Uniform Distribution Theory **8** (2013), no.1, 73–87

uniform
distribution
theory

THE NEAREST INTEGER CONTINUED FRACTION AND THE MOVING AVERAGE ERGODIC THEOREM

Hailiza Haili-Kamarul — Radhakrishnan Nair

ABSTRACT. We use a moving average ergodic theorem to derive various results concerning moving averages of nearest integer continued fractions previously known only for non-moving averages and then derived using the pointwise ergodic theorem.

Communicated by S. Akiyama

1. Introduction

In this paper we apply the moving average theorem of A. Bellow, R. Jones and J. Rosenblatt [BJR] which we state presently to derive some new results about the metric theory of moving averages for the nearest integer continued fractions. The corresponding results for non-moving averages are known and some are classical. For a continued fraction expansion of a real number x there are two standard presentations for it we can give. The first is

$$x \;=\; [c_0; \epsilon_1 c_1, \epsilon_2 c_2, \cdots],$$

and the second

$$x = c_0 \;+\; \cfrac{\epsilon_1}{c_1 + \cfrac{\epsilon_2}{c_2 + \ddots}}.$$

Of course they say the same thing. Here $(c_i)_{n=1}^{\infty}$ is a sequence of integers and $\epsilon_i \;\in\; \{-1, 1\}$. The numbers c_i $(i \;=\; 1, 2, \cdots)$ are called the partial quotients of

2010 Mathematics Subject Classification: 11K50, 28D99.
Keywords: Continued fraction, dynamical system, Lebesgue measure, ergodic.

the expansion and for each natural number n the truncates

$$\frac{P_n}{Q_n} = [c_0; \epsilon_1 c_1, \cdots, \epsilon_n c_n] = c_0 + \cfrac{\epsilon_1}{c_1 + \ddots + \cfrac{\epsilon_n}{c_n}},$$

are called the convergents of the expansion. The expansion is called semi-regular if : (i) c_n is a natural number, for positive n; (ii) $\epsilon_{n+1} + c_{n+1} \geq 1$ and (iii) $\epsilon_{n+1} + c_{n+1} \geq 2$ for infinitely many n if the expansion is itself infinite. Central to the class of semi-regular continued fraction expansions is the regular continued fraction expansion which is also the most familiar and obtained when c_n is a natural number for natural numbers n and ϵ_i takes the value one for all i. Here and henceforth for a real number y let $\lfloor y \rfloor$ denote the the greatest integer less than y and let $\{y\}$ denote its fractional part, that is $y - \lfloor y \rfloor$. Notice that for the regular continued fraction expansion $c_0 = \lfloor x \rfloor$. It is thus convenient and no real restriction to assume x is in $[0, 1)$. If this is done we define the Gauss map

$$Tx = \left\{\frac{1}{x}\right\}, \ x \neq 0 \ ; \ T0 = 0.$$

on $[0, 1)$. We have $c_i(x) = c_1(T^{i-1}x)$ $(i = 1, 2, \cdots)$ – the partial quotients of the regular continued fraction expansion of x. The nearest integer continued fraction expansion is defined by an analogous procedure. Here we set $c_0(x)$ to be the integer nearest to x and we thus also implicitly define $\epsilon_1(x)$. Given this information set Ω to be $(-\frac{1}{2}, \frac{1}{2}) \setminus \mathbf{Q}$ and define the map $S : \Omega \to \Omega$ by

$$Sx = \frac{\epsilon_1}{x} - \left\lfloor |x|^{-1} + \frac{1}{2} \right\rfloor.$$

Then we have $c_i(x) = c_{i-1}(Sx)$ and $\epsilon_i(x) = \epsilon_{i-1}(Sx)$ for $i \geq 1$. Again in this instance $(c_i(x))_{i=1}^{\infty}$ defines the partial quotients of the nearest integer continued fraction expansion. For $n \geq 1$ we always have that $c_n(x) \geq 2$ and that $c_n(x) + \epsilon_{n+1}(x) \geq 1$.

In section 2.1, using the ergodic theory of S and its natural extension we state and prove some results concerning the moving average behaviour of the nearest integer continued fraction expansion. These results extend certain earlier work of G. J. Rieger [Ri]. The sequence $\left(\frac{P_n}{Q_n}\right)_{n=1}^{\infty}$ in the case of the nearest integer continued fraction expansion is a subsequence of the corresponding sequence for the regular continued fraction expansion [Pe p. 168-78]. Recall the inequality

$$\left| x - \frac{P_n}{Q_n} \right| \leq \frac{1}{Q_n^2},$$

which is classical, well known and due to L. J. Dirichlet [HW p.194-8] for regular continued fractions and hence also for the nearest integer continued fractions, their convergents being a subsequence of those of the regular continued fraction expansion. Clearly if for each natural number n we set

$$\phi_n(x) = Q_n^2 \left| x - \frac{P_n}{Q_n} \right|$$

for the nearest integer continued fraction expansion, then for each x the sequence of numbers $(\phi_n(x))_{n=1}^{\infty}$ lies in the interval $[0, g]$ where $g = \frac{1}{2}(\sqrt{5} - 1)$ [Pe p. 168-78]. In section 2.2, extending work of H. Jager [J] and C. Kraaikamp [Kr1] we state and prove results concerning the distribution of the sequence $(\phi_n(x))_{n=1}^{\infty}$ for almost all x with respect to Lebesgue measure. Section 2.3 studies other arithmetic sequences attached to the nearest integer continued fraction expansion. An analogous study of the regular continued fraction appears in [KN].

We now state the moving average ergodic theorem [BJR]. We begin by introducing some notation. Let Z be a collection of points in $\mathbf{Z} \times \mathbf{N}$ and let

$$Z^h = \{(n, k) : (n, k) \in Z \text{ and } k \geq h\},$$

$$Z_\alpha^h = \{(z, s) \in \mathbf{Z}^2 : |z - y| < \alpha(s - r) \text{ for some } (y, r) \in Z^h\}$$

and

$$Z_\alpha^h(\lambda) = \{n : (n, \lambda) \in Z_\alpha^h\}. \qquad (\lambda \in \mathbf{N})$$

Geometrically we can think of Z_α^1 as the lattice points contained in the union of all solid cones with aperture α and vertex contained in $Z^1 = Z$. We say a sequence of pairs of natural numbers $(n_l, k_l)_{l=1}^{\infty}$ is *Stoltz* if there exists a collection of points Z in $\mathbf{Z} \times \mathbf{N}$, and a function $h = h(t)$ tending to infinity with t such that $(n_l, k_l)_{l=t}^{\infty} \in Z^{h(t)}$ and there exist h_0, α_0 and $A > 0$ such that for all integers $\lambda > 0$ we have $|Z_{\alpha_0}^{h_0}(\lambda)| \leq A\lambda$. Our main tool is the following theorem [BJR].

Theorem 1. *Let $(X, \beta, , \mu, T)$ denote a dynamical system, with set X, a σ-algebra of its subsets β, a measure μ defined on the measurable space (X, β) such that $\mu(X) = 1$ and a measurable, measure preserving map T from X to itself. Suppose f is in $L^1(X, \beta, \mu)$ and that the sequence of pairs on natural numbers $(n_l, k_l)_{l=1}^{\infty}$ is Stoltz then if (X, β, μ, T) is ergodic,*

$$m_f(x) = \lim_{l \to \infty} \frac{1}{k_l} \sum_{i=1}^{k_l} f(T^{n_l + i} x),$$

exists almost everywhere with respect to Lebesgue measure.

Note that if $m_{l,f}(x) = \frac{1}{k_l} \sum_{i=1}^{k_l} f(T^{n_l+i} x)$ then

$$m_{l,f}(Tx) - m_{l,f}(x) = k_l^{-1}(f(T^{n_l+k_l+1}) - f(T^{n_l+1} x)).$$

This means that $m_f(Tx) = m_f(x)$ μ almost everywhere. A dynamical system (X, β, μ, T) is called ergodic if given any $A \in \beta$ for which $T^{-1}A := \{x \in X : Tx \in A\} = A$, the set A has either full or null measure. A standard fact in ergodic theory is that if (X, β, μ, T) is ergodic and $m_f(Tx) = m_f(x)$ almost everywhere, then $m_f(x) = \int_X f d\mu$ μ almost everywhere [CFS p.14]. The term Stoltz is used here because the condition on $(k_l, n_l)_{l=1}^{\infty}$ is analogous to the condition required in the classical non-radial limit theorem for harmonic functions also called a Stoltz condition, which suggested the above theorem to the authors of [BJR]. Averages where $k_l = 1$ for all l will be called non-moving. Moving averages satisfying the above hypothesis can be constructed by taking for instance $n_l = 2^{2^l}$ and $k_l = 2^{2^{l-1}}$.

2. Applying ergodic theory to the nearest integer continued fraction expansion

2.1. Average behaviour of convergents

Let $G = \frac{1}{2}(1 + \sqrt{5})$. Also let η denote the measure on $(-\frac{1}{2}, \frac{1}{2})$ given for arbitrary Lebesgue measurable sets $E \subset (-\frac{1}{2}, \frac{1}{2})$ by

$$\eta(E) = \frac{1}{\log G} \int_E \rho(t)dt,$$

where

$$\rho(t) = \tfrac{1}{(G+t)} \text{ if } t \geq 0 \text{ and } \rho(t) = \tfrac{1}{G+t+1} \text{ if } t < 0.$$

It was shown in [Ri] that η is preserved by the map S and that the dynamical system (Ω, β, η, S), where β denotes the σ-algebra of Lebesgue measurable sets is ergodic. This fact via Theorem 1 has a number of arithmetic consequences. Our first theorem is the following

Theorem 2.1. *Suppose we are given two integers c and ϵ with $c \geq 2$, ϵ in $\{-1, 1\}$ and such that $c + \epsilon \geq 2$. Suppose also that $(n_l, k_l)_{l=1}^{\infty}$ is Stoltz. Then if*

$$L(c, \epsilon) = \frac{1}{\log G} \log \frac{(4c + \epsilon - 3 + 2\sqrt{5})(4c + \epsilon - 5 + 2\sqrt{5})}{(4c + \epsilon - 7 + 2\sqrt{5})(4c + \epsilon - 1 + 2\sqrt{5})},$$

$$\lim_{l \to \infty} \frac{1}{k_l} |\{1 \leq i \leq k_l : c_{n_l+i}(x) = c \,;\, \epsilon_{n_l+i} = \epsilon\}| = L(c, \epsilon),$$

almost everywhere with respect to Lebesgue measure.

Proof. Note that x is in

$$B_{c,\epsilon}^{+} \;=\; \left[\frac{4}{4c+\epsilon+\;1},\frac{4}{4c+\epsilon-\;1}\right]$$

if and only if

$$\epsilon_1(x) \;=\; \min(1,c_1(x)) \;=\; \min(\epsilon_2(x),c) \;=\; \epsilon\in\{-1,1\},$$

and that x is in

$$B_{c,\epsilon}^{-} \;=\; \left[\frac{-4}{4c+\epsilon+\;1},\frac{-4}{4c+\epsilon-\;1}\right]$$

if and only if

$$\epsilon_1(x) \;=\; \min(-1,c_1(x)) \;=\; \min(\epsilon_2(x),c) \;=\; \epsilon=\{-1,1\}.$$

Applying Theorem 1 to the characteristic function of the set $B_{c,\epsilon}^{+}\cup B_{c,\epsilon}^{-}$ we see that the required averages converge to

$$\frac{1}{\log G}\int_{B_{c,\epsilon}^{+}\cup B_{c,\epsilon}^{-}}\rho(t)dt=\frac{1}{\log G}\log\frac{(4c+\epsilon-3+2\sqrt{5})(4c+\epsilon-5+2\sqrt{5})}{(4c+\epsilon-7+2\sqrt{5})(4c+\epsilon-1+2\sqrt{5})},$$

as required. □

We have the following general result concerning the average behaviour of the convergents. See [KN] for an analogous result about the the convergents of regular continued fraction expansion.

Theorem 2.2. *Suppose the function F with domain the non-negative real numbers and range the real numbers is continuous and increasing. For each natural number n and arbitrary non-negative real numbers $a_1,\cdots,\ a_n$ we define*

$$M_{F,n}(a_1,\cdots,a_n) \;=\; F^{-1}\left[\frac{1}{n}\sum_{j=1}^{n}F(a_j)\right].$$

Then if the sequence of pairs of natural numbers $(n_l,k_l)_{l=1}^{\infty}$ is Stoltz

$$\lim_{l\to\infty}M_{F,n}(c_{n_l+1}(x),\cdots,\ c_{n_l+k_l}(x)) \;=\; F^{-1}\left[\frac{1}{\log g}\int_{-\frac{1}{2}}^{\frac{1}{2}}F(c_1(t))d\rho(t)\right],$$

almost every where with respect to Lebesgue measure.

Proof. If F in $L^1(\eta)$ the result follows immediately from Theorem 1. If however F is not in $L^1(\eta)$, set

$$f_M(x):=F(c_1(x))\text{ if }F(c_1(x))\le M\text{ and }f_M(x):=M\text{ if }F(c_1(x))>M.$$

This means that for each $M \geq F(1)$ and almost all x with respect to Lebesgue measure we have

$$\begin{aligned}
&\limsup_{l\to\infty} \frac{1}{k_l} \sum_{i=1}^{k_l} F(c_{n_l+i}(x)) \\
&\geq \lim_{l\to\infty} \frac{1}{k_l} \sum_{i=1}^{k_l} f_M(S^{n_l+i}(x)) \\
&= \frac{1}{\log G} \int_{-\frac{1}{2}}^{\frac{1}{2}} f_M(t)d\rho(t),
\end{aligned}$$

which tends to infinity with M, as required. □

Finally in this section we note the following result.

Theorem 2.3. *If the sequence of pairs of natural numbers $(n_l, k_l)_{l=1}^{\infty}$ is Stoltz, then*

$$\lim_{l\to\infty} \frac{1}{k_l} \sum_{i=1}^{k_l} \epsilon_{n_l+i}(x) = \frac{1}{\log G} \log \frac{G^3}{4},$$

almost everywhere with respect Lebesgue measure.

Proof. Note that $\epsilon_{n-1}(Sx) = \epsilon_n(x)$ and application of Theorem 1 gives the result. □

2.2. Average behaviour of the approximation constants $(\phi_n(x))_{n=1}^{\infty}$

In this section we prove a result relating to the distribution of the sequence of pairs $(\phi_{n-1}(x), \phi_n(x))_{n=1}^{\infty}$ for almost all x with respect to Lebesgue measure. Let Γ_1 be the interior of the quadrilateral; with vertices $(0,0)$, $(g,0)$, $(2g^3, g^2)$ and $(0, \frac{1}{2})$ where $g = \frac{1}{2}(\sqrt{5} - 1)$. Also let Γ_{-1} be the interior of the quadrilateral with vertices $(0,0)$, $(g^2, 0)$, $(2g^3, g)$ and $(0, \frac{1}{2})$. Further put $\Gamma = \Gamma_1 \cup \Gamma_{-1}$. In [Kr] it is shown that for all irrational x the sequence is distributed over Γ. Further where $G = \frac{1}{2}(\sqrt{5} + 1)$ set

$$f_1(\alpha, \beta) = \frac{1}{\log G} \frac{1}{\sqrt{1 - 4\alpha\beta}} \text{ if } 1 - 4\alpha\beta > 0$$

and

$$f_{-1}(\alpha, \beta) = \frac{1}{\log G} \frac{1}{\sqrt{1 + 4\alpha\beta}} \text{ if } 1 + 4\alpha\beta > 0.$$

Also set

Set $f(\alpha, \beta) := f_1(\alpha, \beta)$ if $(\alpha, \beta) \in \Gamma_1 \setminus \Gamma_{-1}$; set $f(\alpha, \beta) := f_1(\alpha, \beta) + f_{-1}(\alpha, \beta)$ if $(\alpha, \beta) \in \Gamma_1 \cap \Gamma_{-1}$ and set $f(\alpha, \beta) := f_{-1}(\alpha, \beta)$ if $(\alpha, \beta) \in \Gamma_{-1} \setminus \Gamma_1$.

Then we have the following theorem.

Theorem 2.4. *Suppose A is a Borel subset of the set Γ then if the sequence of pairs of natural numbers $(n_l, k_l)_{l=1}^{\infty}$ is Stoltz we have*

$$\lim_{l\to\infty} \frac{1}{k_l} \sum_{l=1}^{k_l} \chi_A(\phi_{n_l+i-1}(x), \phi_{n_l+i}(x)) \;=\; \int_A f(\alpha,\beta) d\alpha\ d\beta,$$

almost everywhere with respect to Lebesgue measure.

The stationary variant of Theorem 2.4 appears in [Kr]. To prove Theorem 2.4 we need to use the ergodic properties of the map S associated to the nearest integer continued fraction expansion or more accurately its natural extention. In particular we need the following theorem to be found in either [Ri] or [INT]. See also [N].

Theorem 2.5. *Let*

$$D \;=\; \{x \;\in\; \Omega \;:\; x \;\le\; 0\} \times [0, g-1) \cup \{x \;\in\; \Omega \;:\; x \;\ge\; 0\} \times [0, g).$$

Let β denote the σ-algebra of Borel sets in D and let μ be the measure defined on the measurable space (D, β) with Radon - Nikodym derivative $(\log G)^{-1}(1 + xy)^2$ relative to two dimensional Lebesgue measure on D. Also define the map $N : D \to D$ by

$$N(x, y) \;=\; (Sx, (c_1 \;+\; \epsilon_1 y)^{-1}).$$

Then the dynamical system (D, β, μ, N) is ergodic.

The dynamical system (D, β, μ, N) is the natural extention of the nearest integer continued fraction transformation (Ω, β, η, S), defined in the introduction and [Ri]. See [CFS p. 239-41] for a formal discussion of the natural extension. In [Pe p. 168-78] it is shown that if x is irrational then $\frac{Q_{n-1}}{Q_n} \;<\; g$ and if $c_n \;\ge\; 3$ then $\frac{Q_{n-1}}{Q_n} \;<\; g^2$. We readily see that $(S^n x, \frac{Q_{n-1}}{Q_n})$ is in D.

To prove Theorem 2.4 we need to prove the following theorem.

Theorem 2.6. *Suppose A is a Borel subset of the set D. Then if the sequence of pairs of natural numbers $(n_l, k_l)_{l=1}^{\infty}$ is Stoltz we have*

$$\lim_{l\to\infty} \frac{1}{k_l} \sum_{i=1}^{k_l} \chi_A(S^{n_l+i}x, \frac{Q_{n_l+i-1}}{Q_{n_l+i}}) \;=\frac{1}{\log G}\; \int_A \frac{dxdy}{(1 \;+\; xy)^2},$$

almost everywhere with respect to Lebesgue measure.

P r o o f. Note that $\epsilon_n(x) = \epsilon_1(S^{n-1}x)$ and so from the procedure for defining the nearest integer continued fraction expansion we observe the following recurrence relations:

$$P_{-1} = 1 \; ; \; P_0 = 0; \; P_n = c_n P_{n-1} + \epsilon_n P_{n-2} \quad (n = 1, 2, \cdots) \tag{2.1}$$

and

$$Q_{-1} = 1 \; ; \; Q_0 = 0; \; Q_n = c_n Q_{n-1} + \epsilon_n Q_{n-2}. \quad (n = 1, 2, \cdots) \tag{2.2}$$

The analoguous recurrence relation for the regular continued fraction expansion is well known and proved similarly. Using (2.1) and (2.2) and the definition of the map S we readily see that for each natural number n

$$N^n(x, 0) = (S^n x, \frac{Q_{n-1}}{Q_n}). \tag{2.3}$$

Note that

$$\frac{Q_{n-1}}{Q_n} = [0; c_n, \epsilon_n c_{n-1}, \cdots, \epsilon_2 c_1].$$

We also easily check that for any y such that $(x, y) \in D$

$$\lim_{n\to\infty} (N^n(x, 0) - N^n(x, y)) = 0.$$

Let B be the set of x for which the conclusion of Theorem 2.6 fails and let

$$C = \{x \in B : x \leq 0\} \times [0, g-1) \cup \{x \in B : x \geq 0\} \times [0, g).$$

Then for almost all (x, y) in C the sequence $N^n(x, y)$ is not distributed with respect to the measure having Radon - Nikodym derivative $(\log G)^{-1}(1 + xy)^2$ relative to two dimensional Lebesgue measure on D. This is in contradiction to Theorem 1 unless the measure of B is zero as required. □

We now complete the proof of Theorem 2.4. The following discussion also appears in [Kr]. First we note by induction that

$$x - \frac{P_n}{Q_n} = \frac{(-1)^n \epsilon_1(x) \cdots \epsilon_n(x) S^n x}{Q_n(Q_n + Q_{n-1} S^n x)} \qquad (n = 1, 2, \cdots)$$

and from the fact $\epsilon_{n+1} S^n x > 0$ we see that

$$\phi_n(x) = \frac{\epsilon_{n+1} S^n x}{1 + \frac{Q_{n-1}}{Q_n} S^n x} \qquad (n = 1, 2, \cdots)$$

Also because

$$\frac{1}{\epsilon_{n+1} S^n x} = c_{n+1} + S^{n+1} x$$

and

$$\frac{Q_{n+1}}{Q_n} = c_{n+1} + \epsilon_{n+1}\frac{Q_{n-1}}{Q_n},$$

we have

$$\phi_n(x) = \frac{Q_n}{Q_{n-1}} \frac{1}{1 + \frac{Q_n}{Q_{n+1}} S^{n-1}x}.$$

Let

$$(\alpha, \beta) = F(a,b) = (\frac{b}{1 + ab}, \frac{a}{1 + ab})$$

for $ab \neq 1$. Then F has derivative

$$F'(a,b) = \begin{pmatrix} \frac{-b^2}{(1 + ab)^2} & \frac{1}{(1 + ab)^2} \\ \frac{1}{(1 + ab)^2} & \frac{-a^2}{(1 + ab)^2} \end{pmatrix}$$

and Jacobian $J = -(1 - ab)(1 + ab)^{-3}$. Then as a consequence of Theorem 2.6 for almost all x with respect to Lebesgue measure $(F(S^n x), \frac{Q_{n-1}}{Q_n}))_{n=1}^{\infty}$, which is just the sequence $(\phi_n(x), \epsilon_{n+1}\phi_{n+1}(x))_{n=1}^{\infty}$, is distributed over $\Gamma_1 \cup \Gamma_1^*$, with density $\frac{1}{|J|\log G}\frac{1}{(1 + xy)^2}$. Here Γ_1^* is the reflection of Γ_1 in the α axis. Thus from the definition of F we have

$$\frac{1}{|J|\log G}\frac{1}{(1 + ab)^2} = \frac{1}{|J|\log G}\frac{1}{(1 - 4\alpha\beta)^{\frac{1}{2}}}$$

and the other details follow analoguously and so Theorem 2.4 is proved. □

Corollaries of Theorem 2.4

Apply Theorem 2.4 with

$$A = \{(x,y) \in D : x < a \text{ and } y < b\}$$

we have the following moving average analogue of the Doeblin-Lenstra conjecture proved in [BJW].

Corollary 2.7. *Let*

$$G(z) = \begin{cases} \frac{z}{\log G} & if\ z \in [0, 1-g]; \\ \frac{1}{\log G}(z - \frac{z}{1-g} + \log\frac{z}{1-g} + 1) & if\ z \in [1-g, g]. \end{cases}$$

Then if the pair of sequences of natural numbers $(n_l, k_l)_{l=1}^{\infty}$ is Stoltz we have

$$\lim_{l\to\infty} \frac{1}{k_l} |\{1 \leq i \leq k_l : \phi_{n_l+i}(x) \leq z\}| = G(z)$$

for almost all x with respect to Lebesgue measure.

To derive our next result we need the following Lemma [Kr].

Lemma 2.8. *Let $\Delta(a)$ be the interior of the triangle in the (α, β) plane with vertices $(0,0)$, $(0,a)$ and $(a,0)$ with $0 < a \leq 5g - 2$. Set*

$$H_1(a) = \frac{1}{\log G} \int\int_{\Delta(a)\cap\Gamma_1} f(x,y)dxdy = \int_0^a h_1(t)dt$$

and

$$H_{-1}(a) = \frac{1}{\log G} \int\int_{\Delta(a)\cap\Gamma_{-1}} f(x,y)dxdy = \int_0^a h_{-1}(t)dt.$$

Then for a in $[0, 5g - 2]$

$$h_1(a) = \begin{cases} \frac{1}{2\log G}\log\frac{1+a}{1-a} & \text{if } a \in [0,\frac{1}{2}] \\ \frac{1}{2\log G}\log 3 & \text{if } a \in [\frac{1}{2}, g] \\ \frac{1}{2\log G}\log\frac{1+(3g-1)}{1-(3g-1)} - \log(\frac{1+a}{1-a}) & \text{if } a \in (g, 3g-1] \\ 0 & \text{if } a \in (3g-1, 5g-2] \end{cases}$$

and

$$h_{-1}(a) = \begin{cases} \frac{1}{\log G}\arctan a & \text{if } a \in [0, g^2] \\ \frac{1}{\log G}\arctan g^2 & \text{if } a \in (g^2, \frac{1}{2}] \\ \frac{1}{\log G}(\arctan(5g-2) - \arctan a) & \text{if } a \in (\frac{1}{2}, 5g-2]. \end{cases}$$

We have the following result.

Corollary 2.9. *Let $h = h_1 + h_{-1}$. Then if the sequence of pairs of natural numbers $(n_l, k_l)_{l=1}^{\infty}$ is Stoltz*

$$\lim_{l\to\infty} \frac{1}{k_l} |\{1 \leq i \leq k_l : \phi_{n_l+i-1}(x) + \phi_{n_l+i}(x) < a\}| = \int_0^a h(t)dt,$$

almost everywhere with respect to Lebesgue measure.

P r o o f. If $\Delta(a)$ is as in the statement of Lemma 2.8 we have

$$\lim_{l\to\infty} \frac{1}{k_l} |\{1 \leq i \leq k_l : \phi_{n_l+i-1}(x) + \phi_{n_l+i}(x) < a : \epsilon_n = 1\}|$$
$$= \frac{1}{\log G} \int\int_{\Delta(a)\cap\Gamma_1} f(x,y)dxdy,$$

and

$$\lim_{l\to\infty} \frac{1}{k_l} |\{1 \leq i \leq k_l : \phi_{n_l+i-1}(x) + \phi_{n_l+i}(x) < a : \epsilon_n = -1\}|$$
$$= \frac{1}{\log G} \int\int_{\Delta(a)\cap\Gamma_{-1}} f(x,y)dxdy.$$

Thus Corollary 2.9 follows from Lemma 2.8. □

In [Kr] it is shown that for irrational x $-g < \phi_{n-1}(x) - \phi_n(x) < g$ for all positive integers n. We have the following result proved in the same way to Corollary 2.9.

Corollary 2.10. *Let* $k = k_1 + k_{-1}$

$$k_1(a) = \begin{cases} \frac{1}{2\log G}(\log 3 + \log\frac{1+a}{1-a}) & \textit{if } a \in [-\frac{1}{2}, 3g-2] \\ \frac{1}{2\log G}\frac{1}{2}\log 5 & \textit{if } a \in (3g-2, 0] \\ \frac{1}{2\log G}(\frac{1}{2}\log 5 - \log\frac{1+a}{1-a}) & \textit{if } a \in (g, 3g-1] \\ 0 & \textit{if } a \in (g^2, g] \end{cases}$$

and

$$k_{-1}(a) = \begin{cases} \frac{1}{\log G}(\arctan\frac{1}{2} + \arctan a) & \textit{if } a \in [-\frac{1}{2}, 0] \\ \frac{1}{\log G}\arctan\frac{1}{2} & \textit{if } a \in (0, 5g-3] \\ \frac{1}{\log G}(\arctan g - \arctan a) & \textit{if } a \in (5g-3, g]. \end{cases}$$

Then if the sequence of pairs of natural numbers $(n_l, k_l)_{l=1}^{\infty}$ *is Stoltz*

$$\lim_{l\to\infty}\frac{1}{k_l}|\{1 \le i \le k_l : \phi_{n_l+i-1}(x) - \phi_{n_l+i}(x) < a\}| = \int_0^a k(t)dt,$$

almost everywhere with respect to Lebesgue measure.

2.3. Other sequences attached to the nearest integer continued fraction expansion

Theorem 1 has a number of other consequences for the nearest integer continued fraction expansion which we now describe. Let

$$L(z) = \begin{cases} \frac{1}{\log G}(\log(1 + \frac{z}{2}) - \log(1 - \frac{z}{2})) & \text{if } z \in [0, g^2] \\ \frac{1}{\log G}(\log(1 + \frac{z}{2}) - \log(1 - \frac{g^2}{2})) & \text{if } z \in (g^2, g]. \end{cases}$$

Note that L is monotonically increasing, continuous and such that $L(0) = 0$ with $L(g) = 1$. We have the following theorem

Theorem 2.11. *If the sequence of pairs of natural numbers* $(n_l, k_l)_{l=1}^{\infty}$ *is Stoltz then*

$$\lim_{l\to\infty}\frac{1}{k_l}|\{1 \le i \le k_l : \frac{Q_{n_l+i}}{Q_{n_l+i-1}} \le z\}| = L(z),$$

almost everywhere with respect to Lebesgue measure.

P r o o f. Let

$$A(z) = \{(x, y) : (x, y) \in D ; y \leq z\}.$$

We note that

$$\frac{Q_{n-1}}{Q_n} = [0; c_n, \epsilon_n c_{n-1}, \cdots, \epsilon_2 c_1].$$

As a consequence of this observation and (2.3) we see that for $n > n_0(\epsilon)$ and any $(x, y) \in D$ if

$$N^n(x, y) \in A(z - \epsilon)$$

then

$$N^n(x, y) \in A(z).$$

This is equivalent to $\frac{Q_{n-1}}{Q_n} \leq z$ and which in turn implies

$$N^n(x, y) \in A(z + \epsilon).$$

Hence, for almost all x with respect to Lebesgue measure, using the argument used to prove Theorem 2.6 we see that

$$\lim_{l\to\infty} \frac{1}{k_l} |\{1 \leq i \leq k_l : \frac{Q_{n_l+i}}{Q_{n_l+i-1}} \leq z\}| = \mu(A(z)).$$

Computation reveals that $\mu(A(z)) = L(z)$, as required. □

Taking first moments and evaluating $\int_0^g z dL(z)$ we readily have the following theorem.

Theorem 2.12. *If the sequence of pairs of natural numbers $(n_l, k_n)_{l=1}^{\infty}$ is Stoltz, then*

$$\lim_{l\to\infty} \frac{1}{k_l} \sum_{i=1}^{k_l} \frac{Q_{n_l+i}}{Q_{n_l+i-1}} = \frac{1}{\log G}(\sqrt{5} - 2 + 4\log 2 + 2\log(\sqrt{5} - 2)),$$

almost everywhere with respect to Lebesgue measure.

For $z \in [0, \frac{g}{2}]$ let

$$M_1(z) = \frac{1}{\log G}\left(\log(1 + z) - \frac{z}{1+z}\log(2Gz)\right)$$

and for $z \in [0, \frac{g^2}{2}]$ let

$$M_2(z) = \frac{1}{\log G}\left(-\log(1 - z) - \frac{z}{1-z}\log(2Gz)\right).$$

Next define $M : [0, \frac{g}{2}] \to [0, 1]$ that is continuous and monotonically increasing by

$$M(z) = \begin{cases} M_1(z) + M_2(z) & \text{if} z \in [0, \frac{g^2}{2}] \\ M_1(z) + M_2(\frac{g^2}{2}) & \text{if} z \in (\frac{g^2}{2}, \frac{g}{2}]. \end{cases}$$

Further let

$$R_n = \frac{\left|x - \frac{P_n}{Q_n}\right|}{\left|x - \frac{P_{n-1}}{Q_{n-1}}\right|}. \qquad (n = 1, 2, \cdots)$$

In [Pe p. 168-78] it is shown that $0 \leq R_n \leq \frac{g}{2}$ for irrational x. We have the following theorem,

Theorem 2.13. *If the sequence of pairs of natural numbers* $(n_l, k_n)_{l=1}^{\infty}$ *is Stoltz then*

$$\lim_{l\to\infty} \frac{1}{k_l} |\{1 \leq i \leq k_l : R_{n_l+i} \leq z\}| = M(z),$$

almost everywhere with respect to Lebesgue measure.

P r o o f. Let

$$B_+(z) = \{(x, y) \in D : x \geq 0 ; xy \leq z\}$$

and let

$$B_-(z) = \{(x, y) \in D : x < 0 ; |x|y \leq z\}.$$

Then using the fact that

$$R_n = \frac{Q_{n-1}}{Q_n} |N^n x|$$

and the argument used in the proof of Theorem 2.11 we have that

$$\lim_{l\to\infty} \frac{1}{k_l} |\{1 \leq i \leq k_l : R_{n_l+i} \leq z\}| = \mu(B_+(z)) + \mu(B_-(z)).$$

Computation varifies that if $z \in [0, \frac{g}{2}]$

$$\mu(B_+(z)) = M_1(z)$$

and that if $z \in [0, \frac{g^2}{2}]$

$$\mu(B_-(z)) = M_2(z),$$

completing the proof of Theorem 2.13. □

Let $Li_2(x)$ denote the dilogarithm defined for instance in [Ri]. Again taking first moments we have the following result.

Theorem 2.14. *If the sequence of pairs of natural numbers $(n_l, k_l)_{l=1}^{\infty}$ is Stoltz then*

$$\lim_{l\to\infty} \frac{1}{k_l} \sum_{i=1}^{k_l} R_{n_l+i} \;=\; -3 \;-\; \log 2 \;+\; \frac{1}{\log G}\left(\frac{\pi^2}{12} \;+\; \log 4 \;-\; 2\mathrm{Li}_2\left(\frac{g^2}{2}\right)\right),$$

almost everywhere with respect to Lebesgue measure.

Acknowledgment. We thank the referee for his very detailed comments that substantially improved the presentation of the paper.

REFERENCES

[BJR] A. BELLOW – R. JONES – J. ROSENBLATT: *Convergence of moving averages*, Erg. Th. & Dynam. Syst. **10** (1990), no. 1, 43–62.

[BJW] W. BOSMA – H. JAGER – F. WIEDIJK: *Some metrical observations on the approximation by continued fractions*, Indag. Math. **45** (1983), 281-299.

[CFS] I. P. CORNFELD – S. V. FORMIN – YA. G. SINAI: *Ergodic Theory*, Springer Verlag, 1982.

[HW] G. H. HARDY – E. WRIGHT: *An Introduction to Number Theory*, Oxford University Press, 1979.

[INT] S. ITO – H. NAKADA – S. TANAKA: *On the invariant measure for the transformation associated with some real continued fractions*, Keio Engineering reports, **30** (1981), 61–69.

[J] H. JAGER: *Metric results for the nearest integer continued fractions*, Indag. Math. **47** (1985), no. 4, 417-427.

[KN] H. KAMARUL-HAILI – R. NAIR: *On moving averages and continued fractions*, Uniform Distribution Theory **6** (2011), no. 1, 65–78.

[Kr] C. KRAAIKAMP: *The distribution of some sequences connected with the nearest integer continued fraction expansion*, Indag. Math. **49** (1987), 177–191.

[N] H. NAKADA: *Metrical theory for a class of continued fraction transformations and their natural extentions*, Tokyo J. Math. **4** (1981), no. 2, 399-426.

[Pe] O. PERRON: *Die Lehre von der Kettenbrüche"*, Band 1, B-G Teubner, Stuttgart 3, verb. v. etw. Auft.

[Ri] G. J. RIEGER: *Mischung und Ergodizität bei Kettenbrüche" nach nächsten Ganzen*, Journal f. d. reine und angew Math., **310** (1979), 171–181.

Received February 9, 2011
Accepted August 10, 2012

Hailiza Haili-Kamarul
School of Mathematical Sciences
Universiti Sains Malaysia
11800 USM Pulau Pinang
MALAYSIA.
E-mail: hailiza@cs.usm.my

Radhakrishnan Nair
Mathematical Sciences
University of Liverpool
Liverpool L69 7ZL
U.K.
E-mail: nair@liv.ac.uk