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QUANTITATIVE REARRANGEMENT THEOREMS FOR SEQUENCES IN COMPACT SPACES

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ABSTRACT. Every sequence which is dense in a compact metric space X can be rearranged, such that the new sequence is uniformly distributed in X with respect to a given non-negative Borel-measure on X. We give quantitative versions of this result, in the sense that we give estimates for the growth-rate of the permutations in question.

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1. Introduction

Rearrangement theorems play an important role in the theory of uniform distribution. The classical result in that topic is that from John von Neumann, who showed in 1925 [9] that every sequence $(x_n)_{n\in\mathbb{N}}$ which is dense in the unit-interval can be rearranged such that the resulting sequence is uniformly distributed modulo one. That is: There always exists a permutation τ of $\mathbb{N}$ such that $(x_{\tau(n)})_{n\in\mathbb{N}}$ is uniformly distributed modulo one.

There were many generalisations of this theorem both of non-quantitative and of quantitative form. See for example [1], [2], [3], [6], [8] and the monograph [4]. In 1987 Niederreiter [7] showed that all standard rearrangement results which are not of a quantitative form are simple consequences of the following theorem which we will state here in a little bit modified form:

Theorem A ([7]): *Let (X,d) be a compact metric space without isolated points and $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ two sequences dense in X. Then there exists a permutation τ of $\mathbb{N}$ such that*

$$\lim_{n\to\infty} d(x_n, y_{\tau(n)}) = 0.$$

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In [5] the author gave a quantitative version of this theorem in the sense that estimates for the speed of the convergence of $d(x_n, y_{\tau(n)})$ were given and it was shown that now also the quantitative standard results are simple corollaries of the following variant:

Theorem B ([5])**:** *Let (X, d) be a compact metric space without isolated points, $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ two sequences dense in X and $(\epsilon(n))_{n\in\mathbb{N}}$ a sequence of positive reals. Then there exist permutations τ and σ of $\mathbb{N}$ such that*

$$d(x_{\tau(n)}, y_{\sigma(n)}) < \epsilon(n)$$

for all $n \in \mathbb{N}$.

The main aim of this paper now is, to give a quantitative version of Niederreiter's Theorem A in another direction: We will study the function $H(\tau,N) := \frac{\tau(N)}{N}$ for the permutation τ used in Theorem A.
Of course we always have

$$H_{inf} := \liminf_{N\to\infty} H(\tau, N) \leq 1 \leq H_{sup} := \limsup_{N\to\infty} H(\tau, N).$$

It seems further to be intuitively obvious that τ can be chosen such that $H_{inf} = H_{sup} = 1$ if and only if $(x_n)_{n\in\mathbb{N}}$ in some sense is "very near to" $(y_n)_{n\in\mathbb{N}}$ (for example, of course, if $\lim_{n\to\infty} d(x_n, y_n) = 0$), and that $H_{inf} = 0$ and $H_{sup} = \infty$, if the distribution properties of $(x_n)_{n\in\mathbb{N}}$ are "quite different" from the distribution properties of $(y_n)_{n\in\mathbb{N}}$.

In Theorem 1 we show that it is possible to give a non-trivial, in some sense best possible *general* lower bound for $H(\tau, N)$, and we show that it is not possible to give a *general* upper bound for $H(\tau, N)$.

It will become clear from Theorem 1 that upper bounds for $H(\tau, N)$ depend heavily on the distribution properties of the sequence and on the topological properties of the space (X, d) as well.

In Theorem 2, the main result of this paper, we give such upper bounds for $H(\tau, N)$ in dependence on the distribution properties of the sequences and on the topological properties of (X, d).

Further we will give several examples for illustrating the results.

In Chapter 2 we formulate and prove Theorem 1, in Chapter 3 we give several definitions and prove some properties for the metric space (X, d) which are necessary for the formulation and for the proof of Theorem 2 which will be given in Chapter 4. In Chapter 5 we give some illustrating examples for the results of Theorem 2.

2. Formulation and Proof of Theorem 1

In everything what follows, X denotes a compact metric space without isolated points with metric d and with a non-negative normed Borel-measure μ. For simplicity we assume that the support of μ is equal to X. (This restriction is not of importance because of Theorem 1.3, page 177 in [4].)

By Theorem 2.5, page 185 in [4] a sequence $(x_n)_{n\in\mathbb{N}}$ in X can be rearranged to a $\mu-$uniformly distributed sequence in X if and only if $(x_n)_{n\in\mathbb{N}}$ is dense in X.

Here a sequence $(x_n)_{n\in\mathbb{N}}$ is said to be $\mu-$uniformly distributed ($\mu-$ u.d.) if

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(x_n) = \int_X f d\mu$$

for all real-valued continuous functions f on X.

By Theorem 1.2, page 175 in [4] this is equivalent to

$$\lim_{N\to\infty} \frac{A(N;M)}{N} = \mu(M)$$

for all $\mu-$continuity sets $M \subseteq X$. Here $A(N;M) = |\{n|x_n \in M, n = 1,2,\ldots,N\}|$ and M is called $\mu-$continuity set if $\mu(\delta M) = 0$, where δM is the boundary of M.

In the following $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ always are sequences dense in X and τ always is a permutation of $\mathbb{N}$.

As already announced in the introduction, in Theorem 1a we show that it is not possible to give a non-trivial *general* upper bound for $H(\tau,N) = \frac{\tau(N)}{N}$, for permutations τ which rearrange dense sequences to $\mu-$uniformly distributed sequences. This is not a very surprising result. In Theorem 1b we show however that it is possible to give a non-trivial *general* lower bound for $H(\tau,N)$ which, as is shown in Theorem 1c, is essentially best possible.

Theorem 1. a) *For all (arbitrarily fast growing) functions $g : \mathbb{N} \to \mathbb{R}^+$ there is a sequence $(x_n)_{n\in\mathbb{N}}$ dense in the one-dimensional torus such that for all τ for which $(x_{\tau(n)})_{n\in\mathbb{N}}$ is $\mu-$u.d. in X, we have*

$$\limsup_{N\to\infty} \frac{H(\tau,N)}{g(N)} = +\infty.$$

b) *For all (arbitrarily slowly growing) functions $h : \mathbb{N} \to \mathbb{R}^+$ with $\lim\limits_{N\to\infty} h(N) = +\infty$, for all metric spaces X and all sequences $(x_n)_{n\in\mathbb{N}}$ dense in X, there is a permutation τ of $\mathbb{N}$ for which $(x_{\tau(n)})_{n\in\mathbb{N}}$ is $\mu-$u.d. in X and* *such that*

$$\lim_{N\to\infty} H(\tau, N)\cdot h(N) = +\infty.$$

c) *There is a sequence* $(x_n)_{n\in\mathbb{N}}$*, dense in the one-dimensional torus such that for all* τ *for which* $(x_{\tau(n)})_{n\in\mathbb{N}}$ *is* μ*−u.d. in* X *we have*

$$\liminf_{N\to\infty} H(\tau, N) = 0.$$

P r o o f. a) Let $\tilde{g}(x) := x^2\cdot\max_{y\le 3x}(g(y))$ (and note that $\tilde{g}$ is strictly increasing).

Consider a sequence $(x_n)_{n\in\mathbb{N}}$ in the one-dimensional torus (with one-dimensional Lebesgue-measure) $X = T = [0,1)$, dense and such that $x_n \in [0, \frac{1}{2})$ if and only if $n = \tilde{g}(m)$ for an $m \in \mathbb{N}$. Since $(x_{\tau(n)})_{n\in\mathbb{N}}$ is uniformly distributed in T we have

$$\lim_{N\to\infty} \frac{1}{N}\cdot|\{n\le N|\tau(n) = \tilde{g}(m) \text{ for some } m\}| = \frac{1}{2}.$$

Therefore for all N given and large enough there is an $m > \frac{N}{3}$ and an $n \le N$ with $\tau(n) = \tilde{g}(m) \ge \tilde{g}\left(\frac{N}{3}\right) \ge \tilde{g}\left(\frac{n}{3}\right)$.

Since $\tilde{g}$ is strictly increasing hence there are infinitely many n_k with $\tau(n_k) \ge \tilde{g}\left(\frac{n_k}{3}\right)$. Therefore

$$\lim_{k\to\infty}\frac{H(\tau,n_k)}{g(n_k)} = \lim_{k\to\infty}\frac{\tau(n_k)}{n_k\, g(n_k)} \ge$$
$$\ge \lim_{k\to\infty}\frac{\tilde{g}\,(n_k/3)}{n_k\, g(n_k)} \ge \lim_{k\to\infty}\frac{n_k}{2} = \infty.$$

Hence $\limsup_{N\to\infty}\frac{H(\tau,N)}{g(N)} = \infty$.

b) This follows immediately from Theorem 2a and 2b (which is formulated and proved in Section 4) by the fact that for $(y_n)_{n\in\mathbb{N}}$ μ−u.d. in X and $(x_{\tau(n)})_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} d(x_{\tau(n)}, y_n) = 0$, we have that $(x_{\tau(n)})_{n\in\mathbb{N}}$ is μ−u.d. in X.

c) Let $(x_n)_{n\in\mathbb{N}}$ be dense in T with $x_n = 0$ if n is odd. Since $(x_{\tau(n)})_{n\in\mathbb{N}}$ is uniformly distributed, for every $\epsilon > 0$ we have for the sequence $(x_{\tau(n)})_{n\in\mathbb{N}}$ that

$$\lim_{N\to\infty}\left|\frac{A(N;[0,\epsilon))}{N} - \epsilon\right| = 0,$$

hence

$$\limsup_{N\to\infty}\frac{1}{N}\cdot|\{n\le N|\tau(n) \text{ odd }\}| \le \epsilon,$$

therefore $\liminf_{N\to\infty} H(\tau,N) \le 2\epsilon$, and the result follows.

□

3. Some Definitions and Auxiliary Results for Metric Spaces

From Theorem 1 we see that the question how regular $H(\tau, N)$ can be depends heavily on the properties of $(x_n)_{n\in\mathbb{N}}$ and also on the properties of the metric space X.

In order to be able to study this connection, we have to introduce a new density measure for sequences in compact metric spaces. A well-known measure for the density of a sequence is the dispersion of the sequence (see [5] and the references cited there). But, for our purposes we need a quantity which takes more care on the distribution of $(x_n)_{n\in\mathbb{N}}$ in X. This quantity will be called "$B-$upper density function f" and will be defined in the following:

Definition 1. a) Let B be a family of subsets of X such that every element of B is a $\mu-$continuity set and such that a sequence $(x_n)_{n\in\mathbb{N}}$ is dense in X if and only if for all $M \in B$ we have $x_n \in M$ for infinitely many n.
Then we call B a density determining set of X.

b) For a sequence $(x_n)_{n\in\mathbb{N}}$ in X and an $M \subseteq X$ we denote by $i(N; M)$ the index of the $N-$th element of $(x_n)_{n\in\mathbb{N}}$ which is contained in M. We set $i(N; M) = \infty$, if the number of x_n in M is less than N.

c) Let B be a density determining set of X and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. If there is a function f with $f : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that

$$i(N; M) \leq f\left(\frac{N}{\mu(M)}\right)$$

for all $M \in B$ and for all $N \geq N(M)$ then we call f a B-upper density function for $(x_n)_{n\in\mathbb{N}}$.

Remark 1. a) Of course the class of $\mu-$continuity sets with positive measure always is a density determining set in X.

b) If, for example, X is the $s-$dimensional torus T^s, then the family I of boxes of the form $\prod_{i=1}^{s}[a_i, b_i)$ with $0 \leq a_i < b_i \leq 1$ is a density determining set in X.

c) If B is a density determining set and $M \in B$, then $\mu(M) > 0$ since the support of μ is X.

d) Note that $A(i(N; M), M) = N$ if $i(N; M)$ is finite. Therefore for any μ-continuity set M with positive measure and for uniformly distributed sequences $(x_n)_{n\in\mathbb{N}}$ we have the relation $\lim_{N\to\infty} \frac{N}{i(N;M)} = \mu(M)$.

Remark 2. a) If f is an upper density function for $(x_n)_{n\in\mathbb{N}}$ with respect to the class of $\mu-$continuity sets with positive measure, then for all density-determining sets B it is a $B-$upper density function for $(x_n)_{n\in\mathbb{N}}$.

b) Since with f also $\tilde{f} := \max_{y\leq x} f(y)$ is an upper density function for $(x_n)_{n\in\mathbb{N}}$, in the following we always can and will assume that f is monotonically increasing.

We have

Proposition 1. a) Let B be a density-determining set in X.
The sequence $(x_n)_{n\in\mathbb{N}}$ is dense in X if and only if there exists a $B-$upper density function f for $(x_n)_{n\in\mathbb{N}}$.

b) The sequence $(x_n)_{n\in\mathbb{N}}$ is $\mu-$uniformly distributed in X, if and only if for all $\epsilon > 0$ the function $f(x) := (1+\epsilon)\cdot x$ is a $B-$upper density function.

Proof. a) We have $\mu(M) > 0$ for all $M \in B$. If $(x_n)_{n\in\mathbb{N}}$ is not dense in X, then there is an $M \in B$ with $x_n \in M$ for at most finitely many, say $N_0 - 1$ elements. So $i(N_0; M) = \infty$, and thus an upper density function f cannot exist.
If $(x_n)_{n\in\mathbb{N}}$ is dense in X, then by Theorem 2.5, page 185 and by Theorem 1.2, page 178 in [4], there is a permutation τ of $\mathbb{N}$ such that for the sequence $(x_{\tau(n)})_{n\in\mathbb{N}}$ and for all $\mu-$continuity sets M we have $\lim_{N\to\infty} \frac{A(N;M)}{N} = \mu(M)$. Hence for all $N \geq N_1(M)$ for $(x_{\tau(n)})_{n\in\mathbb{N}}$ we have $\frac{A(N;M)}{N} \geq \frac{\mu(M)}{2}$ and therefore for $(x_n)_{n\in\mathbb{N}}$ and for all M with $\mu(M) > 0$:

$$i(N; M) \leq \max_{m \leq \frac{2N}{\mu(M)}} \tau(m) =: f\left(\frac{N}{\mu(M)}\right)$$

for all $N \geq N_1(M)$.
So this f is an upper density function with respect to the class of $\mu-$continuity sets with positive measure and therefore with respect to every density determining set B.

b) If $(x_n)_{n\in\mathbb{N}}$ is $\mu-$uniformly distributed, then obviously we can choose $f(x) = (1+\epsilon)\cdot x$ for every $\epsilon > 0$ as an upper density function for the class of $\mu-$continuity sets with positive measure and therefore with respect to every density determining set B.
Let now for every $\epsilon > 0$, $f(x) = (1+\epsilon)\cdot x$ be an upper density function for the class M_μ of $\mu-$continuity sets with positive measure (and therefore with respect to every density determining set B). Let $M \in M_\mu$ and assume that if $\mu(M) < 1$ then also $\bar{M} := X\backslash M$ is in M_μ.

For $(x_n)_{n\in\mathbb{N}}$ we have for every $\epsilon > 0$ and all $N \geq N(M,\epsilon)$:

$$i(N;M) \leq \frac{N}{\mu(M)} \cdot (1+\epsilon)$$

and therefore for all $N \geq \tilde{N}(M,\epsilon)$:

$$\frac{\mu(M)}{1+\epsilon} \leq \frac{A(N;M)}{N}.$$

Applying this to the complement $\bar{M} \in M_\mu$ of M we get

$$\frac{A(N;M)}{N} \leq \frac{\epsilon+\mu(M)}{1+\epsilon}.$$

Since in Theorem 1.2, page 175 in [4] we can restrict to the $\mu-$continuity sets with positive measure, the result that $(x_n)_{n\in\mathbb{N}}$ is $\mu-$uniformly distributed follows.

□

To illustrate the idea of upper-density functions, we give an example:

Example 1. Let $X = T = [0,1)$ again be the one-dimensional torus with $d(x,y) := \|x-y\|$ (where $\|x\|$ denotes the distance of x to the nearest integer) and with μ the normalised Haar-measure.
We consider sequences in X with various upper or lower distribution functions (compare with Chapter 1.7 in [4]).

a) For the sequence $(x_n)_{n\in\mathbb{N}}$ in T let

$$\psi(x,y) := \liminf_{N\to\infty} \frac{A(N;[x,y))}{N}$$

for $0 \leq x < y < x+1$ and $x < 1$, where for $y > 1$ we define $A(N;[x,y)) := A[N;[x,1)) + A(N;[0,y))$.

If $\inf_{x\in T} \psi(x,x+t) > 0$ for all t $(0 \leq t < 1)$, then for all functions $h : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $\lim_{x\to\infty} h(x) = +\infty$, the function $f(x) := x \cdot h(x)$ is an $I-$upper density function of $(x_n)_{n\in\mathbb{N}}$ (see Remark 1b)).

b) If there is a constant $b > 0$ such that $\psi(x,y) \geq b \cdot (y-x)$ for all $x \leq y$, then for all $\epsilon > 0$ the function $\frac{1+\epsilon}{b} \cdot x$ is an $I-$upper density function for $(x_n)_{n\in\mathbb{N}}$.

c) Let $z(x,y)$ be a distribution function of $(x_n)_{n\in\mathbb{N}}$ with respect to the sequence $0 = N_0 < N_1 < N_2 < \ldots$ of integers (compare with Definition 7.2, page 53 in [4]) that is:

$$\lim_{i\to\infty} \frac{A(N_i;[x,y))}{N_i} = z(x,y) \quad \text{for } 0 \leq x < y < x+1 \quad \text{and } x < 1.$$

Let $Z(t) := \inf_{x\in T} z(x, x+t) > 0$ for all t with $0 < t < 1$, and for $x > 0$ let

$$L(x) := N_{k+1} \text{ for } N_k < x \leq N_{k+1}.$$

Then for all functions $h : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $\lim_{x\to\infty} h(x) = +\infty$, the function $f(x) := L(x \cdot h(x))$ is an $I-$upper density function for $(x_n)_{n\in\mathbb{N}}$.

d) If for a constant $b > 0$ we have $z(x, y) \geq b\cdot(y-x)$ for all $x < y$, then for all $\epsilon > 0$ the function $L(\frac{1+\epsilon}{b} \cdot x)$ is an $I-$upper density function for $(x_n)_{n\in\mathbb{N}}$.

Proof. a) Let $0 \leq t < 1$. Then for all x there is an $N_0(x, t)$ such that for all $N \geq N_0(x, t)$ we have:

$$\frac{A(N; [x, x+t))}{N} \geq \frac{1}{2} \cdot \inf_{x\in T} \psi(x, x+t) =: \phi(t) > 0,$$

and therefore for all $N \geq N_0(x, t)$

$$i(N; [x, x+t)) \leq \frac{N}{\phi(t)}.$$

Let $N_1(t)$ be so large that $\frac{N}{\phi(t)} \leq \frac{N}{t} \cdot h(\frac{N}{t})$ for all $N \geq N_1(t)$, then for all $N \geq N(x, t) := \max(N_0(x, t), N_1(t))$ we have

$$i(N; [x, x+t)) \leq \frac{N}{t} \cdot h\left(\frac{N}{t}\right).$$

b) Let $0 \leq t < 1$. Then for all x and all $\epsilon > 0$ there is an $N_0(x, t, \epsilon)$ such that for all $N \geq N_0(x, t, \epsilon)$ we have:

$$\frac{A(N; [x, x+t))}{N} \geq \frac{1}{1+\epsilon} \cdot \inf_{x\in T} \psi(x, x+t) \geq \frac{1}{1+\epsilon} \cdot b \cdot t,$$

and therefore for all $N \geq N_0(x, t, \epsilon)$

$$i(N; [x, x+t)) \leq \frac{1+\epsilon}{b} \cdot \frac{N}{t}.$$

The result follows.

c) Let $t > 0$ and $k(M)$ be such that $N_{k(M)} \leq M < N_{k(M)+1}$. Then for all x and all $M > M_0(t)$ we have

$$A(N_{k(M)+1}; [x, x+t)) \geq \frac{1}{2} \cdot N_{k(M)+1} \cdot Z(t) > \frac{1}{2} \cdot M \cdot Z(t).$$

For arbitrary $N \geq \frac{Z(t)}{2} \cdot M_0(t) := N_0(t)$ let $M := \frac{2}{Z(t)} \cdot N$. Then

$$A(N_{k(M)+1}; [x, x+t)) \geq \frac{1}{2} \cdot M \cdot Z(t) = N.$$

Hence

$$i(N; [x, x+t)) \leq N_{k(M)+1} = L\left(\frac{2}{Z(t)} \cdot N\right).$$

Let $N_1(t)$ be so large that

$$\frac{2N}{Z(t)} \leq \frac{N}{t} \cdot h\left(\frac{N}{t}\right)$$

for all $N > N_1(t)$.

Then for all $N \geq \max(N_0(t), N_1(t))$ we have

$$i(N; [x, x+t)) \leq L\left(\frac{N}{t} \cdot h\left(\frac{N}{t}\right)\right)$$

and the result follows.

d) Follows in analogy to c) and b).

□

Before we can state the main result in the next chapter, we need another technical result and a further definition.

Proposition 2. Let (X, d) be a compact metric space with non-negative normed Borel-measure μ. For $A \subseteq X$ let $D(A) := \sup_{x,y \in A} d(x, y)$ be the diameter of A. Then there is a sequence $(P_k)_{k \in \mathbb{N}}$ of finite partitions P_k of X with the following properties:

(i) For every $A \in P_k$, P_{k+1} contains a partition of A.

(ii) $\mu(A) \neq 0$ for all $A \in P_k$.

(iii) A is $\mu-$continuous for all $A \in P_k$.

(iv) $\lim_{k\to\infty}(\max_{A \in P_k} D(A)) = 0$.

For any such $(P_k)_{k \in \mathbb{N}}$ the set $B = \bigcup_{k=1}^{\infty} P_k$ is a density determining set in X.

Proof. Let $P_0 := \{X\}$ and assume that $P_1, P_2, \ldots, P_k$ already are defined. Let $A \in P_k$ be arbitrary and $D_k := \max_{A \in P_k} D(A)$.
Note that each point of X has a basis of neighbourhoods consisting of μ--continuity sets. We choose for every x in the closure of A an open neighbourhood U_x with diameter at most $\frac{D_k}{3}$ that is a μ-continuity set. There exists a finite subset $\{U_{x_i} \mid i = 1, \ldots, s\}$ that forms a finite sub-covering of closure of A.

We denote $A_i := U_{x_i}$. Next we consider the finite set A^* of subsets of A which is defined by

$$A^* = \left\{ A \cap \bigcap_{i \le t} A_i \cap \bigcap_{t < i \le s} \bar{A}_i \,|\, 1 \le t \le s \right\} \backslash \{\phi\},$$

where $\bar{A}_i$ is the complement of A_i in X.
A^* forms a partition of A consisting of μ-continuity sets, and the diameter of each of the elements of A^* is less or equal $\frac{D_k}{3}$.
Denote the elements of A^* such that $A^* = \{B_1, \dots, B_u, C_1, \dots, C_v\}$ with $\mu(B_i) > 0$ and $\mu(C_i) = 0$ for all i.
Since for $C := \bigcup_{i=1}^{v} C_i$ we have $\mu(C) = 0$, and because the support of μ is X, C cannot contain an inner point. So for $x \in C$ let

$$i_x := \min\{i \,|\, 1 \le i \le u, d(x, B_i) < \frac{D_k}{12}\}, V_i := \{x \in C \,|\, i_x = i\}$$

and $B'_i := B_i \cup V_i$ for $1 \le i \le u$.
So we get a finite partition $B'_1, \dots, B'_u$ of A with $D(B'_i) \le \frac{D_k}{2}$, with all the B'_i $\mu-$ continuous, and with $\mu(B'_i) \ne 0$.
Since we can do this for every $A \in P_k$, in this way we obtain a new partition P_{k+1} of X. The sequence of partitions generated in this way obviously satisfies conditions i) to iv).

For the second part of the assertion we have to show that for every sequence $(x_n)_{n \in \mathbb{N}}$ in X we have: For every $\epsilon > 0$ and for every $x \in X$ there are infinitely many x_n in $B(x; \epsilon)$, if and only if in every $A \in B$ there are infinitely many x_n.

For $x \in X$ and $\epsilon > 0$ given, let k be such that $D_k < \epsilon$. Take $A \in P_k$ such that $x \in A$, then $A \subseteq B(x; \epsilon)$ and if in A there are infinitely many x_n so there are in $B(x; \epsilon)$.

Assume now that $(x_n)_{n \in \mathbb{N}}$ is dense in X. Every $A \in P_k$ contains an inner point x because $\mu(A) > 0$ and A is $\mu-$continuous. Let $\epsilon > 0$ be such that $B(x; \epsilon) \subseteq A$. Then infinitely many x_n are in $B(x; \epsilon)$ and hence in A. □

Definition 2. a) For a sequence of partitions like in Proposition 2 let

$$u_k := \max \left\{ \frac{\mu(A_1)}{\mu(A_2)} | A_1 \in P_k, A_2 \in P_{k+1}, A_2 \subseteq A_1 \right\}.$$

If there is a constant $c \ge 2$ and a sequence of partitions like above and such that $u_k \le c$ for all $k \in \mathbb{N}$, then we call the metric space X uniformly divisible and c a splitting constant of X.

b) If for a partition-sequence (P_k) like above the system B is a density determining class of X with $\bigcup_{k=1}^{\infty} P_k \subseteq B$, then we say that B is a density determining class over (P_k).

We give an example for the above defined concepts:

Example 2. a) Every compact measureable subset X of $\mathbb{R}^s$ is uniformly divisible with splitting constant $c = 2$ and with respect to Lebesgue-measure.

b) The $s-$dimensional torus T^s is uniformly divisible with $c = 2$ and with all P_k consisting only of intervals of the form $I = \prod_{i=1}^{s} [a_i, b_i)$ with $0 \leq a_i < b_i \leq 1$.

Proof. a) This is obvious, since for every $\mu-$continuity set A of $\mathbb{R}^s$ and every $(a_1, \ldots, a_s) \in \mathbb{R}^s \backslash \{\mathbf{0}\}$ there is at least one hyperplane with normal vector $(a_1, \ldots, a_s) \in \mathbb{R}^s$ which divides A into two $\mu-$continuous parts with the same measure.

b) We can choose

$$P_k := \left\{ \prod_{i=1}^{l} \left[\frac{j_i}{2^{m+1}}, \frac{j_i+1}{2^{m+1}}\right) \times \prod_{i=l+1}^{s} \left[\frac{h_i}{2^m}, \frac{h_i+1}{2^m}\right) \right. \left| j_i = 0, 1, \ldots, 2^{m+1}-1; h_i = 0, 1, \ldots 2^m - 1 \right\}$$

when $k = sm + l$ with m, l non-negative integers.

□

4. Formulation and Proof of Theorem 2 (Main Result)

We are now ready to state and to prove the main result of the paper which essentially answers the question raised in Chapter 1.

Theorem 2. *In the following let $(x_n)_{n\in\mathbb{N}}$ be dense in X, $(y_n)_{n\in\mathbb{N}}$ be $\mu-$uniformly distributed in X and $h : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $\lim_{x\to\infty} h(x) = \infty$ (arbitrarily slowly growing).*
Let $(P_k)_{k\in\mathbb{N}}$ be a sequence of partitions dividing X, and let f be a (monotonically increasing, see Remark 2b) upper density function for $(x_n)_{n\in\mathbb{N}}$ with respect to a density determining class B over (P_k). Then:

a) *There always (even if X is not uniformly divisible) is a permutation τ of $\mathbb{N}$ with*

$$\lim_{N\to\infty} H(\tau,N)h(N) = +\infty \ , \ \limsup_{N\to\infty} H(\tau,N)\frac{N}{f(h(N)\cdot N)} \le 1$$

and

$$\lim_{n\to\infty} d(x_{\tau(n)}, y_n) = 0.$$

b) *If X is uniformly divisible, and (P_k) divides X with a splitting constant c_0, then for all $c > c_0$ there is a permutation τ of $\mathbb{N}$ with*

$$\lim_{N\to\infty} H(\tau,N)\cdot h(N) = +\infty \ , \ \limsup_{N\to\infty} H(\tau,N)\cdot\frac{N}{f(c\cdot N)} \le 1$$

and

$$\lim_{n\to\infty} d(x_{\tau(n)}, y_n) = 0.$$

c) *If X is uniformly divisible, and (P_k) divides X with a splitting constant c_0, and if moreover $(x_n)_{n\in\mathbb{N}}$ is such that for some constants $a_1, a_2 > 0$ and all $A \in B$ we have*

$$\frac{1}{a_1}\mu(A) \le \frac{A(N;A)}{N} \le a_2\mu(A)$$

for all $N \ge N(A)$,
then for all $b_1 > a_1$ and all $b_2 > a_1 \cdot c_0$ there is a permutation τ of $\mathbb{N}$ with

$$\frac{1}{b_1} < \liminf_{N\to\infty} H(\tau,N) \le \limsup_{N\to\infty} H(\tau,N) < b_2$$

and

$$\lim_{n\to\infty} d(x_{\tau(n)}, y_n) = 0.$$

P r o o f. First we prove the upper bound for a) and b):
Let $M_1 < M_2 < \dots$ be a sequence of integers such that for the sequence $(x_n)_{n\in\mathbb{N}}$ we have

(*)
$$i(N;A) \le f\left(\frac{N}{\mu(A)}\right)$$

for all $A \in P_k$ and all $N \ge M_k$,

(**) for an $\epsilon > 0$, which will be chosen later, for the sequence $(y_n)_{n\in\mathbb{N}}$ we have

$$\frac{N}{\mu(A)}\cdot\frac{1}{1+\epsilon} \le i(N;A) \le \frac{N}{\mu(A)}\cdot\frac{1}{1-\epsilon}$$

for all $A \in \bigcup_{i=0}^{k} P_i$ and all $N \ge M_k$,

(***) for every $A \in P_k$ there is an i with $M_k < i \le M_{k+1}$ with $y_i \in A$.

Now we construct a permutation τ:
Assume that $\tau(1), \ldots, \tau(M_k)$ already are defined. Then we define $\tau(M_k+1), \ldots, \tau(M_{k+1})$ step by step such that for all i with $M_k + 1 \leq i \leq M_{k+1}$ we have:

(i) $\tau(i) \neq \tau(j)$ for all $j < i$

(ii) $x_{\tau(i)}$ lies in the same element A of P_k as y_i
(This can be satisfied since A contains inner points.)

(iii) $\tau(i)$ is minimal with the above two properties.

By construction (and because of (***)) we have that τ indeed is a permutation and that $\lim_{i\to\infty} d(y_i, x_{\tau(i)}) = 0$.

Take now any A from P_k.
Because of (**), for every $l = 1, 2, \ldots, M_{k+1} - M_k$ there are at most

$$(1+\epsilon)\cdot(M_k + l)\cdot\mu(A) - (1-\epsilon)\cdot M_k\cdot\mu(A) = (1+\epsilon)\cdot l\cdot\mu(A) + 2\epsilon\cdot M_k\cdot\mu(A)$$

indices i with $M_k + 1 \leq i \leq M_k + l$ such that $y_i \in A$.

Take now any $\tilde{A} \in P_{k-1}$ with $A \subseteq \tilde{A}$. Because of (**) there are at most

$$(1+\epsilon)\cdot M_k\cdot\mu(\tilde{A})$$

indices i with $i = 1, 2, \ldots M_k$ such that $y_i \in \tilde{A}$.
Therefore, by the definition of τ there are at most

$$(1+\epsilon)\cdot M_k\cdot\mu(\tilde{A})$$

indices i with $i = 1, 2, \ldots M_k$ such that $x_{\tau(i)} \in \tilde{A}$ and consequently at most as many such that $x_{\tau(i)} \in A$.
Hence especially there are at most

$$(1+\epsilon)\cdot(M_k + l)\cdot\mu(A) - (1-\epsilon)\cdot M_k\cdot\mu(A) + (1+\epsilon)\cdot M_k\cdot\mu(\tilde{A}) =$$
$$(1+\epsilon)\cdot l\cdot\mu(A) + 2\epsilon\cdot M_k\cdot\mu(A) + (1+\epsilon)\cdot M_k\cdot\mu(\tilde{A})$$

indices i with $i = 1, 2, \ldots M_k + l$ such that $y_i \in A$.

Therefore, since f is monotonically increasing, because of (*) and because of (ii) and (iii), we have for every $l = 1, 2, \ldots l = 1, 2, \ldots, M_{k+1} - M_k$ the following estimate for τ :

$$\begin{aligned}
\tau(M_k + l) &\leq f\left(\frac{(1+\epsilon)\cdot l\cdot\mu(A) + 2\epsilon\cdot M_k\cdot\mu(A) + (1+\epsilon)\cdot M_k\cdot\mu(\tilde{A})}{\mu(A)}\right) \\
&= f\left((1+\epsilon)\frac{\mu(\tilde{A})}{\mu(A)}M_k + 2\epsilon M_k + (1+\epsilon)\cdot l\right) \\
&\leq f\left((1+3\epsilon)\frac{\mu(\tilde{A})}{\mu(A)}(M_k + l)\right).
\end{aligned}$$

Concerning part a) of the Theorem:
Let $\epsilon > 0$ from above be arbitrary and impose on the sequence M_k the additional condition, that for all k the number M_k is so large that for the u_k from Definition 2a we have
$$\min_{y \geq M_k} h(y) \geq (1+3\epsilon)u_k.$$
Then we have $\tau(i) \leq f(h(i) \cdot i)$ for all i and the result follows.
Concerning part b) of the Theorem:
We have $\frac{\mu(\tilde{A})}{\mu(A)} \leq c_0$. Choose the $\epsilon > 0$ from above so small that $(1+3\epsilon) \cdot c_0 < c$. Then $\tau(i) \leq f(c \cdot i)$ for all i large enough, and the result follows.

Now we prove the lower bound for a) and b):
Let $\mu_k := \min_{A \in P_k} \mu(A)$. Then, by (**) in every $A \in P_k$ there are at least $(1-\epsilon)\mu_k(M_k + l)$ of the elements $y_1, \ldots, y_{M_k+l}$.
For i with $M_k < i < M_{k+1}$ because of (iii) therefore we have $\tau(i) \geq (1-\epsilon)\mu_k \cdot i$.
Let $\epsilon > 0$ from above be arbitrary and impose on the sequence M_k the additional condition that for all k the number M_k is so large that
$$(1-\epsilon)\mu_k \geq \big(\min_{y \geq M_k} h(y)\big)^{-\frac{1}{2}}.$$
Then for all i with $M_k < i \leq M_{k+1}$ we have
$$\tau(i) \geq i \cdot \big(\min_{y \geq M_k} h(y)\big)^{-\frac{1}{2}} \geq i \cdot \big(h(i)\big)^{-\frac{1}{2}}.$$
Hence for all N large enough
$$H(\tau, N) \cdot h(N) \geq \big(h(N)\big)^{\frac{1}{2}},$$
and the result follows.
Concerning part c) of the Theorem:
We take τ like above.
The upper estimate for τ follows immediately from b).

Concerning the lower bound, choose $\epsilon > 0$ such that $\frac{1-2\epsilon}{a_2+\epsilon} \geq \frac{1}{b_1}$, and impose four further conditions on the sequence (M_k), namely

α) For the sequence $(x_n)_{n \in \mathbb{N}}$ we have
$$\frac{A(N; A)}{N} \leq (a_2 + \epsilon) \cdot \mu(A)$$
for all $A \in P_k$ and all $N \geq M_{k-1}$.

β) For the sequence $(y_n)_{n\in\mathbb{N}}$ we have

$$\frac{A(N;A)}{N} \geq (1-\epsilon)\cdot\mu(A)$$

for all $A \in P_k$ and all $N \geq M_{k-1}$.

γ) For all k we have

$$M_{k+1} > \frac{M_k}{\epsilon\mu_k},$$

which implies that for all $l \geq M_{k+1}$ we have

$$\mu(A)\cdot l\cdot(1-\epsilon) - M_k \geq (1-2\epsilon)\cdot l\cdot\mu(A).$$

δ) $\left[M_{k+1}\frac{1}{b_1}\right] > M_{k-1}$ which implies that for all $l \geq M_{k+1}$ we have

$$\left[\frac{(1-2\epsilon)l}{a_2+\epsilon}\right] > M_{k-1}.$$

Then we have:

For $k \geq 1$ let $M_{k+1} \leq l < M_{k+2}$. By β) in every $A \in P_k$ there are at least $\mu(A)\cdot l\cdot(1-\epsilon)$ of the y_n with $n = 1, 2, \ldots, l$.
Further, by α) and δ), in A there are at most $(1-2\epsilon)\cdot l\cdot\mu(A)$ of the x_n with $n = 1, 2, \ldots, \left[\frac{(1-2\epsilon)l}{a_2+\epsilon}\right]$.
Therefore, and because of γ), we have that for all $j < \left[\frac{(1-2\epsilon)l}{a_2+\epsilon}\right]$ there is an i with $M_k < i \leq l$ and with $\tau(i) = j$.
Therefore we have $\tau(l) \geq \frac{(1-2\epsilon)l}{a_2+\epsilon}$ for all $M_{k+1} \leq l < M_{k+2}$, and the result follows.

□

5. Some Examples

For the illustration of the results of Theorem 2 we give some examples.

Example 3. For all $b_1 > \frac{e}{e-1}$ and all $b_2 > 2e-2$ there is a permutation τ of $\mathbb{N}$ with $\frac{1}{b_1}N < \tau(N) \leq b_2 N$ for all N, such that the sequence $(\{\log\tau(n)\})_{n\in\mathbb{N}}$ is uniformly distributed in $[0,1)$.

Proof. For the sequence $(\{\log\tau(n)\})_{n\in\mathbb{N}}$ we have (see [4], pages 58 - 59):

$$\liminf_{N\to\infty}\frac{A(N;[x,y))}{N} = \frac{e^y - e^x}{e^x(e-1)} \geq \frac{1}{e-1}(y-x)$$

and

$$\limsup_{N\to\infty} \frac{A(N;[x,y))}{N} = \frac{e(e^y - e^x)}{e^y(e-1)} \leq \frac{e}{e-1}(y-x)$$

for all $0 \leq x < y < x+1$; $0 \leq x < 1$.
By Example 2b and Theorem 2c there is a permutation σ with $\frac{1}{b_1}N < \sigma(N) \leq b_2N$ for all $N > N_0$ for some N_0, and such that $(\{\log \sigma(n)\})_{n\in\mathbb{N}}$ is uniformly distributed in $[0,1)$.
We will define now τ in the following way:

Let $i_1 < i_2 < \ldots i_k \leq N_0$ be the indices $\leq N_0$ for which $\sigma(i_l) =: j_l > N_0$ and let $N_0 < r_1 < r_2 < \ldots < r_k$ be the indices $\geq N_0$ for which $\sigma(r_l) =: s_l \leq N_0$. Then we define:

$$\tau(i) := i \text{ for } i \leq N_0$$

$$\tau(r_l) := j_l \text{ for } l = 1,2,\ldots,k$$

$$\tau(i) := \sigma(i) \text{ otherwise.}$$

Since we have changed only finitely many elements, the sequence $(\{\log \tau(n)\})_{n\in\mathbb{N}}$ remains uniformly distributed in $[0,1)$. Further:

$$\frac{1}{b_1}r_l < \sigma(r_l) = s_l \leq N_0 \leq \tau(r_l) = j_l = \sigma(i_l) < b_2 i_l \leq b_2 r_l,$$

and therefore $\frac{1}{b_1}N < \tau(N) \leq b_2N$ for all N. □

Example 4. Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be dense in X with (strictly increasing) upper density functions f respectively g.
Let $h : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $\lim_{x\to\infty} h(x) = +\infty$ (arbitrarily slowly growing).
Then there is a permutation τ of $\mathbb{N}$ with:

$$\frac{g^{-1}(N)}{h(N)} \leq \tau(N) \leq f(h(N)\cdot N)$$

for all $N \geq N_0$ and with

$$\lim_{n\to\infty} d(x_{\tau(n)}, y_n) = 0.$$

P r o o f. Without loss of generality we may assume that h is monotonically increasing. Choose any strictly increasing function $h^* : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with the following properties:

(i) h^* and $\frac{x}{h^*(x)}$ are strictly increasing and tend to infinity.

(ii) Let the function $\tilde{h}$ be defined such that $\tilde{h}(x)\cdot x$ is the inverse of $\frac{x}{h^*(x)}$, then we demand that $h^*\left(\left\lceil \tilde{h}(x)\cdot x \right\rceil\right) \cdot \left\lceil \tilde{h}(x)\cdot x \right\rceil \leq h(x)\cdot x$ for all x large enough.

(iii)

$$h(x) \geq \frac{g^{-1}(x)}{g^{-1}\left(\left[\frac{x}{h^*(x)}\right]\right)} \cdot h^*\left(g^{-1}\left(\left[\frac{x}{h^*(x)}\right]\right)\right)$$

for all x large enough.

Note that it indeed is no problem to choose h^* in this way, since $\tilde{h}$ is the 'smaller', the 'smaller' h^* is, and (for (iii)) since $\frac{h^*(x)}{x}$ is decreasing in x. Let $\breve{h}(x) := \min(h^*(x), \tilde{h}(x))$. Note that $\tilde{h}$ is monotonically increasing to infinity and so also $\breve{h}$ is monotonically increasing with $\lim_{N\to\infty} \breve{h}(N) = \infty$. Further let $(z_n)_{n\in\mathbb{N}}$ be a sequence, uniformly distributed in X. Then by Theorem 2a there are permutations σ and ρ such that (condition (*))

$$\frac{\sigma(N)}{N}\breve{h}(N) \geq 1 \quad \text{and} \quad \frac{\sigma(N)}{f(\breve{h}(N)\cdot N)} \leq 1$$

for all $N \geq N_0$ and $\lim_{n\to\infty} d(x_{\sigma(n)}, z_n) = 0$ and (condition (**))

$$\frac{\rho(N)}{N}\breve{h}(N) \geq 1 \quad \text{and} \quad \frac{\rho(N)}{g(\breve{h}(N)\cdot N)} \leq 1$$

for all $N \geq N_0$ and $\lim_{n\to\infty} d(y_{\rho(n)}, z_n) = 0$.
Hence with $\tau := \sigma \circ \rho^{-1}$ we have

$$\lim_{n\to\infty} d(x_{\tau(n)}, y_n) = 0.$$

Now, first we derive upper and lower estimates for $\rho^{-1}(N)$:
By (**) we have

$$\rho(N) \geq \frac{N}{\breve{h}(N)} \geq \frac{N}{h^*(N)}$$

for all $N \geq N_0$. Let now $M \geq \frac{N_0}{h^*(N_0)}$ be given and N' such that $\frac{N'}{h^*(N')} \leq M < \frac{N'+1}{h^*(N'+1)}$. Then

$$\rho(N') \geq \frac{N'}{h^*(N')}$$

and hence (since $\rho(N') \in \mathbb{N}$)

$$\rho(N') \geq M.$$

For all $N > N'$ we have

$$\rho(N) \geq \frac{N}{h^*(N)} \geq \frac{N'+1}{h^*(N'+1)} \quad \text{and hence} \quad \rho(N) \geq \left\lceil \frac{N}{h^*(N)} \right\rceil > M.$$

Consequently,

$$\left\lceil \tilde{h}(M)\cdot M \right\rceil \geq \rho^{-1}(M) \quad \text{for all} \quad M \geq \frac{N_0}{h^*(N_0)}.$$

On the other hand, by (**) we have

$$\rho(N) \leq g\big(\breve{h}(N)\cdot N\big) \leq g\big(\tilde{h}(N)\cdot N\big)$$

for all $N \geq N_0$. Hence

$$N \leq \rho^{-1}\left(\left\lceil g(\tilde{h}(N)\cdot N)\right\rceil\right)$$

for all N with $\tilde{h}(N)\cdot N \geq \tilde{h}(N_0)\cdot N_0$. Like above from this we conclude that

$$g^{-1}\left(\left\lfloor \frac{M}{h^*(M)}\right\rfloor\right) \leq \rho^{-1}(M) \quad \text{for all} \quad M \geq g(\tilde{h}(N_0)N_0).$$

Altogether

(***)
$$g^{-1}\left(\left\lfloor \frac{M}{h^*(M)}\right\rfloor\right) \leq \rho^{-1}(M) \leq \left\lceil \tilde{h}(M)\cdot M\right\rceil \quad \text{for all} \quad M \geq M_0.$$

Therefore for τ, on the one hand, by (*), (***) and by (ii), for N large enough we have:

$$\begin{aligned}\tau(N) = \sigma\big(\rho^{-1}(N)\big) \leq f\Big(\breve{h}\big(\rho^{-1}(N)\big)\rho^{-1}(N)\Big) \leq \\ \leq f\left(h^*\left(\left\lceil \tilde{h}(N)\cdot N\right\rceil\right)\cdot\left\lceil \tilde{h}(N)N\right\rceil\right) \leq f\big(h(N)\cdot N\big).\end{aligned}$$

On the other hand, by (**), (***) and by (iii) we have

$$\tau(N) = \sigma(\rho^{-1}(N)) \geq \frac{\rho^{-1}(N)}{h^*(\rho^{-1}(N))} \geq \frac{g^{-1}\left(\left\lfloor \frac{N}{h^*(N)}\right\rfloor\right)}{h^*\left(g^{-1}\left(\left\lfloor \frac{N}{h^*(N)}\right\rfloor\right)\right)} \geq \frac{g^{-1}(N)}{h(N)}$$

for all N large enough, and the result follows. $\square$

Example 5. Let X be uniformly divisible with a splitting constant c_0 with respect to (P_k) and let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be dense in X with

$$\frac{1}{a_1}\mu(A) \leq \frac{A(N;A)}{N} \leq a_2\cdot\mu(A)$$

for the sequence $(x_n)_{n\in\mathbb{N}}$, and

$$\frac{1}{a_3}\mu(A) \le \frac{A(N;A)}{N} \le a_4 \cdot \mu(A)$$

for the sequence $(y_n)_{n\in\mathbb{N}}$ and for all $A \in B$ (B a density determining class over (P_k)), all $N \ge N(A)$ and constants $a_1, a_2, a_3, a_4 > 0$.
Then for all $b_1 > a_2 a_3 c_0$ and $b_2 > a_1 a_4 c_0$, there is a permutation τ of $\mathbb{N}$ with

$$\frac{1}{b_1} < \liminf_{N\to\infty} H(\tau, N) \le \limsup_{N\to\infty} H(\tau, N) < b_2$$

and

$$\lim_{n\to\infty} d(x_{\tau(n)}, y_n) = 0.$$

Proof. This is an easy corollary of Theorem 2c. □

Example 6. Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be dense in the one-dimensional torus with asymptotic distribution functions F respectively G (compare with Chapter 1.7. in [4]), that is with:

$$\lim_{N\to\infty} \frac{A(N;[x,y))}{N} = F(x,y)$$

for the sequence $(x_n)_{n\in\mathbb{N}}$, respectively with

$$\lim_{N\to\infty} \frac{A(N;[x,y))}{N} = G(x,y)$$

for the sequence $(y_n)_{n\in\mathbb{N}}$, for all $0 \le x < y \le x+1$ and $x < 1$.
Let $a_1, a_2, a_3, a_4 > 0$ be constants with

$$\frac{1}{a_1}(y-x) \le F(x,y) \le a_2(y-x)$$

and

$$\frac{1}{a_3}(y-x) \le G(x,y) \le a_4(y-x)$$

for all $0 \le x < y \le 1$.
Then for all $b_1 > a_2 a_3$ and $b_2 > a_1 a_4$ there is a permutation τ of $\mathbb{N}$ with

$$\frac{1}{2b_1} < \liminf_{N\to\infty} H(\tau, N) \le \limsup_{N\to\infty} H(\tau, N) < 2b_2$$

and such that $(x_{\tau(n)})_{n\in\mathbb{N}}$ has G as asymptotic distribution function.

Proof. This is an easy corollary of Example 5. □

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