Uniform Distribution Theory **8** (2013), no.2, 29–38

uniform
distribution
theory

HYBRIDIZATION OF VAN DER CORPUT SEQUENCES AND POLYNOMIAL WEYL SEQUENCES

Shu Tezuka

ABSTRACT. In this paper, we consider the hybridization of van der Corput sequences and polynomial Weyl sequences, which includes the former as one extreme of base polynomial of degree one, and the latter as the other extreme of base polynomial of degree infinity. We show that between these two extremes the hybridization provides one-dimensional low-discrepancy sequences for base polynomial of any positive degree, and thereby leads to a new construction of digital $(0,1)$-sequences.

Communicated by Oto Strauch

1. Introduction

For a set of points $P_N = \{X_0, X_1, \dots, X_{N-1}\}$ in the unit interval $[0,1]$, let $A(P_N; I)$ denote the number of points contained in a given interval $I \subseteq [0,1]$. Then, the (star) discrepancy D_N is defined by

$$D_N = \sup_{0 \le \alpha \le 1} \left| \frac{A(P_N; [0,\alpha))}{N} - \alpha \right|.$$

For more detail on the discrepancy theory and its applications, see [2, 5, 7, 8, 9]. In this paper, we consider the discrepancy of a special class of infinite sequences in $[0,1]$, the so-called digital $(t,1)$-sequences. First, we need the following definitions:

2010 Mathematics Subject Classification: Primary 65D30.
Keywords: low-discrepancy sequences, polynomial Weyl sequences, van der Corput sequences, $(t,1)$-sequences.
This research was supported by KAKENHI(22540141).

Definition 1. Let $b \geq 2$ be an integer. A b-ary interval E is defined by

$$E = \left[\frac{a}{b^d}, \frac{a+1}{b^d} \right)$$

with integers $d \geq 0$ and integers $0 \leq a < b^d$.

Definition 2. Let $0 \leq t \leq m$ be an integer. A point set P_N of $N = b^m$ points in $[0, 1]$ is called a $(t, m, 1)$-net if $A(P_N; E) = b^t$ for every b-ary interval E of length b^{t-m}.

Definition 3. Let $t \geq 0$ be an integer. A $(t, 1)$-sequence $X_0, X_1, \dots,$ in base b is a sequence of points in $[0, 1]$ if for all integers $j \geq 0$ and $m > t$, the point set consisting of the $[X_n]_{b,m}$ with $jb^m \leq n < (j + 1)b^m$ is a $(t, m, 1)$-net in base b, where $[X]_{b,m}$ denotes the b-ary m-digit truncation of X.

The s-dimensional generalization of $(t, m, 1)$-nets and $(t, 1)$-sequences is called as (t, m, s)-nets and (t, s)-sequences, respectively, whose comprehensive exposition can be found in Dick and Pillichshammer [2]. We have the following theorem for the one-dimensional case.

Theorem 1. *For any $N > 1$, the discrepancy of the first N points of a $(t, 1)$- -sequence in base b satisfies*

$$D_N \leq c(t, b) \frac{\log N}{N},$$

where $c(t, b)$ is a constant depending only on t and b.

Note that $c(t, b) \leq c(u, b)$ for any $u > t$ because $(t, 1)$-sequences are always $(u, 1)$-sequences. Hereafter, we assume that b is a prime power. The standard construction of digital $(t, 1)$-sequences is described as follows:

Definition 4. Let $B = \{0, 1, \dots, b-1\}$. Denote bijections $\lambda_i : GF(b) \to B$ and $\psi_i : B \to GF(b)$ for $i = 1, 2, \dots$. We assume that $\lambda_i(0) = \psi_i(0) = 0$, $(i \geq 1)$. Let c_{ij}, $(i, j \geq 1)$, be in $GF(b)$. We define a sequence of points $X_n, n = 0, 1, \dots,$ as

$$X_n = \sum_{i=1}^{\infty} x_{ni} b^{-i},$$

where

$$x_{ni} = \lambda_i \left(\sum_{j=1}^{\infty} c_{ij} \psi_j \big(a_j(n) \big) \right),$$

for $i \geq 1$. Here $n = \sum_{j=1}^{\infty} a_j(n) b^{j-1}$ is the digit expansion of n in base b.

The matrix $C = (c_{ij})$ is called the generator matrix of a digital $(t, 1)$-sequence in base b. Remark that the van der Corput sequence is identical to the sequence whose generator matrix is the (infinite) identity matrix.

We denote

$$\mathbf{c}_m(l) = (c_{m,1}, \dots, c_{m,l}) \in GF(b)^l,$$

and let

$$C(d; l) = \{\mathbf{c}_m(l) \mid 1 \le m \le d\}.$$

Define $\rho(C; l)$ as the maximal integer d such that $C(d; l)$ is linearly independent over $GF(b)$, where if $\mathbf{c}_1(l)$ is a zero vector, then $\rho(C; l) = 0$. Then we have the following result [1].

Theorem 2. *The sequence given in Definition 4 is a strict $(t, 1)$-sequence in base b, where*

$$t = \sup_{l \ge 1} \bigl(l - \rho(C; l)\bigr).$$

This result tells us that in order to construct low-discrepancy sequences, we need a generator matrix C such that $\rho(C; l)$ is as large as possible for all $l \ge 1$. The optimal case is $t = 0$, i.e., $(0, 1)$-sequences. The van der Corput sequence is an example of $(0, 1)$-sequences because $l = \rho(C; l)$ for all $l \ge 1$.

One approach to the construction of $(t, 1)$-sequences is based on the formal Laurent series expansions over finite fields. Denote $S(z) \in GF\{b, z\}$ by

$$S(z) = \sum_{j=w}^{\infty} x_j z^{-j},$$

where all $x_j \in GF(b)$ and w is an integer. Hereafter, we use the following notations: $[S(z)]$ means the polynomial part of $S(z) \in GF\{b, z\}$ and $[S(z)]_{p(z)} \overset{\text{def}}{=} [S(z)] \pmod{p(z)}$ with $0 \le \deg([S(z)]_{p(z)}) < \deg(p)$. We denote the discrete exponential valuation of $S(z)$ by $\nu(S)$. If $S(z)$ is a polynomial, we have $\nu(S) = \deg(S)$. We define the $e \times e$ Hankel matrix $H_e(S)$ for a formal Laurent series $S(z) = \sum_{j=1}^{\infty} x_j z^{-j}$ in the following way:

$$H_e(S) = \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_e \\ x_2 & x_3 & x_4 & \cdots & x_{e+1} \\ & \cdots & \cdots & \cdots & \\ & \cdots & \cdots & \cdots & \\ x_{e-1} & x_e & x_{e+1} & \cdots & x_{2e-2} \\ x_e & x_{e+1} & x_{e+2} & \cdots & x_{2e-1} \end{pmatrix}.$$

We define

$$\tau\bigl(H_e(S)\bigr) = \max_{1 \le l \le e} \Bigl(l - \rho\bigl(H_e(S); l\bigr)\Bigr).$$

Larcher and Niederreiter [4] first defined and investigated the polynomial Weyl sequence, whose generator matrix is given by the Hankel matrix $H_\infty(S)$, where $S(z)$ is an irrational formal Laurent series as a given parameter for the sequence. (To be precise, they considered a more general case, i.e., the polynomial analogue of Kronecker sequences, which are the multi-dimensional version of Weyl sequences.) They proved the next lemma, which is useful for proving our main result later.

Lemma 1. *Let $p(z)$ and $q(z)$ are polynomials over $GF(b)$ with $\deg(q) < \deg(p)$ and $\gcd(p, q) = 1$. Let the partial quotients in the continued fraction expansion over $GF(b)$ of $q(z)/p(z)$ be $g_1(z)$, ..., $g_K(z)$, that is,*

$$\frac{q(z)}{p(z)} = \cfrac{1}{g_1(z) + \cfrac{1}{g_2(z) + \ddots + \cfrac{1}{g_K(z)}}}.$$

Let $d_k = \sum_{i=1}^{k} \deg(g_i)$ for $k = 1, \dots, K$ and $d_0 = 0$. Then we have

$$\rho(H_\infty(q/p); l) = \begin{cases} d_k, & \text{if } d_k \le l < d_{k+1} \text{ for some } 0 \le k < K; \\ d_K, & \text{otherwise.} \end{cases}$$

Note that $d_K = \deg(p)$.

This lemma implies that if a pair of polynomials $(q(z), p(z))$ satisfies that the degrees of partial quotients in the expansion of $q(z)/p(z)$ are all ones, then we have $\tau(H_e(q/p)) = 0$, where $e = \deg(p)$. In this context, Fibonacci polynomials over $GF(b)$ were defined as follows [9]:

Definition 5. A polynomial $F_n(z)$ over $GF(b)$ is called a Fibonacci polynomial of degree n over $GF(b)$ if there exist degree-one polynomials $g_1(z), \dots, g_n(z)$ over $GF(b)$ such that

$$F_i(z) = g_i(z)F_{i-1}(z) + F_{i-2}(z),\ 1 \le i \le n,$$

where

$$F_{-1}(z) = 0 \quad \text{and} \quad F_0(z) = 1.$$

2. Main Results

We define a class of infinite sequences in $[0, 1]$ based on the radical inverse function with respect to polynomial arithmetic over $GF(b)$.

Definition 6. Let n be a nonnegative integer whose digit expansion in base b is $n = a_{m+1}(n)b^m + \cdots + a_2(n)b + a_1(n)$, where $m = \lfloor \log_b n \rfloor$. Denote $v_n(z) = n_m z^m + \cdots + n_1 z + n_0$, where $n_i = \psi_{i+1}(a_{i+1}(n))$, for $i = 0, 1, \ldots, m$. Let $p(z)$, the so-called base polynomial, be an arbitrary non-constant polynomial over $GF(b)$. Then $v_n(z)$ can be represented in terms of $p(z)$ as follows: Let $e = \deg(p)$ and $s = \lfloor m/e \rfloor$.

$$v_n(z) = r_s(z)p(z)^s + \cdots + r_1(z)p(z) + r_0(z),$$

where $r_i(z) = [v_n(z)/p(z)^i]_{p(z)}, 0 \le i \le s$. Note that $e > \deg(r_i)$ for all $0 \le i \le s$. We define the following function $\phi_{p,q} : GF[b, z] \to GF\{b, z\}$:

$$\phi_{p,q}(v_n(z)) = \frac{[q(z)r_0(z)]_{p(z)}}{p(z)} + \cdots + \frac{[q(z)r_{s-1}(z)]_{p(z)}}{p(z)^s} + \frac{[q(z)r_s(z)]_{p(z)}}{p(z)^{s+1}}, \quad (1)$$

where $q(z)$ is a polynomial with $\deg(q) < \deg(p)$ satisfying $\gcd(p, q) = 1$.

Then, a one-dimensional sequence in $[0, 1]$ is defined as follows:

$$X_n = \sigma(\phi_{p,q}(v_n(z))) \quad \text{for} \quad n = 0, 1, \ldots, \quad (2)$$

where σ is a mapping from $GF\{b, z\}$ to the real field defined by $\sigma(\sum_{j=1}^{\infty} x_j z^{-j}) = \sum_{j=1}^{\infty} \lambda_j(x_j)b^{-j}$.

We should remark that the (original) van der Corput sequence corresponds to the special case of $\phi_{p,q}(v_n(z))$ with $q(z) = 1$ and $p(z) = z$, and that polynomial van der Corput sequences introduced in [10, 11] correspond to the special case with $q(z) = 1$ and any $p(z)$. From the above definition, the next lemma immediately follows, which shows that the function $\phi_{p,q}(v_n(z))$ has the linearity.

Lemma 2. *Let $v_n(z) = n_m z^m + \cdots + n_1 z + n_0$. We have*

$$\phi_{p,q}(v_n(z)) = \sum_{i=0}^{m} n_i \phi_{p,q}(z^i).$$

Proof.
For $j = 0, 1, \ldots, m$, and $i = 0, 1, \ldots, \lfloor j/e \rfloor$, denote $r_i^{(j)}(z) = [z^j/p(z)^i]_{p(z)}$. Then we have

$$r_i(z) = n_m r_i^{(m)}(z) + \cdots + n_1 r_i^{(1)}(z) + n_0.$$

Thus,

$$\begin{aligned} \phi_{p,q}(v_n(z)) &= \sum_{i=0}^{s} \frac{[q(z)r_i(z)]_{p(z)}}{p(z)^{i+1}} \\ &= \sum_{j=0}^{m} n_j \sum_{i=0}^{\lfloor j/e \rfloor} \frac{[q(z)r_i^{(i)}(z)]_{p(z)}}{p(z)^{i+1}} \end{aligned}$$

$$= \sum_{j=0}^{m} n_j \phi_{p,q}(z^j).$$

The second equality comes from $r_i^{(j)}(z) = 0$ for $i = \lfloor j/e \rfloor + 1, \dots, s(= \lfloor m/e \rfloor)$. The proof is complete. $\square$

Therefore, for $m = 0, 1, \dots$, the coefficients in the formal Laurent series expansion of $\phi_{p,q}(z^m)$ constitute the $(m+1)$-th column vector of the generator matrix of the sequence (2). The following lemma shows that $\phi_{p,q}(z^m)$, $m = 0, 1, \dots$, satisfy a certain linear recurrence relation.

Lemma 3. *For a given $m \geq e$, let $s = \lfloor m/e \rfloor$, and denote $p(z)^s = d_{es} z^{es} + d_{es-1} z^{es-1} + \cdots + d_0$ with $d_{es} \neq 0$. Then, we have*

$$\nu \left(\sum_{j=0}^{es} d_j \phi_{p,q}\left(z^{m-es+j}\right) \right) < -es.$$

P r o o f. From equation (1), we denote for $m \geq 0$

$$\phi_{p,q}(z^m) = \frac{[q(z) r_0^{(m)}(z)]_{p(z)}}{p(z)} + \cdots + \frac{[q(z) r_{s-1}^{(m)}(z)]_{p(z)}}{p(z)^s} + \frac{[q(z) r_s^{(m)}(z)]_{p(z)}}{p(z)^{s+1}},$$

where $r_i^{(m)}(z) = [z^m / p(z)^i]_{p(z)}$, $0 \leq i \leq s$, and $s = \lfloor m/e \rfloor$. Note that for $0 \leq i < s$, we have $z^{m-es} p(z)^s = 0 \pmod{p(z)^{i+1}}$, i.e., $d_{es} z^m = -(d_{es-1} z^{m-1} + \cdots + d_0 z^{m-es}) \pmod{p(z)^{i+1}}$. Let $h = m - es$, then for $0 \leq i < s$,

$$\begin{aligned} [q(z) r_i^{(m)}(z)]_{p(z)} &= [q(z) [z^m / p(z)^i]_{p(z)}]_{p(z)} \\ &= -\sum_{j=0}^{es-1} d_{es}^{-1} d_j [q(z) [z^{h+j} / p(z)^i]_{p(z)}]_{p(z)}. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} -\sum_{j=0}^{es-1} d_{es}^{-1} d_j \phi_{p,q}(z^{m-es+j}) &= -\sum_{j=0}^{es-1} d_{es}^{-1} d_j \sum_{i=0}^{s} \frac{[q(z)[z^{h+j}/p(z)^i]_{p(z)}]_{p(z)}}{p(z)^{i+1}} \\ &= \sum_{i=0}^{s-1} \frac{[q(z) r_i^{(m)}(z)]_{p(z)}}{p(z)^{i+1}} + \frac{\epsilon(z)}{p(z)^{s+1}} \\ &= \phi_{p,q}(z^m) + \frac{\epsilon(z) - [q(z) r_s^{(m)}(z)]_{p(z)}}{p(z)^{s+1}}, \end{aligned}$$

where $\epsilon(z) = -\sum_{j=0}^{es-1} d_{es}^{-1} d_j [q(z)[z^{h+j}/p(z)^s]_{p(z)}]_{p(z)}$.
Since $\nu((\epsilon(z) - [q(z)r_s^{(m)}(z)]_{p(z)})/p(z)^{s+1}) < -es$, the proof is complete. □

Remark 1. From Definition 6, the elements of the m-th column vector of the generator matrix form a linear feedback shift register sequence whose characteristic polynomial is $p(z)^i$, where $i = \lfloor (m-1)/e \rfloor + 1$ for $m \geq 1$. The above lemma implies that the characteristic polynomial of the elements of the m-th row vector of the generator matrix is $p(z)^i$, where $i = \lfloor (m-1)/e \rfloor + 1$ for $m \geq 1$. However, the generator matrix is not necessarily symmetric, contrary to the case of $q(z) = 1$ [10].

We give two examples of sequences defined in (2):

Example 1. Take $p(z) = z^2 + z + 1$ and $q(z) = z$ from $GF[2, z]$. Then, the generator matrix is given by

$$C = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & \cdots \\ 1 & 0 & 1 & 1 & 0 & 1 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 1 & \cdots \\ 1 & 1 & 0 & 0 & 1 & 1 & \cdots \\ 1 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Example 2. Take $p(z) = z^3$ and $q(z) = z^2 + 1$ from $GF[2, z]$. Then, the generator matrix is given by

$$C = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The next lemma plays an important role in proving our main result.

Lemma 4. *Let C be the generator matrix of a $(t, 1)$-sequence. Then, we have*

$$\rho(LCU; l) = \rho(C; l)$$

for $l = 1, 2, \ldots$, where L is any nonsingular lower triangular (infinite) matrix, and U is any nonsingular upper triangular (infinite) matrix.

P r o o f. Let $A^{(l)}$ denote the upper-left $l \times l$ submatrix of an infinite matrix A. Then, for $l = 1, 2, \dots$, we have

$$(LCU)^{(l)} = L^{(l)} C^{(l)} U^{(l)},$$

from the assumptions on L and U. Since $L^{(l)}$ and $U^{(l)}$ are nonsingular for all $l \geq 1$, the claim follows. $\square$

We now present our main result:

Theorem 3. *The sequence defined in* (2) *is a strict* $(\tau(H_e(q/p)), 1)$*-sequence in base* b*, where* $e = \deg(p)$*.*

P r o o f. First, we should observe that equation (1) is rewritten as follows:

$$\phi_{p,q}(v_n(z)) = \sum_{i=0}^{s} \left\{ \frac{q(z) r_i(z)}{p(z)} \right\} \frac{1}{p(z)^i},$$

where $\{S(z)\}$ means the fractional part of $S(z)$. Thus,

$$\phi_{p,q}(z^m) = \sum_{i=0}^{s} \left\{ \frac{q(z) r_i^{(m)}(z)}{p(z)} \right\} \frac{1}{p(z)^i}, \tag{3}$$

where $s = \lfloor m/e \rfloor$.

Let $P_0 = I$. For $s = 1, 2, \dots$, we denote a nonsingular lower triangular matrix P_s by

$$P_s = \begin{pmatrix} c_{es} & 0 & \dots & 0 & 0 \\ c_{es+1} & c_{es} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{e(s+1)-1} & c_{e(s+1)-2} & \cdots & c_{e(s+1)-3} & c_{es} \end{pmatrix},$$

where $1/p(z)^s = \sum_{i=es}^{\infty} c_i z^{-i}$. Also for $s = 0, 1, \dots$, we denote a nonsingular upper triangular matrix R_s by

$$R_s = \begin{pmatrix} b_{e-1}^{(s)} & b_{e-2}^{(s)} & \dots & b_1^{(s)} & b_0^{(s)} \\ 0 & b_{e-1}^{(s)} & \dots & b_2^{(s)} & b_1^{(s)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & b_{e-1}^{(s)} \end{pmatrix},$$

where $r_s^{(e(s+1)-1)}(z) = b_{e-1}^{(s)} z^{e-1} + \cdots + b_1^{(s)} z + b_0^{(s)}$. We should notice that compared to the term at $i = s$ of equation (3), P_s corresponds to $1/p(z)^s$ and R_s does to $r_s^{(m)}(z)$, $es \le m < e(s+1)$.

From Lemma 3 and Remark 1, by repeating elimination procedures in rows as well as in columns of the generator matrix for the sequence defined in (2), without any interchange of rows and of columns, we arrive at the following matrix:

$$\begin{pmatrix} W_0 & O & O & \dots \\ O & W_1 & O & \dots \\ O & O & W_2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{4}$$

where the diagonal submatrices are defined by

$$W_s = P_s H_e(q/p) R_s,$$

for $s = 0, 1, \dots$ Note that $H_e(q/p)$ is nonsingular because $\gcd(p, q) = 1$. Since both P_s and R_s are nonsingular, from Lemmata 1 and 4 as well as the definition of τ, the proof is complete. □

We have the following corollary.

Corollary 1. *If $p(z)$ and $q(z)$ are consecutive Fibonacci polynomials, i.e., $p(z) = F_e(z)$ and $q(z) = F_{e-1}(z)$, where $e = \deg(p)$, then the sequence defined in (2) is a $(0,1)$-sequence in base b.*

P r o o f. From the definition of Fibonacci polynomials in Definition 5, it follows that the matrix in (4) can be further reduced, without any interchange of rows and of columns, to the (infinite) identity matrix. □

3. Discussions

It should be emphasized that a polynomial van der Corput sequence with base polynomial $p(z)$ is a strict $(t,1)$-sequence with $t = \deg(p) - 1$, while the hybridization can produce $(0,1)$-sequences regardless of the degree of base polynomial. Since polynomial van der Corput sequences correspond to the hybridization with $q(z) = 1$, the sequences produced by the hybridization can be viewed as polynomial analogues of generalized van der Corput sequences (see, e.g., [3]). Therefore, polynomial analogues of generalized Halton sequences can be constructed based on the hybridization using distinct irreducible base polynomials, for which we can exploit a nice theorem by Mesirov and Sweet [6] that for the most practical case of $GF(2)$ any irreducible polynomial is a Fibonacci polynomial.

REFERENCES

[1] Dick, J.—Niederreiter, H.: *On the exact t-value of Niederreiter and Sobol' sequences*, Journal of Complexity, **24** (2008), 572–581.

[2] Dick, J.—Pillichshammer, F.: *Digital Nets and Sequences. Discrepancy Theory and Quasi-Monte Carlo Integration*. Cambridge University Press, Cambridge, 2010.

[3] Faure, H.: *Good permutations for extreme discrepancy*, Journal of Number Theory, **42** (1992), 47–56.

[4] Larcher, G.—Niederreiter, H.: *Kronecker-type sequences and nonarchimedean Diophantine approximations*, Acta Arithmetica, **63** (1993), 379–396.

[5] Lemieux, C.: *Monte Carlo and Quasi-Monte Carlo Sampling*, Springer, 2009.

[6] Mesirov, J. P.—Sweet, M. M.: *Continued fraction expansions of rational expressions with irreducible denominators in characteristic 2*, Journal of Number Theory, **27** (1987), 144–148.

[7] Novak, E.—Woźniakowski, H.: *Tractability of Multivariate Problems, Volume II: Standard Information for Functionals*, European Mathematical Society, 2010.

[8] Strauch, O.—Porubský, Š: *Distribution of Sequences: A Sampler*, Peter Lang, 2005.

[9] Tezuka, S.: *Uniform Random Numbers: Theory and Practice*, Kluwer Academic Publishers, 1995.

[10] Tezuka, S.: *Polynomial arithmetic analogue of Halton sequences*, ACM Transactions on Modeling and Computer Simulation, **3** (1993), 99–107.

[11] Tezuka, S.—Tokuyama, T.: *A note on polynomial arithmetic analogue of Halton sequences*, ACM Transactions on Modeling and Computer Simulation, **4** (1994), 279–284.

Received April 3, 2012
Accepted December 10, 2012

Shu Tezuka
Institute of Mathematics for Industry,
Kyushu University, 744 Motooka,
Nishi-ku, Fukuoka-shi, Fukuoka-ken,
Japan 819-0395
E-mail: tezuka@imi.kyushu-u.ac.jp