Uniform Distribution Theory **8** (2013), no.2, 39–47

uniform distribution theory

DISTRIBUTION OF ZEROS OF L-FUNCTIONS ON THE CRITICAL LINE

Kamel Mazhouda — Sami Omar

ABSTRACT. In this paper, we prove under the Riemann hypothesis that the zeros of L-functions from the Selberg class are uniformly distributed on the critical line when the conductor is sufficiently large and the degree of these L-functions grows with the analytic conductor.

Communicated by Christian Mauduit

1. Introduction

There is an intimate connection between the distribution of the nontrivial zeros of an L-function and the distribution of prime numbers. Many number theorists are interested in the horizontal distribution of zeros. However, there is also much interest in studying the distribution of the imaginary parts of the zeros (the vertical distribution). This phenomenon has been investigated by Lang [2], Tsfasman and Vlădut [7] and Zykin [8] in the case of the zeta functions of number fields and function fields. The extended Selberg class $\mathcal{S}^{\#}$ consists of Dirichlet series

$$F(s)=\sum_{n=1}^{+\infty}\frac{a(n)}{n^s},\qquad Re(s)>1$$

satisfying the following hypothesis.

- **Analytic continuation:** there exists a non negative integer m such that $(s-1)^m F(s)$ is an entire function of finite order. We denote by m_F the smallest integer m which satisfies this condition.

2010 Mathematics Subject Classification: 11M06,11M26, 11M45, 30D35.
Keywords: L-functions, Selberg class, Weil explicit formula, Riemann hypothesis, Distribution of zeros.

- **Functional equation:** for $1 \leq j \leq r$, there are positive real numbers Q_F (the conductor), λ_j and there are complex numbers μ_j, ω with $Re(\mu_j) \geq 0$ and $|\omega| = 1$, such that

$$\phi_F(s) = \omega\overline{\phi_F(1-\overline{s})},$$

where

$$\phi_F(s) = F(s)Q_F^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j).$$

The Selberg class $\mathcal{S}$ [6] is the set of functions $F \in \mathcal{S}^{\#}$ satisfying the two more following axioms:

- **Ramanujan hypothesis:** $a(n) = O(n^{\epsilon})$;
- **Euler product:** $F(s)$ satisfies

$$F(s) = \prod_p \exp\left(\sum_{k=1}^{+\infty} \frac{b(p^k)}{p^{ks}}\right)$$

with suitable coefficients $b(p^k)$ satisfying $b(p^k) = O(p^{k\theta})$ for some $\theta < \frac{1}{2}$.

It is expected that for every function in the Selberg class the analogue of the Riemann hypothesis holds, i.e, that all non trivial (non-real) zeros lie on the critical line $Re(s) = \frac{1}{2}$. The degree of $F \in \mathcal{S}$ is defined by

$$d_F = 2\sum_{j=1}^{r} \lambda_j.$$

The logarithmic derivative of $F(s)$ has also the Dirichlet series expression

$$-\frac{F'}{F}(s) = \sum_{n=1}^{+\infty} \Lambda_F(n) n^{-s}, \qquad Re(s) > 1,$$

where $\Lambda_F(n) = b(n)\log n$ is the generalized von Mangoldt function. Let

$$q_F = \frac{(2\pi)^{d_F} Q_F^2}{\beta}, \quad \text{where} \quad \beta = \prod_{j=1}^{r} \lambda_j^{-2\lambda_j},$$

be the analytic conductor of $F \in \mathcal{S}$. If $N_F(T)$ counts the number of zeros of $F(s) \in \mathcal{S}$ in the rectangle $0 \leq Re(s) \leq 1$, $0 < Im(s) \leq T$ (according to multiplicities) one can show by standard contour integration the formula

$$N_F(T) = \frac{d_F}{2\pi} T \log T + c_1 T + O\left(\log T\right), \tag{1}$$

where

$$c_1 = \frac{1}{2\pi}\Big(\log q_F - d_F\big(\log(2\pi)+1\big)\Big)$$

in analogy to the Riemann-von Mangoldt formula for Riemann's zeta-function $\zeta(s)$, the prototype of an element in $\mathcal{S}$.

We will call a sequence $\{F_k(s)\}_{k=1,2,\dots\infty}$ of L-functions from the Selberg class a family if $d_k = d_{F_k}$ tends to infinity. A family $\{F_k(s)\}_{k=1,2,\dots,\infty}$ of L-functions from the Selberg class is called asymptotically exact if the limit

$$\lim_{k\to+\infty} \frac{\log q_{F_k}}{\log q_{F_k} + \log d_k} = \eta$$

exists. It is called asymptotically bad if $\eta = 0$ and asymptotically good otherwise.

In this paper, we prove under the Riemann hypothesis that the zeros of L-functions in the Selberg class are uniformly distributed on the critical line $Re(s) = 1/2$ when the conductor is sufficiently large and the degree of these L-functions grows with the analytic conductor. Our main tool is the Weil explicit formula in conjunction with the theory of distributions.

To do so, let $\{F_k\}_{k=1,2,\dots,\infty}$ be an asymptotically exact family of functions in the Selberg class of degree d_k. For each F_k we define the measure

$$\mathcal{H}_{F_k} = \frac{2\pi}{\log q_{F_k}} \sum_{F_k(\rho)=0} \delta_{t(\rho)}, \tag{2}$$

where $t(\rho) = \frac{1}{i}\left(\rho - \frac{1}{2}\right)$, and ρ runs over all non trivial zeros of the functions F_n. Here δ_a denotes the atomic (Dirac) measure at a. Assuming the Riemann hypothesis, $t(\rho)$ is real and $\mathcal{H}_{F_k}$ is a discrete measure on $\mathbb{R}$.

2. The Weil explicit formula

The Weil explicit formula will be used in the proof of our main result (Theorem 1) below.

Proposition 1. [3, 4] *Let $f : \mathbb{R} \to \mathbb{C}$ satisfy the following conditions:*

- *f is normalized,*

$$2f(x) = f(x+0) + f(x-0), \quad x \in \mathbb{R}.$$

- *There is a number* $b > 0$ *such that*
$$V_{\mathbb{R}}\left(f(x)e^{\left(\frac{1}{2}+b\right)|x|}\right) < \infty,$$
where $V_{\mathbb{R}}(.)$ *denotes the total variation on* $\mathbb{R}$.
- *For all* $1 \le j \le r$*, the function* $G_j(x) = f(x)e^{-ix\frac{Im(\mu_j)}{\lambda_j}}$ *satisfies*
$$G_j(x) + G_j(-x) = 2f(0) + O(|x|).$$

Let $F(s) \in \mathcal{S}$*. Then,*

$$\begin{aligned}\sum_{\rho} H(\rho) &= m_F\big(H(0) + H(1)\big) + 2f(0)\log Q_F \\ &+ \sum_{j=1}^{r}\int_{0}^{+\infty}\left\{\frac{2\lambda_j G_j(0)}{x} - \frac{e^{\left[\left(1-\frac{\lambda_j}{2}-Re(\mu_j)\right)\frac{x}{\lambda_j}\right]}}{1-e^{-\frac{x}{\lambda_j}}}\big(G_j(x)+G_j(-x)\big)\right\}e^{-\frac{x}{\lambda_j}}dx \\ &- \sum_{n=1}^{\infty}\left[\frac{\Lambda_F(n)}{\sqrt{n}}f(\log n) + \frac{\overline{\Lambda_F(n)}}{\sqrt{n}}f(-\log n)\right], \end{aligned} \tag{3}$$

where

$$H(s) = \int_{-\infty}^{+\infty} f(x)e^{(s-1/2)x}dx \quad et \quad \sum_{\rho} H(\rho) = \lim_{T\to\infty}\sum_{|Im(\rho)|<T} H(\rho).$$

For more details about the proof of the above proposition, see for example the paper [1] of Barner. Using [1, Proposition page.146] the integral in the sum in the third term in the right-hand side of (3) can be written as follows

$$\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\widehat{G_j}(x)+\widehat{G_j}(-x)}{2}\psi\left(\frac{\lambda_j}{2} + Re(\mu_j) + i\lambda_j x\right)dx,$$

where

$$\widehat{G_j}(x) = \int_{-\infty}^{+\infty} G_j(t)e^{itx}dt$$

is the Fourier transform of G_j and $\psi(s) = \frac{\Gamma'}{\Gamma}(s)$.

3. Some known results from the theory of distributions

Let us recall several standard notations and facts from the theory of distributions [5]. Let $S = S(\mathbb{R})$ be the space of complex valued infinitely differentiable functions on $\mathbb{R}$ which are rapidly decreasing together with all their derivatives.

This vector space is naturally equipped with a standard topology, so that the Fourier transform is a topological automorphism of S. Its topological dual S' is called the space of tempered distributions. By duality, the Fourier transform is also defined on S' and it is also a topological automorphism there. The space S' is contained in the space D' of all distributions, which is the topological dual of the space D of complex valued infinitely differentiable functions with compact support on $\mathbb{R}$. The space of measures $\mathcal{M}$ is the topological dual of the space of complex valued continuous functions with compact support on $\mathbb{R}$. Of course, $\mathcal{M} \subset D'$. The space of measures $\mathcal{M}$ contains the cone of positive measures $\mathcal{M}_+$ i.e., of those measures whose value at a positive real-valued function is positive. The space of distributions D' also contains the cone of positive distributions D'_+. It is known that $D'_+ = \mathcal{M}_+$. The intersection $\mathcal{M}_{sl} = \mathcal{M} \cap S'$ is called the space of measures of slow growth. The criterion for a measure μ to be of slow growth is that for some positive integer k the integral

$$I_k = \int_{-\infty}^{+\infty} \frac{d\mu}{(x^2+1)^k}$$

converges. Therefore, using the fact that the series $\sum_{\rho \neq 0,1} \frac{1}{\rho^2}$ converges [1], we deduce that the measure $\mathcal{H}_{F_k}$ defined above (equation (2)) is a measure of slow growth.

Note that the Fourier transform is defined on S and S' and it is a topological automorphism on these spaces. Because D is dense in S then $\widehat{D}$ is also dense in $\widehat{S}$. To prove that $\mathcal{H}_{F_k}$ is a measure of slow growth it suffices to prove that it is defined on a dense subset and it is continuous on this dense subset in the topology of S. In other words, to prove that a sequence of measures of slow growth converges to a measure of slow growth it suffices to prove its convergence on a dense subset to a continuous measure on this dense subset.

4. Main result and proof

Now we are ready to formulate the main result of this paper, expressing the limit distribution of zeros of L-functions in the Selberg class.

Theorem 1. *Let $\{F_k\}_{k=1,2,\ldots,\infty}$ be a family of functions in the Selberg class of degree d_k such that $q_{F_k} \to \infty$. Under the Riemann hypothesis and in the space of measures of slow growth on $\mathbb{R}$, the limit*

$$\mathcal{H} = \lim_{k\to\infty} \mathcal{H}_k = \lim_{d_k\to\infty} \mathcal{H}_{F_k}$$

exists.

P r o o f. To prove Theorem 1, we apply the formula (3) only to functions from $S(\mathbb{R})$ which clearly satisfy the other conditions of Proposition 1.

Let F_k be a function in $\mathcal{S}$ of degree d_k. Take any $h \in \widehat{D}$ and let $h = \widehat{f}$, $f \in D$. We have $h(t) = H(\frac{1}{2} + it)$. The function f satisfies the above conditions of Proposition 1, and, dividing by $\log q_{F_k}$, we get

$$\begin{aligned}\mathcal{H}_{F_k}(h) &= \frac{2\pi}{\log q_{F_k}} \sum_{F_k(1/2+it)=0} h(t) \\ &= 2\pi \left[\frac{m_{F_k}(H(0)+H(1))}{\log q_{F_k}}\right] + 2\pi f(0)\frac{\log Q^2_{F_k}}{\log q_{F_k}} \\ &\quad - \frac{2\pi}{\log q_{F_k}} \sum_{n=1}^{\infty} \left[\frac{\Lambda_{F_k}(n)}{\sqrt{n}} f(\log n) + \frac{\overline{\Lambda_{F_k}(n)}}{\sqrt{n}} f(-\log n)\right] \\ &\quad + \sum_{j=1}^{r} \frac{2}{\log q_{F_k}} \int_{-\infty}^{+\infty} \frac{\widehat{G_j}(t) + \widehat{G_j}(-t)}{2} \psi\left(\frac{\lambda_j}{2} + Re(\mu_j) + i\lambda_j t\right) dt,\end{aligned}$$

where

$$G_j(t) = f(t)e^{-it\frac{Im(\mu_j)}{\lambda_j}} \quad \text{and} \quad \widehat{G}_j(t) = h\left(t - \frac{Im(\mu_j)}{\lambda_j}\right).$$

Taking a subsequence $\{F_k\}$, we can assume that

$$\lim_{k\to+\infty} \frac{\log q_{F_k}}{\log q_{F_k} + \log d_k} = \eta$$

exists. Let fix h and tend d to infinity. If $F \in \mathcal{S}$ and $\epsilon > 0$ is fixed, then $|a_n| \le c(\epsilon)n^{\epsilon}$ implies that $|b(p^v| \le c(\epsilon)\,(2^v - 1)\, p^{v\epsilon}/v$. Using the last inequality and that the integral converges uniformly as $h(t) \in S$, we obtain

$$\begin{aligned}\mathcal{H}(h) &= 2\pi f(0)\eta \\ &\quad + 2\sum_{j=1}^{r} \int_{-\infty}^{+\infty} \frac{h\left(t - \frac{Im(\mu_j)}{\lambda_j}\right) + h\left(-t + \frac{Im(\mu_j)}{\lambda_j}\right)}{2} \lim_{k\to+\infty} \frac{\psi\left(\frac{\lambda_j}{2} + Re(\mu_j) + i\lambda_j t\right)}{\log q_{F_k} + \log d_k} dt \\ &= 2\pi f(0)\eta + 2\sum_{j=1}^{r} \int_{-\infty}^{+\infty} \frac{h(t)+h(-t)}{2} \lim_{k\to+\infty} \frac{\psi\left(\frac{\lambda_j}{2} + \mu_j + i\lambda_j t\right)}{\log q_{F_k} + \log d_k} dt \\ &= 2\pi f(0)\eta + 2\int_{-\infty}^{+\infty} \frac{h(t)+h(-t)}{2} \lim_{k\to+\infty} \sum_{j=1}^{r} \left[\frac{\psi\left(\frac{\lambda_j}{2} + \mu_j + i\lambda_j t\right)}{\log q_{F_k} + \log d_k}\right] dt.\end{aligned}$$

Using the Stirling formula, we have

$$\psi(z) = \log z + O\left(\frac{1}{|z|}\right).$$

Then

$$\lim_{k\to+\infty}\sum_{j=1}^{r}\left[\frac{\psi\left(\frac{\lambda_j}{2}+\mu_j+i\lambda_j t\right)}{\log q_{F_k}+\log d_k}\right]=\lim_{k\to+\infty}\frac{1}{2}\left(\frac{\log d_k}{\log q_{F_k}+\log d_k}\right)=\frac{1}{2}(1-\eta).$$

Recall that $2\pi H(0)=\int_{-\infty}^{+\infty}\psi(t)dt$. Therefore, we deduce

$$\mathcal{H}=2\pi H(0)\eta+2\pi H(0)(1-\eta)=2\pi H(0)=\int_{-\infty}^{+\infty}\psi(t)dt$$

and Theorem 1 follows. □

Remarks.

- Theorem 1 implies that the zeros of an L-functions in the Selberg class are asymptotically uniformly distributed for any asymptotically bad family.
- Theorem 1 implies that any fixed interval around $Re(s)=1/2$ contains zeros of F_n when q_{F_n} is sufficiently large.

Examples.

- Let $F=\{K_i\}_{i=1,2,\ldots,\infty}$ be an asymptotically exact family of number fields. For each K_i, we define the measure

$$\mathcal{H}=\frac{\pi}{g_i}\sum_{\zeta_{K_i}(\rho)=0}\delta_{t(\rho)},$$

  where $g_i=g_{K_i},\ t(\rho)=(\rho-1/2)/i$, and ρ runs over all non-trivial zeros of the zeta-function $\zeta_{K_i}(s)$. By Theorem 1, we have

$$\mathcal{H}=\lim_{i\to+\infty}\mathcal{H}_i=\lim_{i\to+\infty}\mathcal{H}_{K_i}$$

  exists. The last result was proved also by Tsfasman & Vlǎdut [7, Theorem 5.2 and §5].
- Let

$$\xi_N(s)=N^{gs/2}(2\pi)^{-gs}\Gamma^g\left(\frac{k-1}{2}+s\right)L_N\left(\frac{k-1}{2}+s\right)$$

  where the positive integer weight $k>2$, g is the dimension of the space $S_k(N,\chi)$, Γ is the Gamma function, the Hecke L-function L_N has an Euler product that we omit and χ is a Dirichlet character of modulus N with $\chi(-1)=(-1)^k$. Here, $S_k(N,\chi)$ is the space of all cusp forms of weight k and character χ for the Hecke congruence subgroup $\Gamma_0(N)$ of level N.

In particular, $f \in S_k(N,\chi)$ is holomorphic in the upper half plane and transforms as

$$f[(az+b)/(cz+d)] = \chi(d)(cz+d)^k f(z)$$

for all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, where

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) | c = 0 \mathrm{mod} N \right\}.$$

To each newform $g(z)$ we associate the mesure

$$\mathcal{H}_g = \frac{2\pi}{\log q_g} \sum_{L_g(\rho)=0} \delta_{t(\rho)},$$

where $t(\rho) = (\rho - 1/2)/i$, and ρ runs over all non-trivial zeros of the zeta-function

$$L_g(s) \quad \text{and} \quad q_g = N\left(\frac{k-1}{2}+3\right)\left(\frac{k+1}{2}+3\right).$$

Using Theorem 1 above and assuming the Riemann hypothesis, we deduce that for any family $\{g_i(z)\}_{i=1,2,\ldots,\infty}$ of primitive forms with $q_{g_i} \to +\infty$,

$$\mathcal{H} = \lim_{i\to+\infty} \mathcal{H}_i = \lim_{i\to+\infty} \mathcal{H}_{g_i}$$

exists in the space of measures of slow growth on $\mathbb{R}$ and it is equal to the measure with density 1.

Acknowledgements. The second author is supported by the King Khalid University, research grant No. KKU-S027-33.

REFERENCES

[1] Barner, K.: *On A. Weil's explicit formula*, J. Reine Angew. Math. **323** (1981), 139–152.

[2] Lang, S.: *On the zeta function of number fields*, Invent. Math. **12** (1971), 337–345.

[3] Omar, S.—Mazhouda, K.: *Le critère de Li et l'hypothèse de Riemann pour la classe de Selberg*, J. Number Theory **125** (2007), no. 1, 50–58.

[4] Omar, S.—Mazhouda, K.: *Erratum et addendum à l'article, Le critère de Li et l'hypothèse de Riemann pour la classe de Selberg [J. Number Theory 125 (2007) 50–58]*, J. Number Theory **130** (2010), no. 4, 1109–114.

[5] Schwartz, L.: *Théorie des Distributions*, Hermann, Paris, (1966).

[6] Selberg, A.: *Old and new conjectures and results about a class of Dirichlet series*, presented at the Amelfi Conference on Number Theory, September 1989, in: Collected Papers, vol. **II**, Springer-Verlag, (1991).

[7] TSFASMAN M. A.—VLĂDUT, S. G.: *Infinite global fields and the generalized BrauerSiegel theorem*, Moscow Mathematical Journal, **2** (2002), no. 2, 329–402.

[8] ZYKIN, A.: *Asymptotic properties of zeta functions over finite fields* (preprint).

Received November 11, 2012
Accepted January 2, 2013

Kamel Mazhouda
Faculty of Sciences of Monastir
Department of Mathematics
5000 Monastir
Tunisia
E-mail: kamel.mazhouda@fsm.rnu.tn

Sami Omar
Faculty of Sciences of Tunis
Department of Mathematics
2092 Campus
Universitaire El Manar
Tunis, Tunisia
E-mail: sami.omar@fst.rnu.tn