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uniform distribution theory

ON THE HAUSDORFF DIMENSION OF SIMPLY NORMAL NUMBERS TO DIFFERENT BASES

Antonis Bisbas

ABSTRACT. Let $a \geq 2$ be an integer and n be the greatest integer such that a is a n-th power of an integer. We prove that the Hausdorff dimension of the set of numbers which are simply normal to base a, not simply normal to any base $a^{m/n}$ with $m > n$, not simply normal to any base $a^{m/n}$ with $m < n$, m does not divide n and normal to each algebraic base multiplicatively independent to base a, is equal to 1. This extends a previous result of P. Hertling [9] who proved that a set bigger than the above set has the cardinality of the continuum. For the proof we use the results of [2] and [3] and some probability measures constructed by an inhomogeneous Markov process.

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1. Introduction

Let $r \geq 2$ be an integer. A number $x \in [0,1]$ is called simply normal to the base r if in the r-adic expansion of x, each r-adic digit occurs with the asymptotic frequency $1/r$. A number $x \in [0,1]$ is called normal to algebraic base $r > 1$, if the sequence $r^n x$, $n \in \mathbb{N}$, is uniformly distributed modulo 1. In the case where r is an integer, there is an equivalent definition in terms of the frequencies of r-adic digits of x, see [11]. We say that two numbers $r, s > 1$ are multiplicatively independent ($r \not\sim s$), if $\log s / \log r$ is irrational. In [13] Schmidt proved that normality to integer base $r > 1$ implies normality to integer base $s > 1$ if r, s are multiplicatively dependent integers, that is, if r and s are coprime powers of the same rational integer. He also proved that if $r > 1$ and $s > 1$ are multiplicatively independent integers, then there exist uncountably many real numbers which are normal to base s but not even simply normal to base r, (see also Cassels [7]). It is easy to see that simple normality to base r^n implies simple normality to base r, see also [9].

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Let N_r be the set of those $x \in [0,1]$ that are normal to base s for any algebraic number $s > 1$ with $r \not\sim s$. It is clear that $N_r = N_{r^n}$, $n \in \mathbb{N}$. Let S_r be the set of real numbers which are simply normal to base r and S'_r be its complement to the set of real numbers. We denote by I_n the set of natural numbers which are ≥ 2, smaller than n and does not divide n.

Let $a, b \geq 2$ be two integers bases. The following theorem is proved in [9]:

Theorem A. *If there is no $n \in \mathbb{N}$ such that $a = b^n$, then the cardinality of the set of real numbers that are simply normal to base a but not to base b (that is of the set $S_a \cap S'_b$) is equal to the cardinality of continuum.*

In this paper we extend Theorem A and simultaneously prove another proof of the above result. We prove the following:

Theorem 1. *Let $a \geq 2$ be an integer. Let n be the greatest integer such that a is a n-th power of an integer. The Hausdorff dimension of the set of numbers which are simply normal to base a, not simply normal to any base $a^{m/n}$ with $m > n$, not simply normal to any base $a^{m/n}$ with $m \in I_n$ and normal to each algebraic base multiplicatively independent to base a, is equal to 1.*

Note. I would like to thank Prof. Y. Bugeaud for all his fruitful comments which improved the presentation of the paper and for pointing me that in his recently published paper [6] he independently proved that the Hausdorff dimension of the set of Theorem A is equal to 1.

Using the previous notation we have that our Theorem gives that the Hausdorff dimension of the set $S_a \cap (\cap_{m>n} S'_{a^{m/n}}) \cap (\cap_{m \in I_n} S'_{a^{m/n}}) \cap N_a$ is equal to 1. For the proof of Theorem 1 we construct Markov measures as in [2], [3] and [4]. We describe these measures. Let $P^{(0)} = (p_i^{(0)})$ with $p_i^{(0)} = \frac{1}{r}$, $i = 0, 1, \ldots, r-1$, be the vector of length r. We suppose that we have a Markov chain with state space $\{0, 1, \ldots, r-1\}$ given by the sequence of $r \times r$ transition matrices

$$P^{(n)} = \left(p_{ij}^{(n)}\right), \quad i, j \in \{0, 1, \ldots, r-1\}, \quad n = 1, 2, \ldots.$$

Let $E_{n,k}(x) = \left[\frac{k}{r^n}, \ \frac{k+1}{r^n}\right)$, $n = 1, 2, \ldots,\ k = 0, 1, \ldots,\ r^n - 1$, be the interval in the n-th generation of the r-adic partition of $[0,1]$ which contains $x \in [0,1]$. We write the r-adic expansion of x by

$$x = \sum_{j=1}^{\infty} \frac{\varepsilon_j}{r^j}, \quad \varepsilon_j = \varepsilon_j(x) \in \{0, 1, \ldots, r-1\}.$$

We can define a probability measure μ on $[0,1]$, by its values on $E_{n,k}(x)$:

$$\mu\big(E_{n,k}(x)\big) = p^{(0)}_{\varepsilon_1} \prod_{j=1}^{n-1} p^{(j)}_{\varepsilon_j,\varepsilon_{j+1}}. \tag{1}$$

Some results concerning the Hausdorff dimension of the above measures are given in [3] and for the study of their Fourier transforms in [4] as well. In [3] we proved that the above measure is concentrated on a set with Hausdorff dimension equal to

$$\liminf_{n\to\infty} \frac{\log \mu(E_{n,k}(x))}{-n\log r} = \liminf_{N\to\infty} \frac{-1}{N\log r} \sum_{n=1}^{N} \sum_{i,j=0}^{r-1} C(i,n-1)\, p^{(n)}_{ij} \log p^{(n)}_{ij}, \ \ \mu\text{- a.e.} \tag{2}$$

where

$$C(i,n-1) = P^{(0,n-1)} L_i, \ \ P^{(0,n-1)} = \prod_{\nu=0}^{n-1} P^{(\nu)} \quad \text{and} \quad L_i = (0,\dots,1,\dots,0)^T,$$

(1 is in the position i), $i = 0,\dots,r-1$. Notice that we assume that $0^0 = 1$.

We also use a law of large numbers based on a Davenport-Erdös and LeVeque Theorem [5] Lemma 1.8, (see also [3] p. 27, [8]) and we refer this for the convenience of the reader.

Proposition 1. *Let μ be a measure on $T = [0,1]$. Let f_j, $j \in \mathbb{N}$, be a bounded sequence of measurable functions defined on $[0,1]$. If the series*

$$\sum_{N=1}^{\infty} \frac{1}{N} \int_T \left| \frac{1}{N} \sum_{j=1}^{N} \left(f_j - \int_T f_j \right) \right|^2 d\mu$$

converges, then $\mu-$almost everywhere

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \left(f_j - \int_T f_j \right) = 0.$$

In order to apply the above Proposition we use the delta index $\delta(P^{(n)})$ of the matrix $P^{(n)}$,

$$\delta\left(P^{(n)}\right) = \frac{1}{2} \max_{i,k} \sum_{j=0}^{r-1} \left| p^{(n)}_{ij} - p^{(n)}_{kj} \right|.$$

2. Proof of Theorem 1

Let $r \geq 2$, $m \geq 2$ and n be integers such that $a = r^n$, $m > n$ or $m \in I_n$.

Proof. We consider the numbers $p_i \in (0,1)$, $i = 0,1,\dots,r-1$, such that $p_0 < \frac{1}{r}$ and $\sum_{i=0}^{r-1} p_i = 1$. We require the probabilities to be positive in order to apply directly the results about the normality from [2], otherwise if we would like to have a larger class of measures we can use [[2], p. 32]. We define the transition matrices $P = (p_{ij})$, $i,j = 0,1,\dots,r-1$, with $p_{ij} = p_{(i+j)\text{mod r}}$ and $Q = (q_{ij}), i,j = 0,1,\dots,r-1$, with $q_{ij} = \frac{1}{r}$. We note that the symmetry of P is crucial in order to check the normality from Proposition 1. Let $P^{(\nu)} = (p_{ij}^{(\nu)}),\ i,j \in \{0,1,\dots,r-1\}$, $\nu = 1,2,\dots$, be the sequence of symmetric transition matrices such that

$$P^{(\nu)} = \begin{cases} Q, & \text{for } \nu \neq nk, \\ P, & \text{for } \nu = nk, \end{cases} \tag{3}$$

where $k = 1,2,\dots$. We consider a measure as in (1). Let

$$x = \sum_{j=1}^{\infty} \frac{\varepsilon_j^{(\kappa)}(x)}{(r^\kappa)^j},$$

where $\varepsilon_j^{(\kappa)}(x) \in \{0,\dots,r^\kappa - 1\}$, be the r^κ-adic expansion of $x \in [0,1]$. Given $k \in \{0,\dots,r^\kappa - 1\}$, we can write $k = \sum\limits_{i=1}^{\kappa} c_i r^{i-1}$, $c_i \in \{0,1,\dots,r-1\}$. Let ${}_{\kappa,k,j} = \{x \in [0,1] : \varepsilon_j^{(\kappa)}(x) = k\}$, and $X_{E_{\kappa,k,j}}$ be the characteristic function on the set $E_{\kappa,k,j}$. By the construction of the matrices $P^{(\nu)}$, (see (3)) we have

$$\mu(E_{\kappa,k,j}) = \left(\frac{1}{r},\dots,\frac{1}{r}\right)\left[\prod_{\nu=1}^{\kappa(j-1)} P^{(\nu)}\right] L_{c_\kappa}\left[\prod_{i=1}^{\kappa-1} p_{c_{\kappa-i+1}c_{\kappa-i}}^{(j-1)\kappa+i}\right], \quad \kappa > 1 \tag{4}$$

and for $\kappa = 1$

$$\mu(E_{1,k,j}) = \left(\frac{1}{r},\dots,\frac{1}{r}\right)\left[\prod_{\nu=1}^{j-1} P^{(\nu)}\right] L_{c_1} = \frac{1}{r}. \tag{5}$$

I. Simply normal to base r^n. By (4) for $\kappa = n > 1$ and by (3) we have

$$\mu(E_{n,k,j}) = \left(\frac{1}{r},\dots,\frac{1}{r}\right)\left[\prod_{\nu=1}^{n(j-1)} P^{(\nu)}\right] L_{c_n}\left[\prod_{i=1}^{n-1} p_{c_{n-i+1}c_{n-i}}^{(j-1)n+i}\right] = \frac{1}{r^n}$$

because in the set $\{(j-1)n+1, (j-1)n+2, \ldots, (j-1)n+(n-1)\}$ there is no multiple of n. In order to apply Proposition 1 with $f_j = X_{E_{n,k,j}}$ we observe that we have to estimate the differences

$$\int (X_{E_{n,k,j}} X_{E_{n,k,l}})\, d\mu - \int X_{E_{n,k,j}}\, d\mu \int X_{E_{n,k,l}}\, d\mu, \quad j < l.$$

We have

$$\begin{aligned}
\mu(E_{n,k,j} \cap E_{n,k,l}) &= \left(\frac{1}{r}, \ldots, \frac{1}{r}\right) \left[\prod_{\nu=1}^{n(j-1)} P^{(\nu)}\right] L_{c_n} \left[\prod_{i=1}^{n-1} p^{(j-1)n+i}_{c_{n-i+1}c_{n-i}}\right] \\
&\quad . \, L^T_{c_1} \left[\prod_{\nu=nj}^{n(l-1)} P^{(\nu)}\right] L_{c_n} \left[\prod_{i=1}^{n-1} p^{(l-1)n+i}_{c_{n-i+1}c_{n-i}}\right] \\
&= \frac{1}{r^n}. \, L^T_{c_1} \left[\prod_{\nu=nj}^{n(l-1)} P^{(\nu)}\right] L_{c_n} \frac{1}{r^{n-1}} \\
&= \frac{1}{r^{2n-1}}. \, L^T_{c_1} \left[\prod_{\nu=nj}^{n(l-1)} P^{(\nu)}\right] L_{c_n}
\end{aligned}$$

and so

$$\mu(E_{n,k,j} \cap E_{n,k,l}) - \mu(E_{n,k,j})\mu(E_{n,k,l}) = \frac{1}{r^{2n-1}} \left(L^T_{c_1} \left[\prod_{\nu=nj}^{n(l-1)} P^{(\nu)}\right] L_{c_n} - \frac{1}{r} \right).$$

If $j < l-1$, then at least one matrix in the product $\prod\limits_{\nu=nj}^{n(l-1)} P^{(\nu)}$ is equal to Q, $(n > 1)$ and so the product is equal to Q. Therefore the above difference is equal to 0 and by Proposition 1 (we note that the terms with $j = l-1$ do not affect the convergence) we take

$$\lim_{\to +\infty} \frac{1}{N} \sum_{j=1} X_{n,k,j}(x) = \frac{1}{r^n}, \quad \mu - a.e.,$$

so μ - almost all the numbers in $[0,1]$ are simply normal to base r^n, $\mu(S_{r^n}) = 1$. In the case where $\kappa = n = 1$ we use (5). For $j < l$ we have

$$\begin{aligned}
\mu(E_{1,k,j} \cap E_{1,k,l}) &= \left(\frac{1}{r}, \ldots, \frac{1}{r}\right) \left[\prod_{\nu=1}^{j-1} P^{(\nu)}\right] L_{c_1} \, L^T_{c_1} \left[\prod_{\nu=j}^{l-1} P^{(\nu)}\right] L_{c_1} \\
&= \frac{1}{r}. \, L^T_{c_1} \left[\prod_{\nu=j}^{l-1} P^{(\nu)}\right] L_{c_1}
\end{aligned}$$

and so

$$\begin{aligned}\mu(E_{1,k,j}\cap E_{1,k,l})-\mu(E_{1,k,j})\mu(E_{1,k,l}) &= \frac{1}{r}\left(L_{c_1}^T\left[\prod_{\nu=j}^{l-1}P^{(\nu)}\right]L_{c_1}-\frac{1}{r}\right)\\ &= \frac{1}{r}\left(L_{c_1}^T P^{l-1-j}L_{c_1}-\frac{1}{r}\right).\end{aligned}$$

It is well known that if the transition matrix P is symmetric it is diagonalizable with one eigenvalue the number 1 and the others with absolutely value less than or equal to 1. From this observation we have that the matrix P^{l-1-j} tends to a symmetric transition matrix as the difference $l-j$ tends to infinity with geometric order. By the construction of P we have that $\delta(P)<1$ and using the property $\delta(P^{l-1-j})\le(\delta(P))^{l-1-j}$, see [10], we conclude that the matrix $\lim\limits_{l-j\to\infty}P^{l-1-j}$ is stationary (delta index is equal to 0). Therefore $\lim\limits_{l-j\to\infty}P^{l-1-j}=Q$ with geometric order, since the limit is a stationary symmetric transition matrix. So the difference

$$\int(X_{E_{\kappa,k,j}}X_{E_{\kappa,k,l}})\,d\mu-\int X_{E_{\kappa,k,j}}\,d\mu\int X_{E_{\kappa,k,l}}\,d\mu,\quad j\neq l$$

tends to 0 geometrically as $j-l\to\infty$ and so the requirements of Proposition 1 are satisfied and the result is complete.

II. No simply normality to base r^m.

(i) Suppose that $n<m$. We apply (4) for $\kappa=m$, and we take for simplicity $k=0$. Then by (3) we have

$$\mu(E_{m,0,j})=\frac{1}{r}\left[\prod_{i=1}^{m-1}p_{00}^{(j-1)m+i}\right]=\frac{1}{r^{m-c(j)}}p_0^{c(j)}=\frac{1}{r^m}(r\,p_0)^{c(j)},$$

where $c(j)$ is the number of multiples of n in the set of numbers $\{(j-1)m+i,$ $i=1,\dots,m-1\}$. Since $n<m$, we have $c(j)\ge 1$ and $c(j)$ depends on n and m. We have

$$\begin{aligned}\mu(E_{m,0,j}\cap E_{m,0,l}) &= (\frac{1}{r},\dots,\frac{1}{r})\left[\prod_{\nu=1}^{m(j-1)}P^{(\nu)}\right]L_0\left[\prod_{i=1}^{m-1}p_{00}^{(j-1)m+i}\right]\\ &\qquad .\ L_0^T\left[\prod_{\nu=mj}^{m(l-1)}P^{(\nu)}\right]L_0\left[\prod_{i=1}^{m-1}p_{00}^{(l-1)m+i}\right]\end{aligned}$$

$$= \frac{1}{r^m}(r\,p_0)^{c(j)} L_0^T \left[\prod_{\nu=mj}^{m(l-1)} P^{(\nu)}\right] L_0 \frac{1}{r^{m-1}}(r\,p_0)^{c(j)}$$

$$= \left(\frac{1}{r^m}(r\,p_0)^{c(j)}\right)^2 r\,L_0^T \left[\prod_{\nu=mj}^{m(l-1)} P^{(\nu)}\right] L_0.$$

Since $m > n$, for $l-1 > j$ there is at least one matrix equal to Q in the product $\prod_{\nu=mj}^{m(l-1)} P^{(\nu)}$ and so $\prod_{\nu=mj}^{m(l-1)} P^{(\nu)} = Q$. Therefore $\mu(E_{m,0,j} \cap E_{m,0,l}) = \mu(E_{m,0,j})\mu(E_{m,0,l})$, $l-1 > j$ and by using Proposition 1 we have

$$\lim_{\to+\infty} \frac{1}{N}\left(\sum_{j=1} X_{m,0,j}(x) - \frac{1}{r^m}(r\,p_0)^{c(j)}\right) = 0, \quad \mu - a.e..$$

Since $c(j) \geq 1$, by the assumption $p_0 < \frac{1}{r}$, we take

$$\mu\Big(\Big\{x\,:\, \lim_{N\to+\infty} \frac{1}{N}\sum_{j=1}^{N} X_{m,0,j}(x) = \frac{1}{r^m}\Big\}\Big) = 0,$$

i.e. almost all the numbers are not simply normal to the base r^m, so $\mu(S'_{r^m}) = 1$.

(ii) Suppose that $m \in I_n$. For $\kappa = m > 1$ and for $k = 0$, we take

$$\mu(E_{m,0,j}) = \frac{1}{r^{m-c(j)}} p_0^{c(j)}$$

as in the previous case. We proceed as before and take

$$\mu(E_{m,0,j} \cap E_{m,0,l}) = \left(\frac{1}{r^m}(r\,p_0)^{c(j)}\right)^2 r\,L_0^T \left[\prod_{\nu=mj}^{m(l-1)} P^{(\nu)}\right] L_0,$$

$\prod_{\nu=mj}^{m(l-1)} P^{(\nu)} = Q,\ n > m,\ l-1 > j$. Thus $\mu(E_{m,0,j} \cap E_{m,0,l}) = \mu(E_{m,0,j})\mu(E_{m,0,l})$ and so by Proposition 1 we have

$$\lim_{\to+\infty} \frac{1}{N}\sum_{j=1}\big(X_{m,0,j}(x) - \mu(E_{m,0,j})\big) = 0, \quad \mu - a.e.$$

Since $n > m$ and m does not divide n, we observe that for some j, it is possible to have $\mu(E_{m,0,j}) = \frac{1}{r^m}$, (i.e., $c(j) = 0$), take for example $j = 1$. For some other j, (for example we take j such that $n \in \{(j-1)m+1, \ldots, (j-1)m+m-1\}$), we have

$$\mu(E_{m,0,j}) = \frac{p_0}{r^{m-1}} < \frac{1}{r^m}, \text{ since } p_0 < \frac{1}{r}.$$

We observe that in the set $\{\mu(E_{m,0,j}),\ j = 1, 2, \ldots, N\}$, we have $[\frac{m-(n,m)}{n}N]$ terms with $\mu(E_{m,0,j}) = \frac{p_0}{r^{m-1}}$ and for the remaining terms $\mu(E_{m,0,j}) = \frac{1}{r^m}$. So by the above relation we take

$$\mu\Big(\Big\{x : \lim_{N\to+\infty} \frac{1}{N}\sum_{j=1}^{N} X_{m,0,j}(x) = \frac{1}{r^m}\Big\}\Big) = 0,$$

proving that μ - almost all the numbers are not simply normal to the base r^m. Therefore $\mu(S'_{r^m}) = 1$ for $m \in I_n$.

By [2] these Markov measures satisfy $\mu(N_r) = \mu(N_{r^n}) = 1$. So

$$\mu\big(S_a \cap (\cap_{m>n} S'_{a^{m/n}}) \cap (\cap_{m\in I_n} S'_{a^{m/n}}) \cap N_a\big) = 1.$$

Finally we calculate the Hausdorff dimension of the above intersection of sets. By (2) and (3) we obtain

$$\begin{aligned}
\liminf_{N\to\infty} \frac{\log \mu(E_{N,k}(x))}{-N\log r} &= \liminf_{N\to\infty} \frac{-1}{N\log r} \sum_{\nu=1}^{N} \frac{1}{r} \sum_{i,j=0}^{r-1} p_{ij}^{(\nu)} \log p_{ij}^{(\nu)}, \ \mu\text{- a.e.} \\
&= -\frac{1}{n\log r} \sum_{i=0}^{r-1} p_j \log p_j - \frac{n-1}{n\log r} \log\frac{1}{r}, \ \mu\text{- a.e.} \\
&= -\frac{1}{n\log r} \sum_{i=0}^{r-1} p_j \log p_j + \frac{n-1}{n}, \ \mu\text{- a.e.}
\end{aligned}$$

From this and a well known result, (see [1], Theorem 14.1) the Hausdorff dimension of our set is at least equal to

$$1 - \frac{1}{n}\Big(1 + \frac{1}{\log r}\sum_{i=0}^{r-1} p_j \log p_j\Big).$$

The probabilities $p_0, \ldots, p_{r-1}$ satisfy the hypothesis $p_0 < \frac{1}{r}$. This means that if the numbers p_j tend to $\frac{1}{r}$ then the above number tends to 1 as we want. □

As a consequence we give another proof of Theorem A.

P r o o f o f T h e o r e m A. In case where a, b are multiplicatively independent the proof is given in [9] and emerges from [13]. In the other and more interesting case where a, b are multiplicatively dependent, we use Theorem 1 and the fact that we can write $a = r^n$, $b = r^m$, with $m > n$, or with $m \in I_n$. We observe that our set is of positive Hausdorff dimension and so it is uncountable. □

Remark. In the case where a and b are multiplicatively independent then the Hausdorff dimension of the set of numbers which are normal to base a and not to base b is equal to 1. This is a partial result of a more general one which is given in [12].

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Antonis Bisbas
Technological Education Institute
of West Macedonia
School of Technological Applications
General Sciences Department
Kila 50100, Kozani
GREECE
http://gen.teikoz.gr/$^{\sim}$bisbas/
E-mail: bisbas@kozani.teikoz.gr