

A NOTE ON THE DUFFIN-SCHAEFFER CONJECTURE

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ABSTRACT. Given a sequence of real numbers $\{\psi(n)\}_{n \in \mathbb{N}}$ with $0 \leq \psi(n) < 1$, let $W(\psi)$ denote the set of $x \in [0, 1]$ for which $|xn - m| < \psi(n)$ for infinitely many coprime pairs $(n, m) \in \mathbb{N} \times \mathbb{Z}$. The purpose of this note is to show that if there exists an $\epsilon > 0$ such that $\sum_{n \in \mathbb{N}} \psi(n)^{1+\epsilon} \cdot \frac{\varphi(n)}{n} = \infty$, then the Lebesgue measure of $W(\psi)$ equals 1.

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1. Introduction to the Duffin-Schaeffer conjecture

Given arithmetic functions $\psi : \mathbb{N} \rightarrow [0, \infty)$ and $\omega : \mathbb{N} \rightarrow \mathbb{N}$ with ω increasing, let $W(\psi, \omega)$ denote the set of $x \in [0, 1]$ for which $|x \cdot \omega(n) - m| < \psi(n)$ for infinitely many coprime pairs $(\omega(n), m) \in \mathbb{N} \times \mathbb{Z}$. If ω is the identity map then we simply write $W(\psi)$ for $W(\psi, \omega)$. In this note we will study a long-standing problem of Duffin and Schaeffer ([2]), who conjectured

$$\sum_{n \in \mathbb{N}} \psi(n) \cdot \frac{\varphi(\omega(n))}{\omega(n)} = \infty \Leftrightarrow \mathcal{M}(W(\psi, \omega)) = 1,$$

where φ denotes Euler's totient function, \mathcal{M} is the Lebesgue measure on \mathbb{R} . Since the necessity part of this conjecture follows from the first Borel-Cantelli lemma (see e.g. [4, Theorem 2.4]), we need only focus on the sufficiency one.

There are many partial results towards the sufficiency part of the Duffin-Schaeffer conjecture, for example, by assuming any of the following additional conditions:

- A1: ω is the identity map, $n \mapsto n\psi(n)$ is non-increasing (Khintchine [6]);
- A2: ω is the identity map, $\psi(n) \leq \frac{c}{n}$ for some $c > 0$ (Vaaler [12]);

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- A3: $(\omega(m), \omega(n)) = 1$ for any $m \neq n$, ψ is arbitrary (Strauch [11]);
- A4: $\{\omega(n)\}$ is a lacunary sequence, ψ is arbitrary (Harman [3]);
- A5: $\psi(n) \geq c(\frac{\varphi(\omega(n))}{\omega(n)})^R$ for some $c, R > 0$ (Harman [3]).

We should also note the following beautiful breakthroughs:

- PV: The higher-dimensional Duffin-Schaeffer conjecture, also known as the Sprindžuk conjecture ([10]), was solved by Pollington and Vaughan ([7, 8]);
- BH: $\sum_{n \in \mathbb{N}} \psi(n) = \infty \Rightarrow$ Hausdorff dimension of $W(\psi) = 1$ (Baker-Harman [4, Theorem 10.7]).

Recently, Haynes, Pollington and Velani ([5, Corollary 1], see also [4]) established the Duffin-Schaeffer conjecture by assuming the following extra divergence condition:

- HPV1: $\sum_{n \in \mathbb{N}} (\frac{\psi(n)}{n})^{1+\epsilon} \cdot \varphi(n) = \infty$.

Then as an immediate application of the HPV1 theorem and the Beresnevich-Velani Transference Principle ([1, Theorem 2]), they also studied the Hausdorff dimensional Duffin-Schaeffer conjecture ([1, Conjecture 2]) and were able to improve the BH theorem to ([5, Theorem 2]):

- HPV2: $\sum_{n \in \mathbb{N}} (\frac{\psi(n)}{n})^{1-\epsilon} \cdot \varphi(n) = \infty (\forall \epsilon > 0) \Rightarrow$ Hausdorff dim. of $W(\psi) = 1$.

The purpose of this note is to adapt some ideas of Harman in [3] to improve the HPV1 theorem as follows:

THEOREM 1.1. *Let $\psi : \mathbb{N} \rightarrow [0, 1)$ be any function. Then $\mathcal{M}(W(\psi)) = 1$ if there exists an $\epsilon > 0$ such that*

$$\sum_{n \in \mathbb{N}} \psi(n)^{1+\epsilon} \cdot \frac{\varphi(n)}{n} = \infty.$$

Before proceeding to the proof of Theorem 1.1 in the next section, let us first give some remarks and corollaries.

REMARK 1.2. We point out by constructing an example that the function ψ assumed in Theorem 1.1 cannot be extended to all non-negative arithmetic functions. Since $\{\frac{\varphi(n)}{n} : n \in \mathbb{N}\}$ is dense in $(0, 1)$ ([9]), there exists a sequence of positive integers $n_2 < n_3 < n_4 < \dots$ so that

$$\frac{\varphi(n_k)}{n_k} \sim \frac{1}{k \cdot (\log k)^{2+\frac{\epsilon}{2}}},$$

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where $\epsilon > 0$ is any prescribed real number. We then define $\psi(n_k) = \log k$ and note

$$\begin{aligned} \sum_k \psi(n_k)^{1+\epsilon} \cdot \frac{\varphi(n_k)}{n_k} &\sim \sum_k \frac{1}{k \cdot (\log k)^{1-\frac{\epsilon}{2}}} = \infty, \\ \sum_k \psi(n_k) \cdot \frac{\varphi(n_k)}{n_k} &\sim \sum_k \frac{1}{k \cdot (\log k)^{1+\frac{\epsilon}{2}}} < \infty. \end{aligned}$$

Applying not Theorem 1.1 but the first Borel-Cantelli lemma gives $\mathcal{M}(W(\psi))=0$.

COROLLARY 1.3 (HPV1). *Let $\psi : \mathbb{N} \rightarrow [0, \infty)$ be any function. Then $\mathcal{M}(W(\psi)) = 1$ if there exists an $\epsilon > 0$ such that*

$$\sum_{n \in \mathbb{N}} \left(\frac{\psi(n)}{n} \right)^{1+\epsilon} \cdot \varphi(n) = \infty.$$

Proof. We divide \mathbb{N} into two parts \mathbb{X}, \mathbb{Y} , that is, $n \in \mathbb{X}$ or $n \in \mathbb{Y}$ according as $\psi(n) < 1$ or not, and have two cases to consider.

CASE 1: Suppose $\sum_{n \in \mathbb{X}} \left(\frac{\psi(n)}{n} \right)^{1+\epsilon} \cdot \varphi(n) = \infty$. Obviously, $\sum_{n \in \mathbb{X}} \psi(n)^{1+\epsilon} \cdot \frac{\varphi(n)}{n} = \infty$. Thus $\mathcal{M}(W(\psi)) = 1$ follows from applying Theorem 1.1.

CASE 2: Suppose $\sum_{n \in \mathbb{Y}} \left(\frac{\psi(n)}{n} \right)^{1+\epsilon} \cdot \varphi(n) = \infty$. In this case it is easy to prove that $\sum_{n \in \mathbb{Y}} \frac{\psi(n)}{n} \cdot \varphi(n) = \infty$. Applying Harman's Condition A5 gives $\mathcal{M}(W(\psi)) = 1$.

This finishes the proof. □

REMARK 1.4. Similar to the proof of the HPV2 theorem, one may expect that as an application of Theorem 1.1 and the Beresnevich-Velani Transference Principle the following proposition might be true:

- $\sum_{n \in \mathbb{N}} \frac{\psi(n)^{1-\epsilon}}{n^{1-2\epsilon}} \cdot \varphi(n) = \infty$ ($\forall \epsilon > 0$) \Rightarrow Hausdorff dimension of $W(\psi) = 1$.

We remark that even if the above proposition is true, it cannot give any genuine improvement of the HPV2 theorem. The reason is as follows: If there are infinitely many $n \in \mathbb{N}$ so that $\psi(n) \geq 1$, then by the BH theorem $W(\psi)$ has full Hausdorff dimension. Thus to study what sufficient condition can guarantee $W(\psi)$ has full Hausdorff dimension, we may assume without loss of generality that $\psi(n) < 1$ for all $n \in \mathbb{N}$. Suppose this is the case, then $\sum_{n \in \mathbb{N}} \frac{\psi(n)^{1-\epsilon}}{n^{1-2\epsilon}} \cdot \varphi(n) = \infty$ implies $\sum_{n \in \mathbb{N}} \frac{\psi(n)^{1-2\epsilon}}{n^{1-2\epsilon}} \cdot \varphi(n) = \infty$. Now we can apply the HPV2 theorem to ensure $W(\psi)$ has full Hausdorff dimension.

COROLLARY 1.5. *Let $\psi, \gamma : \mathbb{N} \rightarrow [0, \infty)$ be any functions with $\limsup_{n \rightarrow \infty} \frac{\log \gamma(n)}{\log n} \leq 0$. Then $W(\psi)$ has full Hausdorff dimension if*

$$\sum_{n \in \mathbb{N}} \psi(n) \cdot \gamma(n) = \infty.$$

PROOF. As discussed in Remark 1.4, to prove Corollary 1.5 we may assume without loss of generality that $\psi(n) < 1$ for all $n \in \mathbb{N}$. Since $\limsup_{n \rightarrow \infty} \frac{\log \gamma(n)}{\log n} \leq 0$, we observe that for any $\epsilon > 0$,

$$\psi(n) \cdot \gamma(n) \leq \left(\frac{\psi(n)}{n} \right)^{1-\epsilon} \cdot \varphi(n)$$

holds for sufficiently large n . According to the HPV2 theorem we obtain that $W(\psi)$ has full Hausdorff dimension. This finishes the proof. \square

2. Proof of the main result

A THEOREM OF HARMAN. *Let ψ_1, \dots, ψ_k ($k \geq 2$) be functions of $n \in \mathbb{N}$, taking values in $[0, 1]$. Write*

$$\theta(n) = \prod_{j=1}^k \psi_j(n), \tag{1}$$

and suppose for some positive reals δ and K , that for each n with $\theta(n) \neq 0$,

$$\max_{1 \leq j \leq k} \frac{\theta(n)}{\psi_j(n)} \leq K \cdot \theta(n)^\delta. \tag{2}$$

A theorem of Harman ([3, Thm. 2]) on the higher-dimensional Duffin-Schaeffer conjecture claims that if

$$\sum_{n \in \mathbb{N}} \theta(n) \cdot \left(\frac{\varphi(n)}{n} \right)^k = \infty, \tag{3}$$

then $\mathcal{M}_k(W(\psi_1, \dots, \psi_k)) = 1$, where \mathcal{M}_k is the k -dimensional Lebesgue measure, $W(\psi_1, \dots, \psi_k)$ is the set of $(x_1, \dots, x_k) \in [0, 1]^k$ for which

$$|nx_j - m_j| < \psi_j(n) \quad (n, m_j) = 1, \quad j = 1, \dots, k \tag{4}$$

for infinitely many $(n, m_1, \dots, m_k) \in \mathbb{N} \times \mathbb{Z}^k$.

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Proof of Theorem 1.1. Obviously, we may assume that $0 < \epsilon < 1$. We divide \mathbb{N} into two parts \mathbb{X}, \mathbb{Y} , that is, $n \in \mathbb{X} \Leftrightarrow \psi(n) \leq \left(\frac{\varphi(n)}{n}\right)^{2/\epsilon}$, $n \in \mathbb{Y} \Leftrightarrow \psi(n) > \left(\frac{\varphi(n)}{n}\right)^{2/\epsilon}$. Since $\psi(n) < 1$ for each $n \in \mathbb{N}$, we can deduce from $\sum_{n \in \mathbb{N}} \psi(n)^{1+\epsilon} \cdot \frac{\varphi(n)}{n} = \infty$ that $\sum_{n \in \mathbb{N}} \psi(n) \cdot \frac{\varphi(n)}{n} = \infty$. Now we have two cases to consider.

CASE 1: Suppose $\sum_{n \in \mathbb{Y}} \psi(n) \cdot \frac{\varphi(n)}{n} = \infty$. Then according to Harman's Condition A5, we have $\mathcal{M}(W(\psi)) = 1$.

CASE 2: Suppose $\sum_{n \in \mathbb{Y}} \psi(n) \cdot \frac{\varphi(n)}{n} < \infty$. Obviously, $\sum_{n \in \mathbb{Y}} \psi(n)^{1+\epsilon} \cdot \frac{\varphi(n)}{n} < \infty$. We define $\Psi(n) = \psi(n)$ if $n \in \mathbb{X}$, and $\Psi(n) = 0$ if $n \in \mathbb{Y}$. It is easy to check that for any $n \in \mathbb{N}$,

$$\Psi(n) \leq \left(\frac{\varphi(n)}{n}\right)^{2/\epsilon},$$

and

$$\sum_{n \in \mathbb{N}} \Psi(n)^{1+\epsilon} \cdot \frac{\varphi(n)}{n} = \infty.$$

Now it is time to apply Harman's theorem introduced in the beginning of this section. Let $k = 2$ and let $\psi_1(n) = \Psi(n)$, $\psi_2(n) = \Psi(n)^\epsilon \cdot \frac{n}{\varphi(n)}$. Then for any $n \in \mathbb{N}$ with $\theta(n) \neq 0$, equivalently, $\Psi(n) \neq 0$, one has

$$\frac{\theta(n)}{\psi_1(n)} = \Psi(n)^\epsilon \cdot \frac{n}{\varphi(n)} \leq \Psi(n)^{\epsilon/2} = (\Psi(n)^{1+\epsilon})^{\frac{\epsilon}{2(1+\epsilon)}} \leq \theta(n)^{\frac{\epsilon}{2(1+\epsilon)}}, \quad (5)$$

and similarly,

$$\frac{\theta(n)}{\psi_2(n)} = \Psi(n) = (\Psi(n)^{1+\epsilon})^{\frac{1}{1+\epsilon}} \leq \theta(n)^{\frac{1}{1+\epsilon}}. \quad (6)$$

Noting $\epsilon \in (0, 1)$, and $\theta(n) = \Psi(n)^{1+\epsilon} \cdot \frac{n}{\varphi(n)} \leq \Psi(n)^{1+\frac{\epsilon}{2}} < 1$, we have

$$\max_{1 \leq j \leq 2} \frac{\theta(n)}{\psi_j(n)} \leq \theta(n)^{\frac{\epsilon}{2(1+\epsilon)}}. \quad (7)$$

Observe also that $\psi_1(n) < 1$, $\psi_2(n) = \Psi(n)^\epsilon \cdot \frac{n}{\varphi(n)} \leq \Psi(n)^{\frac{\epsilon}{2}} < 1$, and

$$\sum_{n \in \mathbb{N}} \theta(n) \cdot \left(\frac{\varphi(n)}{n}\right)^2 = \sum_{n \in \mathbb{N}} \Psi(n)^{1+\epsilon} \cdot \frac{\varphi(n)}{n} = \infty. \quad (8)$$

With these preparations we apply Harman's theorem to get $\mathcal{M}_2(W(\psi_1, \psi_2)) = 1$. Now we note an elementary relation, that is, $W(\psi_1, \psi_2) \subset W(\psi_1) \times W(\psi_2)$. As an immediate consequence, $\mathcal{M}(W(\psi_1)) = 1$. Since $\psi_1 \leq \psi$, we have $W(\psi_1) \subset W(\psi)$, which gives $\mathcal{M}(W(\psi)) = 1$. This finishes the proof of Theorem 1.1. \square

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