 + 3C_1 - 2a)\tfrac{T}{2\pi} + \mathtt{E}(T), \tag{2}$$

where C_n are the Stieltjes constants [2] and

$$E(T) = \begin{cases} \mathcal{O}\left(T^{\frac{1}{2}+\epsilon}\right) & \text{under Riemann hypothesis,} \\ \mathcal{O}\left(Te^{-C\sqrt{\log T}}\right) & \text{unconditionally.} \end{cases}$$

Using formula (2), they conclude that the main term describes how the values $\zeta(1/2+it)$ approach the value a in the complex plane on average.

Our main result is stated in the following:

Theorem 1. 1) If X is an integer ≥ 1, then

$$\sum_{\rho_a;\ 0<\gamma_a\leq T} \zeta'(\rho_a)X^{\rho_a}$$
$$= \left(\frac{1}{2}-a\right)\frac{T}{2\pi}\log^2\frac{T}{2\pi} + (2a + C_0 - 1 - \log X)\frac{T}{2\pi}\log\frac{T}{2\pi}$$
$$+\left(1 - 2a - C_0 - C_0^2 + 3C_1 + \sum_{X=mn}\Lambda(n)\log m - -(C_0 - 1 + \frac{1}{2}\log X)\log X\right)\frac{T}{2\pi}$$
$$+\mathcal{O}\left(Te^{-C\sqrt{\log T}}\right). \tag{3}$$

[1] In fact, in the proof of (2) Garunkštis and Steuding used the following formula established and recently corrected by Fujii in [2]

$$\sum_{0<\gamma\leq T}\zeta'(\rho) = \frac{T}{4\pi}\log^2\left(\frac{T}{2\pi}\right) + \frac{T}{2\pi}\log\left(\frac{T}{4\pi}\right).(C_0 - 1) + \frac{T}{2\pi}(1 - C_0 - C_0^2 + 3C_1) + O\left(Te^{-C\sqrt{\log T}}\right),$$

where the summation is over all nontrivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$.

[2] The Stieltjes constants are given by the Laurent series expansion of $\zeta(s)$ at $s = 1$, $\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty}(-1)^n\frac{C_n}{n!}(s-1)^n$. For example, $C_0 = \lim_{N\longrightarrow+\infty}\left(\sum_{n=1}^{N}\frac{1}{n} - \log N\right)$.

2) If $X=\frac{h}{q}$ is a positive rational number with $q>1,\ h>0$ and $gcd(h,q)=1$, then

$$\sum_{\rho_a;\ 0<\gamma_a\le T}\zeta'(\rho_a)X^{\rho_a}=-\frac{aT}{2\pi}\log^2\frac{T}{2\pi}$$
$$+\frac{T}{2\pi q}\log\frac{T}{2\pi q}\left\{2qa-\sum_{p|q}\frac{\log p}{p-1}-\log q+\sum_{d|q,d\neq q}\Lambda(d)\frac{P(d)}{P(d)-\psi_0(P(d))}\right\}$$
$$+\frac{T}{2\pi q}\left[2aq-(C_0-1)\log q+C_1+\sum_{p|q}\left(\frac{p\log^2 p}{(p-1)^2}+\frac{\log p}{p-1}\right)\right]$$
$$+\frac{T}{2\pi q}\left[-\sum_{c=1}^{q-1}\frac{\mu(q/(q,c))}{\varphi(q/(q,c))}\left(B_0\left(\frac{c}{q}\right)\log q-B_1\left(\frac{c}{q}\right)\right)+\sum_{d|q}\Lambda(d)\left(\log d+\frac{\log P(d)}{P(d)-1}\right)\right]$$
$$+\frac{T}{2\pi q}\left[-\sum_{d|q,d\neq q}\Lambda(d)\left(\frac{P(d)}{P(d)-\psi_0(P(d))}\log d+\right.\right.$$
$$\left.\left.+\frac{P(d)}{P(d)-\psi_0(P(d))}+\frac{P(d)\psi_0(P(d))\log P(d)}{(P(d)-\psi_0(P(d)))^2}\right)\right]$$
$$+\frac{T}{2\pi q}\left[\log X\left(\sum_{p|q}\frac{\log p}{p-1}+\log q-\sum_{d|q,d\neq q}\Lambda(d)\frac{P(d)}{P(d)-\psi_0(P(d))}\right)\right]+\mathcal{O}\left(Te^{-C\sqrt{\log T}}\right),$$

where φ is the Euler function, μ is the Möbius function, p denotes prime numbers, C_0 and C_1 are the constants in equation (2),

$$P(n)=\begin{cases} p & \text{if } n=p^k \text{ where } p \text{ is a prime number } p \text{ and } k \text{ is an integer} \ge 1,\\ 2 & \text{otherwise,}\end{cases}$$

ψ_0 is the principal character mod $\frac{q}{d}$, $B_0(\omega)$ and $B_1(\omega)$ are the constants of the Hurwitz zeta function expansion at $s=1$ which, for $0<\omega\le 1$, is given by

$$Z(s,\omega)=\sum_{l=0}^{\infty}\frac{1}{(l+\omega)^s}=\omega^{-s}+\frac{1}{s-1}+B_0(\omega)+B_1(\omega)(s-1)+\cdots$$

3) If X is a positive irrational number satisfying $|X-\frac{\widehat{h}}{\widehat{q}}|\le\frac{1}{\widehat{q}^2}$, with integers $\widehat{q},\ \widehat{h}\ge 1$ and $gcd(\widehat{q},\widehat{h})=1$, then

$$\sum_{\rho_a;\ 0<\gamma_a\le T}\zeta'(\rho_a)X^{\rho_a}=\frac{aT}{2\pi}\left[-\log^2\frac{T}{2\pi}+2\log\frac{T}{2\pi}-2\right]$$
$$+\ \mathcal{O}\left[\left(\frac{T}{\sqrt{\widehat{q}}}+T^{9/10}+\sqrt{T}\sqrt{\widehat{q}}\right)D\log^9 T\right],$$

where

$$D=\max_{k\le\frac{T}{2\pi X}}\sum_{d|k}1. \tag{4}$$

Remarks.

- When $X = 1$, we obtain Garunkštis and Steuding's result given by equation (2). Namely, when $a = 0$, we obtain Fujii's result [2, Theorems 1, 2 and 3].
- Under the Riemann hypothesis, the error term in Theorem 1 can be refined as in Garunkštis and Steuding's paper [5] to $\mathcal{O}\left(T^{\frac{1}{2}+\epsilon}\right)$.

P r o o f. Let X be a fixed positive number and a be a complex number. We write $s = \sigma + it$, $\rho_a = \beta_a + i\gamma_a$, where σ, t, β_a and γ_a are real numbers. By the theorem of residues (or Cauchy's theorem), we get

$$\sum_{\rho_a;\ 0<\gamma_a\le T} \zeta'(\rho_a) X^{\rho_a} = \frac{1}{2i\pi}\int_{\mathcal{R}} \frac{\zeta'^2(s)}{\zeta(s)-a} X^s ds, \tag{5}$$

where $\mathcal{R}$ is a rectangular contour in counterclockwise direction which will be specified below. In view of the formula (1), for any large $T_0 \ge 0$, there exists a real number $T \in [T_0, T_0+1[$ such that

$$min_{\rho_a} |T - \gamma_a| \ge \frac{1}{\log T}.$$

We shall distinguish the cases $a \ne 1$ and $a = 1$. Let us suppose $a \ne 1$. We choose $B = \log(T)$ and $b = 1 + \frac{1}{\log T}$. Since $\zeta(\sigma + it) = 1 + o(1)$, then for $\sigma \to +\infty$ there are no a-points in the half plane $Re(s) > B - 1$. Moreover, since there are only finitely many trivial a-points to the right of the line $Re(s) = 1 - b$, we may suppose that there are no a-points on the line segments $[B+i, B+iT]$ and $[1-b+i, 1-b+iT]$ (by varying b slightly if necessary). We also assume that there are no a-points on the line $[1-b+i, B+i]$ (if there is an a-points on this line we always can slightly shift this line). Thus we may choose $\mathcal{R}$ as the rectangular with vertices $1-b+iB$, $B+i$, $B+iT$, $1-b+iT$ at the expense of a small error for disregarding the finitely many nontrivial a-points below $Im(s) = 1$ and for counting finitely many trivial a-points to the right of $Re(s) = 1-b$. Hence, formula (5) can be written as

$$\begin{aligned}\sum_{\rho_a;\ 0<\gamma_a\le T} \zeta'(\rho_a) X^{\rho_a} &= \frac{1}{2i\pi}\left\{\int_{B+i}^{B+iT} + \int_{B+iT}^{1-b+iT} + \int_{1-b+iT}^{1-b+i}\right\} \frac{\zeta'^2(s)}{\zeta(s)-a} X^s ds \\ &\quad + \frac{1}{2i\pi}\int_{1-b+i}^{B+i} \frac{\zeta'^2(s)}{\zeta(s)-a} X^s ds + \mathcal{O}(1) \\ &= I_1 + I_2 + I_3 + I_4 + \mathcal{O}(1). \end{aligned} \tag{6}$$

For $\sigma \geq 1-b$, we have (see [5, Equation (14)]

$$\frac{\zeta'^2(\sigma+iT)}{\zeta(\sigma+iT)-a} \ll T^{\frac{1-\sigma}{2}+\epsilon}.$$

This leads to

$$I_2 + I_4 \ll T^{\frac{1}{2}+\epsilon}.$$

To estimate I_1 we proceed as follows

$$\begin{aligned} I_1 &= \frac{1}{2\pi}\int_1^T \frac{\zeta'^2(B+it)}{\zeta(B+it)-a} X^{B+it} dt \\ &\ll X^B T^{-2\log 2}\log T \\ &\ll T^{-2\log 2}\log T. \end{aligned} \tag{7}$$

To estimate the integral I_3 given by

$$I_3 = \frac{1}{2i\pi}\int_{1-b+i}^{1-b+iT} \frac{\zeta'^2(s)}{\zeta(s)-a} X^s ds,$$

we use the same argument as in [5]. By noting that, there exists a positive constant $A = A(a)$ such that

$$\left|\frac{a}{\zeta(s)}\right| < \frac{1}{2}, \qquad |t| \geq A,$$

for $s \in [1-b+iA, 1-b+iT]$, we get

$$\frac{1}{\zeta(s)-a} = \frac{1}{\zeta(s)}\left(1 + \frac{a}{\zeta(s)} + \sum_{k=2}^{+\infty}\left(\frac{a}{\zeta(s)}\right)^k\right). \tag{8}$$

From

$$\frac{1}{2i\pi}\int_{1-b+i}^{1-b+iA} \frac{\zeta'^2(s)}{\zeta(s)-a} X^s ds = \frac{X^{1-b}}{2\pi}\int_1^A \frac{\zeta'^2(1-b+it)}{\zeta(1-b+it)-a} X^{it} dt \ll 1$$

and equation (8), we deduce that

$$\begin{aligned} &-\frac{1}{2i\pi}\int_{1-b+i}^{1-b+iT} \frac{\zeta'(s)}{\zeta(s)-a} X^s ds \\ &= -\frac{1}{2i\pi}\int_{1-b+iA}^{1-b+iT} \left\{\frac{\zeta'^2}{\zeta}(s) + a\left(\frac{\zeta'}{\zeta}(s)\right)^2 + \frac{\zeta'^2}{\zeta}(s)\sum_{k=2}^{+\infty}\left(\frac{a}{\zeta(s)}\right)^k\right\} X^s ds + O(1) \\ &= J_1 + J_2 + J_3 + O(1). \end{aligned} \tag{9}$$

For the integral J_3, we have

$$\begin{aligned} J_3 &= -\frac{a}{2i\pi}\int_{1-b+iA}^{1-b+iT}\left(\frac{\zeta'}{\zeta}(s)\right)^2\sum_{l=1}^{+\infty}\left(\frac{a}{\zeta(s)}\right)^l X^s ds \\ &= -\frac{aX^{1-b}}{2\pi}\int_A^T\left(\frac{\zeta'}{\zeta}(1-b+it)\right)^2\sum_{l=1}^{+\infty}\left(\frac{a}{\zeta(1-b+it)}\right)^l X^{it}dt. \end{aligned}$$

Hence

$$J_3 \ll T\log^2 T\sum_{l=1}^{+\infty}\left(\frac{\log T}{\sqrt{T}}\right)^l \ll \sqrt{T}\log^3 T. \tag{10}$$

The integral

$$J_1 = -\frac{1}{2i\pi}\int_{1-b+iA}^{1-b+iT}\frac{\zeta'^2}{\zeta}(s)X^s ds$$

can be rewritten in two ways

$$J_1 = -\frac{1}{2i\pi}\int_{\mathcal{R}}\frac{\zeta'^2}{\zeta}(s)X^s ds + O\left(T^{\frac{1}{2}+\epsilon}\right)$$

and

$$J_1 = \sum_{\rho;\ 0<\gamma\le T}\zeta'(\rho)X^\rho + O\left(T^{\frac{1}{2}+\epsilon}\right), \tag{11}$$

where $\rho = \beta + i\gamma$ stands for the nontrivial zeros of ζ. The last sum was evaluated by Fujii [2, Lemma 1] who obtained for all fixed positive number X

$$\begin{aligned} &\sum_{\rho;\ 0<\gamma\le T}\zeta'(\rho)X^\rho \\ &= -\Theta(X)\log(X)\left(\frac{1}{2}\frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{1}{2\pi}\frac{T}{2\pi} + \frac{i\pi}{4}\frac{T}{2\pi}\right) \\ &+ \Theta(X)\frac{T}{2\pi}\sum_{X=mn}\Lambda(n)\log m + X\sum_{k\le\frac{T}{2\pi X}}\log^2(k)e^{2i\pi kX} \\ &+ \frac{1}{2}X\log(X)\sum_{k\le\frac{T}{2\pi X}}\log(k)e^{2i\pi kX} - \left(\frac{1}{2}X\log^2 X - \frac{i\pi}{4}X\log(X)\right)\sum_{k\le\frac{T}{2\pi X}}e^{2i\pi kX} \\ &- X\sum_{k\le\frac{T}{2\pi X}}\sum_{mn=k}\Lambda(n)\log(m)e^{2i\pi kX} + \mathcal{O}\left(\sqrt{T}\log^3 T\right), \end{aligned} \tag{12}$$

where $\Theta(X)$ is defined by

$$\Theta(X) = \begin{cases} 1 & \text{if } X \text{ is an integer} \ge 1 \\ 0 & \text{otherwise.} \end{cases}$$

Applying the functional equation of ζ in the following form

$$\zeta(s) = \Delta(s)\zeta(1-s),$$

with $\Delta(s) = 2^s\pi^{s-1}\Gamma(1-s)\sin(\pi s/2)$, we get

$$\begin{aligned} J_2 &= -\frac{a}{2i\pi}\int_{1-b+iA}^{1-b+iT}\left(\frac{\Delta'}{\Delta}(s) - \frac{\zeta'}{\zeta}(1-s)\right)^2 X^s ds \\ &= -\frac{a}{2i\pi}\int_{1-b+iA}^{1-b+iT}\left\{\left(\frac{\Delta'}{\Delta}(s)\right)^2 - 2\frac{\Delta'}{\Delta}(s)\frac{\zeta'}{\zeta}(1-s) + \left(\frac{\zeta'}{\zeta}(1-s)\right)^2\right\} X^s ds \\ &= K_1 + K_2 + K_3. \end{aligned} \tag{13}$$

We have

$$\begin{aligned} K_2 &= \frac{a}{i\pi}\int_{1-b+iA}^{1-b+iT}\frac{\Delta'}{\Delta}(s)\frac{\zeta'}{\zeta}(1-s)X^s ds \\ &= \frac{a}{\pi}\int_A^T\frac{\Delta'}{\Delta}(1-b+it)\frac{\zeta'}{\zeta}(b-it)X^{1-b+it}dt \\ &= \frac{a}{\pi}X^{1-b}\sum_{m=2}^{+\infty}\frac{\Lambda(m)}{m^b}\int_A^T\left(\log\frac{t}{2\pi} + \mathcal{O}\left(\frac{1}{t}\right)\right)e^{it\log Xm}dt. \end{aligned} \tag{14}$$

With integration by parts, we show that the integral in equation (10) is $\mathcal{O}(\log T)$. It follows that

$$K_2 \ll \log T X^{1-b}\sum_{m=2}^{+\infty}\frac{\Lambda(m)}{m^b} \ll \log T\left|\frac{\zeta'}{\zeta}(b)\right| \ll \log^2(T). \tag{15}$$

Similarly, we obtain

$$\begin{aligned} K_3 &= -\frac{a}{2\pi}\int_A^T\left(\frac{\zeta'}{\zeta}(b-it)\right)^2 X^{1-b+it}dt \\ &= -\frac{a}{2\pi}X^{1-b}\sum_{m,\ n=2}^{+\infty}\frac{\Lambda(m)\Lambda(n)}{(mn)^b}\int_A^T e^{it\log Xmn}dt \\ &\ll \sum_{m,\ n=2}^{+\infty}\frac{\Lambda(m)\Lambda(n)}{(mn)^b}\left|\int_A^T e^{it\log Xmn}dt\right| \\ &\ll \log^2(T). \end{aligned} \tag{16}$$

Finally, we have

$$\begin{aligned} K_1 &= -\frac{a}{2\pi}X^{1-b}\int_A^T \left(-\log\frac{t}{2\pi}+O\left(\frac{1}{t}\right)\right)^2 X^{it}dt \\ &= -\frac{a}{2\pi}X^{1-b}\int_A^T \log^2\frac{t}{2\pi}X^{it}dt+O\left(\log T\right) \\ &= -\frac{aT}{2\pi}\log^2\frac{T}{2\pi}+\frac{aT}{\pi}\log\frac{T}{2\pi}-\frac{aT}{\pi}+O\left(\log^2 T\right). \end{aligned} \tag{17}$$

Hence, from equations (13), (15), (16) and (17), we get

$$J_2 = -\frac{aT}{2\pi}\log^2\frac{T}{2\pi}+\frac{aT}{\pi}\log\frac{T}{2\pi}-\frac{aT}{\pi}+O\left(\log^2 T\right). \tag{18}$$

Further, inserting equations (10), (11), (12) and (18) in (9), we obtain an estimate of the integral I_3. Replacing all the estimates for the integrals I_1, I_2, I_3 and I_4 in equation (6), we deduce

$$\begin{aligned} &\sum_{\rho_a;\ 0<\gamma_a\le T} \zeta'\left(\rho_a\right)X^{\rho_a} \\ &= -\frac{aT}{2\pi}\log^2\frac{T}{2\pi}+\frac{aT}{\pi}\log\frac{T}{2\pi}-\frac{aT}{\pi}+\sum_{0<\gamma\le T}\zeta'(\rho)X^{\rho}+O\left(\log^2 T\right) \\ &= -\frac{aT}{2\pi}\log^2\frac{T}{2\pi}+\frac{aT}{\pi}\log\frac{T}{2\pi}-\frac{aT}{\pi} \\ &\quad -\Theta(X)\log\left(X\right)\left(\frac{1}{2}\frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{1}{2\pi}\frac{T}{2\pi}+\frac{i\pi}{4}\frac{T}{2\pi}\right) \\ &\quad +\Theta(X)\frac{T}{2\pi}\sum_{X=mn}\Lambda(n)\log m + X\sum_{k\le\frac{T}{2\pi X}}\log^2\left(k\right)e^{2i\pi kX} \\ &\quad +\frac{1}{2}X\log\left(X\right)\sum_{k\le\frac{T}{2\pi X}}\log\left(k\right)e^{2i\pi kX} \\ &\quad -\left(\frac{1}{2}X\log^2 X-\frac{i\pi}{4}X\log\left(X\right)\right)\sum_{k\le\frac{T}{2\pi X}}e^{2i\pi kX} \\ &\quad -X\sum_{k\le\frac{T}{2\pi X}}\sum_{mn=k}\Lambda(n)\log\left(m\right)e^{2i\pi kX}+\mathcal{O}\left(\sqrt{T}\log^3 T\right). \end{aligned} \tag{19}$$

To finish the proof of Theorem 1 for $a\neq 1$, we note that the assertions (1), (2) and (3) can easily be deduced from the last equation and Theorems 1, 2 and 3 of [2].

For $a = 1$, we consider the function $l(s) = 2^s(\zeta(s) - 1)$ in place of $\zeta(s) - a$. Furthermore, we have

$$\frac{l'}{l}(s) = \log 2 + \frac{\zeta'(s)}{\zeta(s) - 1}.$$

This implies that the constant term does not contribute by integration over a closed contour and we use the same argument as in the case $a \neq 1$. $\square$

Concluding Remarks.

In [3], Fujii showed that under the Riemann hypothesis, for $X > 1$, we have

$$\lim_{T\to\infty} \frac{1}{T/2\pi} \sum_{0<\gamma\leq T} \left[X^{1/2+i\gamma}\left(L(1/2+i\gamma,\chi) - 1\right) - \xi(X)\right] = \begin{cases} M(X,\chi) & \text{if} \quad X \text{is rational}, \\ 0 & \text{if} \quad X \text{is irrational}, \end{cases}$$

where $L(s,\chi)$ is some Dirichlet L-function with a primitive Dirichlet character with modulus ≥ 3, $\xi(X)$ and $M(X,\chi)$ are some constants.

Therefore, it is an interesting question to

- refine the error terms in Theorem 1 under the Riemann hypothesis.
- extend Theorem 1 to the Dirichlet L-functions (this is a work in progress) and its higher derivatives.
- extend Theorem 1 to other classes of Dirichlet L-functions (automorphic L--functions or the Selberg class with some further conditions) and its higher derivatives.

These problems will be considered in a sequel to this article.

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