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uniform distribution theory

ON THE LIMIT DISTRIBUTION OF CONSECUTIVE ELEMENTS OF THE VAN DER CORPUT SEQUENCE

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ABSTRACT. Recently, Fialová and Strauch, Uniform Distribution Theory, 6(1):101-125, 2011, calculated the asymptotic distribution function (adf) of the two-dimensional sequence $(\phi_b(n), \phi_b(n+1))_{n\geq 0}$, where $(\phi_b(n))_{n\geq 0}$ denotes the van der Corput sequence in base b. In the present paper we solve the general problem asking for the limit distribution of $(\phi_b(n), \phi_b(n+1), \ldots, \phi_b(n+s-1))_{n\geq 0}$. We use the fact that the van der Corput sequence can be seen as the orbit of the origin under the ergodic von Neumann-Kakutani transformation.

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1. Introduction

In the open problem collection on the web site of *Uniform distribution theory* the following problem is stated:

> *Let* $(\phi_b(n))_{n\geq 0}$ *denote the van der Corput sequence in base* b*. Find the distribution of the sequence* $(\phi_b(n), \phi_b(n+1), \ldots, \phi_b(n+s-1))_{n\geq 0}$ *in* $[0,1)^s$*.*[1]

The case $s = 2$ has recently been solved by Fialová and Strauch [3]. They showed that every point $(\phi_b(n), \phi_b(n+1))_{n\geq 0}$ lies on the line segment

$$y = x - 1 + \frac{1}{b^k} + \frac{1}{b^{k+1}}, \quad x \in \left[1 - \frac{1}{b^k}, 1 - \frac{1}{b^{k+1}}\right]$$

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[1]Problem 1.12 in the open problem collection as of 11. December 2011 (http://www.boku.ac.at/MATH/udt/unsolvedproblems.pdf)

for $k \geq 0$. Furthermore they could give an explicit formula for the asymptotic distribution function $g(x,y)$ of $(\phi_b(n), \phi_b(n+1))_{n\geq 0}$ to calculate the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |\phi_b(n) - \phi_b(n+1)| = \int_0^1 \int_0^1 |x-y| dg(x,y) = \frac{2(b-1)}{b^2}$$

previously demonstrated by Pillichshammer and Steinerberger [13]. They also noted that the adf of $(\phi_b(n), \phi_b(n+1))_{n\geq 0}$ is a copula.
In this article we solve the problem for the sequence $(\phi_b(n), \phi_b(n+1), \ldots, \phi_b(n+s-1))_{n\geq 0}$ for $s > 2$. A multi-dimensional extension of the van der Corput sequence $(\phi_b(n))_{n\geq 0}$, is given by the so-called Halton sequence, $(\phi_{b_1}(n), \phi_{b_2}(n), \ldots, \phi_{b_s}(n))_{n\geq 0}$ which is uniformly distributed if and if the bases $b_i 1 \leq i \leq s$ are co-prime (see [7]). These sequences are well-studied objects in discrepancy theory, since they belong to the class of so-called low discrepancy sequences. For classical results in discrepancy theory, on low discrepancy sequences and the van der Corput sequence see e.g. [1], [2] or [9].

Recently, several authors investigated the ergodic properties of low discrepancy sequences, see e.g. [6] and [8]. In the case of van der Corput sequences this can be done using the so-called von Neumann-Kakutani transformation, which will be discussed in the second section.
The outline of this article is as follows: in the second section we define the van der Corput sequence and the von Neumann-Kakutani transformation and recall their basic properties. In the third section we state our main results on the distribution of $(\phi_b(n), \phi_b(n+1), \ldots, \phi_b(n+s-1))_{n\geq 0}$.

2. van der Corput sequence and von Neumann-Kakutani transformation

Let $b \in \mathbb{N}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Then for every $n \in \mathbb{N}_0$, we can write

$$n = \sum_{i\geq 0} n_i b^i$$

where $n_i \in \{0, 1, \ldots, b-1\}, i \geq 0$. The above sum is called b-adic representation of n. The n_i are uniquely determined and at most a finite number of n_i are non-zero. Furthermore, every real $x \in [0,1)$ has a b-adic representation of the following form

$$x = \sum_{i\geq 0} x_i b^{-i-1} \tag{1}$$

where $x_i \in \{0, 1, \ldots, b-1\}, i \geq 0$. We call x a b-adic rational if $x = ab^{-c}$, where a and c are positive integers and $0 \leq a < b^c$. For all b-adic integers there are exactly two representations of the form (1), one where $x_i = 0, i \geq i_0$ and one where $x_i = b-1, i \geq i_0$ for sufficiently large $i_0 \in \mathbb{N}$. If we restrict ourselves to representations with $x_i \neq b-1$ for infinitely many i, then the coefficients x_i in (1) are uniquely determined for all $x \in [0,1)$.
For $n \in \mathbb{N}_0$ we define the so-called radical-inverse function or Monna map $\phi_b(n) \colon \mathbb{N}_0 \to [0,1)$ by

$$\phi_b(n) = \phi_b\left(\sum_{i\geq 0} n_i b^i\right) := \sum_{i \geq 0} n_i b^{-i-1}.$$

Note that $\phi_b(n)$ maps $\mathbb{N}_0$ to the set of b-adic rationals in $[0,1)$, and therefore the image of $\mathbb{N}_0$ under $\phi_b(n)$ is dense in $[0,1)$.

Definition 2.1. *The van der Corput sequence in base* b *is defined as* $(\phi_b(n))_{n\geq 0}$.

It is a classical result that the van der Corput sequence is uniformly distributed in $[0,1)$, see e.g. [9]. Furthermore, its s-dimensional extension, the Halton sequence given by $(\phi_{b_1}(n), \ldots, \phi_{b_s}(n))_{n\geq 0}$ for co-prime bases $b_i, 1 \leq i \leq s$, is uniformly distributed on $[0,1)^s$. Properties of the van der Corput and the Halton sequence are very well-understood, since they are so-called low discrepancy sequences, which are central objects in Quasi-Monte Carlo integration.
A second approach to define the van der Corput sequence is by using the von Neumann-Kakutani transformation $T_b \colon [0,1) \to [0,1)$. For any integer $b \geq 2$ the inductive construction of T_b is as follows: at first $[0,1)$ is split into b intervals $I_i^1 = \left[\frac{i}{b}, \frac{i+1}{b}\right)$ for $i = 0, 1, \ldots b-1$. Then the transformation $T_{1,b} \colon \left[0, \frac{b-1}{b}\right) \mapsto \left[\frac{1}{b}, 1\right)$ is defined as translation of I_i^1 into I_{i+1}^1 for $i = 0, 1, \ldots, b-1$. The next step is to divide all intervals I_i^1 into b subintervals of the form $I_i^2 = \left[\frac{i}{b^2}, \frac{i+1}{b^2}\right)$ for $i = 0, 1, \ldots b^2 - 1$. Transformation $T_{2,b} \colon \left[0, \frac{b^2-1}{b^2}\right) \mapsto \left[\frac{1}{b^2}, 1\right)$ is given as the extension of $T_{1,b}$ which translates $I^2_{b^2-b+i}$ into $I^2_{b^2-b+i+1}$ for $i = 0, 1, \ldots, b-1$. Such a construction is called splitting-and-stacking-construction and is illustrated in Figure 1 for $b = 2$. Finally we define the von Neumann-Kakutani transformation as $T_b = \lim_{n\to\infty} T_{n,b}$. A plot of the transformation T_2 is given in Figure 2. By an observation of Lambert [10], [11] (see also Hellekalek [7]) the van der Corput sequence in base b is exactly the orbit of the origin under T_b, which means that

$$(T_b^n 0)_{n\geq 0} = (\phi_b(n))_{n\geq 0}, \quad b \geq 2, \tag{2}$$

where $T_b^n x$ denotes the value of x under after n iterations of T_b.
For a proof of the ergodicity and measure-preserving properties of the von

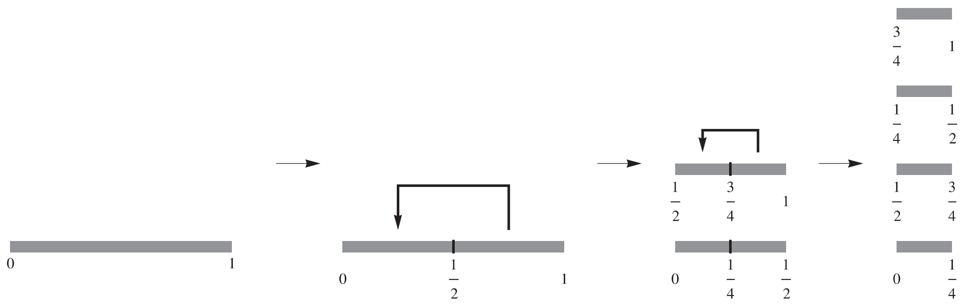

FIGURE 1. The first two steps of a splitting-and-stacking-construction in base $b = 2$.

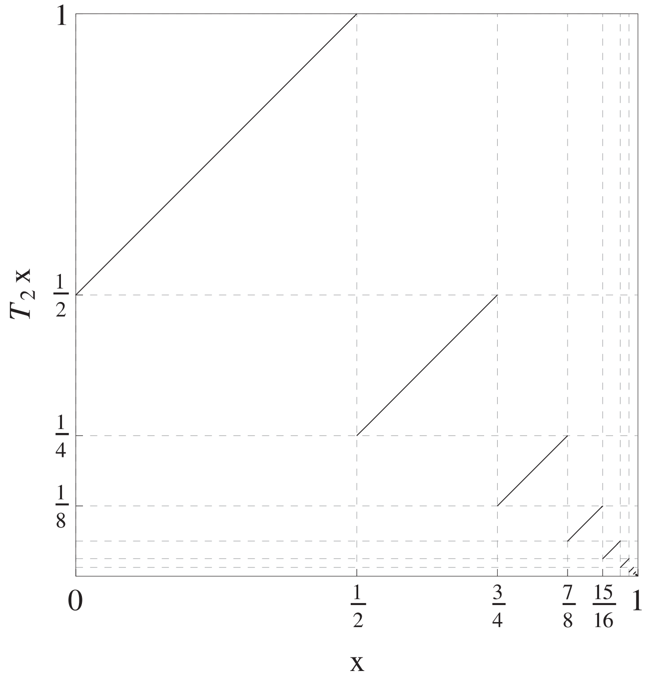

FIGURE 2. The von Neumann-Kakutani transformation in base $b = 2$.

Neumann-Kakutani transformation, see e.g. [4] or [5]. It follows from the ergodicity of the von Neumann-Kakutani transformation that $(T_b^n x)_{n \geq 0}$ is uniformly distributed for almost every $x \in [0, 1)$. Furthermore, it can be shown that the von Neumann-Kakutani transformation is uniquely ergodic, which implies that

$(T_b^n x)_{n\geq 0}$ is uniformly distributed for every $x \in [0,1)$, see e.g. [6]. Moreover, Pagés [12] showed that the orbit of the von Neumann-Kakutani transformation starting at an arbitrary point $x \in [0,1)$ is a low discrepancy sequence. Another possible generalization of the van der Corput sequence is the so-called randomized van der Corput sequence $(T_b^n X)_{n\geq 0}$ where X is uniformly distributed on $[0,1)$, see [14].

Recently, Fialová and Strauch solved the problem of calculating the limit distribution of the sequence $(\phi_b(n), \phi_b(n+1))_{n\geq 0}$. They also concluded that the limit distribution is a copula. We consider the multi-dimensional extension of this problem. By (2)

$$(\phi_b(n), \phi_b(n+1))_{n\geq 0} = (T_b^n 0, T_b^{n+1} 0)_{n\geq 0} = (T_b^n 0, T_b(T_b^n 0))_{n\geq 0}.$$

By the fact that $(T_b^n 0)_{n\geq 0}$ is uniformly distributed on $[0,1)$ one can show that $(\phi_b(n), \phi_b(n+1))_{n\geq 0}$ is uniformly distributed on

$$\Gamma = \{(x,y) : y = T_b x\}.$$

Note that Γ coincides with the graph of the von Neumann-Kakutani transformation in Figure 2. In the next section we use this approach to find the limit distribution of $(\phi_b(n), \phi_b(n+1), \ldots, \phi_b(n+s-1))_{n\geq 0}$ for arbitrary $s \geq 2$.

3. The limit distribution of consecutive elements of the van der Corput sequence

In the sequel we assume that b, s are fixed. Let T denote the von Neumann-Kakutani transformation in base b as described in Section 2. We define a map $\gamma(t)\colon [0,1) \to [0,1)^s$ by setting

$$\gamma(t) := \begin{pmatrix} t \\ Tt \\ T^2 t \\ \vdots \\ T^{s-1} t \end{pmatrix}$$

and

$$\Gamma := \{(x_1, x_2, \ldots, x_s) \in [0,1]^s : x_i = T^{i-1} x_1, i = 2, \ldots, s\} = \{\gamma(t) : t \in [0,1)\}.$$

The Lebesgue measure λ_1 on $[0,1)$ induces a measure ν on Γ by setting

$$\nu(A) = \lambda_1(\{t : \gamma(t) \in A\}), \quad A \subset \Gamma.$$

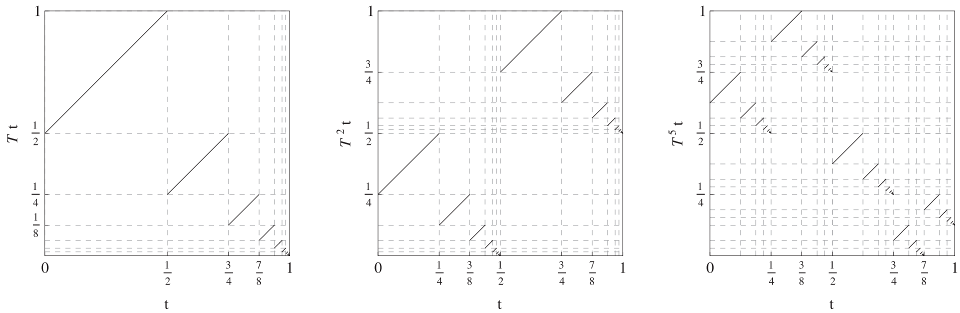

FIGURE 3. Function graphs of Tt, T^2t and T^5t. These curves appear as the two-dimensional projections of Γ for large s.

Furthermore, ν induces a measure μ on $[0,1)^s$ by embedding Γ into $[0,1)^s$. More precisely for every measurable subset $B \subseteq [0,1)^s$ we set

$$\mu(B) = \nu(B \cap \Gamma).$$

Theorem 3.1. *The limit measure of* $(\phi_b(n), \phi_b(n+1), \ldots, \phi_b(n+s-1))_{n\geq 0}$ *is* μ.

Proof. As mentioned in Section 2, we can rewrite

$$\begin{aligned}(\phi_b(n), \phi_b(n+1), \ldots, \phi_b(n+s-1))_{n\geq 0} &= (T^n 0, T^{n+1}0, \ldots, T^{n+s-1}0)_{n\geq 0} \\ &= (T^n 0, T(T^n 0), \ldots, T^{s-1}(T^n 0))_{n\geq 0}.\end{aligned}$$

Since $(T^n 0)_{n\geq 0}$ is uniformly distributed on $[0,1)$ and T is a measure-preserving transformation with respect to λ_1, it follows immediately that $(T^i(T^n 0))_{n\geq 0}$ is uniformly distributed on $[0,1)$ for $i = 1, \ldots, s-1$. Moreover, by construction $(T^n 0, T(T^n 0), \ldots, T^{s-1}(T^n 0))_{n\geq 0} \in \Gamma$ for all $n \geq 0$.
Now consider a Jordan measurable set $B \in [0,1)^s$. We define the empirical measure of the first N points of $(T^n 0, \ldots, T^{s-1}(T^n 0))_{n\geq 0}$ as

$$\mu_N(B) = \frac{1}{N}\#\{0 \leq n \leq N : (T^n 0, T(T^n 0), \ldots, T^{s-1}(T^n 0)) \in B\}.$$

We have

$$\begin{aligned}\lim_{N\to\infty} \mu_N(B) &= \lim_{N\to\infty} \frac{1}{N}\#\{0 \leq n \leq N : (T^n 0, T(T^n 0), \ldots, T^{s-1}(T^n 0)) \in B\} \\ &= \lim_{N\to\infty} \frac{1}{N}\#\{0 \leq n \leq N : (T^n 0, T(T^n 0), \ldots, T^{s-1}(T^n 0)) \in B \cap \Gamma\} \\ &= \lim_{N\to\infty} \frac{1}{N}\#\{0 \leq n \leq N : T^n 0 \in \text{ Projection}_{x_1}(B \cap \Gamma)\} \\ &= \lambda_1(\text{ Projection}_{x_1}(B \cap \Gamma))\end{aligned}$$

$$= \nu(B \cap \Gamma) = \mu(B)$$

where the fourth equation holds since $(T^n 0)_{n \geq 0}$ is uniformly distributed on $[0, 1)$ and since the map $t \to Tt$ is a bijection, and where $\text{Projection}_{x_1}(A)$ denotes the projection of A onto its first coordinate. □

Remark 3.1. *Note that the measure μ is a copula on $[0, 1]^s$ for every s since every distribution function of a multi-dimensional sequence $(x_n^1, \ldots, x_n^s)_{n \geq 0}$ is a copula if the sequences $(x_n^1)_{n \geq 0}, \ldots, (x_n^s)_{n \geq 0}$ are uniformly distributed on $[0, 1]$.*

Remark 3.2. *The set Γ is a collection of countably many line segments in $[0, 1)^s$, which are parallel to the diagonal. Informally speaking Theorem 3.1 means that $(\phi_b(n), \phi_b(n + 1), \ldots, \phi_b(n + s - 1))_{n \geq 0}$ is uniformly distributed on Γ.*

Remark 3.3. *By the unique ergodicity of T, the conclusion of Theorem 3.1 holds also for the sequence $(T^n x, T(T^n x), \ldots, T^{s-1}(T^n x))_{n \geq 0}$ for arbitrary $x \in [0, 1)$.*

Remark 3.4. *Another class of uniformly distributed sequences which can be seen as the orbits of certain points under an ergodic transformation are sequences of the form $(\{n\alpha\})_{n \geq 0}$, where $\{x\}$ denotes the fractional part of x and α is irrational. In this case the corresponding transformation $\widehat{T}$ is simply the rotation $\widehat{T}\colon x \mapsto x + \alpha \mod 1$. It can easily be shown that the limit distribution of consecutive elements $(\{n\alpha\}, \{(n+1)\alpha\}, \ldots, \{(n+s-1)\alpha\})_{n \geq 0}$ is the uniform distribution on the curve $\widehat{\Gamma}$ which is given by*

$$\widehat{\Gamma} := \{(t, \widehat{T}t, \ldots, \widehat{T}^{s-1}t), t \in [0, 1)\}.$$

However, since in this case the transformation $\widehat{T}$ has a particularly simple structure, the same result can also be easily obtained using analytic arguments.

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