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BADLY APPROXIMABLE SYSTEMS OF LINEAR FORMS IN ABSOLUTE VALUE

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ABSTRACT. In this paper we show that the set of mixed type badly approximable simultaneously small linear forms is of maximal dimension. As a consequence of this theorem we settle the conjecture stated in [10].

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1. Introduction

Let $X = (x_{ij}) \in \mathbb{R}^{mn}$ be a point identified as an $m \times n$ matrix. Throughout, m and n are fixed natural numbers. Let

$$q_1 x_{1i} + q_2 x_{2i} + \ldots + q_m x_{mi} \qquad (1 \leq i \leq n),$$

be a system of n linear forms in m variables written more concisely as $\mathbf{q}X$. The classical result of Dirichlet [4] states that for any point $X \in \mathbb{I}^{mn}$, there exist infinitely many integer points $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^n \times \mathbb{Z}^m \setminus \{\mathbf{0}\}$ such that

$$\|\mathbf{q}X\| = |\mathbf{q}X - \mathbf{p}| := \max_{1 \leq i \leq n} |q_1 x_{1i} + q_2 x_{2i} + \ldots + q_m x_{mi} - p_i| < |\mathbf{q}|^{-\frac{m}{n}}, \qquad (1)$$

where $|\mathbf{q}|$ denotes the supremum norm i.e. $|\mathbf{q}| := \max\{|q_1|, |q_2|, \ldots, |q_m|\}$, and $\|\mathbf{q}X\|$ denotes the distance from $\mathbf{q}X$ to the nearest vector with integer coordinates in the supremum norm.

The right hand side of (1) may be sharpened by a constant $c(m, n)$ but the best permissible values for $c(m, n)$ are unknown except for $m = n = 1$. A point $X \in \mathbb{R}^{mn}$ is said to be *badly approximable* if the right hand side of (1) cannot be improved by an arbitrary positive constant. Denote the set of all such points

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as $\mathbf{Bad}(m,n)$; that is $X \in \mathbf{Bad}(m,n)$ if there exists a constant $C(X) > 0$ such that

$$\|\mathbf{q}X\| > C(X)|\mathbf{q}|^{-\frac{m}{n}} \quad \text{for all } \mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}.$$

The Khintchine–Groshev theorem [8, 11] implies that $\mathbf{Bad}(m,n)$ is of mn–dimensional Lebesgue measure zero and a result of Schmidt [12] states that $\dim \mathbf{Bad}(m,n) = mn$, where $\dim A$ denotes the Hausdorff dimension of the set A – see [7] for the definition of Hausdorff dimension.

An alternative set to consider is obtained by replacing the distance to the nearest integer vector $\|\cdot\|$ by the usual supremum norm. The Hausdorff dimension of the set

$$\mathbf{Bad}^*(m,n) = \{X \in \mathbb{I}^{mn} : |\mathbf{q}X| > C(X)|\mathbf{q}|^{-\frac{m}{n}+1} \quad \text{for all } \mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}\}$$

is discussed in [10] where it is proved that $\mathbf{Bad}^*(m,1) = m$.

It can be easily seen that the above set is obtained from $\mathbf{Bad}(m,n)$ by simply imposing the additional condition that $\mathbf{p} = \mathbf{0}$. It is a natural question to ask what happens for a less restrictive condition on the allowed set of vectors $\mathbf{p}$. Such a condition gives rise to a range of hybrids between the set $\mathbf{Bad}(m,n)$ and $\mathbf{Bad}^*(m,n)$, where some coordinates of the image are considered in absolute value and some in the distance to nearest integer.

Let $A \subseteq \mathbb{Z}^n$ be a lattice of integer vectors in a linear subspace of dimension u, where u is an integer and $0 \le u \le n$. A consequence of the Dirichlet type theorem established by Dickinson in [2] is the following statement.

Lemma (Dickinson). *Suppose that $m + u > n$. For each $X \in \mathbb{R}^{mn}$ there exist infinitely many non-zero integer vectors $(\mathbf{p},\mathbf{q}) \in A \times \mathbb{Z}^m$ such that*

$$|\mathbf{q}X - \mathbf{p}| < C|\mathbf{q}|^{-\frac{m+u}{n}+1}.$$

In view of this lemma, it is natural to consider the following badly approximable set. Let $\mathrm{Bad}_A(m,n)$ denote the set of $X \in \mathbb{R}^{mn}$ for which there exists a constant $C(X) > 0$ such that

$$|\mathbf{q}X - \mathbf{p}| > C(X)|\mathbf{q}|^{-\frac{m+u}{n}+1} \quad \forall \ (\mathbf{p},\mathbf{q}) \in A \times \mathbb{Z}^m \setminus \{\mathbf{0}\}. \tag{2}$$

The set $\mathrm{Bad}_A(m,2)$ is related to an exceptional set associated with the linearization of germs of complex analytic diffeomorphisms of $\mathbb{C}^m$ near a fixed point and is the m-dimensional version of the Schröder's functional equation. In this case, one considers complex numbers $z_1,\ldots,z_m$ with $z_j = x_j + iy_j$. Consider the set of vectors $(x_1,\ldots,x_m,y_1,\ldots,y_m) = (\mathbf{x},\mathbf{y}) \in \mathbb{R}^{2m}$. Optimal control over the small denominators arising is attained when $\max\{|\mathbf{q}\cdot\mathbf{x}|, \|\mathbf{q}\cdot\mathbf{y}\|\}$ is as large as possible for all $\mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$. Evidently, this is an example of a set such as the above with $A = \mathbb{Z}^m$ under a natural embedding into $\mathbb{Z}^{2m}$. We refer [1, 5, 6] for further details.

In the present note, as a consequence of the following theorem we prove a conjecture from [10].

Theorem 1. *The Hausdorff dimension of* $\mathrm{Bad}_A(m,n)$ *is maximal. If* $m+u \leq n$, *the Lebesgue measure of* $\mathrm{Bad}_A(m,n)$ *is full.*

Remarks.

(i) Clearly when $u = 0$, $\mathrm{Bad}_A(m,n)$ is identified with the set $\mathrm{Bad}^*(m,n)$ of [10]. It not only settles the conjecture but also disagrees with the 'final comment' of the paper that for $m < n$ the set $\mathrm{Bad}^*(m,n)$ is zero dimensional.

(ii) As a consequence of the Khintchine–Groshev type theorem established in [3, 9] that for $m + u > n$,

$$|\,\mathrm{Bad}_A(m,n)|_{mn} = 0,$$

where $|\,.\,|_k$ denotes k-dimensional Lebesgue measure.

(iii) As a consequence of our theorem, it is clear that for $m + u \leq n$ the complementary set to $\mathrm{Bad}_A(m,n)$ should be of zero mn dimensional Lebesgue measure. This set corresponds to the set of well approximable systems of linear forms in the present setup, and the consequence stated here is indeed proved in [3, 9].

(iv) An obvious corollary of the above theorem is that $\dim \mathrm{Bad}_A(m,n) = mn$.

(v) We conjecture that the set $\mathrm{Bad}_A(m,n)$ is in fact winning for the Schmidt game, which is the case for the usual set of badly approximable systems of linear forms (see [12]). This is a stronger statement than the property of having maximal Hausdorff dimension. We have chosen not to pursue this issue in the present note, although we believe that our methods should give some insight into the matter.

2. Proof of Theorem 1

Initially, we reduce the problem to the case when $A = \mathbb{Z}^u \times \{(0, ..., 0)\}$. To see that it suffices to consider this case, let $\{v_1, \dots, v_u\}$ be a basis of the lattice A, extend the basis to a basis of $\mathbb{R}^n$ by adding vectors $\{v_{u+1}, \dots, v_n\}$ and consider the map sending v_i to e_i, the i'th standard basis vector of $\mathbb{R}^n$. Evidently, this change of basis is a linear automorphism of $\mathbb{R}^n$ and hence bi-Lipschitz. Since such maps preserve Hausdorff dimension as well as the properties of being null/full with respect to Lebesgue measure, it suffices to consider the problem in the image of the map. The inequalities defining the problem are also unchanged up to modifying the constant $C(X)$ by a factor depending only on A.

With the particular case $A = \mathbb{Z}^u \times \{(0, ..., 0)\}$ in mind, the defining inequalities take a particularly pleasing form. Namely, $X \in \mathrm{Bad}_A(m, n)$ if and only if

$$\left|(\mathbf{p}, \mathbf{q})\begin{pmatrix} I_u & 0 \\ X_u & \tilde{X} \end{pmatrix}\right| \geq C(X)\,|\mathbf{q}|^{-\frac{m+u}{n}+1}, \qquad (3)$$

for all $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^u \times \mathbb{Z}^m \setminus \{\mathbf{0}\}$, where X is the matrix $(X_u,\ \tilde{X})$. Put differently, we want an appropriate lower bound on the system of linear forms

$$\max\left(\|\mathbf{q}\cdot\mathbf{x^{(1)}}\|, \dots, \|\mathbf{q}\cdot\mathbf{x^{(u)}}\|, |\mathbf{q}\cdot\mathbf{x^{(u+1)}}|, \dots, |\mathbf{q}\cdot\mathbf{x^{(n)}}|\right),$$

where $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^u \times \mathbb{Z}^m \setminus \{\mathbf{0}\}$.

We now split the proof into two parts depending upon the choices of m, u and n.

First, by way of motivating our proof when $m + u \leq n$, we discuss the case when $u = 0$ and $m = n$. In this case, we consider the inequalities

$$|\mathbf{q}X| \geq C(X)$$

for all $\mathbf{q} \in \mathbb{Z}^m$. It was already remarked in [10] that unless there is linear dependence among the columns of X, this is trivially satisfied. Hence, in this simple sub-case,

$$\mathbb{R}^{m(m+u)} \setminus \{X \in \mathbb{R}^{n^2} : \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)} \text{ are linearly dependent}\} \ \subseteq\ \mathrm{Bad}_A(m, n).$$

In other words, the set $\mathrm{Bad}_A(m, n)$ contains the set of matrices of full rank, which is full with respect to Lebesgue measure. Hence, $\mathrm{Bad}_A(m, m + u)$ is full. This clearly implies that $\dim \mathrm{Bad}_A(m, m + u) = m(m + u)$, which proves our main theorem in the particular case $m + u = n$.

Note that in fact we get the stronger inequality

$$|\mathbf{q}X| > C(X)|\mathbf{q}| \quad \forall\ \mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\} \qquad (4)$$

in the set considered. This follows as an invertible matrix can only distort the unit cube by a specified amount. This is much stronger than the defining inequality of the set $\mathrm{Bad}_A(m, n)$ in this special case. This feature also applies to the more general setting when $m+u \leq n$ and further underlines the quantitative difference between the case $m + u \leq n$ and the converse $m + u > n$.

We now give a full proof in the case $m + u \leq n$. We will argue much in the spirit of the above. For a generic $X \in \mathbb{R}^{mn}$, the matrix $\tilde{X}$ in (3) has full rank.

Performing Gaussian elimination on the columns of a matrix of the form of (3) implies the existence of an invertible $(n \times n)$-matrix $E(X)$ such that

$$\begin{pmatrix} I_u & 0 \\ X_u & \tilde{X} \end{pmatrix} = \begin{pmatrix} I_u & 0 & 0 \\ \hat{X} & I_m & 0 \end{pmatrix} E(X).$$

Applying this matrix from the right to a vector $(\mathbf{p}, \mathbf{q})$, we see that

$$(\mathbf{p}, \mathbf{q}) \begin{pmatrix} I_u & 0 \\ X_u & \tilde{X} \end{pmatrix} = (\mathbf{p}, \mathbf{q}) \begin{pmatrix} I_u & 0 & 0 \\ \hat{X} & I_m & 0 \end{pmatrix} E(X) = \begin{pmatrix} \mathbf{p} + \mathbf{q}\hat{X} \\ \mathbf{q} \\ \mathbf{0} \end{pmatrix}^T E(X).$$

Multiplication by the matrix E on the right hand side only serves to distort the unit cube in the absolute value to a different parallelepiped depending on X. This induces a different norm on the image, but by equivalence of norms on Euclidean spaces, this distortion can be absorbed in a positive constant. In other words,

$$\left| (\mathbf{p}, \mathbf{q}) \begin{pmatrix} I_u & 0 \\ X_u & \tilde{X} \end{pmatrix} \right| \geq C(X) \left| \begin{pmatrix} \mathbf{p} + \mathbf{q}\hat{X} \\ \mathbf{q} \\ \mathbf{0} \end{pmatrix}^T \right| \tag{5}$$

for all $(\mathbf{p}, \mathbf{q}) \in A \times \mathbb{Z}^m \setminus \{\mathbf{0}\}$.

Finally, since the norm is the *supremum* norm, we get

$$\left| \begin{pmatrix} \mathbf{p} + \mathbf{q}\hat{X} \\ \mathbf{q} \\ \mathbf{0} \end{pmatrix}^T \right| \geq |\mathbf{q}| \, .$$

By (5), almost every X is in $\mathrm{Bad}_A(m, n)$, and in fact with the stronger requirement from (4). This completes the proof of Theorem 1 in the case when $m + u \leq n$.

Now we consider the case $m + u > n$. We will apply the following extension of Lemma 2.1 of [10].

Lemma 2. *Let $S \subseteq \mathrm{Mat}_{(m+u-n)\times n}(\mathbb{R})$ of Hausdorff dimension $(m + u - n)n$. Let $\mathcal{X} \subseteq \mathrm{GL}_n(\mathbb{R})$ be a subset of a subspace of dimension $(n - u)n$, such that $\mathcal{X}$ has positive $(n - u)n$–Lebesgue measure. Then, the set*

$$\Lambda = \left\{ \begin{pmatrix} X' \\ \tilde{X}X' \end{pmatrix} \in \mathrm{Mat}_{(m+u)\times n}(\mathbb{R}) : X' \in \mathcal{X}, \tilde{X} \in S \right\}$$

has Hausdorff dimension mn. Here, $\tilde{X}X'$ denotes the product of the two matrices $\tilde{X}$ and X'.

P r o o f o f L e m m a 2. The proof is essentially an adaptation of the proof in [10]. As in that paper, the upper bound is trivial.

Without loss of generality, we will assume that $|\mathcal{X}|_{(n-u)n} < \infty$. If this is not the case, we will replace $\mathcal{X}$ with a subset of $\mathcal{X}$ of positive and finite measure. Suppose now for a contradiction that $\dim \Lambda < mn$ and fix an $\epsilon > 0$. Then, there is a $\delta > 0$ and a cover $\mathcal{C}$ of Λ by hypercubes in $\mathbb{R}^{(m+u)n}$ such that

$$\sum_{C \in \mathcal{C}} \operatorname{diam}(C)^{mn-\delta} < \epsilon.$$

For a fixed $X' \in \mathcal{X}$, define the set

$$B(X') = \left\{ \begin{pmatrix} X' \\ \tilde{X}X' \end{pmatrix} \in \mathrm{Mat}_{(m+u)\times n}(\mathbb{R}) : \tilde{X} \in S \right\}.$$

Note that

$$\mathcal{C}(X') = \left\{ \left(\begin{pmatrix} X' \\ \mathrm{Mat}_{(m+u-n)\times n}(\mathbb{R}) \end{pmatrix} \cap C \right) \in \mathrm{Mat}_{(m+u)\times n}(\mathbb{R}) : C \in \mathcal{C} \right\}$$

is a cover of $B(X')$ by $(m+u-n)n$-dimensional hypercubes in the slice of the larger space $\mathrm{Mat}_{m\times n}(\mathbb{R})$ obtained by fixing the upper matrix to be X'.

As in [10], we define for each $C \in \mathcal{C}$ a function,

$$\lambda_C(X') = \begin{cases} 1 & \text{if } \left(\begin{pmatrix} X' \\ \mathrm{Mat}_{(m+u-n)\times n}(\mathbb{R}) \end{pmatrix} \cap C \right) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

It is easily seen that

$$\int_{\mathcal{X}} \lambda_C(X') dX' \leq \operatorname{diam}(C)^{(n-u)n},$$

where the integral is with respect to the $(n-u)\times n$-dimensional Lebesgue measure on the ambient subspace of $\mathrm{GL}_n(\mathbb{R})$. Also,

$$\sum_{C \in \mathcal{C}(X')} \operatorname{diam}(C)^{(m+u-n)n-\delta} = \sum_{C \in \mathcal{C}} \lambda_C(X') \operatorname{diam}(C)^{(m+u-n)n-\delta}.$$

We integrate the latter expression with respect to X' to obtain

$$\int_{\mathcal{X}} \sum_{C \in \mathcal{C}(X')} \operatorname{diam}(C)^{(m+u-n)n-\delta} dX'$$

$$= \sum_{C \in \mathcal{C}(X'} \int_{\mathcal{X}} \lambda_C(X') dX' \operatorname{diam}(C)^{(m+u-n)n-\delta}$$

$$\leq \sum_{C \in \mathcal{C}(X'} \operatorname{diam}(C)^{mn-\delta} < \epsilon.$$

Since the right hand side is an integral of a non-negative function over a set of positive measure, there must be an $X_0 \in \mathcal{X}$ with

$$\sum_{C \in \mathcal{C}(X_0)} \operatorname{diam}(C)^{(m+u-n)n-\delta} < \frac{\epsilon}{|\mathcal{X}|_{(n-u)n}}.$$

Indeed, otherwise

$$\int_{\mathcal{X}} \sum_{C \in \mathcal{C}(X')} \operatorname{diam}(C)^{(m+u-n)n-\delta} dX' \geq \int_{\mathcal{X}} \frac{\epsilon}{|\mathcal{X}|_{(n-u)n}} dX' = \epsilon.$$

Take such an X_0. Since we have produced a cover of $B(X_0)$, whose $((m+u-n)n-\delta)$-measure is less than ϵ, it follows that $\dim B(X_0) \leq (m+u-n)n-\delta$. The map from $B(X_0)$ to S defined by

$$\begin{pmatrix} X_0 \\ \tilde{X} X_0 \end{pmatrix} \mapsto \begin{pmatrix} I_{n \times n} \\ \tilde{X} \end{pmatrix}$$

is evidently bi-Lipschitz as $X_0 \in \mathcal{X}$ and hence invertible. It follows that $\dim S = \dim B(X_0) \leq (m+u-n)n-\delta$, which contradicts the condition on S and completes the proof. □

Proving the main theorem is now easy in the case $m+u>n$. Let

$$\mathcal{X} = \left\{ X \in \mathrm{GL}_n(\mathbb{R}) : X = \begin{pmatrix} I_u & 0 \\ X' & X'' \end{pmatrix} \right\}.$$

Evidently, the $(n-u)n$-dimensional Lebesgue measure of $\mathcal{X}$ is positive. Let $S = \mathrm{Bad}(m+u-n, n)$ be the usual set of badly approximable systems of n linear forms in $m+u-n$ variables. Schmidt's theorem [12] tells us that $\dim S = (m+u-n)n$.

For $Y \in S$, there is a constant $C(Y) > 0$ such that for any $\mathbf{r} \in \mathbb{Z}^{m+u-n} \setminus \{\mathbf{0}\}$ and any $\mathbf{p} \in \mathbb{Z}^n$,

$$\left| (\mathbf{p}, \mathbf{r}) \begin{pmatrix} I_n \\ Y \end{pmatrix} \right| \geq C(Y) \left| \mathbf{r} \right|^{-\frac{m+u}{n}+1}.$$

Multiplying a matrix $X \in \mathcal{X}$ onto $\binom{I_n}{Y}$, *i.e.*, considering instead the matrix $\binom{I_n}{Y}X$, only changes the constant $C(Y)$ to another positive constant $C(X,Y) > 0$. Hence,

$$\left| \mathbf{q} \begin{pmatrix} X \\ YX \end{pmatrix} \right| \geq C(X,Y) \left| \mathbf{q} \right|^{-\frac{m+u}{n}+1}, \tag{6}$$

for any $\mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$. The matrix in (6) has the form

$$\begin{pmatrix} I_u & 0 \\ X_u & \tilde{X} \end{pmatrix}$$

by choice of $\mathcal{X}$. In other words, using (3) we see that the set Λ arising from Lemma 2 is a subset of $\mathrm{Bad}_A(m,n)$ with some additional 'dummy' coordinates attached in the first u rows. It follows that $\dim \mathrm{Bad}_A(m,n) \geq \dim \Lambda = mn$, which completes the proof.

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