Uniform Distribution Theory **8** (2013), no.1, 165–189

uniform distribution theory

SEVERAL QUESTIONS AND HYPOTHESES CONCERNING LIMIT POLYNOMIALS FOR THE CHACON$_{(3)}$ TRANSFORMATION

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ABSTRACT. We study the weak closure $\mathscr{L} = \mathrm{WCl}(\{\hat{T}^k\})$ of powers of non-singular Chacon transformation with 2-cuts. It is still an open question does $\mathscr{L}$ contain any Markov operator except an orthogonal projector to the constants Θ and some polynomials $P(\hat{T})$? In this paper we calculate a particular set of limit polynomials

$$P_m(\hat{T}) = \lim_{n\to\infty} \hat{T}^{-mh_n}, \qquad m \in \mathbb{Z},$$

where $h_n = (3^n - 1)/2$ are the sequence of heights of towers in a standard rank one representation of the Chacon map. We show that for any $d \geq 2$ the family of limit polynomials contains infinitely many polynomials of degree d. We also formulate hypotheses and open questions concerning the sequence of polynomials P_m and the entire set $\mathscr{L}$.

Communicated by Ilya Shkredov

1. Introduction

Chacon$_{(3)}$ transformation in terms of symbolic dynamics can be defined as a substitution system over the finite alphabet $\mathbb{A} = \{0, 1\}$ via a pair of substitution rules $0 \mapsto 0010$, $1 \mapsto 1$. Starting with an initial word $w_0 = 0$ and applying the substitution transform we construct the sequence of words w_n,

$$w_0 = 0,$$
$$w_1 = 0010,$$

2010 Mathematics Subject Classification: 28D05, 11B83, 12D10.
Keywords: Chacon transformation, spectral invariants, weak closure of powers, zeros of polynomials, irreducibility.
The authors are grateful to E. Janvresse, T. de la Rue, K. Petersen, A. Bufetov and I. Shkredov for discussions and helpful remarks. This work is supported by CNRS (France), RFFI grant No. 11-01-00759-a, RFFI grant No. 12-01-33020 and the grant "Leading Russian scientific schools" No. NSh-5998.2012.1.

$$w_2 = 0010001010010,$$
$$w_3 = 0010001010010001000101001010010001010010,$$
$$\dots$$

and then define an infinite word w_∞ such that each w_n is a prefix of w_∞. Further, considering the closure X of all shifts of w_∞ in the space $\mathbb{A}^\infty$ endowed with the Tikhonov topology we come to a topological dynamical system $(S, X, \mathcal{B})$, where $\mathcal{B}$ is the σ-algebra of Borel sets and T is the shift transformation,

$$T\colon\ \dots, x_0, x_1, \dots, x_j, \dots \ \mapsto\ \dots, x_1, x_2, \dots, x_{j+1}, \dots$$

Let us consider a natural invariant measure μ on the measurable space $(X, \mathcal{B})$ defined as follows. For a finite word w let $\mu([w])$ be the empirical probability of observing w in w_∞, where where $[w]$ is the set encoded by w:

$$[w] \overset{def}{=} \{x \in X\colon\ x_0 = w(0), \dots, x_{\ell-1} = w(\ell - 1)\},$$

$\ell = |w|$ is the length of w and $w(j)$ denotes the letter at position j in w.

Definition 1. The map T considered as a measure-preserving invertible transformation of the probability space $(X, \mathcal{B}, \mu)$ is called *non-singular Chacon transformation with* 2*-cuts* or *Chacon*$_{(3)}$ *transformation* (see [Cha69, Fri70]).

Transformation T has an interesting combination of ergodic properties. The map T is known to be weakly mixing and power weakly mixing [Dan04], but not strongly mixing [Cha69]. It has trivial centralizer [dJ78] and minimal self-joining [dJRS80]. It is also known that the spectral measure σ of Chacon transformation T is singular and its convolutions satisfy the following condition of pairwise singularity [PR],

$$\sigma \perp \sigma * \sigma,$$
$$\sigma * \sigma \perp \sigma * \sigma * \sigma,$$
$$\dots$$
$$\sigma^{*k} \perp \sigma^{*\ell} \qquad \text{for any} \quad k \neq \ell.$$

The study of convolutions of the spectral type measure σ goes back to the Kolmogorov's question concerning the hypothetic group property of spectrum: *is it true that* $\sigma * \sigma \ll \sigma$? This property holds for the discrete part of spectrum, but it is generally false for the singular component. Moreover, now we know many examples of ergodic transformations T such that $\sigma * \sigma \perp \sigma$ (see [Ose69, Ste87, Goo99, dJL92]).

For a survey of problems in modern spectral theory of dynamical systems the reader can refer to [Lem09] and [KT07].

Definition 2. We say that a map T is *mixing* if

$$\mu(T^k A \cap B) \to \mu(A)\,\mu(B) \quad \text{as} \quad k \to \infty,$$

for any measurable sets A and B, and we call T *weakly mixing* if the convergence holds for a subsequence k_j.

Both mixing and weak mixing properties can be described in spectral terms.

Definition 3. Let $\hat{T}$ be the unitary *Koopman operator*, associated with T and acting in the separable Hilbert space $H = L^2(X, \mu)$ by the following rule

$$\hat{T}\colon\ f(x) \mapsto f(Tx).$$

A sequence of bounded linear operators $\mathcal{A}_j\colon\ H \to H$ in a Hilbert space H *converges weakly* to $\mathcal{A}$ if for any $f, g \in H$,

$$\langle \mathcal{A}_j f,\, g\rangle \to \langle \mathcal{A} f,\, g\rangle\,, \qquad j \to \infty.$$

Let Θ denote the orthogonal projector to constants,

$$(\Theta f)(x) \equiv \int_X f(z)\, d\mu(z).$$

A transformation T is weakly mixing if and only if

$$T^{k_j} \to \Theta$$

for some subsequence k_j. It means that Θ is in the weak closure $\mathscr{L} = \mathrm{WCl}(\{\hat{T}^k\})$ of powers $\hat{T}^k$.

2. Limit polynomials

In our investigation [PR] to prove the pairwise singularity of the convolutions σ^{*k} we used the following observation.

Lemma 4. *In the weak close of powers $\mathscr{L} = \mathrm{WCl}(\{\hat{T}^k\})$ for Chacon transformation T one can find an infinite family of non-trivial square polynomials*

$$Q_m(\hat{T}) = \frac{(3^s - 1)\mathbb{I} + 2(3^s + 1)\hat{T} + (3^s - 1)\hat{T}^2}{4 \cdot 3^s}$$

for $m = 3^s + 1$ and, moreover,

$$Q_m(\hat{T}) = \lim_{n\to\infty} \hat{T}^{m h_n - l_s},$$

where $l_s = h_s = (3^s - 1)/2$ and $\mathbb{I}$ is the identity operator.

In order to understand this phenomenon let us consider a simpler case.

Lemma 5. *There exists a sequence $k_j \to \infty$ such that*

$$\hat{T}^{k_j} \to \frac{\mathbb{I} + \hat{T}}{2}.$$

Proof. Another way to define Chacon transformation as a measure-preserving transformation is to use the concept of rank one transformation.

Definition 6. Let T be a measure-preserving transformation of a probability space $(X, \mathcal{B}, \mu)$. Then T is called *rank one transformation* if there exists a sequence of Rokhlin tower partitions

$$\xi_j = \{B_j, TB_j, T^2B_j, \ldots, T^{h_n-1}B_j, E_j\}$$

of the phase space such that $\mu(E_j) \to 0$ and for any measurable set A one can find ξ_j-measurable sets A_j approximating A: $\mu(A_j \triangle A) \to 0$ as $j \to \infty$.

In fact, Chacon transformation is rank one and can be constructed using so-called cutting-and-stacking construction.

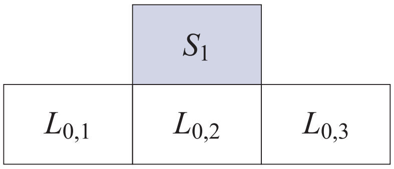

Figure 1. Chacon$_{(3)}$ transformation: several steps in the cutting-and-stacking construction: $n = 1$.

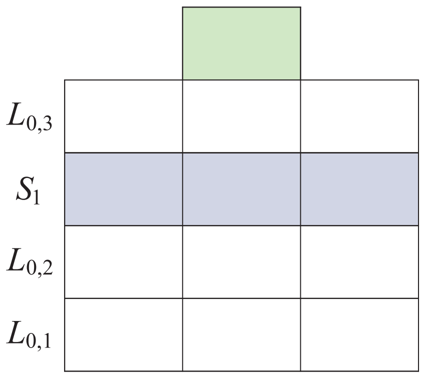

Figure 2. Cutting-and-stacking construction: $n = 2$.

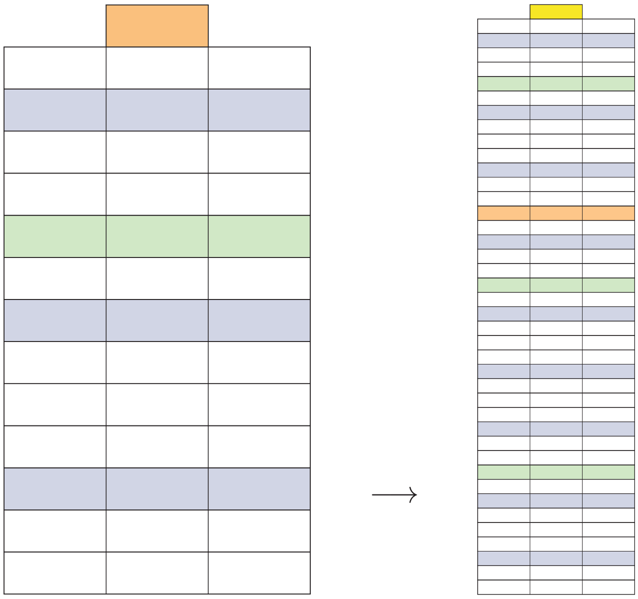

FIGURE 3. Cutting-and-stacking construction: $n = 3$ and $n = 4$.

CONSTRUCTION 7. We start with a unit segment $[0, 1]$ interpreted as a Rokhlin tower U_0 of height $h_0 = 1$. Then we cut this segment twice, in three equal parts

$$L_{1,0} = [0, 1/3), \quad L_{1,1} = [1/3, 2/3), \quad L_{1,2} = [2/3, 1],$$

and add one additional "level", a segment S_1 of length $1/3$ which is drawn above the middle part $[1/3, 2/3)$ (see Fig. 1),

$$\begin{matrix} & S_1 & \\ L_{1,0} & L_{1,1} & L_{1,2}\,. \end{matrix}$$

Now we stack all these segments in the natural order: $L_{1,0}\, L_{1,1}\, S_1\, L_{1,2}$ and we get the next Rokhlin tower U_1 of height $h_1 = 4$ (see Fig. 2). In other words, we assume that

$$L_{1,0} \xrightarrow{T} L_{1,1} \xrightarrow{T} S_1 \xrightarrow{T} L_{1,2},$$

and T will be defined on $L_{1,2}$ on the next steps of the construction. We repeat the same procedure with the new tower: we cat it in three equal columns, put one additional level to the top of the middle column and stack together (Fig. 1–3).

At each step of the construction we have a Rokhlin tower U_n of height $h_n = (3^n - 1)/2$. It can be easily checked that this sequence serves as an approximating sequence of Rokhlin towers in the definition of rank one transformation.

Note that if we draw all the additional level above the corresponding subcolumns without restacking the tower U_n at the step n, we come to the following representation of the Chacon map (see Fig. 4).

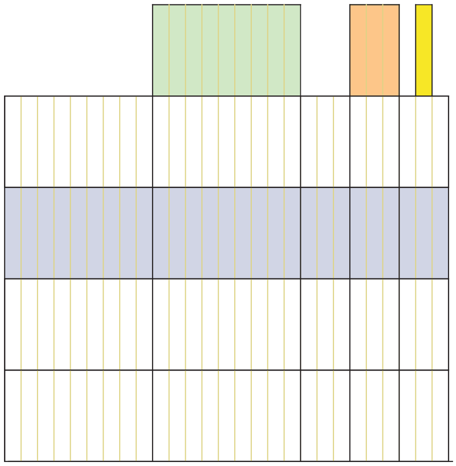

FIGURE 4. Chacon$_{(3)}$ transformation drawn without restacking.

CONSTRUCTION 8. Let us consider a compact group of 3-adic integers $\Gamma = \mathbb{Z}_{(3)}$. We associate Γ with the set of one-sided 3-adic sequences

$$y = (y_1, y_2, \dots, y_k, \dots), \qquad y_k \in \{0, 1, 2\}.$$

As a measure space Γ is isomorphic to the unit segment $[0,1]$ by the mapping

$$y \mapsto \sum_{k=1}^{\infty} \frac{1}{3^k} y_k.$$

It follows easily from the cutting-and-stacking construction that Chacon map T is the integral transformation over the adding machine transformation

$$S\colon \Gamma \to \Gamma\colon\ y \to y + 1$$

acting on the base level of the tower U_n identified with Γ with the ceiling function $r_n(y) = h_n + \phi_0(y)$ (see Fig. 5),

$$\phi_0(y) = \begin{cases} 0 & \text{if } y = \mathtt{22\dots20*}, \\ 1 & \text{if } y = \mathtt{22\dots21*}, \end{cases}$$

where $*$ indicates any symbol in alphabet $\{0, 1, 2\}$ if put inside a block, and any infinite sequence of symbols if it ends the block.

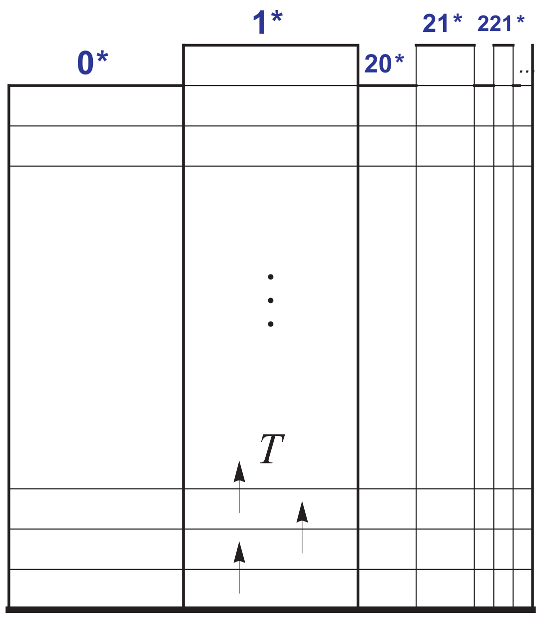

FIGURE 5. Chacon$_{(3)}$ transformation: cutting-and-stacking construction and cocycle $\phi_0(y)$.

Now we are ready to finish the proof of Lemma 5. It can be easily checked that any measurable function $f \in L^2(X, \mu)$ is approximated by functions constant on levels of a tower in the rank one representation (e.g., see [Fer97]). So, without loss of generality we may assume that f is constant on the levels of some tower U_{n_0}. Then f is constant on levels of any tower U_n with $n > n_0$. Let us partition U_n for each $n \geq n_0$ into sets $U_n^{(0)}$ and $U_n^{(1)}$ according to the value of the cocycle $\phi_0(y)$, where y is considered as a point in the base of U_n. We see that

$$f\big(T^{h_n} x\big) = f(x) \quad \text{if} \quad x \in U_n^{(0)}$$

and

$$f\big(T^{h_n} x\big) = f\big(T^{-1} x\big) \quad \text{if} \quad x \in U_n^{(1)}$$

for all points $x \in U_n$ except the first level B_n of the tower U_n (observe that $\mu(B_n) \to 0$). Thus,

$$\hat{T}^{h_n} \to \frac{\mathbb{I} + \hat{T}^{-1}}{2}$$

in the weak topology, since $\mu(U_n^{(0)}) = \mu(U_n^{(0)}) = 1/2$, and applying conjugation we complete the proof. □

Analyzing the effect used in the proof we see that Lemma 5 can be easily extended in the following way. Given $m \in \mathbb{N}$ let us consider the sum

$$\phi_0^{(m)} = \phi_0(y) + \phi_0(Sy) + \cdots \phi_0\big(S^{m-1}y\big)$$

and define the corresponding distribution ρ_m of the values of $\phi_0^{(m)}$. Actually, ρ_m is the measure on $\mathbb{Z}$ with a finite support, and $\rho_m(A) = \lambda((\phi_0^{(m)})^{-1}(A))$, where λ is the Haar probability measure on Γ.

Lemma 9. *For any $m \in \mathbb{N}$ the sequence $\hat{T}^{-mh_n}$ converges weakly to a polynomial $P_m(\hat{T})$ depending on $\hat{T}$, and*

$$P_m(\hat{T}) \stackrel{def}{=} \lim_{n\to\infty} \hat{T}^{-mh_n} = \int_{\mathbb{Z}} \hat{T}^k \, d\rho_m(k) = \int_{\Gamma} \hat{T}^{\phi_0^{(m)}(y)} \, d\lambda(y).$$

The scheme of the proof can be found in [PR], and the idea can be explained as follows. Passing the tower U_n m times we count (in addition to mh_n) the values of the cocycle $\phi_0(y)$ at the points

$$\phi_0(y), \quad \phi_0(Sy), \quad \ldots \quad \phi_0\big(S^{m-1}y\big).$$

Thus, $\hat{T}^{-mh_n}$ converges weakly to the weighted sum of powers $\hat{T}^k$ with respect to the distribution ρ_m.

Let us compute several first polynomials $P_n(\hat{T})$:

$$\begin{aligned}
P_1(\hat{T}) &= \frac{1}{2}(\mathbb{I} + \hat{T}),\\
P_2(\hat{T}) &= \frac{1}{6}\big(\mathbb{I} + 4\hat{T} + \hat{T}^2\big),\\
P_3(\hat{T}) &= \frac{1}{2}\big(\hat{T} + \hat{T}^2\big),\\
P_4(\hat{T}) &= \frac{1}{9}\big(2\hat{T} + 5\hat{T}^2 + 2\hat{T}^3\big),\\
P_5(\hat{T}) &= \frac{1}{18}\big(\hat{T} + 8\hat{T}^2 + 8\hat{T}^3 + \hat{T}^4\big).
\end{aligned}$$

Since the weak closure $\mathrm{WCl}(\{\hat{T}^j\})$ is invariant under multiplication by $\hat{T}^s$ for any $s \in \mathbb{Z}$ we can reduce the polynomials $P_m(\hat{T})$ dividing by the smallest power l_m of $\hat{T}$ in $P_m(\hat{T})$. Set

$$\widetilde{P}_m(z) = z^{-l_m} \cdot P_m(z).$$

Let us represent $\widetilde{P}_m(z)$ in the form

$$\widetilde{P}_m(z) = a_{m,0} + a_{m,1}z + \cdots + a_{m,d(m)}z^{d(m)},$$

where $d(m) = \deg \widetilde{P}_m(z)$.

Lemma 10. *The coefficients $a_{m,j} \in \mathbb{Q}$ satisfy the following Markov property*

$$\sum_{j=0}^{d(m)} a_{m,j} = 1, \quad \textit{and} \quad a_{m,j} \geq 0.$$

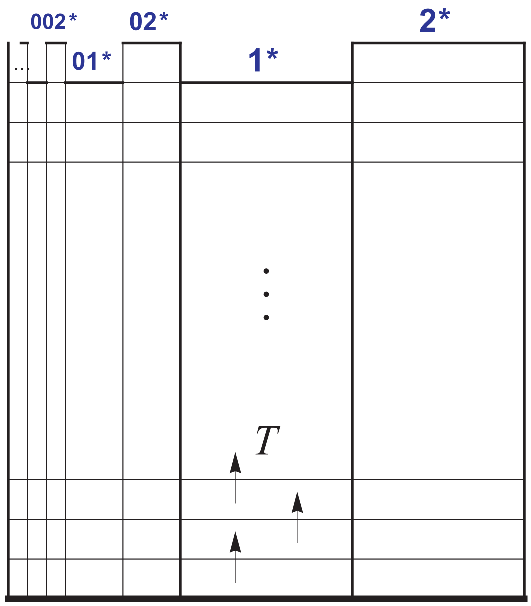

Figure 6. Chacon$_{(3)}$ map after the coordinate change $y \mapsto y + 1$ and the graph of the cocycle $\phi(y)$.

In Table 1 of the Appendix we list the first 122 polynomials $\widetilde{P}_m(z)$.

Let us discuss several remarks explaining the structure of this table. First, for simplicity of calculations we apply the transform $y \mapsto y + 1$ to the base of the tower U_n and consider the following cocycle $\phi(y)$ instead of $\phi_0(y)$ (see Fig. 6),

$$\phi(y) = \begin{cases} 0 & \text{if } y = \texttt{00} \ldots \texttt{01*}, \\ 1 & \text{if } y = \texttt{00} \ldots \texttt{02*}. \end{cases}$$

The function $\phi(y)$ is more convenient for calculation of the iterates $\phi(S^k y)$.

Lemma 11 (see [PR]). *For any power 3^ℓ of three we have*

$$\phi^{(3^\ell)}(y) = \begin{cases} 0 & \textit{if } y = *^\ell(\texttt{0})\texttt{1*}, \\ 1 & \textit{if } y = *^\ell(\texttt{0})\texttt{1*}, \end{cases}$$

where the notation (0) *is used for any sequence of zeros (including empty sequence), and* $*^{\ell}$ *denotes an arbitrary word of length* ℓ*. An equivalent way how we can state this property of the cocycle is to say that*

$$\phi^{(3^{\ell})}(y) = \phi(Ay),$$

where A *is the non-invertible left shift:*

$$A(y_1 y_2 \cdots y_k \cdots) = y_2 y_3 \cdots y_{k+1} \cdots$$

It follows immediately from this lemma that polynomials $\widetilde{P}_m(z)$ repeat after multiplication by 3,

$$P_{3m}(z) = P_m(z).$$

Theorem 12. *For any* $d \in \mathbb{N}$ *the family* $\{\widetilde{P}_m\}$ *contains infinitely many polynomials of degree* d*.*

Proof. The theorem is based on the following observation. If we consider functions $\phi^{(m_1)}(y)$ and $\phi^{(m_2)}(S^k y)$ as random variables defined on Γ, then $\phi^{(m)}(y)$ and $\phi(y)$ are almost independent. This means that for any $\varepsilon_0 > 0$ and any pair of sets $B_1 = \{y\colon\ \phi^{(m_1)}(y) = v_1\}$ and $B_2^{(k)} = \{y\colon\ \phi^{(m_2)}(S^k y) = v_2\}$ we have

$$\left|\mu(B_1 \cap B_2^{(k)}) - \mu(B_1)\mu(B_2^{(k)})\right| < \varepsilon$$

for sufficiently big k. Thus, extending the proof of Lemma 4 we see that configurations

$$m(\ell_1, \ldots, \ell_{d-1}) = \mathtt{1}\mathtt{0}^{\ell_1}\mathtt{1}\mathtt{0}^{\ell_2}\mathtt{100}\ \ldots\ \mathtt{0}^{\ell_{d-1}}\mathtt{1}_3$$

generates for sufficiently big ℓ_j polynomials $P_{(\ell_j)}(z)$ of degree d such that

$$\lim_{\ell_j \to \infty} P_{(\ell_j)}(\hat{T}) = \frac{1}{2^d}(\mathbb{I} + \hat{T})^d.$$

Here $\alpha^1\alpha^2 \cdots \alpha_3^N$ (a sequence of digits with index "3") stands for the 3-adic expansion of an integer number. □

To illustrate the construction used in the proof let us consider configuration

$$m(\ell_1, \ell_2) = \mathtt{1}\mathtt{0}^{\ell_1}\mathtt{1}\mathtt{0}^{\ell_2}\mathtt{1}_3 = \mathtt{1}\overbrace{\mathtt{00}\ \ldots\ \mathtt{0}}^{\ell_1}\mathtt{1}\overbrace{\mathtt{000}\ \ldots\ \mathtt{0}}^{\ell_2}\mathtt{1}_3\,.$$

Set

$$p^{-[i,j]} = p^{-i} + p^{-i-1} + \cdots + p^{-j},$$

and notice that

$$3^{-[1,\infty]} = \lim_{j \to \infty} 3^{-[1,j]} = 1/2\,.$$

Lemma 13. *$\mathcal{P}_{m(\ell_1,\ell_2)}$ is a self-reciprocal polynomial,*

$$\mathcal{P}_{m(\ell_1,\ell_2)} = \gamma z^3 + (1/2-\gamma)z^2 + (1/2-\gamma)z + \gamma,$$

where

$$\gamma = 3^{-[1,\ell_1]}3^{-[1,\ell_2]} + 3^{-[1,\ell_1]}3^{-(\ell_2+1)} + 3^{-(\ell_1+1)}3^{-[1,\ell_2]}.$$

P r o o f. The proof of this lemma is very close to that of Lemma 4. □

In the next section we formulate a set of hypotheses concerning the properties of the limit polynomials $\widetilde{P}_m(z)$. In Hypothesis 1 we conjecture that all polynomials $\widetilde{P}_m(z)$ are self-reciprocal, i.e., they have coefficients $a_{m,j}$ symmetric with respect to the transform $j \mapsto d(m)-j$, where $d(m) = \deg \widetilde{P}_m$,

$$\widetilde{P}_m(z) = \sum_{j=0}^{d(m)} a_{m,j}z^j, \qquad a_{m,j} = a_{m,d(m)-j}.$$

In other words, the sequence $a_{m,j}$ is symmetric.

Note that Hypothesis 1 stated in section 3 implies that, whenever $d(m)$ is *odd*, the point (-1) is always a root of $\widetilde{P}_m$. Nevertheless, we could ask is it the only way to factorize $\widetilde{P}_m$?

Theorem 14. *The family of limit polynomials $\mathcal{P}_m(z)$ contains infinitely many cubic polynomials for which $R_m(z) = (z+1)^{-1}\widetilde{P}_m(z)$ are irreducible over $\mathbb{Q}$.*

We use in the proof the following theorem.

Lemma 15 (Eisenstein's criterion)**.** *Consider a polynomial $P \in \mathbb{Q}[z]$,*

$$P(z) = a_n z^n + \cdots + a_1 z + a_0,$$

and suppose that there exists a prime number p sush that

$$p \nmid a_n, \qquad p^2 \nmid a_0,$$
$$p \mid a_j \quad \text{for} \quad j = 0, 1, \ldots, n-1.$$

Then $P(z)$ is irreducible over $\mathbb{Q}$.

P r o o f o f T h e o r e m 14. Indeed, consider cubic polynomials given by configurations $10^{\ell_1}10^{\ell_2}1$ with $\ell_1 = \ell_2$ (see Table 2 of the Appendix). With a simplified notation $\ell = \ell_1$ we have

$$\mathcal{P}_{m(\ell,\ell)}(z) = \frac{(3a^2+2a)(z^3+1) + (3^{2\ell+1} - 3a^2 - 2a)(z^2+z)}{2\cdot 3^{2\ell+1}},$$

where

$$3^{-[1,\ell]} = \frac{1}{3} + \cdots + \frac{1}{3^\ell} = \frac{a}{3^\ell}, \qquad \gcd(a,3) = 1.$$

Next, let us apply the transform $z = -1 + w$ to $\mathcal{P}_{m(\ell,\ell)}$. We get a new polynomial

$$P^*(w) = \big(3a^2 + 2a\big)w^3 + \big(3^{2\ell+1} - 8a - 12a^2\big)(w-1).$$

What are the common divisors of

$$X = 3a^2 + 2a \quad \text{and} \quad Y = 3^{2\ell+1} - 8a - 12a^2\ ?$$

We have

$$Y + 4X = 3^{2\ell+1}$$

and, at the same time,

$$X = a(3a+2)$$

is factorized in two numbers, both are relatively prime to 3. Thus, taking any prime divisor of Y and applying Eisenstein's criterion we see that $P^*(w)$ is irreducible over $\mathbb{Q}$. □

It is interesting to remark that the quadratic polynomials given in Lemma 4 are factorized over $\mathbb{Q}$, thus, to see that there exists irreducible polynomial $\widetilde{P}_m(z)$ we have to consider a particular example:

$$\widetilde{P}_2(z) = \frac{1}{6}\big(z^2 + 4z + 1\big).$$

Substituting $z = -1 + w$ we get a polynomial

$$P^*(w) = \frac{1}{6}\big(w^2 + 2z - 2\big).$$

We can apply Eisenstein's criterion to P^*, since 2 divides all the coefficient except the coefficient in z^2, and 4 do not divide -2.

3. Questions and hypotheses

Hypothesis 1. The limit polynomials $\widetilde{P}_m(z)$ are *self-reciprocal*, that is

$$\widetilde{P}_m(z) = \sum_{k=0}^{d(m)} a_k z^k, \qquad a_k = a_{d(m)-k}\,.$$

Corollary. *If Hypothesis 1 is true, then -1 is a root of a polynomial $\widetilde{P}_m(z)$, whenever $d(m) \in 2\mathbb{Z}+1$.*

We have to mention that most questions below presume or at least require Hypothesis 1 for a particular m.

Definition 16. Consider two configurations of the same length

$$c_1c_2\dots c_N \quad \text{and} \quad c_1'c_2'\dots c_N',$$

with $c_j, c_j' \in \{0,1,2\}$. We say that m' is *conjugate* to m and write $m' = m^*$ if $c_j' = c_{N+1-j}$.

Hypothesis 2. The polynomials $\widetilde{P}_m(z)$ and $\widetilde{P}_{m^*}(z)$ coincide for any pair of conjugate configurations m and m^*.

It can be observed from Table 1 that some polynomials coincide even for non-conjugate configurations, for example, for $m = 10 = \mathtt{101}_3$ and $m' = 26 = \mathtt{222}_3$.

Question 3. Which pairs of polynomials $\widetilde{P}_m(z)$ and $\widetilde{P}_{m'}(z)$ coincide?

Let $|m|_3$ be the length of the 3-adic expansion of m if $m \notin 3\mathbb{Z}$, and let $|m|_3 = |3^{-1}m|_3$ otherwise.

Hypothesis 4. $P_m^{\mathbb{Z}}(z) = 2{\cdot}3^{|m|_3}{\cdot}\widetilde{P}_m(z)$ is a polynomial with integer coefficients,

$$P_m^{\mathbb{Z}} = b_{m,0} + b_{m,1}z + \dots + b_{m,d(m)}z^d.$$

The greatest common divisor of $b_{m,j}$ is 1 or 2.

This is a well-known fact that the set of all weak limits of powers $\mathscr{L}$ is a semigroup. Thus, it is a natural question: can we get a polynomial $P_m(z)$ as a product of two different elements of $\mathscr{L}$?

Hypothesis 5. The polynomial $\widetilde{P}_m(z)$ has two or more factors which are not $(z+1)$ if and only if (see Table 3)

$$|m|_3 \in 2\mathbb{Z} \quad \text{and} \quad m = m^*.$$

For example, the polynomial

$$\widetilde{P}_{68}(z) = \frac{1}{81}\big(3+5z+z^2\big)\big(1+5z+3z^2\big)$$

corresponds to a symmetric configuration $68 = \mathtt{2112}_3$.

In particular, if Hypothesis 5 is true, then the roots r_j of a polynomial $P_m^{\mathbb{Z}}(z)$ starting with $z^d + \dots$ are algebraic integers.

Hypothesis 6. All roots of any $\widetilde{P}_m(z)$ are real numbers (*Lee—Yang property*).

Remark 17. It follows directly from Hypothesis 1 as well as the definition of the polynomials $\widetilde{P}_m(z)$ that the roots of $\widetilde{P}_m(z)$ must be negative, and they appear in pairs: r and r^{-1}.

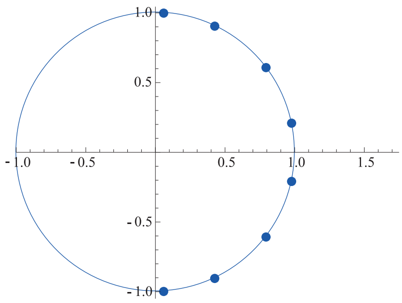

FIGURE 7. Roots of the polynomials Q_{1094}.

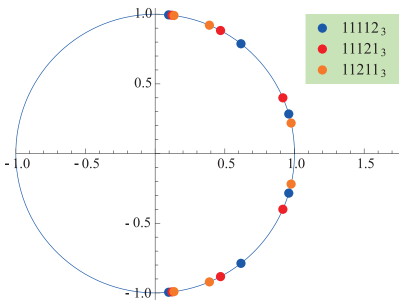

FIGURE 8. Roots of the polynomials Q_{122}, Q_{124} and Q_{130}.

Now, if we assume Hypotheses 1, 5 and 6, then applying to our polynomials the transformation

$$z = \kappa_1(z) = i\frac{z-1}{z+1}$$

mapping $\mathbb{R}$ to the unit circle in the complex plane, we can define the *dual polynomials*

$$Q_m(w) = \widetilde{P}_m\big(\kappa_1(z)\big).$$

HYPOTHESIS 7. The polynomials $Q_m(w)$ are self-reciprocal polynomials having all roots λ_j on the unit circle and in the right-half plane:

$$|\lambda_j| = 1, \qquad \operatorname{Re}\lambda_j > 0.$$

Let us consider, for example, the polynomials:

$$P^{\mathbb{Z}}_{122}(z) = z^6 + 26z^5 + 120z^4 + 192z^3 + 120z^2 + 26z + 1,$$
$$P^{\mathbb{Z}}_{124}(z) = z^6 + 23z^5 + 119z^4 + 200z^3 + 119z^2 + 23z + 1,$$
$$P^{\mathbb{Z}}_{130}(z) = z^6 + 22z^5 + 120z^4 + 200z^3 + 120z^2 + 22z + 1$$

corresponding to the configurations:

$$122 = \texttt{11112}_3, \qquad 124 = \texttt{11121}_3, \qquad 130 = \texttt{11211}_3.$$

The dual polynomials $Q_m(w)$ are

$$Q_{122}(w) = \tfrac{-2i}{486}\big(35w^6 - 117w^5 + 209w^4 - 250w^3 + 209w^2 - 117w + 35\big),$$
$$Q_{124}(w) = \tfrac{-i}{486}\big(77w^6 - 232w^5 + 415w^4 - 496w^3 + 415w^2 - 232w + 77\big),$$
$$Q_{130}(w) = \tfrac{-2i}{486}\big(39w^6 - 117w^5 + 205w^4 - 250w^3 + 205w^2 - 117w + 39\big),$$

and the root of these polynomials are shown on Fig. 8.

Question 8. What is the asymptotic behaviour of the distributions ρ_m?

Remark. In the proof of Theorem 12 we consider, for a given degree d, a set of polynomials corresponding to configurations

$$m = \texttt{1}\ \texttt{0}^{\ell_1}\ \texttt{10}^{\ell_2}\ \texttt{1}\ \ldots\ \texttt{0}^{\ell_{d-1}}\ \texttt{1}_3,$$

where ones are separated by long sequences of zeroes. These configurations generate sums $\phi^{(m)}$ which are reduced to sums of d almost independent random variables, and, in particular,

$$\widetilde{P}_m(z) \to \frac{1}{2^d}(1+z)^d, \qquad \ell_j \to \infty.$$

Thus, it is easy to see that the corresponding distributions ρ_m converge to the binomial distribution.

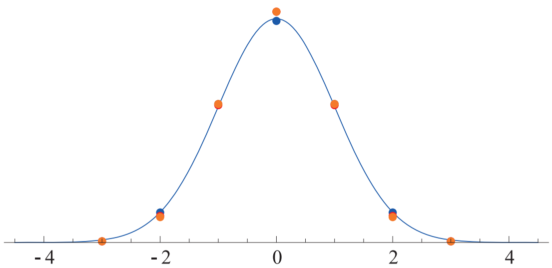

Figure 9. The distributions ρ_{122}, ρ_{124} and ρ_{130} and the normal distribution.

Question 9. Is it true that the distributions ρ_m, centered and scaled, converge to the normal distribution as $d(m) \to \infty$ independently on the structure of $\widetilde{P}_m(z)$? (see Fig. 9).

Hypothesis 10. The first polynomial $\widetilde{P}_m(z)$ of degree d is observed at index

$$m = \frac{3^{d-1}+1}{2}.$$

This means that $\deg \widetilde{P}_m = d$ and $\deg \widetilde{P}_s < d$ for $s < m$. Moreover, if $m \in 2\mathbb{Z}$, then the corresponding polynomials $P^{\mathbb{Z}}_m(z)$ are irreducible (over $\mathbb{Q}$) monic self-reciprocal polynomials. If m is odd, the same is true for $(z+1)^{-1}P^{\mathbb{Z}}_m(z)$. The roots r_j of $P^{\mathbb{Z}}_m(z)$ are algebraic integers, moreover, $r_j \in \mathbb{R}$.

Remark 18. In particular, if Hypothesis 10 is true, then the following estimate holds

$$d(m) \le 1 + \log_3(2m-1).$$

Question 11. How $d(m)$ depends on m?

Question 12. Is it true that no one polynomial $\widetilde{P}_m(z)$ divides another polynomials in $\mathbb{Q}[z]$, and any $\widetilde{P}_m(z)$ is never a product of different polynomials $\widetilde{P}_{m'_k}(z)$ of smaller degree?

Question 13. Is it true that for any m operator $\widetilde{P}_m(\boldsymbol{T})$ is not a product of different operators $A_j \in \mathscr{L}$ in the weak closure of powers of Chacon transformation $\hat{T}$?

Question 14. Can we find $\widetilde{P}_m(z)$ which is an isolated point in the semigroup generated by all $\{\widetilde{P}_{m'}\}$, and can we find $\widetilde{P}_m(\boldsymbol{T})$ which is an isolated point in $\mathscr{L}$?

Question 15. Is it true that the set $\mathscr{L}$ contains operators $\sum_j a_j \hat{T}^j$, where infinitely many $a_j \neq 0$? Is it possible to find among elements $V \in \mathscr{L}$ operators of the form

$$V = \varkappa\Theta + \sum_j a_j \hat{T}^j, \qquad \varkappa \neq 0\,?$$

The following well-known question still has no answer as well.

Question 16. Is Chacon$_{(3)}$ transformation $\varkappa$-mixing, which means that there exists $V \in \mathscr{L}$ such that

$$V = \varkappa\Theta + V_2, \quad \varkappa \neq 0, \quad V_2 \neq 0\,?$$

Hypothesis 17. There exists a global constant $\varepsilon_0 > 0$ such that the following is true. Among the polynomials $P_m(\hat{T})$ as well as in the set $\mathscr{L}$ there is no polynomials $\sum_j a_j \hat{T}^j$ satisfying the property $|a_{j+1}/a_j - 1| < \varepsilon_0$ for any j, whenever $a_{j+1}, a_j > 0$.

Question 18. How we can describe the entire set $\mathscr{L}$ for Chacon$_{(3)}$ transformation?

4. Appendix: The limit polynomials

Table 1. The first 122 limit polynomials $\widetilde{P}_m(z)$.

The columns of this table indicate: the number m, 3-adic expansion of m (configuration), and the polynomial $\widetilde{P}_m(z)$. We mark by * the indexes corresponding to configurations $\mathtt{111}\ldots\mathtt{12}_3$. We skip symmetrical configurations like $\mathtt{112}_3 \sim \mathtt{211}_3$ following Hypothesis 2 which is verified to be true in this interval.

Configuration m	**Polynomial** $\widetilde{P}_m(z)$
$1^* = \mathtt{1}_3$	$\widetilde{P}_1(z) = \widetilde{P}_3(z) = \widetilde{P}_9(Z) = \cdots = \frac{1}{2}(1+z)$
$2^* = \mathtt{2}_3$	$\widetilde{P}_2(z) = \widetilde{P}_6(z) = \cdots = \frac{1}{6}(1+4z+z^2)$
$4 = \mathtt{11}_3$	$\widetilde{P}_4(z) = \frac{1}{9}(2z^2+5z+2)$
$5^* = \mathtt{12}_3$	$\widetilde{P}_5(z) = \frac{1}{18}(z^3+8z^2+8z+1)$
$8 = \mathtt{22}_3$	$\widetilde{P}_8(z) = \frac{1}{9}(2z^2+5z+2)$
$10 = \mathtt{101}_3$	$\widetilde{P}_{10}(z) = \frac{1}{54}(13z^2+28z+13)$
$11 = \mathtt{102}_3$	$\widetilde{P}_{11}(z) = \frac{1}{54}(4z^3+23z^2+23z+4)$
$13 = \mathtt{111}_3$	$\widetilde{P}_{13}(z) = \frac{1}{54}(5z^3+22z^2+22z+5)$
$14^* = \mathtt{112}_3$	$\widetilde{P}_{14}(z) = \frac{1}{54}(z^4+13z^3+26z^2+13z+1)$
$16 = \mathtt{121}_3$	$\widetilde{P}_{16}(z) = \frac{1}{54}(z^4+12z^3+28z^2+12z+1)$

Configuration m	**Polynomial** $\widetilde{P}_m(z)$
$17 = \mathtt{122}_3$	$\widetilde{P}_{17}(z) = \frac{1}{54}\big(4z^3 + 23z^2 + 23z + 4\big)$
$20 = \mathtt{202}_3$	$\widetilde{P}_{20}(z) = \frac{1}{54}\big(z^4 + 12z^3 + 28z^2 + 12z + 1\big)$
$23 = \mathtt{212}_3$	$\widetilde{P}_{23}(z) = \frac{1}{54}\big(5z^3 + 22z^2 + 22z + 5\big)$
$26 = \mathtt{222}_3$	$\widetilde{P}_{26}(z) = \frac{1}{54}\big(13z^2 + 28z + 13\big)$
$28 = \mathtt{1001}_3$	$\widetilde{P}_{28}(z) = \frac{1}{81}\big(20z^2 + 41z + 20\big)$
$29 = \mathtt{1002}_3$	$\widetilde{P}_{29}(z) = \frac{1}{162}\big(13z^3 + 68z^2 + 68z + 13\big)$
$31 = \mathtt{1011}_3$	$\widetilde{P}_{31}(z) = \frac{1}{162}\big(17z^3 + 64z^2 + 64z + 17\big)$
$32 = \mathtt{1012}_3$	$\widetilde{P}_{32}(z) = \frac{1}{81}\big(2z^4 + 20z^3 + 37z^2 + 20z + 2\big)$
$34 = \mathtt{1021}_3$	$\widetilde{P}_{34}(z) = \frac{1}{162}\big(4z^4 + 39z^3 + 76z^2 + 39z + 4\big)$
$35 = \mathtt{1022}_3$	$\widetilde{P}_{35}(z) = \frac{1}{162}\big(16z^3 + 65z^2 + 65z + 16\big)$
$38 = \mathtt{1102}_3$	$\widetilde{P}_{38}(z) = \frac{1}{162}\big(5z^4 + 39z^3 + 74z^2 + 39z + 5\big)$
$40 = \mathtt{1111}_3$	$\widetilde{P}_{40}(z) = \frac{1}{81}\big(3z^4 + 20z^3 + 35z^2 + 20z + 3\big)$
$41^* = \mathtt{1112}_3$	$\widetilde{P}_{41}(z) = \frac{1}{162}\big(z^5 + 19z^4 + 61z^3 + 61z^2 + 19z + 1\big)$

Configuration m	**Polynomial** $\widetilde{P}_m(z)$
$43 = 1121_3$	$\widetilde{P}_{43}(z) = \frac{1}{162}\big(z^5 + 17z^4 + 63z^3 + 63z^2 + 17z + 1\big)$
$44 = 1122_3$	$\widetilde{P}_{44}(z) = \frac{1}{81}\big(2z^4 + 20z^3 + 37z^2 + 20z + 2\big)$
$47 = 1202_3$	$\widetilde{P}_{47}(z) = \frac{1}{162}\big(z^5 + 16z^4 + 64z^3 + 64z^2 + 16z + 1\big)$
$50 = 1212_3$	$\widetilde{P}_{50}(z) = \frac{1}{162}\big(5z^4 + 39z^3 + 74z^2 + 39z + 5\big)$
$52 = 1221_3$	$\widetilde{P}_{52}(z) = \frac{1}{81}\big(2z^4 + 18z^3 + 41z^2 + 18z + 2\big)$
$53 = 1222_3$	$\widetilde{P}_{53}(z) = \frac{1}{162}\big(13z^3 + 68z^2 + 68z + 13\big)$
$56 = 2002_3$	$\widetilde{P}_{56}(z) = \frac{1}{81}\big(2z^4 + 18z^3 + 41z^2 + 18z + 2\big)$
$59 = 2012_3$	$\widetilde{P}_{59}(z) = \frac{1}{162}\big(z^5 + 17z^4 + 63z^3 + 63z^2 + 17z + 1\big)$
$62 = 2022_3$	$\widetilde{P}_{62}(z) = \frac{1}{162}\big(4z^4 + 39z^3 + 76z^2 + 39z + 4\big)$
$68 = 2112_3$	$\widetilde{P}_{68}(z) = \frac{1}{81}\big(3z^4 + 20z^3 + 35z^2 + 20z + 3\big)$
$71 = 2122_3$	$\widetilde{P}_{71}(z) = \frac{1}{162}\big(17z^3 + 64z^2 + 64z + 17\big)$
$80 = 2222_3$	$\widetilde{P}_{80}(z) = \frac{1}{81}\big(20z^2 + 41z + 20\big)$
$82 = 10001_3$	$\widetilde{P}_{82}(z) = \frac{1}{486}\big(121z^2 + 244z + 121\big)$

Configuration m	**Polynomial** $\widetilde{P}_m(z)$
$83 = 10002_3$	$\widetilde{P}_{83}(z) = \frac{1}{486}\left(40z^3 + 203z^2 + 203z + 40\right)$
$85 = 10011_3$	$\widetilde{P}_{85}(z) = \frac{1}{486}\left(53z^3 + 190z^2 + 190z + 53\right)$
$86 = 10012_3$	$\widetilde{P}_{86}(z) = \frac{1}{486}\left(13z^4 + 121z^3 + 218z^2 + 121z + 13\right)$
$88 = 10021_3$	$\widetilde{P}_{88}(z) = \frac{1}{486}\left(13z^4 + 120z^3 + 220z^2 + 120z + 13\right)$
$89 = 10022_3$	$\widetilde{P}_{89}(z) = \frac{1}{486}\left(52z^3 + 191z^2 + 191z + 52\right)$
$91 = 10101_3$	$\widetilde{P}_{91}(z) = \frac{1}{486}\left(56z^3 + 187z^2 + 187z + 56\right)$
$92 = 10102_3$	$\widetilde{P}_{92}(z) = \frac{1}{486}\left(17z^4 + 120z^3 + 212z^2 + 120z + 17\right)$
$94 = 10111_3$	$\widetilde{P}_{94}(z) = \frac{1}{486}\left(21z^4 + 121z^3 + 202z^2 + 121z + 21\right)$
$95 = 10112_3$	$\widetilde{P}_{95}(z) = \frac{1}{486}\left(4z^5 + 61z^4 + 178z^3 + 178z^2 + 61z + 4\right)$
$97 = 10121_3$	$\widetilde{P}_{97}(z) = \frac{1}{486}\left(4z^5 + 56z^4 + 183z^3 + 183z^2 + 56z + 4\right)$
$98 = 10122_3$	$\widetilde{P}_{98}(z) = \frac{1}{486}\left(16z^4 + 121z^3 + 212z^2 + 121z + 16\right)$
$100 = 10201_3$	$\widetilde{P}_{100}(z) = \frac{1}{243}\left(8z^4 + 60z^3 + 107z^2 + 60z + 8\right)$
$101 = 10202_3$	$\widetilde{P}_{101}(z) = \frac{1}{486}\left(4z^5 + 55z^4 + 184z^3 + 184z^2 + 55z + 4\right)$

Configuration m	**Polynomial** $\widetilde{P}_m(z)$
$103 = \mathtt{10211}_3$	$\widetilde{P}_{103}(z) = \frac{1}{486}\left(4z^5 + 59z^4 + 180z^3 + 180z^2 + 59z + 4\right)$
$104 = \mathtt{10212}_3$	$\widetilde{P}_{104}(z) = \frac{1}{243}\left(10z^4 + 60z^3 + 103z^2 + 60z + 10\right)$
$106 = \mathtt{10221}_3$	$\widetilde{P}_{106}(z) = \frac{1}{486}\left(16z^4 + 117z^3 + 220z^2 + 117z + 16\right)$
$107 = \mathtt{10222}_3$	$\widetilde{P}_{107}(z) = \frac{1}{486}\left(52z^3 + 191z^2 + 191z + 52\right)$
$110 = \mathtt{11002}_3$	$\widetilde{P}_{110}(z) = \frac{1}{486}\left(17z^4 + 117z^3 + 218z^2 + 117z + 17\right)$
$112 = \mathtt{11011}_3$	$\widetilde{P}_{112}(z) = \frac{1}{243}\left(11z^4 + 60z^3 + 101z^2 + 60z + 11\right)$
$113 = \mathtt{11012}_3$	$\widetilde{P}_{113}(z) = \frac{1}{486}\left(5z^5 + 61z^4 + 177z^3 + 177z^2 + 61z + 5\right)$
$115 = \mathtt{11021}_3$	$\widetilde{P}_{115}(z) = \frac{1}{486}\left(5z^5 + 59z^4 + 179z^3 + 179z^2 + 59z + 5\right)$
$116 = \mathtt{11022}_3$	$\widetilde{P}_{116}(z) = \frac{1}{243}\left(10z^4 + 60z^3 + 103z^2 + 60z + 10\right)$
$119 = \mathtt{11102}_3$	$\widetilde{P}_{119}(z) = \frac{1}{486}\left(6z^5 + 61z^4 + 176z^3 + 176z^2 + 61z + 6\right)$
$121 = \mathtt{11111}_3$	$\widetilde{P}_{121}(z) = \frac{1}{486}\left(7z^5 + 65z^4 + 171z^3 + 171z^2 + 65z + 7\right)$
$122^* = \mathtt{11112}_3$	$\widetilde{P}_{122}(z) = \frac{1}{486}\left(z^6 + 26z^5 + 120z^4 + 192z^3 + 120z^2 + 26z + 1\right)$

Table 2. Several remarkable limit polynomials $\widetilde{P}_m(z)$ for $m \le 1094$.

the first occurrence of degree d
$m = 1 \ = 1_3$ $\widetilde{P}_1(z) = \widetilde{P}_3(z) = \widetilde{P}_9(Z) = \cdots = \frac{1}{2}(1+z)$
$m = 2 \ = 2_3$ $\widetilde{P}_2(z) = \widetilde{P}_6(z) = \cdots = \frac{1}{6}\big(1+4z+z^2\big)$
$m = 5 \ = 12_3$ $\widetilde{P}_5(z) = \frac{1}{18}\big(z^3+8z^2+8z+1\big)$
$m = 14 \ = 112_3$ $\widetilde{P}_{14}(z) = \frac{1}{54}\big(z^4+13z^3+26z^2+13z+1\big)$
$m = 41 \ = 1112_3$ $\widetilde{P}_{41}(z) = \frac{1}{162}\big(z^5+19z^4+61z^3+61z^2+19z+1\big)$
$m = 122 \ = 11112_3$ $\widetilde{P}_{122}(z) = \frac{1}{486}\big(z^6+26z^5+120z^4+192z^3+120z^2+26z+1\big)$
$m = 365 \ = 111112_3$ $\widetilde{P}_{365}(z) = \frac{1}{1458}\big(z^7+34z^6+211z^5+483z^4+483z^3+211z^2+34z+1\big)$
$m = 1094 \ = 1111112_3$ $\widetilde{P}_{1094}(z) = \frac{1}{4374}\big(z^8+43z^7+343z^6+1050z^5+1500z^4+1050z^3+343z^2+43z+1\big)$

Similar configurations
$m = 122 \ = 11112_3$ $\widetilde{P}_{122}(z) = \frac{1}{486}\big(z^6 + 26z^5 + 120z^4 + 192z^3 + 120z^2 + 26z + 1\big)$
$m = 124 \ = 11121_3$ $\widetilde{P}_{124}(z) = \frac{1}{486}\big(z^6 + 23z^5 + 119z^4 + 200z^3 + 119z^2 + 23z + 1\big)$
$m = 130 \ = 11211_3$ $\widetilde{P}_{130}(z) = \frac{1}{486}\big(z^6 + 22z^5 + 120z^4 + 200z^3 + 120z^2 + 22z + 1\big)$
$m = 148 \ = 12111_3$ $\widetilde{P}_{148}(z) = \frac{1}{486}\big(z^6 + 23z^5 + 119z^4 + 200z^3 + 119z^2 + 23z + 1\big)$
$m = 202 \ = 21111_3$ $\widetilde{P}_{202}(z) = \frac{1}{486}\big(z^6 + 26z^5 + 120z^4 + 192z^3 + 120z^2 + 26z + 1\big)$
Irreducible up to a root (-1) *cubic polynomials*
$m = 91 \ = 10101_3$ $\widetilde{P}_{91}(z) = \frac{1}{486}\big(56z^3 + 187z^2 + 187z + 56\big)$
$m = 253 \ = 100101_3$ $\widetilde{P}_{253}(z) = \frac{1}{1458}\big(173z^3 + 556z^2 + 556z + 173\big)$
$m = 739 \ = 1000101_3$ $\widetilde{P}_{739}(z) = \frac{1}{4374}\big(524z^3 + 1663z^2 + 1663z + 524\big)$

$$m = 757 \;\; = \mathtt{1001001}_3$$
$$\widetilde{P}_{757}(z) = \frac{1}{4374}\big(533z^3 + 1654z^2 + 1654z + 533\big)$$

Table 3. Non-irreducible polynomials $\widetilde{P}_m(z)$ for $m \le 244$ (up to a root -1).

Index m	**Configuration**	**Factorization of $\widetilde{P}_m(z)$**
4	$\mathtt{11}_3$	$\widetilde{P}_4(z) = \frac{1}{9}(2+z)(1+2z)$
8	$\mathtt{22}_3$	$\widetilde{P}_8(z) = \frac{1}{9}(2+z)(1+2z)$
28	$\mathtt{1001}_3$	$\widetilde{P}_{28}(z) = \frac{1}{81}(5+4z)(4+5z)$
40	$\mathtt{1111}_3$	$\widetilde{P}_{40}(z) = \frac{1}{81}\big(3+5z+z^2\big)\big(1+5z+3z^2\big)$
52	$\mathtt{1221}_3$	$\widetilde{P}_{52}(z) = \frac{1}{81}\big(2+6z+z^2\big)\big(1+6z+2z^2\big)$
56	$\mathtt{2002}_3$	$\widetilde{P}_{56}(z) = \frac{1}{81}\big(2+6z+z^2\big)\big(1+6z+2z^2\big)$
68	$\mathtt{2112}_3$	$\widetilde{P}_{68}(z) = \frac{1}{81}\big(3+5z+z^2\big)\big(1+5z+3z^2\big)$
80	$\mathtt{2222}_3$	$\widetilde{P}_{80}(z) = \frac{1}{9}(5+4z)(4+5z)$
244	$\mathtt{111111}_3$	$\widetilde{P}_{244}(z) = \frac{1}{729}(14+13z)(13+14z)$

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Received July 20, 2012
Accepted November 13, 2012

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