INTERSECTING LINES MODULO ONE

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ABSTRACT. A straight line in the plane of gradient α reduces modulo the integer lattice to a set of parallel lines across the unit square. When α is irrational, these lines are dense in the unit square. Two straight lines with different irrational gradients α and β give two dense families of parallel lines, whose intersections are uniformly distributed in the unit square. The discrepancy bounds involve continued fractions, or linear forms in α and β.

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1. Introduction

We consider a straight line in the plane of gradient α, reduced modulo the integer lattice to a set of parallel lines across the unit square. When α is rational, the same set of line segments is repeated, but when α is irrational, these lines are dense in the unit square. Two straight lines with different irrational gradients α and β give two dense families of parallel lines in the unit square, whose intersections are also dense. We have not found in the literature the statement that these intersection points are uniformly distributed in the unit square.

THEOREM 1. *Let I and J be line segments in the plane of lengths k and ℓ with different irrational gradients α and β respectively. As k and ℓ tend to infinity, the pairs of points P on I and Q on J for which $\overrightarrow{PQ}$ is an integer vector tend to uniform distribution modulo the lattice of integer points.*

Theorem 1 comes from a discrepancy bound whose statement requires more notation. For particular line segments I and J, the vectors $\overrightarrow{PQ}$ with P on I and Q on J form a parallelogram $X_0X_1X_3X_2$ with sides X_0X_1 and X_2X_3 equal and parallel to I, and X_0X_2 and X_1X_3 equal and parallel to J. The discrete set of points P and Q in Theorem 1 are indexed by the integer vectors (m, n)

in the parallelogram. A short calculation shows that P and Q have coordinates of the form

$$\left(\frac{n - m\beta}{\beta - \alpha} + x_0, \frac{\alpha(n - m\beta)}{\beta - \alpha} + y_0\right), \left(\frac{n - m\alpha}{\beta - \alpha} + x_0, \frac{\beta(n - m\alpha)}{\beta - \alpha} + y_0\right), \quad (1.1)$$

which reduce to the same point $(u(m, n), v(m, n))$ modulo one.

The number of intersections is unchanged when we reflect the two given lines in the lines $x = 0$, $y = 0$, or $y = x$. Hence we can suppose that $\beta \geq \max(|\alpha|, 1/|\alpha|)$, and that the two given lines are oriented in the direction of x increasing.

Let N be the number of integer points (m, n) in the parallelogram $X_0 X_1 X_3 X_2$. For $0 < \kappa < 1$ and $0 < \lambda < 1$, let $N(\kappa, \lambda)$ be the number of integer points in the parallelogram for which $0 \leq u(m, n) \leq \kappa$, and $0 \leq v(m, n) \leq \lambda$. We estimate the difference $D(\kappa, \lambda) = N(\kappa, \lambda) - \kappa\lambda N$.

We use the standard notation $[t]$ for the integer part of the real number t, $\{t\} = t - [t]$ for the fractional part, $\rho(t) = \frac{1}{2} - \{t\}$ for the row-of-teeth function and $\|t\|$ for the distance of t from the nearest integer, the 'norm of t modulo one'. Let α and β have continued fractions $[a_0; a_1, a_2, \dots]$ and $[a'_0; a'_1, a'_2, \dots]$ with convergents p_r/q_r and p'_s/q'_s respectively. We write the partial quotient growth function as

$$E(\alpha, K) = a_1 + \cdots + a_r + \frac{K}{q_r}, \quad (1.2)$$

where r is the least index with $q_{r+1} > K$, and we define $E(\beta, K)$ similarly. A term will be called $O(E)$ when it lies between $-BE$ and BE for some numerical constant B which does not involve the parameters such as α, β, k, ℓ and μ that describe the line segments I and J, and B could be calculated explicitly.

The number of lattice points in a polygon has been well studied [2, 7, 8, 9, 10].

PROPOSITION 1. *The number of integer points in the parallelogram $X_0 X_1 X_3 X_2$ is*

$$N = \frac{(\beta - \alpha)k\ell}{\sqrt{(1 + \alpha^2)(1 + \beta^2)}} + O\left(E\left(\alpha, \frac{k}{\sqrt{1 + \alpha^2}}\right) + E\left(\beta, \frac{\ell}{\sqrt{1 + \beta^2}}\right)\right)$$

$$+ O(\beta - \alpha). \quad (1.3)$$

The first term in (1.3) is the area of the parallelogram. To illustrate our approach, we state Proposition 2, used in the proof of Theorem 4.

PROPOSITION 2. *Let g and h be integers, not both 0, and let*

$$\theta(g, h) = \frac{g + h\alpha}{\beta - \alpha}. \quad (1.4)$$

Let $H \geq 3$ be an integer. Then

$$D(\kappa, \lambda) = O\left(\frac{N}{H}\right) + O\left(\frac{k + \ell}{\min \|\theta(g, h)\|}\right), \tag{1.5}$$

where the minimum is over $-2H \leq g \leq 2H$, $-2H \leq h \leq 2H$, with g and h not both zero. The bound also holds with $\|\theta(g, h)\|$ replaced by $\|\beta\theta(g, h)\|$.

THEOREM 2. *Let H be a positive integer with*

$$H \geq \max\left(3, \frac{(\beta - \alpha)^2}{2(1 + \beta)(2 + \beta)|\alpha|}\right). \tag{1.6}$$

Let

$$Q = \frac{2(1 + \beta)H}{\beta - \alpha}. \tag{1.7}$$

Suppose that neither α nor β is a rational number with denominator $q \leq Q$. Let A be a bound for the early partial quotients in the continued fractions for α and β, with $a_{r+1} \leq A$ whenever $q_r \leq Q$ and $a'_{s+1} \leq A$ whenever $q'_s \leq Q$. Suppose that k and ℓ are so large that the number of intersection points N satisfies

$$\frac{(\beta - \alpha)N}{2(1 + \beta)(2 + \beta)(k + \ell)H \log^2 H} > A + 2. \tag{1.8}$$

Then we have the discrepancy bound

$$D(\kappa, \lambda) = O\left(\frac{N}{H}\right). \tag{1.9}$$

Stronger discrepancy bounds can be obtained from stronger Diophantine approximation conditions on α and β. The most helpful condition is that of bounded partial quotients.

THEOREM 3. *Let α and β be distinct irrational numbers whose continued fraction expansions*

$$\alpha = [a_0; a_1, a_2, \dots] \quad and \quad \beta = [a'_0; a'_1, a'_2, \dots]$$

have

$$a_i \leq A \quad and \quad a'_i \leq A \quad for\ i \geq 1.$$

Then we have the discrepancy bound

$$D(\kappa, \lambda) = O\left(\sqrt{\frac{A\beta(k + \ell)N \log N}{\beta - \alpha}}\right) + O\left(\beta^2(k + \ell) \log^2 N\right). \tag{1.10}$$

The most interesting conditions involve bounds for linear forms.

Theorem 4. *Let Let α and β be distinct real numbers. If the Linear Forms Condition*

$$\|m\alpha + n\beta\| \geq \frac{1}{B\big(\max(|m|, |n|)\big)^{\mu}} \tag{1.11}$$

holds with some $B \geq 2$ and some $\mu \geq 2$ for all integers in the range

$$|m|, |n| \leq \left(\frac{5\beta N}{B(\beta - \alpha)^2(k + \ell)}\right)^{\frac{1}{\mu+1}}, \tag{1.12}$$

and if k and ℓ are sufficiently large in terms of α, β and B, then we have

$$D(\kappa, \lambda) = O\left(\left(\frac{B(5\beta)^{\mu}(k + \ell)N^{\mu}}{(\beta - \alpha)^{\mu-1}}\right)^{\frac{1}{\mu+1}}\right). \tag{1.13}$$

If the Inverse Linear Forms Condition

$$\left\|\frac{m}{\alpha} + \frac{n}{\beta}\right\| \geq \frac{1}{B'\big(\max(|m|, |n|)\big)^{\mu}} \tag{1.14}$$

holds with some $B' \geq 2$ and some $\mu \geq 2$ for all integers in the range

$$|m|, |n| \leq \left(\frac{|\alpha|(3 + 2|\alpha|)\beta^2 N}{B'(\beta - \alpha)^2(k + \ell)}\right)^{\frac{1}{\mu+1}}, \tag{1.15}$$

and if k and ℓ are sufficiently large in terms of α, β and B', then we have

$$D(\kappa, \lambda) = O\left(\left(\frac{B'(3 + 2|\alpha|)^{\mu}\beta^{\mu-1}(k + \ell)N^{\mu}}{|\alpha|(\beta - \alpha)^{\mu-1}}\right)^{\frac{1}{\mu+1}}\right). \tag{1.16}$$

Suppose that both conditions hold for

$$|m|, |n| \leq \frac{5\beta N^{\mu}}{\beta - \alpha}, \tag{1.17}$$

and that k and ℓ are sufficiently large in terms of α, β, B and B'. Then we have

$$D(\kappa, \lambda) = O\left(\left(D(k + \ell)^{\frac{2\mu-1}{\mu}}N^{2\mu+1}\right)^{\frac{1}{2\mu+3}}\right) + O\left(\left(D'(k + \ell)N^{\mu-1}\log N\right)^{\frac{1}{\mu}}\right), \tag{1.18}$$

where the coefficients D and D' are constructed from α, β, B and B'.

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This question arose in the study of the set of integer points inside a plane oval curve [3, 4, 5, 6]. In [4] and [5] we consider the expected number of vertices of the convex hull. After some simplifications, we need to sum the discrepancies for a sequence of pairs of line segments in different orientations. I would like to thank Professors A. Haynes, G. Larcher, R. Nair, J. Schoissengeier and R. F. Tichy for suggestions and encouragement with this problem.

2. Weyl Sums

Our starting point is the Erdős-Turán-Koksma theorem in two dimensions (Drmota, Tichy [1, Theorem 1.21]).

LEMMA 2.1 (the Erdős-Turán-Koksma Theorem). *Let (x_μ, y_μ) be a sequence of N points in the plane, which reduce modulo the unit lattice to points (u_μ, v_μ) in the unit square. Let H be a positive integer. Let κ and λ be given in $0 \leq \kappa \leq 1$, $0 \leq \lambda \leq 1$. Then the number of solutions of the double condition*

$$0 \leq u_\mu \leq \kappa, \qquad 0 \leq v_\mu \leq \lambda$$

can be expressed using bounds for the Weyl sums

$$S(g, h) = \sum_\mu e(g x_\mu + h y_\mu)$$

as

$$\kappa \lambda N + O\left(\frac{N}{H}\right) + O\left(\sum_g^* \sum_h r(g, h) |S(g, h)|\right),$$

*where the conditions * of summation are*

$$-H \leq g \leq H, \quad -H \leq h \leq H, \quad \text{either } g \neq 0 \text{ or } h \neq 0,$$

and

$$r(g, h) = \frac{1}{(1 + |g|)(1 + |h|)}.$$

We apply Lemma 2.1 to the set of N points

$$(x(m, n), y(m, n)) = \left(\frac{n - m\beta}{\beta - \alpha}, \frac{\alpha(n - m\beta)}{\beta - \alpha}\right) \tag{2.1}$$

indexed by the integer points (m, n) in the parallelogram S, $X_0 X_1 X_3 X_2$. The Weyl sums are

$$S(g, h) = \sum_m \sum_n e\left(\frac{(g + h\alpha)(n - m\beta)}{\beta - \alpha}\right) = \sum_m \sum_n e\big(\theta(g, h)(n - m\beta)\big), \quad (2.2)$$
$$ {}_{(m,n) \in S} \phantom{e\left(\frac{(g + h\alpha)(n - m\beta)}{\beta - \alpha}\right)} {}_{(m,n) \in S}$$

in the notation (1.4) of Proposition 2.

The Weyl sums are linear in both summands m and n, but when we treat them as repeated sums, the limits of summatiom for the inner sum (over n, say) are $n_1(m)$ and $n_2(m)$ dependent on the outer summand m. The sides of the parallelogram S have equations of the form $y = \alpha x + \gamma$, $y = \beta x + \gamma$, so in the different ranges for m, the upper limits of summation have the forms

$$n_2(m) = [\alpha m + \gamma], \qquad n_2(m) = [\beta m + \gamma], \qquad (2.3)$$

and the lower limits of summation have the forms

$$n_1(m) = -[-\alpha m - \gamma], \qquad n_1(m) = -[-\beta m - \gamma]. \qquad (2.4)$$

Similarly if we make the summation over m the inner summation, the limit of the sum over m are piecewise of the forms

$$m_2(n) = \left[\frac{n}{\alpha} + \gamma\right], \qquad m_2(n) = \left[\frac{n}{\beta} + \gamma\right], \qquad (2.5)$$

$$m_1(n) = -\left[-\frac{n}{\alpha} - \gamma\right], \qquad m_1(n) = -\left[-\frac{n}{\beta} - \gamma\right]. \qquad (2.6)$$

We consider the indices g and h of the Weyl sum (2.2) as fixed, and write θ for $\theta(g, h)$. Performing the inner sum over n explicitly, we have

$$\sum_{n=n_1(m)}^{n_2(m)} e\big(\theta(n - m\beta)\big) = \frac{1}{B(g, h)}\left(e\Big(\theta\big(n_2(m) + 1 - m\beta\big)\Big) - e\Big(\theta\big(n_1(m) - m\beta\big)\Big)\right),$$
$$(2.7)$$

where the denominator is

$$B(g, h) = e(\theta) - 1 = e\left(\frac{g + h\alpha}{\beta - \alpha}\right) - 1 = e\left(\frac{g + h\beta}{\beta - \alpha}\right) - 1. \qquad (2.8)$$

Similarly if we perform the sum over m explicitly as the inner sum, then

$$\sum_{m=m_1(n)}^{m_2(n)} e\left(\theta(n - m\beta)\right) = \frac{1}{B'(g, h)}\left(e\Big(\theta\big(n - m_2(n)\beta - \beta\big)\Big) - e\Big(\theta\big(n - m_1(n)\beta\big)\Big)\right),$$
$$(2.9)$$

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where the denominator is

$$B'(g,h) = e(-\beta\theta) - 1 = e\left(-\frac{(g+h\alpha)\beta}{\beta-\alpha}\right) - 1 = e\left(-\frac{(g+h\beta)\alpha}{\beta-\alpha}\right) - 1. \quad (2.10)$$

When the labels g and h of the Fourier coefficients are small with respect to k and ℓ, the lengths of the parallelogram sides, then we look for cancellation in the outer sum as well. We dissect the parallelogram into regions in which the outer sums take the form

$$\sum_m e\big(n(m)\theta - m\beta\theta\big), \quad (2.11)$$

where we take $n = n_0$, a constant, or $n_1(m)$ or $n_2(m) + 1$ as in (2.3) and (2.4), and the inner sum is over n, and regions in which the inner sum is over m, and the outer sum takes the corresponding form

$$\sum_n e\big(n\theta - m(n)\beta\theta\big), \quad (2.12)$$

where we take $m = m_0$, a constant, or $m_1(n)$ or $m_2(n) + 1$ as in (2.5) and (2.6).

The simplest cases are $n(m) = n_0$ in (2.11), when the outer sum in (2.11) is analogous to the inner sum in (2.9), and $m(n) = m_0$ in (2.12), when the outer sum in (2.12) is analogous to the innner sum in (2.7).

The next simplest cases are $n_2(m) = [\beta m + \gamma]$ in (2.3), when

$$\theta(n_2(m) + 1 - m\beta) = \theta\left(\rho(\beta m + \gamma) + \gamma - \frac{1}{2}\right), \quad (2.13)$$

and $n_1(m) = [-\beta m - \gamma]$ in (2.4), when

$$\theta(n_1(m) + 1 - m\beta) = \theta\left(-\rho(-\beta m - \gamma) + \gamma + \frac{1}{2}\right). \quad (2.14)$$

If β is not a rational number of small height, then the values of $\rho(\beta m + \gamma)$ are uniformly distributed in an interval of length 1. For θ small, the numbers in (2.13) and (2.14) will be uniformly distributed in an interval of length $|\theta|$. When θ is large, we get uniform distribution mod 1 to an accuracy $O(1/|\theta|)$.

However, when $n_2(m) = [\alpha m + \gamma]$ in (2.3), then we get

$$\theta(n_2(m) + 1 - m\beta) = \theta\left(\rho(\alpha m + \gamma) + \alpha m + \gamma - \frac{1}{2} - \beta m\right)$$

$$= \theta\left(\rho(\alpha m + \gamma) + \gamma - \frac{1}{2}\right) - (g + h\alpha)m.$$

This is the difference of two sequences which are uniformly distributed except for certain values of g and h. We avoid this case, and the similar case when $n_1(m) = -[-\alpha m - \gamma]$ in (2.4).

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When we sum over m first, then in (2.5)

$$m_2(n) = \left[\frac{n}{\beta} + \gamma\right] = \frac{n}{\beta} + \gamma + \rho\left(\frac{n}{\beta} + \gamma\right) - \frac{1}{2},$$

and

$$\theta\big(n - \beta m_2(n)\big) = \theta\left(\beta\rho\left(\frac{n}{\beta} + \gamma\right) - \beta\gamma + \frac{\beta}{2}\right),$$

with similar formulae when $m_1(n) = -[-\gamma - n/\beta]$ in (2.6). These sums replace β by $1/\beta$ and θ by $\beta\theta$. Although there is symmetry between α and β, with the argument in (2.2) as

$$\frac{(g + h\alpha)(n - m\beta)}{\beta - \alpha} = \frac{(g + h\beta)(n - m\alpha)}{\beta - \alpha} + gm - hn,$$

taking the point P rather than the point Q in (1.1) distinguishes between α and β once we sum.

We arrange the sums so that the second case in (2.3), (2.4), (2.5)and (2.6) occurs with a linear term in βn or βm, and the first case occurs with a linear term in αn or αm. There are various cases, depending on the ordering of the coordinates (x_i, y_i) of the vertices X_i of the parallelogram of summation. For example, when $\beta > \alpha > 0$, $x_0 < x_1 \leq x_2 < x_3$ we introduce extra endpoints $P_1 = (x_1, y_0)$, $P_2 = (x_2, y_3)$ outside the parallelogram, and $Q_1 = (x_1, x_3 + \beta(x_1 - x_0))$ on the side $X_0 X_2$, $Q_2 = (x_2, x_3 - \beta(x_3 - x_2))$ on the side $X_1 X_3$. For $x_0 \leq m < x_1$ we sum n over $y_0 \leq n \leq n_2(m)$ (triangle $X_0 P_1 Q_1$). For $x_1 \leq m \leq x_2$ we sum n over $n_1(m) \leq n \leq n_2(m)$ (parallelogram $X_1 Q_2 X_2 Q_1$). For $x_2 < m \leq x_3$ we sum n over $n_2(m) < n \leq y_3$ (triangle $P_2 Q_2 X_3$). We must correct by subtracting sums over the triangles $P_1 X_0 X_1$ and $P_2 X_2 X_3$. In these triangles we sum over m first. For $y_0 \leq n < y_1$ we subtract the sum over m in $m_2(n) < m < x_1$ (triangle $P_1 X_0 X_1$). For $y_2 < n \leq y_3$ we subtract the sum over m in $x_2 < m < m_1(n)$ (triangle $P_2 X_2 X_3$).

There is another possible dissection with $P_1 Q_1$ and $P_2 Q_2$ horizontal, P_1 on the line $x = x_0$ and P_2 on the line $x = x_3$. In this case we sum over m first in the parallelogram $X_1 Q_2 X_2 Q_1$ and over n first in the triangles $P_1 X_0 X_1$ and $P_2 X_2 X_3$, and subtract the sums (over m first) in the triangles $P_1 Q_1 X_0$ and $P_2 Q_2 X_3$.

LEMMA 2.2 (outer sums). *Let*

$$L = x_3 - x_0 = \frac{k}{\sqrt{1 + \alpha^2}} + \frac{\ell}{\sqrt{1 + \beta^2}}, \quad L' = y_3 - y_0 = \frac{\alpha k}{\sqrt{1 + \alpha^2}} + \frac{\beta\ell}{\sqrt{1 + \beta^2}}.$$

Let g and h be given, with $\theta = \theta(g, h)$ non-zero in (2.2). Then the outer sums (2.11) with $n(m) = n_2(m) + 1$ in (2.13) and with $n(m) = n_1(m)$ in (2.14)

have size

$$O\left(\frac{L}{|\theta|}\right) + O\left(\sqrt{|\theta|E(\beta, L)L}\right). \tag{2.15}$$

Similarly the outer sums (2.12) with $m(n) = m_2(n) + 1$ and $m(n) = m_1(n)$ taking the second choices in (2.5) and (2.6) have size

$$O\left(\frac{L'}{\beta|\theta|}\right) + O\left(\sqrt{\beta|\theta|E\left(\frac{1}{\beta}, L'\right)L'}\right). \tag{2.16}$$

P r o o f. To discuss the sum in (2.11), we pick an integer $f \geq 3$. We put $T = |\theta|$. The bound in (2.15) is trivial when $T < 1$, so we may assume that $T \geq 1$. In this proof let I be the integer with $I - 1 < fT \leq I$. We cover the unit interval $(-1/2, 1/2]$ with I subintervals

$$U_i = \left(-\frac{1}{2} + \frac{i-1}{fT}, -\frac{1}{2} + \frac{i}{fT}\right].$$

The last interval U_I extends outside the interval $(-1/2, 1/2]$. We replace the right-hand endpoint of U_I by $1/2$. When $\rho(\beta m + \gamma)$ lies in U_i, then

$$\theta(\rho(\beta m + \gamma) + \gamma - 1/2)$$

lies in a corresponding interval $V_i = \theta(U_i + \gamma - 1/2)$. Here $V_1, \ldots, V_I$ are consecutive intervals, all but the last of length $1/f$. Let $\bar{V}_i$ be the reduction modulo one of the interval V_i, so that $\bar{V}_i = \bar{V}_{i+f}$ for $i + f \leq I - 1$.

The summand m in (2.11) runs through some interval of $k' \leq L$ consecutive integers. Let N_i be the number of values of m in the sum for which $\rho(\beta m + \gamma)$ lies in U_i. The trivial estimate is

$$N_i = O\left(\frac{k'}{fT}\right). \tag{2.17}$$

The Ostrowski discrepancy bound (Lemma 3.1 in the next section) gives

$$N_i = \frac{k'}{fT} + O\left(E(\beta, k')\right) \tag{2.18}$$

for $i = 1, \ldots, I - 1$, and

$$N_I = k'\left(1 - \frac{I-1}{fT}\right) + O\left(E(\beta, k')\right). \tag{2.19}$$

For $\rho(\beta m + \gamma)$ in the interval U_i we have

$$e\left(\theta\left(\rho(\beta m + \gamma) + \gamma - \frac{1}{2}\right)\right) = e\left(\theta\left(\frac{i}{fT} + \gamma - 1\right)\right) + O\left(\frac{1}{f}\right)$$
$$= e\left(\pm\frac{i}{f} + \theta(\gamma - 1)\right) + O\left(\frac{1}{f}\right),$$

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where the $\pm$ sign is taken from the sign of θ. This approximation depends only on $i \bmod f$. For $i = 1, \ldots, f - 1$ we have

$$\sum_{\substack{j=1 \\ j \equiv i \bmod f}}^{I} N_i = \left(T + O(1)\right)\left(\frac{k'}{fT} + O(E(\beta, k'))\right)$$

$$= \frac{k'}{f} + O\left(\frac{k'}{fT}\right) + O(TE(\beta, k')). \qquad (2.20)$$

For $i = f$ the last interval is shorter, and (2.17) gives an extra error term of the same size as the first error term on the right of (2.20). Hence (2.20) holds for $i = f$ also. The sum in (2.11) is

$$\sum_{i=1}^{f} \left(e\left(\pm\frac{i}{f} + \theta(\gamma - 1)\right) + O\left(\frac{1}{f}\right)\right) \sum_{\substack{j=1 \\ j \equiv i \bmod f}}^{I} N_i$$

$$= \sum_{i=1}^{f} e\left(\pm\frac{i}{f} + \theta(\gamma - 1)\right)\left(\frac{k'}{f} + O\left(\frac{k'}{fT}\right) + O(TE(\beta, k'))\right) + O\left(\frac{1}{f} \cdot \frac{Ik'}{fT}\right)$$

$$= O\left(\frac{k'}{T}\right) + O(fTE(\beta, k')) + O\left(\frac{k'}{f}\right),$$

since the main terms involve a full set of fth roots of unity, which cancels. In the order of magnitude terms we use $k' \leq L$. The Ostrowski bound $E(\beta, x)$ is an increasing function, so $E(\beta, k') \leq E(\beta, L)$. We choose

$$f = \left\lceil \sqrt{\frac{L}{TE(\beta, L)}} \right\rceil$$

unless $TE(\beta, L) > 9L$, when the bound (2.15) is trivially true.

The bound (2.16) for the outer sum is over n is proved similarly. $\qquad \square$

LEMMA 2.3 (bounds for Fourier coefficients). *When θ is not an integer, then we have*

$$S(g, h) = O\left(\frac{L}{\|\theta\|}\right). \qquad (2.21)$$

When $\beta\theta$ is not an integer, then we have

$$S(g, h) = O\left(\frac{L'}{\|\beta\theta\|}\right). \qquad (2.22)$$

When neither θ nor $\beta\theta$ is an integer, then we have the bounds

$$S(g, h) = O\left(\frac{1}{\|\theta\|}\left(\frac{L}{|\theta|} + \sqrt{|\theta|E(\beta, L)L}\right)\right) + O\left(\frac{1}{\|\theta\|\,\|\beta\theta\|}\right), \qquad (2.23)$$

and

$$S(g, h) = O\left(\frac{1}{\|\beta\theta\|}\left(\frac{L'}{|\beta\theta|} + \sqrt{|\beta\theta|E\left(\frac{1}{\beta}, L'\right)L'}\right)\right) + O\left(\frac{1}{\|\theta\|\,\|\beta\theta\|}\right). \quad (2.24)$$

There are corresponding bounds in terms of α in which θ is replaced by $\beta\theta$ and we use $\|\theta + h\| = \|\theta\|$ and $\|\alpha(\theta + h)\| = \|\beta\theta\|$.

P r o o f. First we notice that in (2.8)

$$|B(g, h)| = |e(\theta) - 1| = 2|\sin \pi\theta| \geq 4\|\theta\|,$$

and similarly in (2.10)

$$|B'(g, h)| = |e(-\beta\theta) - 1| = \geq 4\|\beta\theta\|.$$

Treating the outer sums trivially, we deduce (2.21) from (2.7) and (2.22) from (2.9).

Our main results (2.23) and (2.24) use the dissection of the double sum over the parallelogram $X_0X_1X_3X_2$. We continue to treat the case when $\beta > \alpha > 0$ and $x_0 < x_1 \leq x_2 < x_3$. In the first dissection given, the outer sums over m are estimated by (2.15) of Lemma 2.2. The outer sums over m on X_0P_1 and P_2X_3 have the form

$$\sum_m e(\theta n_0 - \beta\theta m) = O\left(\frac{1}{\|\beta\theta\|}\right). \qquad (2.25)$$

The inner sums over n are of the form (2.7), and of size

$$O\left(\frac{1}{|B(g, h)|}\right) = O\left(\frac{1}{\|\beta\theta\|}\right). \qquad (2.26)$$

There is also part of the dissection where the inner sum is over m, estimated as in (2.25), and the outer sums are over n on P_1X_1 and X_2P_2, estimated as in (2.26).

Similarly in the alternative dissection, the outer sums over n on X_0Q_1, Q_1X_2, X_1Q_2 and X_2Q_3 are all bounded by (2.16). The outer sums over n on X_0P_1 are of type (2.26), and the part of the dissection with inner sums over n has outer sums over m on P_1X_1 and X_2P_2 of type (2.25). $\qquad\square$

LEMMA 2.4 (combining bounds). *Let $\Delta = \|\theta\|$, $\Delta' = \|\beta\theta\|$. If Δ and Δ' are not both 0, then we have*

$$S(g, h) = O\left(\min\left(\frac{L}{\Delta}, \frac{L'}{\Delta'} \right) \right). \tag{2.27}$$

If Δ and Δ' are both non-zero, and h is non-zero, then

$$S(g, h) = O\left(\frac{1}{\Delta\Delta'} \right) + O\left(\frac{L}{\Delta|h|} \right)$$

$$+ O\left(\frac{1}{\Delta} \sqrt{\frac{|g| + \beta|h|}{\beta - \alpha} \left(E(\alpha, L) + E(\beta, L) \right) L} \right). \tag{2.28}$$

If Δ and Δ' are both non-zero, and g is non-zero, then

$$S(g, h) = O\left(\frac{1}{\Delta\Delta'} \right) + O\left(\frac{L'}{\Delta'|g|} \right)$$

$$+ O\left(\frac{1}{\Delta'} \sqrt{\frac{\beta(|g| + |\alpha h|)}{\beta - \alpha} \left(E\left(\frac{1}{\alpha}, L'\right) + E\left(\frac{1}{\beta}, L'\right) \right) L'} \right). \tag{2.29}$$

P r o o f. The first bound (2.27) follows at once from (2.21) and from (2.22) of Lemma 2.3.

To obtain (2.28), we note that $\max(|\theta|, |\theta + h|) \geq |h|/2$. If the maximum is $|\theta|$, then we use (2.23). If the maximum is $|\theta + h|$, then we use (2.23) with β replaced by α, θ replaced by $\theta + h$. We have

$$\max(|\theta|, |\theta + h|) = \frac{1}{\beta - \alpha} \max\left(|g + \beta h|, |g + \alpha h| \right) \leq \frac{|g| + \beta|h|}{\beta - \alpha}.$$

We obtain (2.29) similarly from (2.24), using

$$\frac{|g|}{2} \leq \max\left(|\beta\theta|, |\beta\theta - g| \right) = \frac{1}{\beta - \alpha} \max\left(|\alpha g + \alpha\beta h|, |\beta g + \alpha\beta h| \right).$$

$\square$

LEMMA 2.5 (linear structure). *Let*

$$\eta = \min \|\theta(g, h)\|, \qquad \eta' = \min \|\beta\theta(g, h)\|, \tag{2.30}$$

with the minima taken over $-2H \leq g \leq 2H$, $-2H \leq h \leq 2H$, with g and h not both 0. Then in the range $-H \leq g \leq H$, $-H \leq h \leq H$, the values of $\theta(g, h)$ reduced modulo one are spaced at least η apart, and the values of $\beta\theta(g, h)$

reduced modulo one are spaced at least η' apart, and in the notation of Lemma 2.1 we have

$$\sum_g{}^*\sum_h \frac{r(g,h)}{\|\theta(g,h)\|} = O\left(\frac{1}{\eta}\right), \quad \sum_g{}^*\sum_h \frac{r(g,h)}{\|\beta\theta(g,h)\|} = O\left(\frac{1}{\eta'}\right). \tag{2.31}$$

P r o o f. The identity $\theta(g,h)-\theta(g',h') = \theta(g-g',h-h')$ gives a relation between the fractional parts of the form

$$\{\theta(g,h)\} - \{\theta(g',h')\} = \{\theta(g-g',h-h')\} + c, \tag{2.32}$$

$$\{\beta\theta(g,h)\} - \{\beta\theta(g',h')\} = \{\beta\theta(g-g',h-h')\} + d, \tag{2.33}$$

where the integers c and d are 0 or ± 1. For $|g| \leq H$, $|h| \leq H$ we have $|g - g'| \leq 2H$, $|h - h'| \leq 2H$, so the right hand sides of (2.32) and (2.33) have absolute values at least η and η', respectively. We deduce the spacing property of Lemma.

The first sum in (2.31) is largest when an ordering of the pairs (g,h) for which $\|\theta(g,h)\|$ is increasing is also an ordering for which $(1+|g|)(1+|h|)$ is increasing, so that $r(g,h)$ is decreasing. In the ordering in which $\|\theta(g,h)\|$ is increasing, the kth pair has

$$\|\theta(g,h)\| \geq \left[\frac{k+1}{2}\right]\eta \geq \frac{k\eta}{2}.$$

By the simple upper bound in Dirichlet's divisor problem, for $K \geq 2$ there are at most $4K(1 + \log K)$ pairs (g,h), not both 0, with $(1 + |g|)(1 + |h|) \leq K$. Hence in the ordering in which $(1 + |g|)(1 + |h|)$ is increasing, the kth pair has

$$r(g,h) = \frac{1}{(1+|g|)(1+|h|)} \geq \frac{1+\log k}{4k}.$$

Hence the first sum in (2.31) is

$$\leq \sum_{k=1}^{\infty} \frac{2}{k\eta} \cdot \frac{1+\log k}{4k} = O\left(\frac{1}{\eta}\right),$$

which establishes the first estimate in (2.31). We establish the second estimate similarly. □

P r o o f   o f   P r o p o s i t i o n   2. We obtain Proposition 2 by substituting (2.27) of Lemma 2.4 and (2.31)) of Lemma 2.5 into Lemma 2.1. The lengths L and L' in Lemma 2.4 are the projections of the parallelogram onto the x and y-axes, both bounded by $k + \ell$. □

LEMMA 2.6 (sums of Koksma coefficients). *In the notation of Lemma 2.1 we have*

$$\sideset{}{^*}\sum_{g}\sum_{h} r(g,h) \leq 25 \log^2 H. \tag{2.34}$$

For $\mu \geq 2$ we have

$$\sideset{}{^*}\sum_{g}\sum_{h} r(g,h) \left(\max(|g|,|h|)\right)^{\mu-1} \leq 6H^{\mu-1} \log H. \tag{2.35}$$

P r o o f. When $H \geq 3$ we have

$$\sideset{}{^*}\sum_{g}\sum_{h} r(g,h) = \sum_{g}\sum_{h} \frac{1}{(1+|g|)(1+|h|)} \leq \left(1 + 2\sum_{1}^{H} \frac{1}{k}\right)^2$$

$$\leq (3 + 2\log H)^2 \leq 25 \log^2 H.$$

Let $K = \max(|g|,|h|)$, $k = \min(|g|,|h|)$. Then we have

$$\sideset{}{^*}\sum_{g}\sum_{h} r(g,h)\left(\max(|g|,|h|)\right)^{\mu-1} = 2\sum_{K=1}^{H} \frac{K^{\mu-1}}{K+1}\left(1 + 2\sum_{k=1}^{K} \frac{1}{k+1}\right)$$

$$\leq 2H^{\mu-1}(2\log H + 1) \leq 6H^{\mu-1} \log H$$

$\square$

3. Diophantine Approximation

Our first lemma is a standard result in the theory of $\{n\alpha\}$ sequences. It is implicit in Ostrowski [7]. This form of the bound corresponds to [1, Corollary 1.64] by Drmota and Tichy. A precise bound is given by Sós [9].

LEMMA 3.1 (the Ostrowski discrepancy bound). *Let α be a real number with continued fraction expansion (finite or infinite) $[a_0; a_1, a_2, \ldots]$ having convergents p_r/q_r. Then for any interval I containing K consecutive integers, and for any real γ, we have*

$$\sum_{m \in I} \rho(\alpha m + \gamma) = O\left(E(\alpha, K)\right),$$

where the partial quotient growth function $E(\alpha, K)$ was defined in the introduction.

Proposition 1, which is also a standard result, is deduced immediately from Lemma 3.1.

LEMMA 3.2. *Suppose that the Linear Forms Condition*

$$\|m\alpha + n\beta\| \geq \frac{1}{B\big(\max(|m|, |n|)\big)^{\mu}}, \tag{3.1}$$

where $B \geq 1$ and $\mu \geq 2$ are real numbers, holds for

$$|m|, |n| \leq k + \ell,$$

and

$$k + \ell \geq B. \tag{3.2}$$

Then in Lemma 3.1 we have

$$E(\alpha, L), E(\beta, L), E(\alpha, L'), E(\beta, L') = O\left(B^{\frac{1}{\mu}}(k + \ell)^{1 - \frac{1}{\mu}}\right). \tag{3.3}$$

If (3.1) *holds for*

$$|m|, |n| \leq \frac{k + \ell}{|\alpha|} + 1, \tag{3.4}$$

and if

$$k + \ell \geq \max\left((4B)^{\mu - 1}, \frac{2^{\mu^2}B}{|\alpha|^{\mu - 1}}\right), \tag{3.5}$$

then

$$E\left(\frac{1}{\alpha}, L\right), E\left(\frac{1}{\alpha}, L'\right) = O\left(B^{\frac{1}{\mu}}\left(\frac{k + \ell}{|\alpha|} + 1\right)^{1 - \frac{1}{\mu}}\right). \tag{3.6}$$

There are corresponding bounds for $E(1/\beta, L)$ and $E(1/\beta, L')$, where we change α to β in (3.4), (3.5) and (3.6).

There are corresponding deductions from the Inverse Linear Forms Condition

$$\left\|\frac{m}{\alpha} + \frac{n}{\beta}\right\| \geq \frac{1}{B'\big(\max(|m|, |n|)\big)^{\mu}}, \tag{3.7}$$

in which we change α and β to $1/\alpha$ and $1/\beta$ or to $1/\beta$ and $1/\alpha$ respectively in (3.3), (3.4), (3.5) and (3.6).

P r o o f. Let p_r/q_r be a convergent in the continued fraction for α. The condition (3.1) gives

$$\frac{1}{a_{r+1}q_r} > \frac{1}{q_{r+1}} \geq |p_r - q_r\alpha| \geq \frac{1}{Bq_r^{\mu}},$$

so that

$$q_{r+1} \leq Bq_r^{\mu}, \qquad a_{r+1} < Bq_r^{\mu - 1}. \tag{3.8}$$

When $q_{r+1} \leq k + \ell$, we use

$$a_{r+1} < \min\left(\frac{q_{r+1}}{q_r}, Bq_r^{\mu-1}\right) \leq \left(Bq_{r+1}^{\mu-1}\right)^{1/\mu}.$$

Let q_R be the denominator with

$$q_R \leq k + \ell \leq q_{R+1} \leq Bq_R^{\mu}. \tag{3.9}$$

Then

$$a_1 + \cdots + a_{R-1} \leq B^{1/\mu}\left(q_2^{1-1/\mu} + \cdots + q_R^{1-1/\mu}\right)$$
$$= O\left(B^{1/\mu}(k+\ell)^{1-1/\mu}\right). \tag{3.10}$$

From (3.9)

$$q_R \geq \left(\frac{k+\ell}{B}\right)^{1/(\mu-1)},$$

$$\frac{k+\ell}{q_R} \leq B^{\frac{1}{\mu-1}}(k+\ell)^{\frac{\mu-2}{\mu-1}} = O\left(B^{\frac{1}{\mu}}(k+\ell)^{\frac{\mu-1}{\mu}}\right), \tag{3.11}$$

by the condition (3.2) of the Lemma. The bounds (3.10) and (3.11) establish (3.3) for $E(\alpha, L)$. The other three bounds in (3.3) follow, similarly.

Similarly if $1/\alpha$ has the continued fraction $[a_0''; a_1'', a_2'', \ldots]$ with convergent p_r''/q_r'', then

$$|p_r''| \leq \frac{q_r''}{|\alpha|} + 1,$$

and when $q_{r+1}'' \leq k + \ell$, then $m = p_r''$ satisfies (3.4), so that

$$\frac{1}{a_{r+1}''q_r''} > \frac{1}{q_{r+1}''} \geq \left|p_r'' - \frac{q_r''}{\alpha}\right| \geq \frac{1}{B|\alpha||p_r''|^{\mu}} \geq \frac{|\alpha|^{\mu-1}}{B(q_r'' + |\alpha|)^{\mu}},$$

and hence,

$$q_{r+1}'' \leq \frac{B(q_r'' + |\alpha|)^{\mu}}{|\alpha|^{\mu-1}}, \qquad a_{r+1}'' \leq \frac{B(q_r'' + |\alpha|)^{\mu}}{|\alpha|^{\mu-1}q_r}.$$

When $q_r'' \leq k + \ell$, we use

$$a_{r+1}'' < \min\left(\frac{q_{r+1}''}{q_r''}, \frac{B(q_r'' + |\alpha|)^{\mu}}{|\alpha|^{\mu-1}q_r''}\right) \leq \frac{q_r'' + |\alpha|}{q_r''}\left(\frac{Bq_{r+1}''^{\mu-1}}{|\alpha|^{\mu-1}}\right)^{1/\mu}.$$

Let q_R'' be the denominator with

$$q_R'' \leq k + \ell \leq q_{R+1}'' \leq \frac{B(q_R'' + |\alpha|)^{\mu}}{|\alpha|^{\mu-1}}.$$

Then

$$a_1'' + \cdots + a_{R-1}'' \leq \frac{B^{1/\mu}}{|\alpha|^{1-1/\mu}} \left(q_2''^{\,1-1/\mu} + \cdots + q_R''^{\,1-1/\mu} \right)$$

$$+ B^{1/\mu}|\alpha|^{1/\mu} \left(\frac{1}{q_2''^{\,1/\mu}} + \cdots + \frac{1}{q_R''^{\,1-1/\mu}} \right)$$

$$= O\left(B^{1/\mu} \left(\frac{k+\ell}{|\alpha|} \right)^{1-1/\mu} \right) + O\left(B^{1/\mu}|\alpha|^{1/\mu} \right), \qquad (3.12)$$

and the first error term absorbs the second by (3.5).

From (3.8)

$$q_R'' + |\alpha| \leq k + \ell + |\alpha| \leq \frac{(B(q_R'' + |\alpha|)^\mu + |\alpha|^\mu)}{|\alpha|^{\mu-1}} \leq \frac{2B(q_R'' + |\alpha|)^\mu}{|\alpha|^{\mu-1}},$$

so that

$$q_R'' + |\alpha| \geq |\alpha| \left(\frac{k+\ell+|\alpha|}{2B} \right)^{\frac{1}{\mu-1}} \geq |\alpha| \left(\frac{1}{2} \left(\frac{k+\ell+|\alpha|}{2B} \right)^{\frac{1}{\mu-1}} + 1 \right), \quad (3.13)$$

and by (3.13) and (3.5)

$$\frac{k+\ell}{q_R''} \leq \frac{k+\ell+|\alpha|}{q_r''} \leq \frac{2(2B)^{\frac{1}{\mu-1}}}{|\alpha|}(k+\ell+|\alpha|)^{1-\frac{1}{\mu-1}}$$

$$\leq B^{\frac{1}{\mu}} \left(\frac{k+\ell+|\alpha|}{|\alpha|} \right)^{1-\frac{1}{\mu}}. \qquad (3.14)$$

We deduce the inequalities of (3.6) from (3.12) and (3.14). The same argument gives the corresponding results involving β, since we have not used the condition $\beta > \alpha$.

We argue similarly from the Inverse Linear Forms Condition (3.7). $\qquad \square$

LEMMA 3.3 (small denominators). *Let Δ and Δ' be real numbers with*

$$0 < \Delta \leq \min\left(\frac{1}{3\beta}, \frac{|\alpha|}{\beta - \alpha} \right), \quad 0 < \Delta' \leq \frac{1}{3}. \qquad (3.15)$$

Suppose that

$$|B(g,h)| \leq 2\sin\pi\Delta, \qquad |B'(g,h)| \leq 2\sin\pi\Delta'. \qquad (3.16)$$

Then there are integers c and d and small real numbers δ and ϵ with $|\delta| \leq \Delta$, $|\epsilon| \leq \Delta'$ for which

$$\alpha = \frac{d - g + \epsilon}{c + h + \delta}, \qquad \beta = \frac{d + \epsilon}{c + \delta}. \qquad (3.17)$$

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If $c = 0$, then $d = 0$, and g and h must both be non-zero. If $c \neq 0$, then

$$|\beta c - d| \leq \left(\frac{1}{2} + \beta\right)\Delta + \frac{3}{2}\Delta',$$

and if

$$1 \leq |c| < \frac{1}{(1 + 2\beta)\Delta + 3\Delta'}, \tag{3.18}$$

then the lowest terms form of d/c is a convergent p'/q' to the continued fraction for β.

If $c = -h$, then $d = g$, and g and h must both be non-zero. If $c \neq -h$, then

$$|\alpha(c + h) - d + g| \leq \left(\frac{1}{2} + |\alpha|\right)\Delta + \frac{3}{2}\Delta',$$

and if

$$1 \leq |c + h| < \frac{1}{(1 + 2|\alpha|)\Delta + 3\Delta'}, \tag{3.19}$$

then the lowest terms form of $(d - g)/(c + h)$ is a convergent p/q to the continued fraction for α.

P r o o f. We write

$$\theta = \frac{g + h\alpha}{\beta - \alpha} = c + \delta, \qquad \beta\theta = \frac{(g + h\alpha)\beta}{\beta - \alpha} = d + \epsilon, \tag{3.20}$$

where c and d are integers, and

$$-1/2 < \delta \leq 1/2, \quad -1/2 < \epsilon \leq 1/2.$$

Dividing these equations gives the expression for β in (3.17). Also

$$\theta + h = c + h + \delta = \frac{g + h\beta}{\beta - \alpha}, \quad \beta\theta - g = d - g + \epsilon = \frac{(g + h\beta)\alpha}{\beta - \alpha}. \tag{3.21}$$

Dividing these equations gives the expression for α in (3.17).

We have

$$|B(g, h)| = |2\sin\theta\, e(\theta)| = 2|\sin\pi\delta|.$$

The angle $\pi\delta$ is in the first or the fourth quadrant, so the first inequality in (3.16) is equivalent to $|\delta| \leq \Delta$. Similarly the inequality for $B'(g, h)$ in (3.16) is equivalent to $|\epsilon| \leq \Delta'$.

If $c = 0$, then $|d| = |\beta\delta - \epsilon| < 1$ by (3.15), so that $d = 0$ also, and $\epsilon = \beta\delta$. From (3.15) we see that if $c = d = g = 0$, then $h = 0$, and if $c = d = h = 0$, then $g = 0$. However the values of g and h in our sum are never both 0.

If $c \neq 0$, then

$$\frac{2|c|}{3} \leq |\theta| \leq \frac{4|c|}{3}, \quad |d| \leq |\beta\theta| + \frac{1}{3} \leq \left(\beta + \frac{1}{2}\right)|\theta|.$$

Hence

$$|\beta c - d| = \left| \frac{c(d + \epsilon)}{c + \delta} - d \right| = \left| \frac{\epsilon c - \delta d}{c + \delta} \right|$$

$$\leq \frac{1}{|\theta|} \left(\frac{3}{2} \Delta' |\theta| + \Delta \left(\frac{1}{2} + \beta \right) |\theta| \right) = \left(\frac{1}{2} + \beta \right) \Delta + \frac{3}{2} \Delta'.$$

The condition $|\beta c - d| < 1/2|c|$ ensures that the lowest terms form p'/q' of d/c is a convergent to the continued fraction for β.

Arguing similarly from (3.21), we see that if $c = -h$, then $d = g$, and if $d = g = 0$, then $h = 0$ also, or if $c = h = 0$, then $g = 0$ also, which is impossible. If $c \neq h$, then we use

$$\alpha(c + h) - (d - g) = \frac{(c + h)(d - g + \epsilon)}{c + h + \delta} - d + g = \frac{\epsilon(c + h) - \delta(d - g)}{c + h + \delta},$$

and we argue similarly. $\qquad\square$

LEMMA 3.4 (approximations to θ). *Let g and h be fixed. Let H and K be parameters with $|g| \leq H$, $|h| \leq H$, and K so large that*

$$\|\theta\| = \Delta < \frac{KL}{N} < \frac{K(k + \ell)}{N},$$

$$\|\beta\theta\| = \Delta' < \frac{KL'}{N} < \frac{K(k + \ell)}{N}, \tag{3.22}$$

where Δ and Δ' satisfy the hypotheses (3.15) of Lemma 3.3. For $x \geq 1$ let

$$A(x) = \frac{(\beta - \alpha)N}{2(1 + \beta)(1 + 2\beta)(k + \ell)x} - 2. \tag{3.23}$$

Suppose that $A(HK) \geq 0$. Then in Lemma 3.3 either $c = d = 0$, or the lowest terms form of d/c is a convergent p'_s/q'_s to the continued fraction for β. If the continued fraction does not terminate at p'_s/q'_s, then the next partial quotient a'_{s+1} satisfies

$$a'_{s+1} \geq \frac{N}{(2 + \beta)q'_s(k + \ell)K} - 2 \geq A(HK). \tag{3.24}$$

Also either $c = -h$ and $d = g$, or the lowest terms form of $(d - g)/(c + h)$ is a convergent p_r/q_r to the continued fraction for α. If the continued fraction does not terminate at p_r/q_r, then the next partial quotient a_{r+1} satisfies

$$a_{r+1} \geq \frac{N}{(2 + \beta)q_r(k + \ell)K} - 2 \geq A(HK). \tag{3.25}$$

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P r o o f. In (3.18) and (3.19) of Lemma 3.3 we have

$$(1 + 2|\alpha|)\Delta + 3\Delta' \leq (1 + 2\beta)\Delta + 3\Delta'$$
$$< \frac{K\big((1+2\beta)L + 3L'\big)}{N} < \frac{(4+2\beta)(k+\ell)K}{N}, \qquad (3.26)$$

and

$$|c| \leq \frac{1}{2} + \frac{(1+\beta)H}{\beta - \alpha} \leq \frac{\beta}{\beta - \alpha} + \frac{(1+\beta)H}{\beta - \alpha} < \frac{2(1+\beta)H}{\beta - \alpha}. \quad (3.27)$$

Hence

$$|c|\big((1+2\beta)\Delta + 3\Delta'\big) < \frac{2(1+\beta)H}{\beta - \alpha} \cdot \frac{2(2+\beta)(k+\ell)K}{N} = \frac{2}{A(HK) + 2} \leq 1,$$
$$(3.28)$$

since $A(HK) \geq 0$. Hence (3.18) of Lemma 3.3 holds, and either $c = d = 0$, or d/c is a convergent to β. The same bounds (3.27) and (3.28) hold with c replaced by $c + h$, so (3.19) holds, and either $c = -h$, $d = g$, or $(d - g)/(c + h)$ is a convergent to α.

Let p_r/q_r and p_{r+1}/q_{r+1} be consecutive convergents to α. In the case r even we have

$$\frac{p_r}{q_r} < \frac{p_r + p_{r+1}}{q_r + q_{r+1}} < \alpha < \frac{p_{r+1}}{q_{r+1}};$$

for r odd the inequalities are reversed. Hence

$$\left| \frac{p_r}{q_r} - \alpha \right| > \left| \frac{p_r + p_{r+1}}{q_r + q_{r+1}} - \frac{p_r}{q_r} \right| = \frac{1}{q_r(q_r + q_{r+1})}$$
$$= \frac{1}{q_r((a_{r+1} + 1)q_r + q_{r-1})} \geq \frac{1}{(a_{r+1} + 2)q_r^2}.$$

If $(d - g)/(c + h)$ reduces in lowest terms to the convergent p_r/q_r, then

$$|\alpha(c + h) - d + g| \geq |\alpha q_r - p_r| \geq \frac{1}{(a_{r+1} + 2)q_r}.$$

By Lemma 3.3 we have

$$\frac{1}{a_{r+1} + 2} \leq q_r |\alpha(c + h) - d + g| \leq q_r \left(\left(\frac{1}{2} + \beta \right) \Delta + \frac{3\Delta'}{2} \right).$$

Substituting the inequality (3.26) gives the first inequality in (3.25). Using $q_r \leq |c + h|$ and substituting (3.26), or more conveniently, (3.28) with c replaced by $c + h$, gives the full inequality of (3.25). The corresponding calculations with the continued fraction for β give (3.24). $\square$

4. Proofs of the Theorems

P r o o f s o f T h e o r e m s 1 a n d 2. We take $K = H \log^2 H$ in Lemma 3.4. The two conditions in (3.22) of Lemma 3.4 imply

$$\frac{1}{\Delta}, \quad \frac{1}{\Delta'} \geq \frac{N}{H(k+\ell)\log^2 H} = \frac{2(1+\beta)(2+\beta)(A(H^2 \log^2 H) + 2)H}{\beta - \alpha}$$

$$> \frac{2(1+\beta)(2+\beta)AH}{\beta - \alpha} \geq \frac{2(1+\beta)(2+\beta)H}{\beta - \alpha}$$

$$\geq \max\left(3 + 3\beta, \frac{\beta - \alpha}{|\alpha|}\right), \tag{4.1}$$

since $A \geq 1$, $\beta - \alpha \leq 2\beta$, and H satisfies (1.6). The conditions (3.15) of Lemma 3.3 follow from (4.1). We recall the notation

$$Q = \frac{2(1+\gamma)H}{\beta - \alpha}$$

of (1.7). By Lemma 3.4, either $d/c = p'_s/q'_s$ for some convergent to β with $a'_{s+1} > A$, so $q'_s > Q$ by the choice of A, or $(d - g)/(c + h) = p_r/q_r$ for some convergent to α with $a_{r+1} > A$, so $q_r > Q$ by the choice of A. But when $|g| \leq H$ and $|h| \leq H$, then (3.27) gives $q'_s \leq |c| \leq Q$, and its analogue for $c + h$ gives $q_r \leq |c + h| \leq Q$. Hence the conditions in (3.22) cannot both hold under the assumptions of Theorem 2, and

$$\min\left(\frac{1}{\Delta}, \frac{1}{\Delta'}\right) < \frac{N}{H(k+\ell)\log^2 H}.$$

By (2.21) or (2.22) of Lemma 2.3, we have

$$S(g, h) = O\left(\frac{N}{H \log^2 H}\right).$$

By (2.34) of Lemma 2.6 we have in Lemma 2.1

$$\sum_{g}^{*}\sum_{h} r(g, h)S(g, h) = O\left(\frac{N}{H}\right).$$

Thus both error terms in Lemma 2.1 are $O(N/H)$, which is the result of Theorem 2. More precisely, there is a constant c_4 which can be calculated from c_1, c_2 and c_3 such that $|D(\kappa, \lambda)| \leq c_4 N/H$ under the conditions of Theorem 2.

To prove Theorem 1, we must show that for any $\epsilon > 0$, we can take $H > c_4/\epsilon$ when k and ℓ are sufficiently large. The choice of H determines $K = H \log^2 H$, and in (3.23) of Lemma 3.4 we have

$$x = HK = H^2 \log^2 H.$$

In Proposition 1, N grows like $k\ell$. As a function of k and ℓ with x fixed, the first term on the right of (3.23) grows like $\min(k, \ell)$. For $\min(k, \ell)$ sufficiently large, $A(H^2 \log^2 H)$ is positive in (3.23), and $|D(\kappa, \lambda)| = O(\epsilon N)$. This establishes Theorem 1. $\qquad\square$

P r o o f   o f   T h e o r e m   3. We use Lemma 3.3 to estimate the Weyl sums $S(g, h)$ in Lemma 2.1 We choose Δ and Δ' as large as possible in (3.15). If either inequality in (3.16) is false, then

$$S(g, h) = O\left(\left(\beta + \frac{\beta - \alpha}{|\alpha|}\right)(k + \ell)\right) = O\big(\beta^2(k + \ell)\big), \qquad (4.2)$$

by (2.17) of Lemma 2.4.

When both inequalities in (3.16) hold, then we write

$$\lambda = \epsilon(c + h) - \delta(d - g), \qquad \mu = \epsilon c - \delta d.$$

There are two cases. When $c \neq 0$, then $d/c = p'/q'$ in its lowest terms, with

$$\left|\frac{\mu}{c(c + \delta)}\right| = \left|\frac{d + \epsilon}{c + \delta} - \frac{d}{c}\right| = \left|\beta - \frac{d}{c}\right| = \left|\beta - \frac{p'}{q'}\right|$$

$$\geq \frac{1}{(A + 1)q'^2} \geq \frac{1}{2Ac^2} \geq \frac{1}{3A|c(c + \delta)|},$$

so that

$$|\mu| \geq 1/3A, \quad \text{and} \quad \max(|\epsilon c|, |\delta d|) \geq \frac{1}{6A}.$$

In the second case when $c + h \neq 0$, we have similarly

$$\max(|\epsilon(c + h)|, |\delta(d - g)|) \geq \frac{1}{6A}.$$

Now

$$|c| = \left|\frac{g + h\alpha}{\beta - \alpha} - \delta\right| \leq \frac{|g| + |h\alpha|}{\beta - \alpha} + \frac{1}{3}, \qquad (4.3)$$

and similarly,

$$|c + h| \leq \frac{|g| + \beta|h|}{\beta - \alpha} + \frac{1}{3}, \qquad (4.4)$$

$$|d| \leq \frac{\beta(|g| + |h\alpha|)}{\beta - \alpha} + \frac{1}{3}, \qquad (4.5)$$

$$|d - g| \leq \frac{|\alpha|(|g| + \beta|h|)}{\beta - \alpha} + \frac{1}{3}, \qquad (4.6)$$

Hence either

$$|\delta| \geq \min\left(\frac{1}{4A}, \frac{\beta - \alpha}{12A\beta(|g| + \beta|h|)}\right) = \frac{\beta - \alpha}{12A\beta(|g| + \beta|h|)},$$

or

$$|\epsilon| \geq \min\left(\frac{1}{4A}, \frac{\beta - \alpha}{12A(|g| + \beta|h|)}\right).$$

Then (2.27) of Lemma 2.4 gives

$$S(g, h) = O\left(\frac{A(|g| + \beta|h|)(k + \ell)}{\beta - \alpha}\right). \tag{4.7}$$

By (2.34) of Lemma 2.6, the case (4.2) contributes

$$O(\beta^2(k + \ell)\log^2 H)$$

to the sum over $r(g, h)S(g, h)$ in Lemma 2.1. By the case $\mu = 2$ in (2.35) of Lemma 2.6, the case (4.7) contributes

$$O\left(\frac{A\beta H(k + \ell)\log H}{\beta - \alpha}\right) \tag{4.8}$$

to the same sum. If

$$(\beta - \alpha)N \geq 9A\beta(k + \ell)\log N, \tag{4.9}$$

we choose

$$H = \left[\sqrt{\frac{(\beta - \alpha)N}{A\beta(k + \ell)\log N}}\right],$$

so the term $O(N/H)$ and the bound (4.8) both become

$$O\left(\sqrt{\frac{A\beta N(k + \ell)\log N}{\beta - \alpha}}\right). \tag{4.10}$$

If (4.9) is false, then the term (4.10) is at least as big as the trivial bound $O(N)$.
$\square$

Proof of Theorem 4. We rearrange the expression for θ in (3.20) to get

$$(c + h)\alpha - c\beta + g = \delta(\beta - \alpha),$$

so if $|\delta| \leq 1/2(\beta - \alpha)$, then

$$\|(c + h)\alpha - c\beta\| = (\beta - \alpha)|\delta|.$$

From (4.3) and (4.4)

$$\max(|c|, |c + h|) \leq \frac{|g| + \beta|h|}{\beta - \alpha} + \frac{1}{3} \leq \frac{|g| + \beta(|h| + 2)}{\beta - \alpha},$$

so that the Linear Forms Condition (1.11) gives

$$|\delta| = \|\theta(g, h)\| \geq \frac{1}{B(\beta - \alpha)}\left(\frac{\beta - \alpha}{|g| + \beta(|h| + 2)}\right)^\mu. \tag{4.11}$$

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The expression raised to the μth power in (4.11) is less than 1, so (4.11) holds trivially when $|\delta| > 1/2(\beta - \alpha)$.

For $|g| \leq 2H$, $|h| \leq 2H$ with g and h not both 0, (4.11) gives

$$|\delta| = \|\theta(g,h)\| \geq \eta \geq \frac{1}{B(\beta - \alpha)} \left(\frac{\beta - \alpha}{5\beta H} \right)^{\mu}.$$

By (2.21) of Lemma 2.3 and (2.31) of Lemma 2.5 we have

$$\sum_g^* \sum_h r(g,h) S(g,h) = O\left((k+\ell) \sum_g^* \sum_h \frac{r(g,h)}{\|\theta(g,h)\|} \right) = O\left(\frac{k+\ell}{\eta} \right)$$

$$= O\left(\frac{B(5\beta H)^{\mu}(k+\ell)}{(\beta - \alpha)^{\mu-1}} \right).$$

Let

$$J = \left[\left(\frac{(\beta - \alpha)^{\mu-1} N}{B(6\beta)^{\mu}(k+\ell)} \right)^{\frac{1}{\mu+1}} \right].$$

When $J \geq 3$ we choose $H = J$ in Lemma 2.1 to obtain the bound (1.13) of Theorem 4. When $J < 3$, then (1.13) follows from the trivial bound $D(\kappa, \lambda) = O(N)$. We have used (1.11) for

$$|c|, |c+h| \leq \frac{6\beta J}{\beta - \alpha} \leq \left(\frac{6\beta N}{B(\beta - \alpha)^2(k+\ell)} \right)^{\frac{1}{\mu+1}}.$$

Similarly we rearrange the expression for $\beta\theta$ in (3.20) to get

$$\frac{d}{\alpha} - \frac{d-g}{\beta} = h - \frac{\epsilon(\beta - \alpha)}{\alpha\beta}.$$

If

$$|\epsilon| \leq \frac{|\alpha|\beta}{2(\beta - \alpha)}, \tag{4.12}$$

then

$$\left\| \frac{d}{\alpha} - \frac{d-g}{\beta} \right\| \leq \frac{(\beta - \alpha)|\epsilon|}{|\alpha|\beta}.$$

From (4.5) and (4.6)

$$\max(|d|, |d-g|) \leq \frac{\beta|g| + |\alpha\beta h|}{\beta - \alpha} + \frac{1}{3} \leq \frac{\beta(|g|+2) + |\alpha\beta h|}{\beta - \alpha}.$$

The Inverse Linear Forms Condition (4.14) gives

$$|\epsilon| = \|\beta\theta(g,h)\| \geq \frac{|\alpha|\beta}{B'(\beta - \alpha)} \left(\frac{\beta - \alpha}{\beta(|g|+2) + |\alpha\beta h|} \right)^{\mu}. \tag{4.13}$$

The expression raised to the μth power in (4.13) is less than 1, so (4.13) holds trivially when (4.12) is false.

For $|g| \leq 2H$, $|h| \leq 2H$ with g and h not both 0, (4.13) gives

$$|\epsilon| = \|\beta\theta(g,h)\| \geq \eta' \geq \frac{|\alpha|\beta}{B'(\beta-\alpha)}\left(\frac{\beta-\alpha}{(3+2|\alpha|\beta H}\right)^{\mu}.$$

By (2.22) of Lemma 2.3 and (2.31) of Lemma 2.5, we have

$$\sum_{g}^{*}\sum_{h} r(g,h)S(g,h) = O\left((k+\ell)\sum_{g}^{*}\sum_{h}\frac{r(g,h)}{\|\beta\theta(g,h)\|}\right) = O\left(\frac{k+\ell}{\eta'}\right)$$

$$= O\left(\frac{B(3+2|\alpha|)^{\mu}\beta^{\mu-1}H^{\mu}(k+\ell)}{|\alpha|(\beta-\alpha)^{\mu-1}}\right).$$

Let

$$J = \left[\left(\frac{|\alpha|(\beta-\alpha)^{\mu-1}N}{B'(3+2|\alpha|)^{\mu}\beta^{\mu-1}(k+\ell)}\right)^{\frac{1}{\mu+1}}\right].$$

When $J \geq 3$ we choose $H = J$ in Lemma 2.1 to obtain the bound (1.16) of Theorem 4. When $J < 3$, then (1.16) follows from the trivial bound $D(\kappa, \lambda) = O(N)$. We have used (1.14) for

$$|d|, |d-g| \leq \frac{(3+2|\alpha|)\beta J}{\beta-\alpha} \leq \left(\frac{|\alpha|(3+2|\alpha|)\beta^2 N}{B'(\beta-\alpha)^2(k+\ell)}\right)^{\frac{1}{\mu+1}},$$

which is the range (1.15).

When both Linear Forms Conditions hold, then we can use (2.28) and (2.29) of Lemma 2.4. We combine these bounds as

$$S(g,h) = O\left(\frac{1}{\|\theta\|\|\beta\theta\|}\right) + O\left((k+\ell)\min\left(\frac{1}{|h|\,\|\theta\|}, \frac{1}{|g|\,\|\beta\theta\|}\right)\right)$$

$$+ O\left(\frac{1}{\|\theta\|}\sqrt{\frac{(|g|+\beta|h|)(E(\alpha,L)+E(\beta,L))L}{\beta-\alpha}}\right)$$

$$+ O\left(\frac{1}{\|\beta\theta\|}\sqrt{\frac{\beta(|g|+|\alpha h|)}{\beta-\alpha}\left(E\left(\frac{1}{\alpha},L'\right)+E\left(\frac{1}{\beta},L'\right)\right)L'}\right). \quad (4.14)$$

Each of these four terms has to be estimated carefully. In the first term we can use Lemma 2.5 either on $\eta = \min\|\theta(g,h)\|$ or on $\eta' = \min\|\beta\theta(g,h)\|$ to obtain

$$\sideset{}{^*}\sum_{g}\sum_{h}\frac{r(g,h)}{\|\theta(g,h)\|\,\|\beta\theta(g,h)\|}=O\left(\frac{1}{\eta\eta'}\right)=O\left(\frac{BB'5^{\mu}(3+2|\alpha|)^{\mu}\beta^{2\mu-1}H^{2\mu}}{|\alpha|(\beta-\alpha)^{2\mu-2}}\right),$$

$$(4.15)$$

by (4.11) and (4.13).

In the third term in (4.14) we use Lemma 3.2 to bound the expression inside the square root by

$$O\left(\frac{\beta H}{\beta-\alpha}.B^{\frac{1}{\mu}}(k+\ell)^{2-\frac{1}{\mu}}\right).$$

Hence by (2.31) of Lemma 2.5 and (4.11), the third term in (4.14) contributes

$$O\left(\frac{B^{1+1/2\mu}5^{\mu}(\beta H)^{\mu+1/2}(k+\ell)^{(2\mu-1)/2\mu}}{(\beta-\alpha)^{\mu-1/2}}\right) \qquad (4.16)$$

towards the sum $\sum\sum^{*}r(g,h)S(g,h)$.

Similarly by the analogue of (3.3) in Lemma 3.2 for the Inverse Linear Forms Condition, we bound the expression inside the square root in the fourth term in (4.14) by

$$O\left(\frac{(1+|\alpha|)\beta H}{\beta-\alpha}.B'^{\frac{1}{\mu}}(k+\ell)^{2-\frac{1}{\mu}}\right).$$

Hence by the bound (4.13) and (2.31) of Lemma 2.5, the fourth term contributes

$$O\left(\frac{B'^{1+1/2\mu}\sqrt{1+|\alpha|}(3+2|\alpha|)^{\mu}\beta^{\mu-1/2}H^{\mu+1/2}(k+\ell)^{(2\mu-1)/2\mu}}{|\alpha|(\beta-\alpha)^{\mu-1/2}}\right) \qquad (4.17)$$

towards the sum $\sum\sum^{*}r(g,h)S(g,h)$. We combine the terms (4.16) and (4.17) as

$$O\left(\sqrt{D}H^{\mu+1/2}(k+\ell)^{(2\mu-1)/2\mu}\right), \qquad (4.18)$$

where

$$D=\left(B^{2+1/\mu}5^{2\mu}\beta^{2}+\frac{B'^{2+1/\mu}(1+|\alpha|)(3+2|\alpha|)^{2\mu}}{|\alpha|^{2}}\right)\left(\frac{\beta}{\beta-\alpha}\right)^{2\mu-1}.$$

The second term in (4.14) presents the most complication. Let $K=\max(g,h)$. Then (4.11) and (4.13) give

$$\min\left(\frac{1}{|h|\,\|\theta(g,h)\|},\frac{1}{|g|\,\|\beta\theta(g,h)\|}\right)$$

$$=O\left(\min\left(\frac{B(|g|+\beta(|h|+2))^{\mu}}{(\beta-\alpha)^{\mu-1}|h|},\frac{B'\beta^{\mu-1}(|g|+2+|\alpha h|)^{\mu}}{|\alpha|(\beta-\alpha)^{\mu-1}|g|}\right)\right)$$

$$=O\left(\left(\frac{B(3\beta)^{\mu}}{(\beta-\alpha)^{\mu-1}}+\frac{B'(2+|\alpha|)^{\mu}\beta^{\mu-1}}{|\alpha|(\beta-\alpha)^{\mu-1}}\right)K^{\mu-1}\right)=O\left(D'K^{\mu-1}\right).$$

Hence by (2.35) of Lemma 2.6, the second term in (4.14) contributes

$$O\big(D'H^{\mu-1}(k+\ell)\log H\big) \tag{4.19}$$

towards the sum $\sum\sum^* r(g,h)S(g,h)$.

We choose H so that all the contributions (4.15), (4.18) and (4.19) are $O(N/H)$. Let

$$J = \left[\left(\frac{N^2}{D(k+\ell)^{2-1/\mu}}\right)^{\frac{1}{2\mu+3}}\right].$$

As before, when $J \geq 3$, we choose $H = J$ in Lemma 2.1 to estimate the term (4.18) as

$$O\left(\left(DN^{2\mu+1}(k+\ell)^{2-\frac{1}{\mu}}\right)^{\frac{1}{2\mu+3}}\right). \tag{4.20}$$

When $J < 3$, then (4.20) follows from the trivial bound $D(\kappa,\lambda) = O(N)$.

The bound (4.15) is of the form

$$O\big(D''H^{2\mu}\big) = O\left(\left(\frac{D''^{2\mu+3}N^{4\mu}}{D^{2\mu}(k+\ell)^{4\mu-2}}\right)^{\frac{1}{2\mu+3}}\right), \tag{4.21}$$

smaller than the estimate (4.20) provided that

$$N = O\left(\left(\frac{D^{2\mu+1}}{D''^{2\mu+3}}\right)^{1/(2\mu-1)}(k+\ell)^{2+1/\mu}\right),$$

which follows from the trivial estimate $N = O((k+\ell)^2)$ provided that $k+\ell$ is sufficiently large in terms of α, β, B and B'.

Let

$$J' = \left[\left(\frac{N}{D'(k+\ell)\log N}\right)^{\frac{1}{\mu}}\right].$$

When $H = J'$, the term (4.19) is $O(N/H)$, which is

$$O\left(\big(D'N^{\mu-1}(k+\ell)\log N\big)^{1/\mu}\right), \tag{4.22}$$

larger than (4.20) when $J' \leq J$, which occurs when

$$N^3 \leq \frac{D'^{2\mu+3}}{D^\mu}.(k+\ell)^4(\log N)^{2\mu+3}. \tag{4.23}$$

The inequality (4.23) can hold when k and ℓ have different orders of magnitude in Proposition 1, so we require the term (4.22). When $H = J' \leq J$, then (4.21) still holds, so the term (4.15) is the smallest.

We have used the Linear Forms Conditions for

$$|m|, |n| \leq \frac{5\beta H}{\beta - \alpha} \leq \frac{5\beta}{\beta - \alpha} \max(J, J') \leq \frac{5\beta N^{\frac{1}{\mu}}}{\beta - \alpha}. \qquad \square$$

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