Uniform Distribution Theory **9** (2014), no.1, 43–77

uniform
distribution
theory

PARTITION FUNCTIONS IN NUMERATION SYSTEMS WITH BOUNDED MULTIPLICITY

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ABSTRACT. For a given increasing sequence of positive integers $A = (a_k)_{k\geq 0}$ and for q, an integer ≥ 2 or eventually $q = \infty$, let $M_{A,q}(n)$ denote the number of representations of a given integer n by sums $\sum_{k\geq 0} e_k a_k$ with integers e_k in $[0, q)$. If $a_0 = 1$, the sequence A constitutes a numeration system for the natural numbers and A takes the name of scale. The partition problem consists in studying the asymptotic behavior of $M_{A,q}(\cdot)$ and its summation function $\Gamma_{A,q}(\cdot)$. In this paper we study various aspects of this problem. In the first part we recall important results and methods developed in the literature with attentions to the binary numeration system, the d-ary numeration system and also the Fibonacci and the m-bonacci scales. These cases show that $M_{A,q}(\cdot)$ can be very irregular. In the second part, miscellaneous general results are proved and we investigate in more details sequences A which grow exponentially. In particular, we generalize a result of Dumont-Sidorov-Thomas in proving that if $a_k \sim c\gamma^k$ (with the only natural restriction $\gamma > 1$) then $\Gamma_{A,q}(x) = x^{\log_\gamma q} H(\log_\gamma x) + o(x^{\log_\gamma q})$ where H is a function strictly positive, continuous, periodic of period 1 and almost everywhere differentiable. The final part is devoted to a particular family of recurrent sequences G called Pisot scales. We prove in that case that for any suitable q, there exists a set $S_{G,q}$ of positive integers with natural density 1 such that $\lim_{s\to\infty,\, s\in S_{G,q}} \log M_{G,q}(s)/\log s$ exists. The proof uses a previous work of D.-J. Feng and N. Sidorov related to the multiplicity of the radix θ-expansions of real numbers using digits $0, 1, \ldots, q-1$, where θ is any positive Pisot number.

Communicated by Oto Strauch

Dedicated to the memory of Gérard Rauzy

2010 Mathematics Subject Classification: 11P99, 28XX.
Keywords: partition function, numeration system, radix expansion, Pisot scale.
† This author is supported by the RGC grant and the Focused Investments Scheme in the CUHK.
†† These authors are partially supported by Aix-Marseille University-CNRS: UMR 7373-École Centrale de Marseille.

1. The partition problem

The study of partitions of integers into parts in $\mathbb{N}$ (the set of all natural numbers) emerged in 1748 with Euler in his famous *Introductio in analysisin infinitorum* [24] where he considered in particular the number $p(n)$ of partitions of n into integer parts with possible repetitions of parts and the number $q(n)$ of partitions into unequal parts. Classical results of Euler consist in formal identities relating the generating functions $f(x) := \sum_{n=0}^{\infty} p(n)x^n$ and $g(x) := \sum_{n=0}^{\infty} q(n)x^n$ to the products $f(x) = \prod_{k=1}^{\infty}(1-x^k)^{-1}$ and $g(x) = \prod_{k=1}^{\infty}(1+x^k)$. Notice that these formulae show that $g(x) = f(x)/f(x^2)$. This combinatory approach has been intensively investigated. Many results have been derived from formal identities relating sums and products as above. For example, the classical relation $\sum_{n=0}^{\infty} q(n)x^n = \prod_{k=0}^{\infty} \frac{1}{1-x^{2k+1}}$, also due to Euler, says that $q(n)$ is the number of partitions of n into odd parts. For an overview of many other formal identities we refer the reader to H.L. Alder [3] and for the traditional theory of partitions we refer to the monograph of G.E. Andrews [4] and the references therein which cover the subject up to 1976.

The partition problem in full generality can be set up as follows. Let $A = (a_k)_{k\geq 0}$ be an increasing sequence of positive integers called parts. We naturally assume that there is no common divisor of all a_k except 1 (otherwise we divide all parts by their greatest common divisor). Such a sequence A will be called a *base of parts*.

A partition of a positive integer n into parts in A is defined as a sum

$$n = e_0 a_0 + \cdots + e_{m-1} a_{m-1}\,, \tag{1}$$

in which the integers e_k, called digits, verify $e_k \geq 0$ and $e_{m-1} \neq 0$. The digit e_k is called the multiplicity (or weight) of the part a_k in (1). Partitions of n are distinguished by their m-tuples $(e_0, \ldots, e_{m-1})$. For a given integer $q \geq 2$, or eventually $q = \infty$, let $M_{A,q}(n)$ denote the number of distinct partitions of n with $e_k(n) < q$ for all $k = 0, \ldots, m-1$. In particular $M_{A,q}(1) = 1$ if $a_0 = 1$ and $M_{A,q}(1) = 0$ otherwise. We also set $M_{A,q}(0) = 1$ and $M_{A,q}(n) = 0$ for negative integers n. The map $M_{A,q}(\cdot)$ will be called the partition function with base of parts A and multiplicity strictly less than q. If $q = 2$, each part a_k in (1) occurs with multiplicities e_k at most 1. If $q = \infty$, each part a_k in (1) may appear with multiplicities e_k at most $\lfloor n/a_k \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer not exceeding x. Obviously $M_{A,q}(\cdot) \leq M_{A,q'}(\cdot)$ if $q \leq q'$ and $M_{A,\infty}(n) = \max_q M_{A,q}(n)$.

In case $a_0 = 1$, the base of parts A will be called a *scale*. A familiar example is the d-adic scale ($d \geq 2$):

$$E_d := (d^n)_{n\geq 0}\,.$$

One interest to introduce the notion of scale is that for any natural number n there exists a unique sequence of nonnegative integers $e_k(n)$ and a unique nonnegative integer $h = h_A(n)$ such that

$$n = e_0(n)a_0 + \cdots + e_h(n)a_h, \quad e_h(n) \neq 0, \tag{2}$$

with $e_j(n) = 0$ if $j > h$ and

$$e_0(n)a_0 + \cdots + e_k(n)a_k < a_{k+1} \tag{3}$$

for all $k \geq 0$ (for an account of various arithmetical, combinatory and dynamical properties about scales, see for example [27, 30, 8, 9]). By construction, $a_h \leq n < a_{h+1}$. The successive digits $e_k(n)$, $k = h, h-1, \ldots, 0$ can be computed in this order, step by step, applying the so-called greedy algorithm. The right member of equality (2) under constraints (3) will be called the standard A-expansion of n.

The integer $e_k(n)$ is then called the standard k-th digit (in the scale A). Of course, one has $e_k(n) < a_{k+1}/a_k$. In case $A = \mathbb{N}$ (the set of natural numbers), we use the traditional notations $q(n)$ for $M_{\mathbb{N},2}(n)$ and $p(n)$ for $M_{\mathbb{N},\infty}(n)$.

The partition problem consists in finding asymptotic formulae for $p(n)$ and $q(n)$ or similar quantities. The first deep progress on this problem emerged in 1918 with G. H. Hardy and S. Ramanujan. In their paper [33], the authors develop various methods to attack the partition problem. One method was elementary, another one used Tauberian theorems developed in [32], but the most powerful method, presently known as the *circle method*, was based on the representation of the coefficients of power series by mean of the integral Cauchy formula. To apply this formula G. H. Hardy and S. Ramanujan took into account that the map $\eta(z) = e^{\frac{i\pi z}{12}}/f(e^{i\pi z})$ ($\Im(z) > 0$) already studied by R. Dedekind [17], is analogous to a modular form, and introduced the Farey dissection of $[0, 1]$. They obtained an accurate asymptotic expansion of $p(n)$ which gives in particular

$$p(n) \sim \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{2n/3}} \tag{4}$$

where here and afterward, for any positive real-valued sequences u and v, the notation $u(n) \sim v(n)$ is set for $\lim_{n\to\infty} u(n)/v(n) = 1$. Also, a large class of partition problems were exhibited in [33], including all partition functions $M_{\mathbb{N},q}$, for which the circle method may be applied in a similar way in order to obtain sharp results. In particular, the authors gave the asymptotic behavior

$$q(n) \sim \frac{1}{4.3^{1/4}n^{3/4}} e^{\pi\sqrt{n/3}}. \tag{5}$$

It was also pointed out in [33] that the number $p^{(k)}(n)$ of partitions of n into parts which are perfect k-th powers does not belong to the class considered above, but for $k = 2$, by mean of a Tauberian argument H. Hardy and S. Ramanujan got an equivalence of $\log p^{(2)}(n)$ and then, using functional relations that exhibit the behavior of the generating function $\sum_{n=0}^{\infty} p^{(k)}(n)z^n = \prod_{n=1}^{\infty}(1 - z^{n^k})^{-1}$ ($|z| < 1$) near the roots of unity they gave but without proof an asymptotic estimate of $p^{(k)}(n)$. This later result was improved by E. M. Wright [58] who found an asymptotic expansion of $p^{(k)}(n)$. Refining Hardy-Ramanujan circle method, H. Rademacher obtained a series expansion of $p(n)$ in [49] (see also the monograph of T. Apostol [5]). Notice that recently, W. De Azevedo Pribitkin [7] derived the exact formula of Rademacher, by substituting the circle method for the computation of the Fourier expansion of $\eta^{-1}(\cdot)$. The Hardy-Ramanujan's Tauberian theorem ([32], Theorem A) is an interesting tool to evaluate the asymptotic behavior of general partition functions by their logarithms. A typical example is the asymptotic formula of N. A. Brigham [11] from which the author derived Hardy-Ramanujan formulae in terms of logarithm for partitions into k-th powers as well as partitions into prime numbers. M. Dutta [20] using a Tauberian theorem from [33] (see also [32], Theorem C) obtained

$$\log M_{\mathbb{N},q}(n) = \pi\sqrt{2/3}\big(n(q-1)/q\big)^{1/2}\big(1 + o(1)\big)$$

for any integer $q \geq 2$. P. Hagis in [31] improved this result, obtaining a convergent infinite series for $M_{\mathbb{N},q}(n)$ using the Hardy-Rademacher-Ramanujan circle method.

The elementary side was explored by P. Erdős in [21]. For example, he got $p(n) \sim an^{-1}e^{cn}$ (with $c = \pi\sqrt{2/3}$) without succeeding to compute the value of a, but a bit later, D. J. Newman [43] proved elementarily that $a = \frac{1}{4\sqrt{3}}$. P. Erdős considered also bases A of parts having natural density $\alpha > 0$ and proved that if the parts have no common prime factor then $\log M_{A,\infty}(n) \sim c\sqrt{\alpha n}$ ($c = \pi\sqrt{2/3}$) and moreover, if $M_{A,2}(n) > 0$ for n large enough, then $\log M_{A,2}(n) \sim c\sqrt{\alpha n/2}$. In addition, the converse in both cases is true: each above asymptotic equivalence implies that A is of density α.

It is natural to investigate asymptotic formulae and expansion series for partition functions with parts selected in some interesting bases of parts. There are numerous papers on this topic. One step in this direction was done by K. Mahler in 1940 [41]. Studying solutions of the functional equation $\frac{f(z+\omega)-f(z)}{\omega} = f(qz)$ ($\omega \neq 0$, $0 < q < 1$) he derived the approximate formula $\log M_r(n) \sim \frac{1}{2}(\log n)^2/\log r$ where $M_r(n)$ is the number of partitions of n with parts in the set E_r of nonnegative powers r^n of a given integer $r \geq 2$. Combining Mellin transform and saddle point method N. G. de Bruijn in [16] improved Malher's result in the following form:

$$\log M_r(rh) = Q_r(\log h, \log\log h) + \psi\left(\frac{\log h - \log\log h}{\log r}\right) + \mathcal{O}\left(\frac{(\log\log h)^2}{\log h}\right) \quad (6)$$

where $Q_r(\cdot,\cdot)$ is a quadratic form and $\psi(\cdot)$ is a periodic function mod 1 and analytical in the trip $|\Im(z)| < \frac{\pi}{2\log r}$.

Another step was marked by A. E. Ingham who proved in [35] a very general Tauberian theorem that can be used to deduce asymptotic formulae for $p(n)$, $q(n)$ but also for partition functions $M_{A,2}(\cdot)$, $M_{A,\infty}(\cdot)$ with various possible bases of parts A including partition of n into integers k_1-th or k_2-th powers. In fact, following Ingham's notation, let $\Lambda = (\lambda_\nu)_{\nu\geq 1}$ be an increasing sequence of positive real numbers and set $\Lambda(u) := \#\{\nu\,;\ \lambda_\nu \leq u\}$. Assuming that $\Lambda(u) = Bu^\theta + R(u)$ $(B > 0,\ \theta > 0)$ and $\int_0^u R(v)v^{-1}dv = b\log u + c + o(1)$ when $u \to \infty$, Ingham obtained (Theorem 2, [35]) asymptotic behavior for the numbers $P(u)$ (resp. $P^*(u)$) of solutions $\lambda_{\nu_1} + \lambda_{\nu_2} + \lambda_{\nu_3} + \cdots < u$ with $\nu_1 < \nu_2 < \nu_3 < \cdots$ (resp. $\nu_1 \leq \nu_2 \leq \nu_3 \leq \cdots$) and also asymptotic behavior of the discrete derivatives $P_h(u) = \frac{1}{h}(P(u) - P(u-h))$, $P_h^*(u) = \frac{1}{h}(P^*(u) - P^*(u-h))$, with monotonic conditions on P_h and P_h^*. The monotonic assumption on P_h was removed by F. C. Auluck and C. B Haselgrove in [6] and also by G. Meinardus [42].

The Tauberian approach of A. E. Ingham stimulated many works, notably those concerning generalizations of the Mahler's problem, in connection with the presence of the periodic term occurring in de Bruijn formula (6). In particular, W. B. Pennington [45] and later W. Schwartz [55] used the Tauberian method of A. E. Ingham combined with the Mellin transform to study unrestricted partitions into parts in $\Lambda = (\lambda_\nu)_{\nu\geq 1}$ (not necessarily with integers) verifying a lacunary condition which, following W. Schwartz, takes the form $\Lambda(u) \sim B(\log u)^\theta$ with $0 < \theta \leq 1$. Notice that the periodic terms in the asymptotic behavior of $M_{A,\infty}$ and $M_{A,2}$ for bases of parts A that satisfy $a_k \sim k$ were also present in the work of Erdős and B. Richmond [23].

Roth and G. Szekeres in 1954 (see [54]) gave a new proof of the asymptotic formulae for $q(n)$ and $p(n)$ using the saddle point method applied to the Cauchy integral formula. In fact, their method can be applied to a rather wide class of partition functions $M_{A,\infty}(n)$ with bases of parts $A = (a_n)_{n\geq 0}$ satisfying the following two conditions:

$$\lim_{k\to\infty} \frac{\log a_k}{\log k} \quad \text{exists;} \quad (7)$$

$$\lim_{k\to\infty} \inf_{\substack{\alpha \\ \frac{1}{2a_k}<\alpha\le\frac{1}{2}}} \left\{\frac{1}{\log k}\sum_{\nu=1}^{k}\|a_\nu\alpha\|^2\right\} = \infty \tag{8}$$

where $\|x\| := \min\{|x-z|\,;\ z\in\mathbb{Z}\}$ for any real number x. In particular, A can be the set $\mathbb{P}$ of prime numbers or the sets $N_f := \{f(k)\,;\ k\in\mathbb{N}\}$, $P_f := \{f(p)\,;\ p\in\mathbb{P}\}$ where f is a polynomial such that $f(\mathbb{N})\subset\mathbb{N}$ and no prime number divides all integers $f(k)$. Readily, if a prime number divides a_ν for all ν large enough then the condition (8) is not satisfied.

Following the method of Roth and Szekeres, L. B. Richmond [52, 53] was able to extend somewhat the Pennington's results about $M_{A,\infty}(n)$. For example, his main theorem (Theorem 2.1 [53]), can be applied to partitions into the Fibonacci numbers and even more, for any base A of parts defined by a linear recurrence. He obtained asymptotic behaviors similar to the de Bruijn's result. Notice that for any infinite part A of $\mathbb{N}$, a theorem of P. T. Bateman and P. Erdős says that the partition function $M_{A,\infty}(\cdot)$ is strictly increasing if and only if removing any element from A, the remaining elements have no common prime divisor. Of course, this theorem does not apply to $A = E_r$ but in that case, setting as above $M_r(\cdot)$ for $M_{E_r,\infty}(\cdot)$, one has readily the equalities

$$M_r(rn) = M_r(rn+1) = \cdots = M_r(rn+r-1)$$

and

$$M_r(rn) = M_r\big(r(n-1)\big) + M_r(r\lfloor n/r\rfloor)$$

which imply that $M_r(\cdot)$ is increasing.

Asymptotic behaviors of partition functions with multiplicity $< q$ have stimulated very a few works probably because these functions can be very irregular according to the choice of parts and q. B. Reznick studied $M_{E_2,q}(\cdot)$ in great details [51]. Notably he showed that $M_{E_2,2k}(\cdot)$ is monotonic but $M_{E_2,2k+1}(\cdot)$ alternates its growth: $M_{E_2,2k+1}(2n) > M_{E_2,2k}(2n+1)$ for $n > k$ and $M_{E_2,2k+1}(2n+1) < M_{E_2,2k+1}(2n+2)$ for all n. Explicit old and new formulae are also given for particular multiplicity q. For example, according to an easily checked relation $M_{E_2,4}(2n) = M_{E_2,4}(n) + M_{E_2,4}(n-1)$, one obtains that $M_{E_2,4}(n) = \lfloor n/2\rfloor$ ([1], 1983, Problem B 2). Using recurrence relations issuing from the product formula verifying by the generating function of $M_{E_2,q}(\cdot)$ (see (18)), B. Reznick proved that there are constants $0<\alpha<\beta$ such that

$$\alpha n^{\log_2 k} \le M_{E_2,2k}(n) \le \beta n^{\log_2 k}\,. \tag{9}$$

When q is equal to a power of 2 a better result is proved: the limit

$$c := \lim_n \frac{M_{E_2,2^m}(n)}{n^{m-1}} \tag{10}$$

exists and $0 < c < \infty$; the odd case $q = 2k+1$ cannot be compared to a power of n like $M_{E_2,2k}(n)$ but there exist $\mu_i(2k+1)$ $(i = 1, 2)$, $0 \leq \mu_1(2k+1) < \mu_2(2k+1)$ and constants $0 < \alpha < \beta$ such that

$$\alpha n^{\mu_1(2k+1)} \leq M_{E_2,2k+1}(n) \leq \beta n^{\mu_2(2k+1)} \tag{11}$$

for n large enough. Explicit calculation of the exponents $\mu_i(2k+1)$ is a difficult problem. It is known that $\mu_2(5) = \log_2(\psi)$ where $\psi = 2.53861\ldots$ is the unique root of the polynomial $X^4 - 2X^3 - 2X^2 + 2X - 1$ in the interval $[2, 3]$ ([19, Proposition 6.16]).

The partition function $M_{E_2,3}(\cdot)$ is of particular interest. It is related to the Fibonacci sequence $F := (F_n)_n$, defined here by

$$F_0 = 1, \quad F_1 = 2 \quad \text{and} \quad F_{k+2} = F_{k+1} + F_k \quad (k \geq 0)\,. \tag{12}$$

The classical relations

$$M_{E_{2,3}}(2n+1) = M_{E_{2,3}}(n) \text{ and } M_{E_{2,3}}(2n) = M_{E_{2,3}}(n) + M_{E_{2,3}}(n-1) \tag{13}$$

show that $M_{E_2,3}(n) = s(n+1)$ where $s(n)$ is the Stern diatonic sequence [57] (see [56] for more details). A straightforward consequence is the equality $M_{E_2,3}(2^n - 1) = 1$, showing that for $k = 1$ in (11) one has $\alpha = 1$ and $\mu_1(3) = 0$. É. Lucas [40] and D. H. Lehmer [38] proved

$$\max\{M_{E_2,3}(n)\,;\ 2^k - 1 \leq n \leq 2^{k+1} - 1\} = F_{k+1}$$

while from [40, 51], $F_k = M_{E_2,3}(b_k)$ where

$$b_k := \begin{cases} \frac{1}{3}(2^{k+2} - 4) & \text{if } k \text{ is even,} \\ \frac{1}{3}(2^{k+2} - 2) & \text{if } k \text{ is odd}\,. \end{cases}$$

Since the Fibonacci numbers F_k constructed from (12) are given by

$$F_k = \frac{\theta^{k+2} - (\theta')^{k+2}}{\theta - \theta'}$$

with $\theta = \frac{1+\sqrt{5}}{2}$ and $\theta' = \frac{1-\sqrt{5}}{2}$ $(= -1/\theta)$, the above calculation implies that $\mu_2(3) = \log_2 \theta$ (see [51, 19]) and in addition

$$\limsup_n M_{E_2,3}(n)/n^{\log_2 \theta} = \frac{3^{\log_2 \theta}}{\sqrt{5}}\,.$$

Explicit value of $M_{E_2,3}(n)$ was computed in [19, Proposition 6.11]:

Let $1^{a_{2s}}0^{a_{2s-1}} \cdots 0^{a_1}1^{a_0}$ be the word obtained from the usual binary expansion n by grouping the 0's and the 1's with $a_k \geq 1$ for $1 \leq k \leq 2s$,

$a_0 \geq 0$ and $a_0 = 0$ if n is even. Then $M_{E_2,3}(n) = q_s$ where q_s is the denominator of the continued fraction

$$\frac{p_s}{q_s} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \, \cfrac{}{a_{2s-1} + \cfrac{1}{a_{2s}}}}}}.$$

The partition functions of d-adic bases E_d with multiplicities q are explored by Y. Protasov in [47]. He proved that, for fixed d and $q \geq d$, there exist positive constants $C_1(d,q)$, $C_2(d,q)$ and nonnegative exponents $\lambda_1(d,q)$, $\lambda_2(d,q)$ such that

$$C_1(d,q)n^{\lambda_1(d,q)} \leq M_{E_d,q}(n) \leq C_2(d,q)n^{\lambda_2(d,q)}. \tag{14}$$

For $d \leq q < 2d$ one has readily $M_{E_d,q}(d^m - 1) = 1$ for any integer $m \geq 1$. Hence, in those cases $C_1(d,q) = 1$, $\lambda_1(d,q) = 0$. In [19] the authors mainly focused on the behavior of the sum $\sum_{n=0}^{N} M_{E_d,q}(n)$. Nevertheless, using the formal equality $\sum_{n\in\mathbb{Z}} M_{E_d,q}(n)x^n = (1+x+x^2+\cdots+x^{q-1})\big(\sum_{n\in\mathbb{Z}} M_{E_d,q}(n)x^{dn}\big)$ they obtained by induction the bound

$$M_{E_d,q}(n) \leq n^{b(d,q)} \tag{15}$$

with $b(d,q) = \log_d \lfloor (q-1+d)/d \rfloor$ and they gave the exact maximal order $\lambda_2(d,q)$ in (14) for $d \leq q \leq d^2$: if $r = r(d,q) := \lfloor (q-1)/d \rfloor$ and $s = s(d,q) := d((q-1)/d - r(d,q))$ then

$$\lambda_2(d,q) = \begin{cases} \log_d(r+1) & \text{if } r \leq s \\ \log_d\left(\frac{r+\sqrt{r^2+4s+4}}{2}\right) & \text{if } r > s\,. \end{cases}$$

Special cases are also considered, namely:

$$\begin{aligned} M_{E_d,d^m}(n) &= \frac{1}{d^{\frac{m(m-1)}{2}}(m-1)!}n^{m-1} + \mathcal{O}(n^{m-2}) \quad &&\text{[19, Proposition 6.7]} \\ \lambda_1(d,du) &= \lambda_2(d,du) = \log_d u &&\text{[19, Proposition 6.8]}\,. \end{aligned}$$

Most of the above results are rediscovered by Y. Protasov in [47] but using a bit different approach that allows him to calculate explicitly the Reznick's values $\mu_1(2k+1)$ and $\mu_2(2k+1)$ for $1 \leq k \leq 6$.

In this paper, we study the asymptotic behavior of $M_{A,q}(\cdot)$ for a family of linear recurrence scale A related to Pisot numbers (see *infra* and Section 3). This family includes in particular the d-adic scales E_d, the Fibonacci base

$F = (F_n)_{n\geq 0}$ (defined above) and more generally the m-bonacci base $F^{[m]}$ defined by $F_n^{[m]} = 2^n$ for $0 \leq n < m$ and

$$F_{n+m}^{[m]} = F_{n+m-1}^{[m]} + F_{n+m-2}^{[m]} + \cdots + F_n^{[m]} \quad (n \geq 0)\,.$$

The calculation of $M_{F,2}(\cdot)$ was intensively studied by L. Carlitz [13, 14] and also by J. Berstel [10] who obtained a general formula of $M_{F,2}(\cdot)$ in terms of products of 2×2 matrices associated with canonical F-expansion. His method was extended by P. Kocábová, Z. Masácová and E. Pelantová [37] to the m-bonacci base $F^{[m]}$.

The rest of this paper is organized as follows. In Section 2 we give some definitions and collect miscellaneous results for general bases A of parts. In particular, conditions on A and q assuming $M_{A,q}(n) \geq 1$ for all $n \geq 1$ are analyzed and partition functions defined from two bases of parts are compared when these bases are related by inclusion up to a finite set of parts. We pay more attention to bases A of parts of *exponential order*, that is, $0 < \alpha\gamma^k \leq a_k \leq \beta\gamma^k$ for some positive constants α and β and all k. The behavior of

$$\Gamma_{A,q}(x) := \sum_{1\leq n<x} M_{A,q}(n) \tag{16}$$

is crudely estimated. A more accurate result, in the spirit of [19], is obtained in the case that $a_k \sim c\gamma^k$ $(k \to \infty)$ (see Theorem 14). A scale $G = (g_n)_{n\geq 0}$ of the form $g_n = c(\theta)\theta^n + \gamma_n$ where θ is a (positive) Pisot number (*i.e.*, a positive algebraic integer over $\mathbb{Q}$ whose all its conjugates except itself have modulus < 1), $c(\theta) \in \mathbb{Q}(\theta)$ and $\lim_n \gamma_n = 0$ will be called a Pisot scale related to θ.

Section 3 is devoted to Pisot scales. Recall that the unconstraint partitions $(q = \infty)$ for the particular Pisot scales $F^{[m]}$ but also more generally for bases defined by linear recurrences are special cases of a general result of B. Richmond [53]. Our main result in Section 3 says that for any Pisot scale G there exists an integer q_0 such that for $q \geq q_0$ there are a subset $S_{G,q}$ of $\mathbb{N}$ of density 1 and a value $\alpha_{G,q} > 0$ such that

$$\lim_{\substack{s\in S_{G,q}\\ s\to\infty}} \frac{\log M_{G,q}(s)}{\log s} = \alpha_{G,q}\,.$$

The proof exploits a previous work of D.-J. Feng and N. Sidorov [26] where the constant $\alpha_{G,q}$ appears in connection with the growth rate of the number of radix θ-expansions of almost all numbers x using digits $0, \ldots, q-1$.

The various results given above for $G = E_2$ and $q = 2k$ lead to $\alpha_{E_2,2k} = \log_2 k$ with $S_{E_2,2k} = \mathbb{N}$. Also, for $d > 2$, one has $\alpha_{E_d,d^m} = m - 1$ and more generally $\alpha_{E_d,du} = \log_d u$ (again with $S_{E_d,q} = \mathbb{N}$). In fact, these values verify $\alpha_{E_d,q} + 1 = \log_d q$ but if d does not divide q it is known that $\alpha_{E_d,q} + 1 < \log_d q$ (see [26]).

2. Miscellaneous results

2.1. Basic notations and definitions

We denote the set of nonnegative integers by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and recall some usual notations and definitions about combinatorics of words. For any integer $q \geq 2$ or for $q = \infty$ the set $\mathcal{S} := \{s \in \mathbb{N}_0\,;\ s < q\}$, called alphabet of letters (or digits), is equipped with the natural order and $\mathcal{S}^*$ denotes the monoid of words on the alphabet $\mathcal{S}$. The empty word in $\mathcal{S}^*$ is denoted by $\wedge$. For any word w set $w^0 = \wedge$ and for any integer $k \geq 1$, w^k is defined inductively by $w^k = w^{k-1}w$. Let $A = (a_k)_k$ (indexed from 0 in increasing order) be a base of parts. For any word $w = w_0 \cdots w_m$ in $\mathcal{S}^*$ (also written as $(w_0, \dots, w_m)$) of length $|w| = m + 1$ we associate the integer

$$\mathrm{val}_{A,q}(w_0 \cdots w_m) = \sum_{j=0}^{m} w_j a_j\,.$$

The word w is called a (A, q)-expansion of $\mathrm{val}_{A,q}(w_0 \cdots w_m)$ with digits (or weights) w_i in $\mathcal{S}$. We also set $\mathrm{val}_{A,q}(\wedge) = 0$ by convention.

The product space $\Omega_q := \mathcal{S}^{\mathbb{N}_0}$ of $\mathcal{S}$-valued sequences $\varepsilon = (\varepsilon_k)_{k\geq 0}$ is endowed with the usual compact product topology. For ε in Ω_q the cylinder set of length n, relative to ε, is defined by

$$[\varepsilon]_n = \{\omega \in \Omega_q\,;\ \varepsilon_0 \cdots \varepsilon_{n-1} = \omega_0 \cdots \omega_{n-1}\}\,.$$

The compact space Ω_q is metrizable and will be equipped with its Borel σ--algebra and the uniform Bernoulli measure μ_q which is defined on cylinder sets by

$$\mu_q([\varepsilon]_n) = \frac{1}{q^n}\,.$$

With these notations, the partition function $M_{A,q} : \mathbb{N} \to \mathbb{N}_0$ can be redefined by

$$M_{A,q}(n) := \#\Big\{\varepsilon \in \Omega_q\,;\ n = \sum_{n=0}^{\infty} \varepsilon_k a_k\Big\}\,. \tag{17}$$

If A is a finite subset of $\mathbb{N}$, we can define a partition function which is also given by (17). Another way to define the partition function $M_{A,q}$ (A finite or infinite) is furnished by the product formula of the generating series

$$f_{A,q}(z) := \sum_{n=0}^{\infty} M_{A,q}(n) z^n$$

of $M_{A,q}(n)$, that is to say

$$f_{A,q}(z) = \prod_{k=0}^{\infty} \left(1 + z^{a_k} + \cdots + z^{(q-1)a_k}\right). \tag{18}$$

This equality can be viewed formally or analytically, the series and the infinite product converging both for complex number z with modulus $|z| < 1$.

Two real valued sequences $(u_n)_n$, $(v_n)_n$ will be said asymptotically equivalent in growth and we shall write $u_n \asymp v_n$ $(n \to \infty)$ if there exist positive constants α and β and an integer n_0 such that

$$\alpha u_n \leq v_n \leq \beta u_n$$

for all $n \geq n_0$.

By convention, letters i, j, k, n will be reserved to denote nonnegative integers and q will be a fixed integer, always greater than or equal to 2. Finally, we currently reserve the notation $G := (g_n)_{n\geq 0}$ for scales.

2.2. Increasing limitation

If the base of parts A increases very fast, the set of integers n such that $M_{A,q}(n) > 0$ has natural density 0 in $\mathbb{N}$. More precisely:

Proposition 1. *Let $A = (a_k)_{k\geq 0}$ be a base of parts and assume that there exists a constant C such that $\frac{a_{k+1}}{a_k} \geq C \geq 2$ for all k. Set for $q \geq 2$,*

$$V_{A,q} := \{k \in \mathbb{N}_0\, ;\ M_{A,q}(k) > 0\}.$$

Then the inequality

$$\frac{1}{n}\#\big(V_{A,q} \cap [0, n)\big) \leq \frac{q}{a_0}\left(\frac{q}{C}\right)^m \tag{19}$$

holds for $q \leq C$ and $a_m \leq n < a_{m+1}$. Moreover, $M_{A,q}(k) \leq 1$ for all k.

Proof. By recurrence one easily derives from $q \leq C$ that

$$(q-1)(a_0 + \cdots + a_\ell) < a_{\ell+1}$$

for all ℓ. Consequently, if $M_{A,q}(k) > 0$ then k admits a (A, q)-expansion $k = e_0(k)a_0 + \cdots + e_h(k)a_h$ and this expansion is nothing but the standard A'--expansion of k, where A' is the scale obtained from A by adding the part 1 if necessary. This means that $M_{A,q}(k) = 1$. Now, for $a_m \leq n < a_{m+1}$ and $0 \leq k < n$ with $M_{A,q}(k) = 1$, one has $h(k) \leq m$. Therefore

$$\frac{1}{n}\#(V_{A,q} \cap [0, n)) \leq \frac{q^{m+1}}{a_m}$$

and the inequality (19) follows from $a_0 C^m \leq a_m$. □

The next proposition determines for a given scale the smallest possible q such that any integer $n \geq 0$ has a (G, q)-expansion.

Proposition 2. *Let G be a scale, then $M_{G,q}(n) \geq 1$ for all integers $n \geq 0$ if and only if $q \geq 1 + \sup_k \frac{g_k - 1}{S_{k-1}}$ where $S_k := g_0 + \cdots + g_k$.*

Proof. Assume that $q < 1 + \sup_k \frac{g_k-1}{S_{k-1}}$; there exists at least one k with $(q-1)(g_0 + \cdots + g_{k-1}) < g_k - 1$, hence $g_k - 1$ cannot have a (G, q)-expansion. The converse is proved by induction. Assume $q \geq 1 + \frac{g_k-1}{S_{k-1}}$ for all $k \geq 1$. In particular $(q-1)g_0 = (q-1) \geq g_1 - 1$, proving that any integer n in $[0, g_1]$ has a (G, q)-expansion. Now suppose that any integer in the interval $[0, g_{k-1}]$ $(k \geq 2)$, admit a (G, q)-expansion and choose n in $(g_{k-1}, g_k]$. If $n = g_k$ we have nothing else to prove, so we assume $n < g_k$. If $n < (q-1)g_{k-1}$ we choose $e_{k-1} \in \{0, \ldots, q-1\}$ by Euclidian division such that $0 \leq n - e_{k-1}g_{k-1} < g_{k-1}$, then $n - e_{k-1}g_{k-1}$ has a (G, q)-expansion by hypothesis and consequently n has also a (G, q)-expansion. Otherwise $(q-1)g_{k-1} \leq n$. We may assume that $n \neq (q-1)S_k$. Since $g_k - 1 \leq (q-1)S_{k-1}$, there exists $s \in \{1, \ldots, k-1\}$ with $(q-1)(g_{k-1} + \cdots + g_{k-s}) \leq n < (q-1)(g_{k-1} + \cdots + g_{k-s} + g_{k-s-1})$. The above reasoning applies to $n' = n - (q-1)(g_{k-1} + \cdots + g_{k-s}) < (q-1)g_{k-s-1}$ leads to a (G, q)-expansion of n' and by the same time a (G, q)-expansion of n. The proof is complete. □

The case where A is not a scale is resolved by the following result.

Proposition 3. *Let A be a base. Assume that $\sup_k \frac{a_k - 1}{a_0 + \cdots + a_{k-1}} < +\infty$ and no prime number divides all the parts a_k. Then there exists a computable integer Q such that $M_{A,Q}(n) \geq 1$ for all n large enough.*

Proof. From the above results, the proposition is clear if A is a scale. If A is not a scale, by the arithmetical property of A, $\gcd(a_0, \ldots, a_m) = 1$ for some integer m. This implies there exist computable integers $K \geq a_0$ and $q' \geq 1$ such that any integer k in the interval $[K, K + a_0 - 1]$ can be written in the form $k = b_0(k)a_0 + \cdots + b_m(k)a_m$ with $0 \leq b_i(k) \leq q'$ for all $i \in \{0, \ldots, m\}$. Now let $G = (g_k)_{k \geq 0}$ be the scale defined by $g_{k+1} = a_k$ $(k \geq 0)$ and take $q = \max_k \lceil \frac{g_{k+1}}{g_k} \rceil$. Then, for any integer $n \geq 0$, the standard G-expansion

$$n = e_0(n)g_0 + e_1 g_1 + \cdots + e_h(n)g_h$$

is a also (G, q)-expansion and the integer $n - e_0(n)$ $(e_0(n) < a_0)$ has a (A, q)-expansion. Therefore $n + K$ has a (A, Q)-expansion with $Q := q + q'$. □

Proposition 3 justifies the following definition

$$q_A := \min\{q\,;\ \exists K,\ \forall n \geq K,\ M_{A,q}(n) \geq 1\}\,.$$

With q_A we associate the integer n_A defined by

$$n_A := \min\{k \in \mathbb{N}\,;\ \forall\, \ell \geq k,\ M_{A,q_A}(\ell) \geq 1\}\,.$$

If A is a scale G, standard G-expansions may have digits greater than q_G. This fact leads us to introduce

$$q_G^* = \max_k \left\lceil \frac{g_{k+1}}{g_k} \right\rceil \tag{20}$$

in other words q_G^* is the maximum plus 1 of the set of digits that occur in standard G-expansions. Obviously $q_G \leq q_G^*$. The equality holds if G increases regularly. Readily, one has from the definitions:

Proposition 4. *Let G be a scale such that there exists a positive integer a verifying the inequalities $ag_n < g_{n+1} \leq (a+1)g_n$ for all indices $n \geq 0$. Then $q_G = q_G^* = a + 1$.*

Example 5. The scale defined by $g_n = \frac{a^{n+1}-1}{a-1}$, $n \geq 0$, satisfies the hypothesis of Proposition 4. This scale arises in combinatorial of graphs and words [12, 2, 28] but also in symbolic dynamics [18, Chap. 4].

There exist scales G such that $q_G < q_G^*$. The following family of scales shows that all values q_G with $2 < q_G < q_G^*$ are possible.

Proposition 6. *Let a and b be integers such that $1 < b < a$ and define the scale $L := \{1, 2, \ldots, 2a-1, 1+a(2a-1), a(2a-1)(b+1), a(2a-1)(b+1)^2, a(2a-1)(b+1)^3, \ldots\}$. Then, $q_L = b+1$ and $q_L^* = a+1$.*

Proof. The equality $q_L^* = a+1$ is clear due to $L_{2a-2} = 2a-1$ and $L_{2a-1} - 1 = aL_{2a-2} < L_{2a-1}$ proving that aL_{2a-2} is the standard L-expansion of $L_{2a-1} - 1$ and the fact that for all $n \geq 0$, by construction, $(a+1)L_n > L_{n+1}$. To prove that $q_L = b+1$ first notice that any integer n in $\{1, \ldots, a(2a-1)b\}$ has an L-expansion $n_0L_0 + \cdots + n_{2a-2}L_{2a-2}$ with digits n_j satisfying $0 \leq n_j \leq b$. Now, any integer m such that $a(2a-1)(b+1)^s < m$ with $s \geq 1$ has, by applying the greedy algorithm, a standard L expansion $e_0L_0 + \cdots + e_tL_t$ with $t \geq 2a+s-1$ and $e_j \leq b$ if $j \leq 2a$. Looking at the remaining standard L-expansion $m' = e_0L_0 + \cdots + e_{2a-1}L_{2a-1}$ one has two possible cases, namely $m' = aL_{2a-1} = L_0 + L_1 + \cdots + L_{2a-2}$ or $e_{i_0} = 1$ for exactly one index $i_0 \in \{0, \ldots, 2a-2\}$ and $e_{2a-1} \leq a$ so that $m' \leq (i_0+1)+(a-1)(2a-1) \leq a(2a-1)$. Hence, in all cases, m' and consequently m can be expanded in the scale L with digits less or equal to b. Set $x(s) := (b-1)\big(1 + \cdots + (2a-1) + 1 + a(2a-1)(1 + (b+1) + \cdots + (b+1)^s)\big)$ then $x(s) = (b-1)\big(a(2a-1) + 1 + a(2a-1)(\frac{(b+1)^{s+1}-1}{b})\big)$ and

$$\lim_{s\to\infty} \frac{x(s)}{a(2a-1)(b+1)^{s+1})} = 1 - \frac{1}{b}\,.$$

Therefore, in particular $(b-1)(L_0+\cdots+L_n) < L_{n+1}-1$ for n large enough so that $q_L > b$ proving that $q_L = b+1$. □

2.3. Comparison theorems

Proposition 7. *Let A and B be disjoint subsets of $\mathbb{N}$, then*

$$M_{A\cup B,q}(n) = \sum_{j=0}^{n} M_{A,q}(n-j)M_{B,q}(j)\,.$$

Proof. Let $f_{A,q}(z)$ and $f_{B,q}(z)$ be the generating functions of $M_{A,q}(\cdot)$ and $M_{B,q}(\cdot)$. The product formulae show immediately that

$$f_{A,q}(z).f_{B,q}(z) = f_{A\cup B,q}(z)\,.$$

□

If A and A' are bases of parts, the inclusion $A\subset A'$ implies readily $M_{A,q}(n) \le M_{A',q}(n)$, $q_A \ge q_{A'}$ and $n_A \ge n_{A'}$. If A is a base of parts which is not a scale, then by Proposition 7,

$$M_{A\cup\{1\},q}(n) = M_{A,q}(n) + M_{A,q}(n-1)+\cdots+M_{A,q}(n-q+1) \quad (n>q)\,.$$

The following result is more interesting:

Proposition 8. *Let A, A' be two bases of parts. Suppose that $A\subset A'$ and $B = A'\setminus A$ is finite. Then*

$$M_{A',q}(n) = \sum_{\ell=0}^{L} M_{A,q}(n-\ell)M_{B,q}(\ell) \tag{21}$$

with $L = (q-1)\sum_{b\in B} b$. In particular

$$\min_{0\le\ell\le L} M_{A,q}(n-\ell) \le M_{A',q}(n) \le q^{\#B} \max_{0\le\ell\le L} M_{A,q}(n-\ell). \tag{22}$$

Proof. Apply Proposition 7. □

This proposition implies, for example, that if $M_{A,q}(n) \asymp n^\alpha$ then $M_{A',q}(n) \asymp n^\alpha$. The converse is not true. A banal counter-example is given by $A = \{d^n\,;\ n\in\mathbb{N}\}$, $A' = A\cup\{1\}$ and $q = du$. In that case $M_{A',q}(n) \asymp n^{\log_d q}$ (see results quoted in Section 1) but $M_{A,q}(n) = 0$ if $n \not\equiv 0 \pmod d$.

The following lemma shows that a sum of the form $\sum_{a\le n\le b} M_{G,q}(n)$ has the same order of growth than $M_{G,q}(a)$ or $M_{G,q}(b)$ when a tends to ∞ with $a-b$ bounded by a constant.

Lemma 9. *Let $G = (g_k)_{k\ge 0}$ be a scale and let $q \ge q_G^*$.*

(i) *Any integer n, $0 < n < g_k$, verifies*

$$\frac{1}{k} \le \frac{M_{G,q}(n)}{M_{G,q}(n-1)} \le k\,.$$

(ii) *There exists a constant C such that all integers $n \ge 2$ verify*

$$\frac{1}{C\log n} \le \frac{M_{G,q}(n)}{M_{G,q}(n-1)} \le C\log n \quad (C \text{ constant, depending on } G)$$

provides $\inf_k \frac{g_k}{g_{k_1}} > 1$.

Proof. (i) Let $0 < n < g_k$. The set of (G,q)-representations of the integer n is the union, for $0 \le h < k$, of the sets

$$\mathcal{S}_{k,h,n} := \{(0,\dots,0,\varepsilon_h,\dots,\varepsilon_{k-1}) \in \mathcal{S}^k\,;\ \varepsilon_h \ne 0 \text{ and } n = \varepsilon_h g_h + \cdots + \varepsilon_{k-1}g_{k-1}\}.$$

To each $\varepsilon \in \mathcal{S}_{k,h,n}$, we associate

$$\varepsilon' := (\varepsilon_0^{(h)},\dots,\varepsilon_{h-1}^{(h)},\varepsilon_h - 1,\dots,\varepsilon_{k-1}),$$

where $(\varepsilon_0^{(h)},\dots,\varepsilon_{h-1}^{(h)})$ corresponds to the standard G-expansion of $g_h - 1$. Since ε' is a representation of $n-1$, one deduces that $M_{G,q}(n-1) \ge \#\mathcal{S}_{k,h,n}$ and

$$kM_{G,q}(n-1) \ge \sum_{0\le h<k} \#\mathcal{S}_{k,h,n} = M_{G,q}(n).$$

Similarly, the set of (G,q)-representations of $n-1$ is the union, for $0 \le h < k$, of the sets

$$\begin{aligned}\mathcal{B}_{k,h,n-1} \ := \ \{&(q-1,\dots,q-1,\varepsilon_h,\dots,\varepsilon_{k-1}) \in \mathcal{S}^k\,;\ \varepsilon_h \ne q-1 \text{ and}\\ &n-1 = (q-1)(g_0+\cdots+g_{h-1}) + \varepsilon_h g_h + \cdots + \varepsilon_{k-1}g_{k-1}\}.\end{aligned}$$

We associate, to each $\varepsilon \in \mathcal{B}_{k,h,n-1}$,

$$\varepsilon' := (q-1-\varepsilon_0^{(h)},\dots,q-1-\varepsilon_{h-1}^{(h)},\varepsilon_h + 1,\dots,\varepsilon_{k-1}),$$

where $(\varepsilon_0^{(h)},\dots,\varepsilon_{h-1}^{(h)})$ corresponds to the standard G-expansion of $g_h - 1$. Since ε' is the representation of n, one deduces $M_{G,q}(n) \ge \#\mathcal{B}_{k,h,n-1}$ and

$$kM_{G,q}(n) \ge \sum_h \#\mathcal{B}_{k,h,n-1} = M_{G,q}(n-1).$$

(ii) Let $n \ge 2$, and k such that $g_{k-1} \le n < g_k$. Denoting

$$m = \inf_{j\ge1} \log(g_j/g_{j-1})$$

one gets

$$\log n \geq \log g_{k-1} = \sum_{j=1}^{k-1} \log(g_j/g_{j-1}) \geq (k-1)m,$$

hence $k \leq 1 + \frac{\log n}{m}$. Since $1 \leq 2\log n$ one obtains $k \leq (2 + \frac{1}{m}) \log n$, and by (i), the inequalities hold with $C = 2 + \frac{1}{m}$. □

2.4. The growth of $M_{A,q}$ and its summation function for exponential base

The following result shows that if the parts a_k grow exponentially, then the growth rate of $M_{A,q}(n)$ is at most of polynomial. More precisely:

Theorem 10. *Let A be a base of parts a_k such that $a_k \asymp \gamma^k$ and let $q \geq \gamma$. Then*

$$\Gamma_q(N) = \sum_{1 \leq n \leq N} M_{A,q}(n) \asymp N^{\log_\gamma q} \quad (N \to \infty)\,.$$

Proof. By assumption there are constants α, β and an integer k_0 such that $0 < \alpha\gamma^k \leq a_k \leq \beta\gamma^k$ if $k \geq k_0$. Assume that N is large enough such that the integer i_N defined by the inequalities $a_{i_N} \leq N < a_{i_N+1}$ verifies $i_N \geq k_0$. Then, any (A,q)-expansion of integer n, $1 \leq n \leq N$, has the form $e_0a_0 + \cdots + e_{i_N}a_{i_N}$. Therefore

$$\Gamma_q(N) \leq q^{i_N+1},$$

while $i_N \leq \frac{\log N/\alpha}{\log \gamma}$, hence $q^{i_N} \leq (N/\alpha)^{\log_\gamma q}$. Let j_N be the smallest integer verifying $N \leq (q-1)(a_0 + \cdots + a_{j_N})$, then

$$\Gamma_q(N) \geq q^{j_N}\,.$$

Now, assume that N is large enough so that $j_N > k_0$. A straightforward calculation leads to a constant B and an integer N_1 such that

$$N \leq (q-1)(a_1 + \cdots + a_{j_N}) \leq B\gamma^{j_N}$$

whenever $N \geq N_1$. Consequently, for such N the inequality $\log(N/B) \leq j_N \log\gamma$ holds and so, $\Gamma_q(N) \geq (N/B)^{\log_\gamma q}$ as expected. □

Remark 11. The above proof can be adapted to show that if $C_1\gamma_1^k \leq a_k$ with $C_1 > 0$ and $\gamma_1 > 1$ (resp. $a_k \leq C_2\gamma_2^k$), then there exists a positive constant B_1 (reps. B_2) such that

$$\Gamma_q(N) \leq B_1 N^{\log_{\gamma_1} q} \quad (\text{resp. } B_2 N^{\log_{\gamma_2} q} \leq \Gamma_q(N))\,.$$

□

The sum in Theorem 10 is closely related to the uniform Bernoulli measure μ_q on Ω_q (cf. Section 2.1). For all real number $x \geq 0$, if t is any integer verifying $x < a_{t+1}$, then

$$\Gamma_q(x) = q^{t+1}\mu_q(\{\varepsilon \in \Omega_q\,;\ \varepsilon_0 a_0 + \cdots + \varepsilon_t a_t \leq x\})\,. \tag{23}$$

In the remaining part of this section we assume that the base A of parts a_k verifies

$$a_k \sim c\gamma^k \quad (k \to \infty)\,. \tag{24}$$

It is convenient in the sequel to set

$$a_k = c\gamma^k + b_k \tag{25}$$

so that the asymptotic condition (24) is equivalent to the following one

$$\lim_{n\to\infty} \frac{1}{\gamma^n} \sum_{0\leq k\leq n} |b_k| = 0\,. \tag{26}$$

Let $Y : \Omega_q \to \mathbb{R}$ be the random variable defined by

$$Y(\varepsilon) := \sum_{i=0}^{\infty} \varepsilon_i \gamma^{-i} \tag{27}$$

and let $\mu_{\gamma,q} := \mu_q \circ Y^{-1}$ be the law of $Y(\cdot)$. The measure $\mu_{\gamma,q}$ is supported on the interval $I_{\gamma,q} := [0, \frac{(q-1)\gamma}{\gamma-1}]$. Let δ_a denote the Dirac measure centered on the point a and set

$$D_n := \frac{1}{q}\sum_{j=0}^{q-1} \delta_{j/\gamma^n}\,,$$

then $\mu_{\gamma,q}$ is also the infinite convolution product $D_0 * D_1 * D_2 * \cdots$ which is the weak limit of the sequence of discrete probabilities

$$\mu_{\gamma,q,n} := D_0 * D_1 * \cdots * D_n\,.$$

Let us recall basic well known properties of $\mu_{\gamma,q}$. First, as a consequence of a theorem of P. Levy [39, Theorem XIII, p. 150], the measure is continuous and from a general theorem of B. Jessen and W. Wintner [36, Theorem 35, p. 86] it is pure, hence $\mu_{\gamma,q}$ is either singular continuous or absolutely continuous (with respect to the Lebesgue measure on $I_{\gamma,q}$). A straightforward computation gives

$$\mu_{\gamma,q} = \frac{1}{q}\sum_{j=0}^{q-1} \mu_{\gamma,q} \circ S_j^{-1} \tag{28}$$

where S_j is the affine contraction on $\mathbb{R}$ defined by

$$S_j(x) = \frac{x}{\gamma} + j\,. \tag{29}$$

In other words, $\mu_{\gamma,q}$ is a self-similar probability on $I_{\gamma,q}$ associated to the IFS $\{S_j\}_{j=0}^{q-1}$ and the probability weight $(\frac{1}{q}, \frac{1}{q}, \dots, \frac{1}{q})$. According to a general result of J. E. Hutchinson [34], $\mu_{\gamma,q}$ is the unique probability measure verifying (28).

One key result for our next study consists in the fact that the sequence of random variables $V_k : \Omega_q \to \mathbb{R}$ defined by

$$V_k(\varepsilon) := \frac{1}{c\gamma^k} \sum_{i=0}^{k} \varepsilon_i a_{k-i}$$

converge uniformly to the $Y(\cdot)$. The following lemma quantifies this convergence:

Lemma 12. *Let A be a base of parts as above. For all k and all ε in Ω_q,*

$$\left| \sum_{i=0}^{k} \varepsilon_i a_{k-i} - c \sum_{i=0}^{\infty} \varepsilon_i \gamma^{k-i} \right| \le \sigma_k$$

with

$$\sigma_k = q\Big(\frac{c}{\gamma - 1} + \sum_{i=0}^{k} |b_i| \Big)\,.$$

The proof is straightforward and left to the reader. Let F_Y and F_k denote respectively the distribution functions of Y and V_k respectively. By definition, for any integer $k \ge 0$ and $x \in [0, c\gamma^k]$,

$$F_Y(x/c\gamma^k) = \mu_q(\{\varepsilon \in \Omega_q\,;\; c\sum_{i=0}^{\infty} \varepsilon_i \gamma^{k-i} \le x\})$$

and

$$F_k(x/c\gamma^k) = \frac{1}{q^{k+1}} \#\big\{(\varepsilon_0, \dots, \varepsilon_k) \in \{0, \dots, q-1\}^k\,;\; \sum_{i=0}^{k} \varepsilon_i a_{k-i} \le x\big\}\,.$$

Clearly a_k belongs to the interval $I_k = [c\gamma^k - \sigma_k, c\gamma^k + \sigma_k]$ and the assumption (26) implies

$$\sigma_k \in o(\gamma^k)\,.$$

Hence, there exists an index k_0 such that the intervals I_k for $k \ge k_0$ are mutually disjoint and the inequality $x \le c\gamma^k + |b_k|$ implies the inequality $x < a_{k+1}$. Therefore

$$\Gamma_q(x) = q^{k+1} F_k(x/c\gamma^k)\,.$$

We are ready to compare $F_Y(x/c\gamma^k)$ and $\Gamma_q(x)$ using Lemma 12.

Lemma 13. *Let $x \in [0,\infty)$ and let $t = t(x)$ be the smallest integer verifying $x \le c\gamma^t$. According to above notations, set $x_t = c\gamma^t - \sigma_t$ and $x_t' = c\gamma^t + \sigma_t$. Then for $t \ge k_0$, one has*

$$F_Y(x_t/c\gamma^t) \le F_t(x/c\gamma^t) \le F_Y(x_t'/c\gamma^t) \tag{30}$$

and

$$F_t(x_t/c\gamma^t) \le F_Y(x/c\gamma^t) \le F_t(x_t'/c\gamma^t)\,. \tag{31}$$

The proof of this lemma is analogous to the one of Lemma 1.2 in [19]. It is an easy consequence of Lemma 12. The next result is the natural generalization of Theorem 2.1 in [19] which considered scales related to Perron numbers.

Theorem 14. *Let A be a base of parts satisfying*

$$a_k \sim c\gamma^k \quad (k \to \infty)$$

with $c > 0$ and $\gamma > 1$. Then one has asymptotically

$$\Gamma_q(x) = x^{\log_\gamma q} H(\log_\gamma x) + o(x^{\log_\gamma q}) \quad (x \to \infty) \tag{32}$$

where $H(\cdot)$ is the continuous periodic function of period 1, strictly positive and a. e. differentiable defined by

$$H(\xi) = q^{\lceil \xi - \log_\gamma c\rceil + 1 - \xi} F_Y(\gamma^{\xi - \lceil \xi - \log_\gamma c\rceil}/c) \quad (\xi \in \mathbb{R})\,.$$

P r o o f. Taking into account the continuity of F_Y, one derives from the above lemma that

$$\Gamma_q(x) = q^{t(x)+1} F_Y(x/c\gamma^{t(x)}) + o(q^{t(x)+1})$$

and the equality (32) holds with $H(\cdot)$ defined by

$$H(\log_\gamma x) = q^{t(x)+1-\log_\gamma x} F_Y(x/c\gamma^{t(x)})\,.$$

The periodicity 1 of $H(\cdot)$ is clear and, by construction, $H(\cdot)$ is continuous except eventually at the reals numbers $\xi_m = m + \log_\gamma c$ ($m \in \mathbb{Z}$). The left and right limits at ξ_m are respectively $H(\xi_m-) = H(\xi_m) = q^{1-\log_\gamma c} F_Y(1)$ and $H(\xi_m+) = q^{2-\log_\gamma c} F_Y(1/\gamma)$. But looking at $F_Y(\cdot)$ one observes the equality $\frac{1}{q} F_Y(1) = F_Y(1/\gamma)$. The continuity of H at ξ_m follows. The differential property of $H(\cdot)$ is inherited from F_Y and the positivity of H follows from $H(\xi) \ge q^{1-\log_\gamma c} F_Y(\gamma^{-1}) > 0$. □

Remark 15. It is not required in Theorem 14 that the sequence of parts a_k verifies a linear recurrence. A typical example is given by any scale (25) with a bounded sequence $(b_k)_k$. Concrete examples are the scale $a_k = 2^{k+1} - 1$ or the scale $a_k = \lceil (3/2)^k \rceil$. A remarkable fact is that the principal term in (32) depends only on γ, c and q. The error term has been improved for Perron numbers in [19]. □

The following result gives a bound for $M_{A,\lfloor\gamma\rfloor+1}(n)$, the proof is elementary.

Proposition 16. *Let A be base of parts satisfying the hypothesis of Theorem 14 with $\gamma > \frac{1+\sqrt{5}}{2}$ and $\gamma \neq 2$. Then $M_{A,\lfloor\gamma\rfloor+1}(n) \in \mathcal{O}(n)$, that is, there exists $C > 0$ such that $M_{A,\lfloor\gamma\rfloor+1}(n) \leq Cn$ for $n \in \mathbb{N}$.*

Proof. From the hypothesis on a_k, given $\varepsilon > 0$ there exists an integer $k_\varepsilon \geq 2$ such that, for $k \geq k_\varepsilon$ and integer $p \geq 1$

$$a_k - p\sum_{i=0}^{k-2} a_i \geq (1-\varepsilon)c\gamma^k - p(1+\varepsilon)c\sum_{i=0}^{k-2}\gamma^i - K \text{ (with } K = p\sum_{i=0}^{k_1} a_i) \quad (33)$$

$$\geq c\gamma^{k-1}\frac{(1-\varepsilon)(\gamma^2-\gamma-p) - 2p\varepsilon}{\gamma-1} - K.$$

Set $q = \lfloor\gamma\rfloor + 1$ for short and choose $p = \lfloor\gamma\rfloor$. If γ is an integer not equal to 2 then $\gamma^2 - \gamma - p > 0$. If γ is not an integer and $\gamma > 2$ then $\gamma^2 - \gamma - p > \gamma^2 - 2\gamma > 0$. If $2 > \gamma > \frac{1+\sqrt{5}}{2}$, then $p = 1$ and again $\gamma^2 - \gamma - p > 0$. Now, take ε such that $(1-\varepsilon)(\gamma^2-\gamma-p) - 2p\varepsilon > 0$ and then choose k_1, $k_1 \geq k_\varepsilon$, such that the right hand side of (33) is positive for $k \geq k_1$. Therefore $k \geq 2$ and

$$a_k > (q-1)\sum_{i=0}^{k-2} a_i\,. \quad (34)$$

Let C be a constant such that inequality $M_{A,q}(n) \leq Cn$ holds for $1 \leq n < a_{k_1}$. We prove by induction the validity of this inequality for all integers n. Suppose $M_{A,q}(n') \leq Cn'$ for all $n' \leq n-1$ and define k by $a_k \leq n < a_{k+1}$. There exists an integer $\nu \geq 0$ such that

$$\nu a_{k-1} + (q-1)\sum_{i=0}^{k-2} a_i < n \leq (\nu+1)a_{k-1} + (q-1)\sum_{i=0}^{k-2} a_i. \quad (35)$$

As $n < a_{k+1}$, each (A,q)-expansion of $n = e_0a_0 + e_1a_1 + \cdots$ (if such expansion exists) satisfies $e_i = 0$ for $i \geq k+1$. If $e_k = 0$ then $e_{k-1} \geq \nu+1$ by (35). To each such (A,q)-expansion of n with $e_k \neq 0$, we associate the (A,q)-expansion of $n - a_k$, replacing the digit e_k by $e_k - 1$. To each (A,q)-expansion of n with $e_k = 0$, we associate the (A,q)-expansion of $n - (\nu+1)a_{k-1}$, replacing e_{k-1} by $e_{k-1} - (\nu+1)$. Hence

$$\begin{aligned} M_{A,q}(n) &\leq M_{A,q}(n-a_k) + M_{A,q}\big(n-(\nu+1)a_{k-1}\big) \\ &\leq C(n-a_k) + C\big(n-(\nu+1)a_{k-1}\big), \end{aligned}$$

and we obtain $M_{A,q}(n) \leq Cn$ from inequalities (34) and(35). □

Remark 17. For the Fibonacci scale F, I. Pushkarev [48] obtained $M_{F,2}(n) \in \mathcal{O}(\sqrt{n})$. This bound is optimal, due to results of L. Carlitz who computed in [14] numerous values of $M_{F,2}(\cdot)$, in particular (with our notations) $M_{F,2}(F_{2n+1}^2 - 1) = F_{2n+1}$. Notice that $F_{2n+1}^2 - 1 = F_2 + F_6 + \cdots + F_{4n+2}$. □

3. Pisot scales

3.1. Definitions and main result

Let θ be a given (positive) Pisot number of degree r, with conjugates $\theta_0 = \theta$, $\theta_1, \ldots, \theta_{r-1}$ and let $P_\theta(X) = X^r - a_1 X^{r-1} - a_2 X^{r-2} - \cdots - a_r$ denote the monic irreducible polynomial of θ over $\mathbb{Q}$. By definition $\theta > 1$ and $|\theta_i| < 1$ for $1 \le i \le r-1$.

Let $G = (g_n)_{n \ge 0}$ be a Pisot scale related to θ, that is, g_n is of the form

$$g_n = c(\theta)\theta^n + \gamma_n, \tag{36}$$

where $c(\theta)$ belongs to $\mathbb{Q}(\theta)$ and $\lim_n \gamma_n = 0$. An equivalent characterization of (36) is given as follows.

Lemma 18. *A scale $G = (g_n)_{n \ge 0}$ is a Pisot scale related to θ if and only if there exists an integer n_0 such that for all $n \ge n_0$, one has*

$$g_{n+1} = a_1 g_n + \cdots + a_r g_{n+1-r}\,. \tag{37}$$

P r o o f. Assume that $G = (g_n)_{n \ge 0}$ is a Pisot scale related to θ of the form (36). Since θ is a Pisot number and $\lim_n \|c(\theta)\theta^n\| = 0$ by assumption, we know from a result of Pisot (see [46] and [15]) that there exists $\nu \in \mathbb{N}$ such that $\theta^\nu c(\theta)$ belongs to the dual of $\mathbb{Z}[\theta]$ (with respect to the bilinear trace map $(x, y) \mapsto \mathrm{Tr}_{\mathbb{Q}(\theta)/\mathbb{Q}}(xy)$). More explicitly, there exits $b(\theta) \in \mathbb{Z}[\theta]$ such that $c(\theta) = \frac{b(\theta)}{\theta^\nu P'(\theta)}$.

Set $\eta_n := \sum_{j=1}^{r-1} c(\theta_j)\theta_j^n$ where $c(\theta_j)$ denote the conjugate of $c(\theta)$ corresponding to θ_j and set $h_n = c(\theta)\theta^n + \eta_n$ for $n \ge 0$. A priori, h_n is a rational number. We claim that h_n is an integer for any n large enough. In fact, let $N := \prod_{j=0}^{r-1} P'(\theta_j)$. Then N is a non zero rational integer and by construction $Nc(\theta)\theta^n$ is an algebraic integer as soon as $n \ge \nu$, so that $h_n \in \frac{1}{N}\mathbb{Z}$. But $\lim_n ||c(\theta)\theta^n|| = 0$ and $\lim_n \eta_n = 0$, hence h_n is a rational integer for n large enough. As a straightforward consequence, there exists an index n_0 such that for all $n \ge n_0$, one has $g_n = h_n$ $(\gamma_n = \eta_n)$ and so

$$g_{n+1} = a_1 g_n + \cdots + a_r g_{n+1-r}\,. \tag{38}$$

Conversely, if a scale G satisfies (38) for all $n \geq n_0$ where θ is a positive Pisot number of minimal polynomial $P(X) = X^r - a_1 X^{r-1} - a_2 X^{r-2} - \cdots - a_r$, then the equality (36) holds with $c(\theta) \in \mathbb{Q}(\theta)$ and $\lim_n \gamma_n = 0$ but we have a stronger property. Due to the fact that G verifies ultimately the recurrence (38), the series $\sum_{n=0}^{\infty} |\gamma_n|$ converges. □

The main result of the paper is the following estimate that says, roughly speaking, that $M_{G,q}(n)$ is in general close to n^{α} for a suitable exponent α:

Theorem 19. *Let G be a Pisot scale associated to the Pisot number θ as above and let $q \geq q_G^*$ where q_G^* is defined by (20). There exist a real number $\alpha_{G,q} \geq 0$ and a subset $S_{G,q}$ of $\mathbb{N}$ of natural density 1, such that*

$$\lim_{\substack{s \in S_{G,q} \\ s \to \infty}} \frac{\log M_{G,q}(s)}{\log s} = \alpha_{G,q} \,. \tag{39}$$

Moreover, if $q > \theta$ one has $\alpha_{G,q} \neq 0$.

We can give bounds of the constant $\alpha_{G,q}$ thanks to Theorem 10:

Corollary 20. *Let $G = (g_k)_{k \geq 0}$ be a scale satisfying $g_k \sim c\theta^k$ and assume that the conclusion (39) of Theorem 19 holds. Then*

$$1 \leq \log_\theta q - \alpha_{G,q}. \tag{40}$$

Proof. Set $\beta = \alpha_{G,q}$ for short. Let $\varepsilon \in (0, \beta/2]$ and define $S_\varepsilon := \{n \in \mathbb{N};\ M_{G,q}(n) \geq n^{\beta - \varepsilon}\}$ and $N(\varepsilon) := \#(S_\varepsilon \cap [1, N])$. Theorem 19 implies there exists an integer K_ε such that $N(\varepsilon) \geq N/2$ for all $N \geq K_\varepsilon$. Therefore

$$\begin{aligned} \sum_{n \in S_\varepsilon \cap [1,N]} M_{G,q}(n) \quad &\geq \sum_{n \in S_\varepsilon \cap [1,N]} n^{\beta - \varepsilon} \\ &\geq \sum_{1 \leq n \leq N(\varepsilon)} n^{\beta - \varepsilon} \geq \frac{1}{(1 + \beta/2) 2^{1 + \beta/2}} N^{\beta - \varepsilon + 1} \end{aligned}$$

and Theorem 10 implies $\log_\theta q \geq \beta - \varepsilon + 1$. The inequality (40) follows. □

Remark 21.

1) If $G = E_d$, obviously $q_G^* = d$ and for $q = d$ one has $M_{E_d,d}(\cdot) \equiv 1$, hence $\alpha_{E_d,d} = 0$.

2) We shall see later that the constant $\alpha_{G,q}$ in Theorem 19 is related to the constant γ in [26, Theorem 1.1] by the equation

$$\alpha_{G,q} = \frac{\gamma}{\log \theta} \,.$$

This already implies that $\alpha_{G,q} > 0$ for $q > \theta$. Notice that in fact, if θ is not an integer the inequality $q \geq q_G^*$ implies $q > \theta$.

3) As we have already pointed out at the end of Section 1, the inequality (40) is in fact an equality for any d-ary base E_d whenever d divide q. But for any other Pisot scale, this inequality is strict according to [26, Proposition 1.4] and the above remark.

Before starting the proof of Theorem 19 we need some preliminary results.

3.2. Discrete subsets of $\mathbb{N}[\theta]$

In this subsection, we assume that θ is a Pisot number and q is an integer $\geq \theta$.

Define

$$\Lambda_n = \Lambda_n^{(q)} := \left\{ \sum_{j=0}^{n-1} \varepsilon_j \theta^j \,;\, \forall\, j \in \mathbb{N}_0\,, \varepsilon_j \in \{0, 1, \ldots, q-1\} \right\} \tag{41}$$

and

$$\Lambda = \Lambda^{(q)} := \bigcup_{n \geq 1} \Lambda_n^{(q)}\,. \tag{42}$$

The following result is well known. The reader is referred to Erdős and Komornik [22, Theorem I, Lemma 2.1] for a proof.

Lemma 22. *There exist two positive constants $C_1 = C_1(\theta, q)$ and $C_2 = C_2(\theta, q)$ such that*

(i) $|a - b| \geq C_1$ *for all a and b, $a \neq b$, in $\Lambda^{(q)} - \Lambda^{(q)}$.*

(ii) *Any interval J in $[0, +\infty)$ of length greater that C_2, contains an element of $\Lambda^{(q)}$.*

(iii) $\#\Lambda_n^{(q)} \asymp \theta^n$ *$(n \to \infty)$.*

Remark 23. We can rephrase Property (i) in Lemma 22 as follows:

For all positive real numbers r the set $B_\theta(r, q) = [-r, r] \cap \left(\Lambda^{(q)} - \Lambda^{(q)}\right)$ is finite.

As it is stated, the property (i) implies that the overlapping iterated function system $\{S_i(x) = x/\theta + j\}_{j=0}^{q-1}$ satisfies the *finite type condition* introduced in ([25, condition (1.1)]), namely, there exists a finite set $\Gamma \subset \mathbb{R}$ such that for all $\varepsilon_0, \ldots, \varepsilon'_{n-1}$ and $\varepsilon'_0, \ldots, \varepsilon'_{n-1}$ in $\mathcal{S}$,

$$\text{either } \theta^n |S_{\varepsilon_0} \circ \cdots \circ S_{\varepsilon_{n-1}}(0) - S_{\varepsilon'_0} \circ \cdots \circ S_{\varepsilon'_{n-1}}(0)| > \delta$$
$$\text{or } \theta^n \big(S_{\varepsilon_0} \circ \cdots \circ S_{\varepsilon_{n-1}}(0) - S_{\varepsilon'_0} \circ \cdots \circ S_{\varepsilon'_{n-1}}(0)\big) \in \Gamma$$

with $\delta = \frac{\theta(q-1)}{\theta-1}$. In the special case considered here, due to

$$\theta^n S_{\varepsilon_0} \circ \cdots \circ S_{\varepsilon_{n-1}}(0) = \sum_{j=0}^{n-1} \varepsilon_j \theta^{n-j} \quad (\in \theta\,\Lambda^{(q)})\,,$$

one can take $\Gamma = [-\delta, \delta] \cap \big(\theta(\Lambda^{(q)} - \Lambda^{(q)})\big)$. Notice that the attractor of $\{S_j\}_{j=0}^{q-1}$ is $[0, \delta]$. □

3.3. Two level sets

We continue to assume that $g_k = c\theta^k + \gamma_k$ where θ is a Pisot number, $c = c(\theta) \in \mathbb{Q}(\theta)$ and $\lim_{k\to\infty} \gamma_k = 0$ (but in fact $\sum_{n=0}^{\infty} |\gamma_n| < +\infty$). Let q be an integer $\geq \theta$. We introduce the following notations for finite or infinite sequences ε with terms ε_k in $\mathcal{S} := \{0, 1, \ldots, q-1\}$:

$$\mathrm{val}_\theta(\varepsilon_0, \ldots, \varepsilon_{k-1}) := \textstyle\sum_{h=0}^{k-1} \varepsilon_h \theta^h\,,$$

$$\mathrm{val}(\varepsilon_1, \ldots, \varepsilon_k) := \textstyle\sum_{h=1}^{k} \frac{\varepsilon_h}{\theta^h}\,,$$

$$\mathrm{val}_\infty\big((\varepsilon_k)_{k\geq 1}\big) := \textstyle\sum_{k=1}^{\infty} \frac{\varepsilon_k}{\theta^k}\,.$$

The two last sums belong to $[0, \rho]$, with

$$\rho := \frac{q-1}{\theta - 1}\,.$$

Recall the previous notation $\mathrm{val}_G(\varepsilon_0, \ldots, \varepsilon_{k-1}) = \sum_{h=0}^{k-1} \varepsilon_h g_h$.

A sequence ε in $\mathcal{S}^{\mathbb{N}}$ such that $x = \mathrm{val}_\infty(\varepsilon)$ will be called a θ-representation of the real number $x \in [0, \rho]$. An element of the set

$$\begin{aligned} \mathcal{E}_k(x) &= \{(\varepsilon_1, \ldots, \varepsilon_k) \in \mathcal{S}^k \;;\; \exists\, \varepsilon' \in [\varepsilon_1, \ldots, \varepsilon_k],\ x = \mathrm{val}(\varepsilon')\} \\ &= \{\varepsilon \in \mathcal{S}^k \;;\; x - \frac{\rho}{\theta^k} \leq \mathrm{val}(\varepsilon) \leq x\} \end{aligned} \tag{43}$$

will be called a (θ, k)-representation of x. The first level set we defined is related to $\mathcal{E}_k(x)$.

Definition 24. *For any nonnegative real number α, denote*

$$E_1(\alpha) := \Big\{x \in [0, \rho] \;;\; \lim_{k\to\infty} \frac{\log \#\mathcal{E}_k(x)}{k \log \theta} = \alpha\Big\}.$$

In the sequel, we need the following key result:

Theorem 25 (D.-J. Feng and N. Sidorov [26], Theorem 1.1)**.** *For any Pisot number θ and integer $q \geq \theta$, there exists a constant $\alpha(\theta, q) \geq 0$ such that the set $E_1(\alpha(\theta, q))$ has Lebesgue measure ρ. Moreover, $\alpha(\theta, q) > 0$ whenever $q > \theta$.*

The second level set we consider is related to the local dimension of the Bernoulli convolution $\nu_{\theta,q}$ associated to θ with multiplicity q and which is the normalization of $\mu_{\theta,q}$ built previously but now defined on $[0,1]$:

$$\nu_{\theta,q}([a,b]) = \mu_q\big(\mathrm{val}_\infty^{-1}([\rho a, \rho b])\big)\,.$$

Definition 26. *For any nonnegative real number α,*

$$E_2(\alpha) := \Big\{x \in [0,1]\ ;\ \lim_{r\to 0} \frac{\log \nu_{\theta,q}([x-r, x+r])}{\log r} = \alpha\Big\}\,.$$

A consequence of Theorem 25 is the following:

Theorem 27 (D.-J. Feng and N. Sidorov [26], Corollary 1.2)**.** *With definitions and notions of Theorem 25, $E_2(\log_\theta q - \alpha(\theta,q))$ has Lebesgue measure 1.*

3.4. Some properties of the number of representations in base θ

In the sequel, the Pisot number θ, the Pisot scale G such as (36) and the integer $q \geq q_G^* \ (\geq \lceil\theta\rceil)$ are fixed. Some notations will be simplified when the reference to G or θ can be omitted without ambiguity. For any nonnegative real number λ let

$$\mathcal{U}_k(\lambda) := \#\Big\{(\varepsilon_0, \dots, \varepsilon_{k-1}) \in \mathcal{S}^k\ ;\ \lambda = \sum_{h=0}^{k-1} \varepsilon_h \theta^h\Big\}.$$

be the number of k-representations of λ in base θ. By construction $\mathcal{U}_k(\lambda)$ is positive if and only if λ belongs to the discrete set Λ_k introduced in (41). In this subsection we compare the number of representations of integers in the scale G and the number of k-representations of positive real numbers in base θ and then deduce relations between the two level sets.

According to the following lemma, the two levels can be expressed by means of the function $\mathcal{U}_k(\cdot)$.

Lemma 28.

(i) *For $x \in [0,\rho]$,*

$$\#\mathcal{E}_k(x) = \sum_{\substack{\lambda\in\Lambda_k\\ \theta^k x-\rho\leq\lambda\leq\theta^k x}} \mathcal{U}_k(\lambda)\,.$$

(ii) *For any $k \in \mathbb{N}$ and $[a,b] \subseteq [0,1]$,*

$$\frac{1}{q^k} \sum_{a\rho\theta^k \leq \lambda \leq b\rho\theta^k - \rho} \mathcal{U}_k(\lambda) \leq \nu_{\theta,q}([a,b]) \leq \frac{1}{q^k} \sum_{a\rho\theta^k - \rho \leq \lambda \leq b\rho\theta^k} \mathcal{U}_k(\lambda).$$

P r o o f. (i) Indeed $\mathcal{E}_k(x)$ is by definition the set of $(\varepsilon_1, \dots, \varepsilon_k)$ in $\mathcal{S}^k$ such that $\varepsilon_1\theta^{k-1} + \dots + \varepsilon_k\theta^0$ belongs to $[\theta^k x - \rho, \theta^k x]$.

(ii) By definition, $\nu_{\theta,q}([a,b])$ is the probability with respect to μ_q (on Ω_q) of the event

$$\rho^{-1}\sum_{h=1}^{\infty}\frac{\varepsilon_h}{\theta^h} \in [a,b]\,. \tag{44}$$

But (44) holds if $\rho^{-1}\sum_{h=1}^{k}\frac{\varepsilon_h}{\theta^h} \in [a, b - \frac{1}{\theta^k}]$ which is equivalent to

$$\varepsilon_k + \varepsilon_{k-1}\theta + \dots + \varepsilon_1\theta^{k-1} \in [a\rho\theta^k, b\rho\theta^k - \rho]\,.$$

Similarly, (44) implies

$$\varepsilon_k + \varepsilon_{k-1}\theta + \dots + \varepsilon_1\theta^{k-1} \in [a\rho\theta^k - \rho, b\rho\theta^k]\,.$$

□

The next lemma gives inequalities between the partition function $M_{G,q}(\cdot)$ and the function $\mathcal{U}_k(\cdot)$.

Lemma 29. *(i) There exists a constant K such that for integer $\ell \geq 1$ and finite sequence $\varepsilon = (\varepsilon_0, \dots, \varepsilon_{\ell-1})$ with terms in $\mathcal{S}$, the integer $n = \mathrm{val}_G(\varepsilon)$ and the real number $\lambda = \mathrm{val}_\theta(\varepsilon)$ verify,*

$$|n - c(\theta)\lambda| \leq K.$$

(ii) For any $k \in \mathbb{N}$ and $0 \leq a \leq b < g_k$,

$$\sum_{a\leq n\leq b} M_{G,q}(n) \leq \sum_{\substack{\lambda\in\Lambda_k\\ c(\theta)^{-1}(a-K)\leq\lambda\leq c(\theta)^{-1}(b+K)}} \mathcal{U}_k(\lambda). \tag{45}$$

(iii) For any $k \in \mathbb{N}$ and $0 \leq a \leq b$,

$$\sum_{\substack{\lambda\in\Lambda_k\\ a\leq\lambda\leq b}} \mathcal{U}_k(\lambda) \leq \sum_{c(\theta)a-K\leq n\leq c(\theta)b+K} M_{G,q}(n). \tag{46}$$

P r o o f. (i) Recall that $g_k = c(\theta)\theta^k + \gamma_k$ with $\sum_{k=0}^{\infty}|\gamma_k| < \infty$. Write $c = c(\theta)$ for short. Since the ε_k in the given ε do not exceed $q-1$ one has $|n - c\lambda| \leq (q-1)\sum_{k=0}^{\infty}|\gamma_k|$.

(ii) Let $n \in [a,b]$. Since $b < g_k$, any sequence ε (with terms in $\mathcal{S}$) such that $n = \mathrm{val}_G(\varepsilon)$ cannot have more than k digits, the number $\lambda = \mathrm{val}_\theta(\varepsilon)$ is in Λ_k. The inequality in (45) holds because $\mathcal{U}_k(\lambda)$ is the number of representations of λ by sequences of k digits; the inequalities $c^{-1}(a-K) \leq \lambda \leq c^{-1}(b+K)$ are a consequence of $n \in [a,b]$ and $|n - c\lambda| \leq K$.

(iii) Let $\lambda \in \Lambda_k$ and $\lambda = \mathrm{val}_\theta(\varepsilon) \in [a,b]$; the integer $n = \mathrm{val}_G(\varepsilon)$ satisfies $|n - c\lambda| \le K$ hence $ca - K \le n \le cb + K$. □

3.5. Proof of Theorem 19

We start by defining the set

$$R(\alpha,\eta,k) := \Big\{x \in [0,\rho]\ ;\ \frac{\log \#\mathcal{E}_k(x)}{k \log \theta} \notin [\alpha-\eta, \alpha+\eta]\Big\}$$

for $\alpha > 0$, $\eta > 0$ and $k \in \mathbb{N}$.

Recall that $\alpha(\theta,q)$ denotes the positive real number such that $E_1(\alpha(\theta,q))$ has full Lebesgue measure in $[0,\rho]$ (Theorem 25). In other words, the Lebesgue measure of $R(\alpha(\theta,q),\eta,k)$ tends to 0 when k tends to infinity.

Lemma 30. *(i) There exists $A > 0$ such that the distance between two distinct elements of $\Lambda_k \cup (\Lambda_k + \rho)$ is at least A.*

(ii) For any $\varepsilon \in \mathcal{S}^k$,

$$\mathrm{val}(\varepsilon) \in R(\alpha,\eta,k) \Rightarrow \Big[\mathrm{val}(\varepsilon), \mathrm{val}(\varepsilon) + \frac{A}{\theta^k}\Big[\subset R(\alpha,\eta,k).$$

(iii) Setting $\Lambda'_k = \frac{1}{\theta^k}\Lambda_k$ we have

$$\lim_{k\to\infty} \frac{1}{\theta^k} \#\Big(\Lambda'_k \cap R(\alpha_{G,q},\eta,k)\Big) = 0.$$

Proof. (i) Let x and y in $\Lambda_k \cup (\Lambda_k+\rho)$. If x and y both belong to Λ_k or $\Lambda_k+\rho$ the result follows from Lemma 22 (i). It remains the case $x \in \Lambda_k$ and $y \in \Lambda_k + \rho$. Recall that $\rho = \frac{q-1}{\theta-1}$. Setting $b_k := \sum_{h=0}^k (q-1)\theta^h$ we arrive to $(\theta-1)x + b_k \in \Lambda_{k+1}^{(2q-1)}$ and $(\theta-1)y + b_k \in \Lambda_{k+1}^{(3q-2)}$. Hence, with the notation of Lemma 22 the property (i) follows with $A = \min\{C_1(\theta,q), C_1(\theta,3q-2)/(\theta-1)\}$.

(ii) The definition of $\mathcal{E}_k(x)$ means that

$$\mathcal{E}_k(x) = \Big\{\varepsilon \in \mathcal{S}^k\ ;\ x \in \Big[\mathrm{val}(\varepsilon), \mathrm{val}(\varepsilon) + \frac{\rho}{\theta^k}\Big]\Big\}.$$

Now real numbers of the form $\mathrm{val}(\varepsilon)$ or $\mathrm{val}(\varepsilon) + \frac{\rho}{\theta^k}$ are elements of the set $\frac{1}{\theta^k}\big(\Lambda_k \cup (\Lambda_k+\rho)\big)$. From (i) the interval $(\mathrm{val}(\varepsilon), \mathrm{val}(\varepsilon) + \frac{A}{\theta^k})$ does not contain any element of $\frac{1}{\theta^k}\big(\Lambda_k \cup (\Lambda_k+\rho)\big)$. Therefore, for all $\varepsilon' \in \mathcal{S}^k$, under the condition $x \in (\mathrm{val}(\varepsilon), \mathrm{val}(\varepsilon) + \frac{A}{\theta^k})$, both relations $\mathrm{val}(\varepsilon) \in [\mathrm{val}(\varepsilon'), \mathrm{val}(\varepsilon') + \frac{\rho}{\theta^k}]$ and $x \in [\mathrm{val}(\varepsilon'), \mathrm{val}(\varepsilon') + \frac{\rho}{\theta^k}]$ are equivalent to the relation $(\mathrm{val}(\varepsilon), \mathrm{val}(\varepsilon) + \frac{A}{\theta^k}) \subset [\mathrm{val}(\varepsilon'), \mathrm{val}(\varepsilon') + \frac{\rho}{\theta^k}]$. Hence, for all $x \in (\mathrm{val}(\varepsilon), \mathrm{val}(\varepsilon) + \frac{A}{\theta^k})$ one has $\mathcal{E}_k(\mathrm{val}(\varepsilon)) = \mathcal{E}_k(x)$. Consequently

$$\mathrm{val}(\varepsilon) \in R(\alpha,\eta,k) \iff x \in R(\alpha,\eta,k)$$

and the implication in (ii) holds.

(iii) Since $\Lambda'_k = \{\mathrm{val}(\varepsilon)\,;\ \varepsilon \in \mathcal{S}^k\}$, property (ii) shows that the Lebesgue measure of $R(\alpha(\theta,q),\eta,k)$ is at least $\frac{A}{\theta^k}\#(\Lambda'_k \cap R(\alpha(\theta,q),\eta,k))$. This ends the proof of the lemma. □

To continue the proof of Theorem 19 we set (see Theorem 25)

$$\beta := \alpha(\theta,q)$$

and we introduce the subset of integers

$$\mathcal{N}_k(\eta) := \{n \in [g_{k-1}, g_k[\ ;\ \frac{\log M_{G,q}(n)}{\log n} \notin [\beta-\eta, \beta+\eta]\}$$

for $\eta > 0$ and $k \in \mathbb{N}$.

To each integer $n \in \mathcal{N}_k(\eta)$ we are going to associate a real number $\varphi(n)$ in Λ_k so that the value $\psi(n) := \theta^{-k}\varphi(n)$ will belong to $\Lambda'_k = \frac{1}{\theta^k}\Lambda_k$. The construction of $\varphi(n)$ depends on whether $\frac{\log M_{G,q}(n)}{\log n} < \beta - \eta$ or $\frac{\log M_{G,q}(n)}{\log n} > \beta + \eta$.

If $\frac{\log M_{G,q}(n)}{\log n} < \beta - \eta$ and $n = \mathrm{val}_G(\varepsilon)$ where ε corresponds to the standard G-expansion of n we define

$$\varphi(n) := \mathrm{val}_\theta(\varepsilon).$$

Write $c = c(\theta)$ for short. Lemma 29 (i) implies

$$|n - c\varphi(n)| \le K$$

and by Lemma 28 (i) and Lemma 29 (iii)

$$\begin{aligned} \#\mathcal{E}_k\big(\psi(n)\big) = \sum_{\substack{\lambda\in\Lambda_k \\ \varphi(n)-\rho\le\lambda\le\varphi(n)}} \mathcal{U}_k(\lambda) \ &\le \sum_{c\varphi(n)-c\rho-K\le m\le c\varphi(n)+K} M_{G,q}(m) \\ &\le \sum_{n-c\rho-2K\le m\le n+2K} M_{G,q}(m)\,. \end{aligned}$$

Using Lemma 9 (ii) and the inequality $M_{G,q}(n) < n^{\beta-\eta}$ we conclude that for n large enough

$$\#\mathcal{E}_k\big(\psi(n)\big) \le M_{G,q}(n)\cdot n^{\frac{\eta}{2}} < n^{\beta-\frac{\eta}{2}} \le \theta^{k(\beta-\frac{\eta}{3})}\,. \tag{47}$$

In the case that $\frac{\log M_{G,q}(n)}{\log n} > \beta+\eta$, Lemma 29 (ii) implies that

$$M_{G,q}(n) \le \sum_{\substack{\lambda\in\Lambda_k \\ c^{-1}(n-K)\le\lambda\le c^{-1}(n+K)}} \mathcal{U}_k(\lambda).$$

By Lemma 22 (i), the number of λ in Λ_k such that $c^{-1}(n-K) \le \lambda \le c^{-1}(n+K)$ is bounded by a constant K_1, so that at least one of these λ satisfies

$$M_{G,q}(n) \le K_1\mathcal{U}_k(\lambda)\,. \tag{48}$$

Let $\varphi(n)$ be the smallest λ verifying (48) and now, using the inequality $M_{G,q}(n) > n^{\beta+\eta}$ one deduces for n large enough:

$$\#\mathcal{E}_k\big(\psi(n)\big) \geq \mathcal{U}_k\big(\varphi(n)\big) > n^{\beta+\frac{\eta}{2}} \geq \theta^{k(\beta+\frac{\eta}{3})}. \tag{49}$$

Finally, collecting (47) and (49) we have proved that for any $n \in \mathcal{N}_k(\eta)$, $\psi(n)$ belongs to $R\Big(\beta, \frac{3}{\eta}, k\Big)$.

According to the inequality $|n - c\varphi(n)| \leq K$, the number of preimages – by the function ψ – of any λ' in Λ'_k is bounded by a constant K_2. Therefore

$$\#\mathcal{N}_k(\eta) \leq K_2 \# \bigg(\Lambda'_k \cap R\Big(\beta, \frac{\eta}{3}, k\Big)\bigg)$$

and, by Lemma 30 (iii),

$$\lim_{k\to\infty} \frac{1}{g_k - g_{k-1}} \#\big(\mathcal{N}_k(\eta)\big) = 0\,.$$

We are ready to finish the proof of Theorem 19.

From above, we can choose a non decreasing sequence of positive integers $(k(j))_{j\geq 1}$ such that for all $j \in \mathbb{N}$ and all $k \geq k(j)$, the inequality

$$\frac{1}{g_k - g_{k-1}} \# \bigg(\mathcal{N}_k\Big(\frac{1}{j}\Big)\bigg) \leq \frac{1}{j} \tag{50}$$

holds. Then, define the set $S_{G,q}$ announced in Theorem 19 by its complement:

$$(S_{G,q})^c := \bigcup_{j\in\mathbb{N}} \ \bigcup_{k(j)\leq k<k(j+1)} \mathcal{N}_k\Big(\frac{1}{j}\Big).$$

Using (50), for all integers $j > 0$, $k \in [k(j), k(j+1)[$ and $n \in [g_{k-1}, g_k[$, we derive the inequality

$$\frac{1}{n}\#((S_{G,q})^c \cap [1,n]) \leq \frac{k(j)}{n} + \frac{\displaystyle\sum_{j'\in[1,j)} \sum_{k'\in[k(j'),k(j'+1))} \frac{g_{k'} - g_{k'-1}}{j'} + \sum_{k'\in[k(j),k)} \frac{g_{k'} - g_{k'-1}}{j} + \frac{g_k - g_{k-1}}{j}}{\displaystyle\sum_{j'\in[1,j)} \sum_{k'\in[k(j'),k(j'+1))} (g_{k'} - g_{k'-1}) + \sum_{k'\in[k(j),k)} (g_{k'} - g_{k'-1})}.$$

This implies both the density of $(S_{G,q})^c$ is 0 and the constant $\alpha_{G,q}$ in Theorem 19 is equal to $\alpha(\theta, q)$. □

Remark 31. The constant $\alpha_{G,q}$ in Theorem 19 essentially depends on θ. In fact, formula (39) is true for any scale $G = (g_n)_{n\geq 0}$ that verifies the recurrent relation (38) from a given rang. In particular, we are free to change a finite numbers or terms g_n but in preserving the structure of scale.

Theorem 19 says in particular that for all $\varepsilon > 0$, the set $\{n \in \mathbb{N}\ ;\ \frac{\log M_{G,\theta}(n)}{\log n} > \alpha_{G,\theta} + \varepsilon\}$ has density 0. We can give more information if G is a scale E_d and q is not a multiple of d.

Corollary 32. *Let d be an integer, $d \geq 2$, and assume that q is not divisible by d. Then, for all ε_1 and ε_2 such that*

$$0 < \varepsilon_1 < \log_d q - \alpha_{E_d,q} - 1 \text{ and } 0 < \varepsilon_2 < \log_d q - \log_d \left\lfloor \frac{q-1+d}{d} \right\rfloor$$

one has

$$\#\left\{n < N\ ;\ \frac{\log M_{E_d,q}(n)}{\log n} > \alpha_{G,q} + \varepsilon_1\right\} \geq N^{\varepsilon_2} \tag{51}$$

for N large enough.

Proof. The possible choice of ε_1 is valid by Remark 21-3. From [19, Theorem 2.1] or Theorem 10, given any $\varepsilon > 0$ one has for N large enough

$$\sum_{n<N} M_{G,q}(n) \geq N^{\alpha_1 - \varepsilon}$$

with $\alpha_1 = \log_d q$. From (15), $M_{G,q}(n) \leq n^{\alpha_2}$ for all n, where $\alpha_2 = \log_d \left\lfloor \frac{q-1+d}{d} \right\rfloor$. Let $A(N)$ denote the left hand side of (51); a simple evaluation gives, as soon as N is large,

$$N^{\alpha_1-\varepsilon} \leq \sum_{n<N} M_{G,q}(n) \leq \big(N - A(N)\big)N^{\alpha+\varepsilon_1} + A(N)N^{\alpha_2}$$

with $\alpha := \alpha_{E_d,q}$, whence

$$N^{\alpha_1-\varepsilon} - N^{\alpha+1+\varepsilon_1} \leq A(N)N^{\alpha_2}.$$

Since $\varepsilon_1 < \alpha_1 - \alpha_{E_d,q} - 1$, by selecting ε such that $\alpha_1 - \varepsilon > \alpha + 1 + \varepsilon_1$ we arrive to

$$N^{\alpha_1-2\varepsilon} \leq A(N)N^{\alpha_2}.$$

But clearly $\alpha_1 > \alpha_2$ so that (51) holds due to $\varepsilon_2 < \alpha_1 - \alpha_2$. □

3.6. A third level set

The third level set, which we will define below, is related to the standard θ-expansion of real numbers $t \geq 0$. Such an expansion

$$t = \sum_{h>k} \frac{\varepsilon_h(t)}{\theta^h} \quad \Big(k \in \mathbb{Z} \text{ determined by } \frac{1}{\theta^{k+1}} \leq t < \frac{1}{\theta^k}\Big)$$

is given by the greedy algorithm which constructs the digits $\varepsilon_h(t) \in \{0, 1, \ldots, \lfloor\theta\rfloor\}$ step by step, starting with ε_{k+1}, so that the following inequalities

$$\sum_{h>s} \frac{\varepsilon_h(t)}{\theta^h} < \frac{1}{\theta^s}$$

are satisfied for all integers $s \geq k$. These inequalities guarantee the uniqueness of the digits. Recall that $0 \leq \varepsilon_h(t) < \theta$ for all $h > k$ and if t belongs to $[0, 1)$ its standard θ-expansion corresponds to the one introduced by A. Rényi [50] and refined by W. Parry [44]. Notice that the Parry θ-expansion of 1 is not the standard one; it starts with $\varepsilon_0(1) = 0$, $\varepsilon_1(1) = \lfloor\theta\rfloor$ and the remainder digits come from the standard θ-expansion of $1 - \lfloor\theta\rfloor/\theta$. For more details on β-expansions ($\beta > 1$) in general we refer to [29].

The integer

$$\mathrm{Ip}(t) := \begin{cases} \varepsilon_{k+1}(t)g_{-k-1} + \cdots + \varepsilon_{-1}(t)g_1 + \varepsilon_0(t)g_0 & \text{if } k < 0\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

will be called the integral part of $t \geq 0$ subordinated to θ and G. Of course, $\mathrm{Ip}(t) = \lfloor t\rfloor$ if θ is an integer.

Definition 33. *For any nonnegative real number α,*

$$E_3(\alpha) := \Big\{x \in (0, 1)\ ;\ \lim_{k\to\infty} \frac{\log M_{G,q}(\mathrm{Ip}(\theta^k x))}{k \log\theta} = \alpha\Big\}.$$

In the above definition, the condition that $x \in (0, 1)$ has no effect because the set of all nonnegative reals x such that $\lim_{k\to\infty} \frac{\log M_{G,q}(\mathrm{Ip}(\theta^k x))}{k\log\theta} = \alpha$ is stable by multiplication by θ.

Proposition 34. *(i) Let G be a Pisot scale associated to the Pisot number θ. For any nonnegative real number α,*

$$E_3(\alpha) = \big(\rho \cdot E_2(\log_\theta q - \alpha)\big) \cap (0, 1)\,.$$

(ii) $E_3(\alpha_{G,q})$ has full Lebesgue measure.

P r o o f. (i) This equality follows from the inequalities lying between

$\nu_{\theta,q}([x - r, x + r])$ and $M_{G,q}(n_k)$, with $\rho x \in (0, 1)$ and $n_k = \mathrm{Ip}(\theta^k \rho x)$.

More precisely, from Lemma 29 (i) one has

$$|n_k - c\theta^k \rho x| \leq K$$

and by Lemmas 28 (ii) and 29 (iii), if $r > 0$ is small enough,

$$\begin{aligned}\nu_{\theta,q}([x-r,x+r]) &\le \frac{1}{q^k}\sum_{|n-c\theta^k\rho x|\le c\theta^k\rho r+c\rho+K} M_{G,q}(n)\\ &\le \frac{1}{q^k}\sum_{|n-n_k|\le c\theta^k\rho r+c\rho+2K} M_{G,q}(n).\end{aligned} \tag{52}$$

We obtain in the same way a lower bound for $\nu_{\theta,q}([x-r,x+r])$:

$$\begin{aligned}\nu_{\theta,q}([x-r,x+r]) &\ge \frac{1}{q^k}\sum_{|n-c\theta^k\rho x|\le c\theta^k\rho r-c\rho-K} M_{G,q}(n)\\ &\ge \frac{1}{q^k}\sum_{|n-n_k|\le c\theta^k\rho r-c\rho-2K} M_{G,q}(n)\end{aligned} \tag{53}$$

with the condition, from Lemma29 (ii), that

$$c\theta^k\rho(x-r)+c\rho+K \le c\theta^k\rho(x+r)-c\rho-K < g_k. \tag{54}$$

The last inequality in (54) is satisfied for r small enough and k large enough, because we have assumed that $\rho x < 1$ and $\lim_{k\to\infty}\frac{g_k}{c\theta^k} = 1$. Now, given r, we choose k verifying

$$c\rho+2K \le c\theta^k\rho r < (c\rho+2K)\theta, \tag{55}$$

so that: $r \asymp \frac{1}{\theta^k}$, the first inequality in (54) is satisfied and moreover the intervals of summation in (53) are nonempty.

Finally we deduce from (52), (53) and Lemma 9 (ii) two constants K_1, K_2 such that

$$\frac{1}{q^k}M_{G,q}(n_k) \le \nu_{\theta,q}([x-r,x+r]) \le \frac{K_1}{q^k}\left(\log n_k\right)^{K_2} M_{G,q}(n_k). \tag{56}$$

Passing to the logarithm and noticing that the choices of r and k (depending on r) verify asymptotically $\log r \sim -k\log\theta$ we derive that

$$y \in E_3(\alpha) \iff \frac{y}{\rho} \in E_2\left(\frac{\log q}{\log\theta} - \alpha) \cap (0, 1/\rho\right);$$

hence the identity $E_3(\alpha) = \rho\cdot E_2(\frac{\log q}{\log\theta} - \alpha)\cap(0,1)$ holds.

(ii) The set $E_2(\log_\theta q - \alpha_{G,q}))$ has full Lebesgue measure by Theorem 27. Consequently, (i) implies that $E_3(\alpha_{G,q})$ has also full Lebesgue measure. □

Acknowledgements. The work was partially done when the first author visited Université d'Aix-Marseille and the second author visited the Chinese University of Hong Kong in 2009. They are indebted to the generous support by these two Universities.

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Received February 10, 2012
Accepted September 19, 2013

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