ON UNIFORM DISTRIBUTION IN PROBABILITY FOR SEQUENCES OF PRODUCTS OF INDEPENDENT RANDOM ELEMENTS IN A COMPACT METRIC GROUP

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ABSTRACT. We consider the concept of uniform distribution in probability for sequences of random elements in a compact metric group. We prove a criterion for uniform distribution in probability for sequences of products of independent random elements. As a corollary, we obtain a criterion for uniform distribution modulo 1 in probability for sequences of sums of independent random variables which generalizes the classical Weyl's criterion.

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1. Introduction

A real sequence $(x_n)_{n=1}^{\infty}$ is called uniformly distributed modulo 1 if for any continuous function f

$$\frac{1}{n}\sum_{k=1}^{n}f(x_k \bmod 1) \to \int_0^1 f(x)dx\,,\, n \to \infty\,.$$

The following classical result of H.Weyl is known [1].

THEOREM 1.1. *A real sequence $(x_n)_{n=1}^{\infty}$ is uniformly distributed modulo 1 if and only if*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} e^{2\pi i m x_k} = 0, \, m = 1, 2, \dots .$$

The cited paper initiated numerous investigations of the concept of uniform distribution modulo 1, its applications and generalizations. A most complete and detailed overview of results obtained up to 1974 can be found in the monograph of Kuipers and Niederreiter [2].

It is natural to ask: what is the condition of uniform distribution modulo 1 for a sequence of random variables? It is clear that the definition of uniform distribution depends on the selected type of convergence. The terms "uniform distribution modulo 1 almost surely" and "uniform distribution modulo 1 almost everywhere" are already used, and a number of results are known.

In 1963, H. Davenport, P. Erdős and W. J. LeVeque proved the following theorem [3].

THEOREM 1.2. *Let $s_1(x)$, $s_2(x)$, ... be a sequence of real functions bounded and integrable for $a \le x \le b$. Let*

$$S(N, x) = \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i m s_n(x)} \, , \, m = 1, 2, \dots ,$$

and

$$I(N) = \int_{a}^{b} |S(N, x)|^2 dx \, .$$

If the series

$$\sum N^{-1} I(N)$$

converges for each integer $m \ne 0$, then the sequence $(s_n(x))_{n=1}^{\infty}$ is uniformly distributed modulo 1 for almost all x in $[a, b]$. On the other hand, given any increasing function $\Phi(M)$ which tends to infinity with M (however slowly), there exists a sequence $(s_n(x))_{n=1}^{\infty}$ which is not uniformly distributed modulo 1 for any x, and which satisfies the inequality

$$\sum_{N=1}^{M} N^{-1} I(N) < \Phi(M) \, .$$

In 1969 on the basis of [3], Holewijn proved a sufficient condition for uniform distribution modulo 1 almost surely for sequences of random variables $(X_n)_{n=1}^{\infty}$ whose distributions of increments $X_{n+m} - X_n$ depend only on $m = 1, 2, \dots$ [4].

Also Holewijn obtained the following generalization of Weyl's criterion [5].

THEOREM 1.3. *Let* $(X_n)_{n=1}^{\infty}$ *be a sequence of independent random variables with characteristic functions* $(\varphi_n)_{n=1}^{\infty}$ *. The sequence* $(X_n)_{n=1}^{\infty}$ *is uniformly distributed modulo* 1 *almost surely if and only if*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \varphi_k(2\pi m) = 0 \,, \, m = 1, 2, \dots \,.$$

In [6, 7], the asymptotic uniformity in distribution (mod 1) of sums of independent random variables was studied. In these papers, uniformity in distribution (mod 1) of a sequence of random variables means convergence of fractional parts of these random variables in distribution to the Lebesgue measure on $[0, 1)$.

In the present paper, we define the concept of uniform distribution in probability. We consider this concept in a natural generalization: for sequences of random elements in a compact metric group.

Note that the concepts of uniform distribution in probability and uniform distribution almost surely are equivalent for sequences of independent random elements (see section 2). However, we show an example of a sequence of sums of independent random variables which is uniformly distributed modulo 1 in probability but whose almost all trajectories are not uniformly distributed modulo 1. We also show an example of a sequence of sums of independent random variables which is asymptotically uniform in distribution (mod 1), but which is not uniformly distributed modulo 1 in probability. These examples are in section 5.

We prove a criterion for uniform distribution for sequences of products of independent random elements. As a corollary, the criterion for uniform distribution modulo 1 in probability for sequences of sums of independent random variables which generalizes the classical Weyl's criterion is obtained.

2. Main result

Let $\langle G, \cdot \rangle$ be a compact metric group.

We say the sequence $(S_k)_{k=0}^{\infty}$ of random elements in G is *uniformly distributed in probability* if for any continuous real function f on G

$$\frac{1}{n+1} \sum_{k=0}^{n} f(S_k) \xrightarrow{\mathbb{P}} \int_{G} f(g) \, h(dg), \text{ as } n \to \infty \,,$$

where h is the Haar measure on G.

The uniform distribution almost surely is defined analogically by replacing the convergence in probability by the convergence almost surely.

Let $(S_k)_{k=0}^{\infty}$ be a sequence of independent random elements uniformly distributed in probability. As G is compact, for any continuous real f the sequence of variances $(\operatorname{Var} f(S_k))_{k=0}^{\infty}$ is uniformly bounded. Hence, the sequence $(f(S_k))_{k=0}^{\infty}$ obeys the strong law of large numbers [8, ch. X.7, p. 259]. Thus, due to separability of the space of continuous real functions on G, the sequence $(S_k)_{k=0}^{\infty}$ is uniformly distributed almost surely. As it is clear that the uniform distribution almost surely implies the uniform distribution in probability, these concepts are equivalent for sequences of independent random elements.

In the present paper, our aim is to study the concept of uniform distribution in probability for sequences with independent increments.

Let $(\xi_k)_{k=1}^{\infty}$ be a sequence of independent random elements in G, and let their distributions be denoted by $(P_k)_{k=1}^{\infty}$.

Consider the sequence of random elements

$$S_k = S_{k-1} \cdot \xi_k\,,\ k = 1, 2, \dots,$$

where S_0 is a random element in G independent of $(\xi_k)_{k=1}^{\infty}$. Let its distribution be denoted by P_0.

The main result of this paper is as follows.

THEOREM 2.1. *The sequence $(S_k)_{k=0}^{\infty}$ is uniformly distributed in probability if and only if the sequence of probability measures*

$$\frac{1}{n} \sum_{k=1}^{n} P_n * P_{n-1} * \dots * P_k\,,\ n = 1, 2, \dots,$$

$*$-weakly converges to the Haar measure h.

Let us note that the condition of independency of the sequence of the increments $(\xi_k)_{k=1}^{\infty}$ is essential, as it is shown by the following example.

Let ξ_1 be a random variable uniformly distributed on $[0, 1)$, and

$$\xi_k = -\xi_{k-1}\,(\operatorname{mod} 1), \text{for } k = 2, 3, \dots.$$

Let $S_0 = 0$, then $S_k = \xi_1$ for odd k, and $S_k = 0$ for even k. It is clear that for such sequence $(S_k)_{k=0}^{\infty}$ the condition of Theorem 2.1 is satisfied, but the conclusion of the theorem is not valid.

3. Preliminaries

Let $(\Omega, F, \mathbb{P})$ be a probabilistic space, where $\Omega = G^{\mathbb{N}}$ is the space of sequences of elements of G with the topology of pointwise convergence, F is the σ-algebra of Borel sets of Ω, and $\mathbb{P}$ is the probability measure defined by the formula $\mathbb{P} = P_0 \times P_1 \times \cdots \times P_k \times \dots$.

Let $C(G)$ denote the Banach space of continuous real functions on G with the uniform norm $\|.\|$, and let $M(G)$ denote the space of Borel measures on G with the topology of $*$-weak convergence.

For $f \in C(G)$ and $\mu \in M(G)$, let us denote

$$\langle f, \mu \rangle = \int f \, d\mu \, .$$

For $s \in G$, let T_s denote the right-shift operator which acts on $f \in C(G)$ by the formula

$$T_s f(g) = f(g \cdot s)$$

for all $g \in G$ (the left-shift operator is defined in a similar way). The adjoint T_s^* acts on $\mu \in M(G)$ by the formula

$$T_s^* \mu(E) = \mu(E \cdot s^{-1})$$

for all Borel sets E of G.

We remind that the Haar measure on the compact group G is the probability measure $h \in M(G)$ which is invariant under right and left shifts and the inverse transformation $g \to g^{-1}$. It is known that on a compact group the Haar measure exists and is unique [9, ch. 1 point 1.2].

Let ρ denote the metric on the group G. Without loss of generality, we consider that ρ is right-invariant. Indeed, for an arbitrary metric ρ' on G, the metric

$$\rho(g_1, g_2) = \int \rho'(g_1 \cdot g, g_2 \cdot g) h(dg) \, , \text{ for } g_1, g_2 \in G \, ,$$

is right-invariant and generates the same topology.

Let us remember that for a random element F from the linear topological space X, the element $\mathbb{E} F \in X$ is called the expectation of F if for any continuous linear functional $\Lambda \in X^*$

$$\Lambda(\mathbb{E} F) = \int (\Lambda F) \, d\mathbb{P} \, .$$

Theorem 3.1 ([10, p. 74]). *Let X be a topological vector space on which X^* separates points, and let $\mathbb{P}$ be a Borel probability measure on a compact Hausdorff*

space Ω. If $F : \Omega \to X$ is continuous, and if the convex hull H of $F(\Omega)$ has compact closure $\overline{H}$ in X, then the integral

$$y = \int\limits_{\Omega} F \, d\mathbb{P}$$

exists. Moreover $y \in \overline{H}$.

We consider the random elements

$$F : \Omega \to X \,,$$

where X is either the space $C(G)$ or $M(G)$, and F is a bounded continuous function. The reader can easily make sure that in this case the conditions of the cited theorem are satisfied.

Let F denote a random element in $C(G)$ with distribution P_F, and let Q denote a random element in $M(G)$ with distribution P_Q. Let us remember that the random elements F and Q are independent if and only if

$$\mathbb{P}\{F \in U, \, Q \in V\} = P_F(U)P_Q(V)$$

for all Borel sets U of $C(G)$ and V of $M(G)$.

For independent random elements F in $C(G)$ and Q in $M(G)$, we have

$$\mathbb{E} \langle F, Q \rangle = \langle \mathbb{E}F, \, \mathbb{E}Q \rangle \,. \tag{1}$$

Indeed, by the Fubini theorem and the definition of expectation,

$$\begin{aligned}
\mathbb{E} \langle F, Q \rangle &= \int\limits_{\Omega} \langle F, Q \rangle \, d\mathbb{P} = \int\limits_{C(G)} \int\limits_{M(G)} \langle F, Q \rangle \, dP_Q dP_F \\
&= \int\limits_{C(G)} \langle F, \mathbb{E}Q \rangle \, dP_F = \langle \mathbb{E}F, \, \mathbb{E}Q \rangle \,.
\end{aligned}$$

Let $M_1(G) \subset M(G)$ denote the space of Borel probability measures on G with the topology of $*$-weak convergence.

It is known that the space $M_1(G)$ with the $*$-weak topology is metrized by the Kantorovich-Rubinstein metric ρ_{KR}. The Kantorovich-Rubinstein distance between measures $\mu, \, \lambda \in M_1(G)$ can be defined by the formula

$$\rho_{KR}(\mu, \, \lambda) = \sup_{\varphi \in \Phi} \left| \int \varphi \, d\mu - \int \varphi \, d\lambda \right| \,,$$

where $\Phi = \{\varphi \in C(G) \mid \forall g_1, g_2 \in G : \ |\varphi(g_1) - \varphi(g_2)| \leq \rho(g_1, g_2)\}$, see [11, ch. VIII, §4] or [12, p. 342]. Vershik [13] traces some of the history of this metric.

Note that as the metric ρ on the group G is right-invariant, the Kantorovich-Rubinstein metric on $M_1(G)$ is invariant under right shifts.

The right side of the formula for ρ_{KR} is invariant under replacement of φ by $-\varphi$ or by $\varphi + c$, $c \in \mathbb{R}$. Therefore, we can omit the module sign and take the supremum over the set $\Phi_0 = \{\varphi \in \Phi \mid \varphi(e) = 0\}$, where e is the identity element of the group G. As the family Φ_0 is equicontinuous, equibounded and closed, it is compact due to the Arzela-Ascoli theorem. Therefore, the supremum is attained, and

$$\rho_{KR}(\mu, \lambda) = \max_{\varphi \in \Phi_0} \left(\int \varphi \, d\mu - \int \varphi \, d\lambda \right)$$

$$= \max_{\varphi \in \Phi_0} \langle \varphi, \mu - \lambda \rangle \, .$$

LEMMA 3.1. *Let $(\mu_n)_{n=1}^{\infty}$ be a sequence of random elements in $M_1(G)$ which converges to the Haar measure h in probability. Then the sequence of elements $(\mathbb{E}\mu_n)_{n=1}^{\infty}$ $*$-weakly converges to h.*

P r o o f. We show that $\rho_{KR}(\mathbb{E}\mu_n, h) \to 0$ as $n \to \infty$.

Due to the properties of the Kantorovich-Rubinstein metric and the definition of the expectation, for each $n \in \mathbb{N}$, we have

$$\rho_{KR}(\mathbb{E}\mu_n, h) = \max_{\varphi \in \Phi_0} \left(\langle \varphi, \mathbb{E}\mu_n \rangle - \langle \varphi, h \rangle \right)$$

$$= \max_{\varphi \in \Phi_0} \left(\mathbb{E}\langle \varphi, \mu_n \rangle - \langle \varphi, h \rangle \right)$$

$$= \max_{\varphi \in \Phi_0} \mathbb{E} \left(\langle \varphi, \mu_n \rangle - \langle \varphi, h \rangle \right)$$

$$\leq \mathbb{E}\rho_{KR}(\mu_n, h) \, .$$

Since the sequence of random variables $(\rho_{KR}(\mu_n, h))_{n=1}^{\infty}$ converges to 0 in probability and is equibounded in $\mathbb{N} \times \Omega$, then the sequence $(\mathbb{E}\rho_{KR}(\mu_n, h))_{n=1}^{\infty}$ converges to 0. And this implies the desired conclusion. $\qquad\square$

Following [14], we define the convolution of measures $\mu, \lambda \in M(G)$ as the measure $\mu * \lambda \in M(G)$ such that for any function $f \in C(G)$

$$\mu * \lambda(f) = \int \int f(g \cdot u) \, \mu(dg) \, \lambda(du) \, .$$

Let us note that for any measure $\mu \in M_1(G)$,

$$\mu * h = h * \mu = h \, ,$$

and

$$\mu * \delta_e = \delta_e * \mu = \mu \, ,$$

where δ_e is the measure of unit mass concentrated at the identity element e of the group G.

Let s be a random element in G with the distribution P_s, then $T_s f$ is a random shift of the function $f \in C(G)$. Let Q be a random element in $M(G)$ independent of s. By virtue of (1),

$$\mathbb{E} \langle T_s f, Q \rangle = \langle \mathbb{E} T_s f, \mathbb{E} Q \rangle .$$

Since

$$\mathbb{E} T_s f = \int f(g \cdot u) P_s(du) ,$$

we have

$$\langle \mathbb{E} T_s f, \mathbb{E} Q \rangle = \int \int f(g \cdot u) \, P_s(du) \, \mathbb{E} Q(dg) ,$$

and by the definition of convolution,

$$\mathbb{E} \langle T_s f, Q \rangle = \langle f, \mathbb{E} Q * P_s \rangle . \tag{2}$$

For $f \in C(G)$, let us denote

$$\widehat{f}(g) = f(g^{-1}) , \text{for any } g \in G.$$

For $Q \in M(G)$, let us denote

$$\widehat{Q}(U) = Q(U^{-1})$$

for any Borel set U of G. It is easy to show that this implies

$$\left\langle f, \widehat{Q} \right\rangle = \left\langle \widehat{f}, Q \right\rangle , \text{for any } f \in C(G) .$$

Let s be a random element in G with the distribution P_s. It is obvious that

$$\widehat{P}_s = P_{s^{-1}} . \tag{3}$$

Consider the operator $\widehat{}$ applied to the convolution of measures $Q_1 , Q_2 \in M(G)$. For any $f \in C(G)$, we have

$$\left\langle f, \widehat{Q_1 * Q_2} \right\rangle = \int \widehat{f}(g) \, Q_1 * Q_2(dg) = \int \int \widehat{f}(s \cdot t) \, Q_1(ds) Q_2(dt)$$

$$= \int \int f(t^{-1} \cdot s^{-1}) \, Q_1(ds) Q_2(dt) = \int \int f(t \cdot s) \, \widehat{Q}_1(ds) \widehat{Q}_2(dt) .$$

And by the definition of convolution,

$$\widehat{Q_1 * Q_2} = \widehat{Q}_2 * \widehat{Q}_1 . \tag{4}$$

The operator $\widehat{}$: $M_1(G) \to M_1(G)$ is continuous. That is, for any sequence of measures $(Q_n)_{n=1}^{\infty}$ which $*$-weakly converges to the measure Q, the sequence

of measures $(\widehat{Q}_n)_{n=1}^\infty$ *-weakly converges to the measure $\widehat{Q}$. Indeed, for any $f \in C(G)$

$$\left\langle f, \widehat{Q}_n \right\rangle = \left\langle \widehat{f}, Q_n \right\rangle \to \left\langle \widehat{f}, Q \right\rangle = \left\langle f, \widehat{Q} \right\rangle .$$

Due to the definition of the operator $\widehat{}$ and the Haar measure properties,

$$\widehat{h} = h . \tag{5}$$

LEMMA 3.2. *Let* $(\rho_n)_{n=1}^\infty$ *be a sequence of nonnegative real numbers which satisfies the inequalities*

$$\rho_{n+1} \le \alpha_n \rho_n + (1 - \alpha_n)\varepsilon_n , \ \ n = 1, 2, \dots ,$$

where $\alpha_n \in (0, 1]$, $\prod_{k=1}^n \alpha_k \to 0$ *and* $\varepsilon_n \to 0$ *as* $n \to \infty$. *Then the sequence* $(\rho_n)_{n=1}^\infty$ *converges to* 0.

P r o o f. Fix $\delta > 0$. Let $N \in \mathbb{N}$ be such that $|\varepsilon_n| < \delta/2$ for all $n > N$. Let us define a sequence $(\rho'_n)_{n=N}^\infty$ by the formulas

$$\begin{aligned} \rho'_N &= \rho_N , \\ \rho'_{n+1} &= \alpha_n \rho'_n + (1 - \alpha_n)\frac{\delta}{2} , \ \ n = N, \dots . \end{aligned}$$

It is clear that $\rho_n \le \rho'_n$ for $n \ge N$.

By virtue of the condition $\prod_{k=1}^n \alpha_k \to 0$ as $n \to \infty$, a natural $N' > N$ such that for all $n > N'$,

$$\prod_{k=N}^n \alpha_k < \frac{\delta}{2 \max\{|\rho'_N - \delta/2|, \ 1\}}$$

exists. Then for $n > N'$,

$$\left| \rho'_{n+1} - \frac{\delta}{2} \right| = \alpha_n \left| \rho'_n - \frac{\delta}{2} \right| = \prod_{k=N}^n \alpha_n \cdot \left| \rho'_N - \frac{\delta}{2} \right| < \frac{\delta}{2} ,$$

and, therefore, $0 \le \rho_{n+1} < \delta$. The arbitrariness of δ implies the desired conclusion. $\qquad\square$

4. Proof of main result

Let us define a random element Q_n in $M_1(G)$ for each $n = 0, 1, \dots$ in the following way

$$Q_n = \frac{1}{n+1} \sum_{k=0}^n \delta_{S_k} ,$$

where δ_x denotes the measure of unit mass concentrated at x.

Note that for each $n = 0, 1, \dots$ and $f \in C(G)$,

$$\frac{1}{n+1} \sum_{k=0}^{n} f(S_k) = \int f \, dQ_n \,. \tag{6}$$

It is clear that the uniform distribution of the sequence $(S_k)_{k=0}^{\infty}$ in probability is equivalent to the convergence of the sequence $(\rho_{KR}(Q_n, h))_{n=1}^{\infty}$ to 0 in probability.

P r o o f   o f   T h e o r e m 2.1. Note that the sequence $(Q_n)_{n=0}^{\infty}$ satisfies the difference equation

$$Q_n = \frac{n}{n+1} Q_{n-1} + \frac{1}{n+1} \delta_{S_n} \,, \ n = 1, 2, \dots \,.$$

Therefore, the sequence $(\tilde{Q}_n)_{n=0}^{\infty} = (T_{S_n^{-1}}^* Q_n)_{n=0}^{\infty}$ satisfies the equation

$$\tilde{Q}_n = \frac{n}{n+1} T_{\xi_n^{-1}}^* \tilde{Q}_{n-1} + \frac{1}{n+1} \delta_e \,, \ n = 1, 2, \dots \,. \tag{7}$$

Since the metric ρ_{KR} and the Haar measure h are invariant under right shifts, for each $n = 0, 1, \dots$,

$$\rho_{KR}(\tilde{Q}_n, \, h) = \rho_{KR}(T_{S_n^{-1}}^* Q_n, T_{S_n^{-1}}^* h) = \rho_{KR}(Q_n, h) \,.$$

Hence, the sequences $(Q_n)_{n=0}^{\infty}$ and $(\tilde{Q}_n)_{n=0}^{\infty}$ either converge or do not converge to h in probability together.

Let $(\tilde{Q}_n^1)_{n=0}^{\infty}$ and $(\tilde{Q}_n^2)_{n=0}^{\infty}$ be different solutions of the equation (7), and the elementary outcome $\omega \in \Omega$ is fixed. Then for each $n = 0, 1, \dots$,

$$
\begin{aligned}
\rho_{KR}(\tilde{Q}_n^2(\omega), \, \tilde{Q}_n^1(\omega)) &= \rho_{KR}\left(\frac{n}{n+1} T_{\xi_n^{-1}}^* \tilde{Q}_{n-1}^2(\omega), \, \frac{n}{n+1} T_{\xi_n^{-1}}^* \tilde{Q}_{n-1}^1(\omega) \right) \\
&= \frac{n}{n+1} \rho_{KR}\left(\tilde{Q}_{n-1}^2(\omega), \, \tilde{Q}_{n-1}^1(\omega) \right) \,.
\end{aligned}
$$

It is clear that $\rho_{KR}(\tilde{Q}_n^2(\omega), \, \tilde{Q}_n^1(\omega))$ converges to 0 uniformly by $\omega \in \Omega$. Hence, if one solution of the equation (7) converges to h in probability, then all its solutions have this property.

First we prove the *necessity* of the theorem condition. If the sequence $(S_k)_{k=0}^{\infty}$ is uniformly distributed in probability, then, according to the notes above, all the solutions of the equation (7) converge to h in probability.

Let $f \in C(G)$, then by virtue of (7), for $n = 1, 2, \dots$, we have

$$\langle f, \tilde{Q}_n \rangle = \frac{n}{n+1} \langle f, T_{\xi_n^{-1}}^* \tilde{Q}_{n-1} \rangle + \frac{1}{n+1} f(e) \,,$$

and by virtue of (2) and (3),

$$\mathbb{E}\langle f, \tilde{Q}_n \rangle \quad = \quad \frac{n}{n+1} \left\langle f, \mathbb{E}\tilde{Q}_{n-1} * \widehat{P}_n \right\rangle + \frac{1}{n+1} f(e).$$

Then

$$\mathbb{E}\tilde{Q}_n = \frac{n}{n+1} \mathbb{E}\tilde{Q}_{n-1} * \widehat{P}_n + \frac{1}{n+1} \delta_e, \ n = 1, 2, \ldots . \tag{8}$$

According to Lemma 3.1, all solutions of this equation $*$-weakly converge to h. In particular, it is true for the solution with the initial condition $\tilde{Q}_0 = \delta_e$, i.e. for the sequence

$$\frac{1}{n+1} \sum_{k=1}^{n} \widehat{P}_k * \widehat{P}_{k+1} * \ldots * \widehat{P}_n + \frac{1}{n+1} \delta_e, \ n = 1, 2, \ldots .$$

Due to (4), the continuity property of the operator $\widehat{}$ and (5), the condition of the $*$-weak convergence of this sequence to the measure h is equivalent to the theorem condition.

Next we prove the *sufficiency*. Assume that

$$\frac{1}{n+1} \sum_{k=1}^{n} \widehat{P}_k * \widehat{P}_{k+1} * \ldots * \widehat{P}_n + \frac{1}{n+1} \delta_e, \ n = 1, 2, \ldots ,$$

$*$-weakly converges to h. As this sequence is a solution of the equation (8), then all its solutions $*$-weakly converge to h.

Since $(\tilde{Q}_n)_{n=1}^{\infty}$ is a solution of the equation (7), then $(\mathbb{E}\tilde{Q}_n)_{n=1}^{\infty}$ is the solution of the equation (8). We subtract the measure h from both sides of the equation (7) and apply a functional $f \in \Phi$, then we have

$$\langle f, \tilde{Q}_n - h \rangle = \frac{n}{n+1} \langle T_{\xi_n^{-1}} f, \tilde{Q}_{n-1} - h \rangle + \frac{1}{n+1} \langle f, \delta_e - h \rangle, \ n = 1, 2, \ldots .$$

We square both sides of the last equation and consider the expectation

$$\begin{aligned}
\mathbb{E}\langle f, \tilde{Q}_n - h \rangle^2 \quad = \quad & \frac{n^2}{(n+1)^2} \mathbb{E}\langle T_{\xi_n^{-1}} f, \tilde{Q}_{n-1} - h \rangle^2 \\
& + \frac{2n}{(n+1)^2} \langle \mathbb{E}T_{\xi_n^{-1}} f, \mathbb{E}\tilde{Q}_{n-1} - h \rangle \cdot \langle f, \delta_e - h \rangle \\
& + \frac{1}{(n+1)^2} \langle f, \delta_e - h \rangle^2 \\
= \quad & \frac{n^2}{(n+1)^2} \mathbb{E}\langle T_{\xi_n^{-1}} f, \tilde{Q}_{n-1} - h \rangle^2 \\
& + \frac{2n+1}{(n+1)^2} \left(\frac{2n}{2n+1} \langle \mathbb{E}T_{\xi_n^{-1}} f, \mathbb{E}\tilde{Q}_{n-1} - h \rangle \langle f, \delta_e - h \rangle \right.
\end{aligned}$$

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$$+ \frac{1}{2n+1} \langle f, \delta_e - h \rangle^2 \Big) \, .$$

This equality implies the inequality

$$\mathbb{E}\langle f, \tilde{Q}_n - h \rangle^2 \;\leq\; \frac{n^2}{(n+1)^2} \, \mathbb{E}\langle T_{\xi_n^{-1}} f, \tilde{Q}_{n-1} - h \rangle^2$$
$$+ \frac{2n+1}{(n+1)^2} \left(\frac{4\|f\|n}{2n+1} \, \rho_{KR}(\mathbb{E}\tilde{Q}_{n-1}, h) + \frac{4\|f\|^2}{2n+1} \right) \, .$$

We take the maximum of both sides over the set $\{T_g f\}_{g \in G}$, then we have

$$\max_{g \in G} \mathbb{E}\langle T_g f, \tilde{Q}_n - h \rangle^2 \;\leq\; \frac{n^2}{(n+1)^2} \, \max_{g \in G} \mathbb{E}\langle T_{\xi_n^{-1}} T_g f, \tilde{Q}_{n-1} - h \rangle^2$$
$$+ \frac{2n+1}{(n+1)^2} \left(\frac{4\|f\|n}{2n+1} \, \rho_{KR}(\mathbb{E}\tilde{Q}_{n-1}, h) + \frac{4\|f\|^2}{2n+1} \right) \, .$$

As the sequence $(\rho_{KR}(\mathbb{E}\tilde{Q}_{n-1}, h))_{n=1}^{\infty}$ converges to 0, the sequence

$$\varepsilon_n(f) = \frac{4\|f\|n}{2n+1} \, \rho_{KR}(\mathbb{E}\tilde{Q}_{n-1}, h) + \frac{4\|f\|^2}{2n+1} \, , \; n = 1, 2, \ldots,$$

also converges to 0.

Let $z_n(f) = \max_{g \in G} \mathbb{E}\langle T_g f, \tilde{Q}_n - h \rangle^2$ for each $n = 1, 2, \ldots$. By virtue of independence of the random elements ξ_n and $\tilde{Q}_{n-1}$,

$$\max_{g \in G} \mathbb{E}\langle T_{\xi_n^{-1}} T_g f, \tilde{Q}_{n-1} - h \rangle^2 = \max_{g \in G} \mathbb{E}\langle T_g f, \tilde{Q}_{n-1} - h \rangle^2 \, ,$$

therefore,

$$z_n(f) \leq \frac{n^2}{(n+1)^2} \, z_{n-1}(f) + \frac{2n+1}{(n+1)^2} \, \varepsilon_n(f) \, , \; n = 1, 2, \ldots.$$

Lemma 3.2 implies that the sequence $z_n(f)$ converges to 0, therefore, the sequence $(\mathbb{E}\langle f, \tilde{Q}_n - h \rangle^2)_{n=1}^{\infty}$ converges to 0. And therefore, by Chebyshev's inequality [8, ch. IX.6, p. 233], the sequence $(\langle f, \tilde{Q}_n - h \rangle)_{n=1}^{\infty}$ converges to 0 in probability.

The set $\{\alpha\varphi \mid \alpha \in \mathbb{R} \, , \, \varphi \in \Phi\}$ coincides with the algebra of all Lipschitz functions. This algebra is a subalgebra of the algebra $C(G)$. Since it separates points of G and contains the constants, according to the Stone-Weierstrass theorem [10, ch. 5, p. 115], it is everywhere dense in $C(G)$. Thus, the sequence $(\langle f, \tilde{Q}_n - h \rangle)_{n=1}^{\infty}$ converges to 0 in probability for any $f \in C(G)$. $\qquad\square$

5. Uniform distribution modulo 1

Let G be the compact metric group $[0; 1)$ with addition modulo 1 and the metric

$$\rho(x, y) = \min\{x - y \,(\mathrm{mod}\,1), y - x \,(\mathrm{mod}\,1)\}\,.$$

Let $(\xi_k)_{k=1}^{\infty}$ be a sequence of independent random variables in G. Let us define the sequence

$$
\begin{aligned}
S_0 &= 0,\\
S_k &= S_{k-1} + \xi_k \,(\mathrm{mod}\,1),\ \ k = 1, 2, \ldots\,.
\end{aligned}
$$

According to the definition and the properties of the Fourier transformation [14, ch. 3, point 3.2], the criteria of the uniform distribution of the sequence $(S_k)_{k=0}^{\infty}$ modulo 1 in distribution can be formulated as the theorem below.

THEOREM 5.1. *The sequence of the random variables $(S_k)_{k=0}^{\infty}$ is uniformly distributed modulo* 1 *in probability if and only if*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \prod_{r=k}^{n} \varphi_r(2\pi m) = 0,\ \textit{for integer } m \neq 0\,, \tag{9}$$

where φ_r is the characteristic function of the random variable ξ_r.

Note that for degenerate random variables the condition (9) is equivalent to the classical Weyl's criterion.

The following example shows that for sums of independent random variables uniform distribution modulo 1 in probability does not imply uniform distribution modulo 1 almost surely. This example is a modification of the example in [3].

Let $(\xi_k)_{k=1}^{\infty}$ be a sequence of independent random variables in G such that

$$\xi_k = \begin{cases} \alpha & \text{with probability } a_k \\ 0 & \text{with probability } 1 - a_k \end{cases},$$

where $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and

$$a_k = \frac{\log[\log_2(k+1)]}{2^{[\log_2(k+1)]}}\,.$$

Let us show that the sequence $(S_k)_{k=0}^{\infty}$ is not uniformly distributed modulo 1 almost surely, but is uniformly distributed modulo 1 in probability.

First we prove that the condition

$$\rho_{KR}(Q_n, h) \to 0 \ \text{ a.s.}$$

is not satisfied.

We denote $A_i = \{\xi_k = 0,\, k = 2^i - 1, \ldots, 2^{i+1} - 2\}$, $i = 1, 2, \ldots$.

$$\mathbb{P}\{A_i\} = \prod_{k=2^i-1}^{2^{i+1}-2} \left(1 - \frac{\log[\log_2(k+1)]}{2^{[\log_2(k+1)]}} \right) = \left(1 - \frac{\log i}{2^i} \right)^{2^i}.$$

Since

$$\lim_{i\to\infty} i \cdot \left(1 - \frac{\log i}{2^i} \right)^{2^i} = 1,$$

the series $\sum_{i=1}^{\infty} \mathbb{P}\{A_i\}$ diverges by the limit comparison test. Hence, as the events $\{A_i\}_{i=1}^{\infty}$ are independent, the probability of realization of infinite number of events of the family $\{A_i\}_{i=1}^{\infty}$ equals 1 according to the second Borel-Cantelli lemma [8, section VIII.3, p. 201].

This means that for almost all $\omega \in \Omega$, for each natural N, the number i such that $2^i - 1 > N$ and $\xi_k(\omega) = 0$ for $k = 2^i - 1, \ldots, 2^{i+1} - 2$ can be found.

For almost all $\omega \in \Omega$ and any $\varphi \in \Phi$, we have

$$
\begin{aligned}
\rho_{KR}(Q_{2^{i+1}-2}(\omega), h) &\geq \langle \varphi, Q_{2^{i+1}-2}(\omega) - h \rangle \\
&= \langle \varphi, \tfrac{1}{2} Q_{2^i-2}(\omega) \rangle + \langle \varphi, \tfrac{1}{2} \delta_{S_{2^{i+1}-2}(\omega)} - h \rangle.
\end{aligned}
$$

For the function $\varphi_i(x) = \max\{0,\, 1/4 - \rho(S_{2^{i+1}-2}(\omega), x)\}$, we have

$$\langle \varphi_i, \tfrac{1}{2} Q_{2^i-2}(\omega) \rangle \geq 0 \text{ and } \langle \varphi_i, \tfrac{1}{2} \delta_{S_{2^{i+1}-2}(\omega)} - h \rangle = \frac{1}{16}.$$

This implies $\rho_{KR}(Q_{2^{i+1}-2}, h) \geq 1/16$ almost surely for infinitely many values of i, and, therefore, almost all trajectories of the sequence $(S_k)_{k=0}^{\infty}$ are not uniformly distributed modulo 1.

The condition of the uniform distribution modulo 1 in probability for the sequence $(S_k)_{k=0}^{\infty}$ is

$$\frac{1}{n} \sum_{k=1}^{n} \prod_{r=k}^{n} \left(1 - a_r + a_r e^{i2\pi m\alpha} \right) \to 0, \text{ for integer } m \neq 0. \tag{10}$$

Obviously, the first two summands are insignificant and can be left out.

Let us denote $z_r = \left(1 - a_r + a_r e^{i2\pi m\alpha} \right)$. Then

$$
\begin{aligned}
\left| \frac{1}{n} \sum_{k=3}^{n} \prod_{r=k}^{n} z_r \right| &\leq \frac{1}{n} \left(\sum_{l=2}^{[\log_2 n]-1} \left| \sum_{k=2^l-1}^{2^{l+1}-2} \prod_{r=k}^{n} z_r \right| + \left| \sum_{k=2^{[\log_2 n]}-1}^{n} \prod_{r=k}^{n} z_r \right| \right) \\
&\leq \frac{1}{n} \left(\sum_{l=2}^{[\log_2 n]-1} \left| \sum_{k=2^l-1}^{2^{l+1}-2} \prod_{r=k}^{2^{l+1}-2} z_r \right| + \left| \sum_{k=2^{[\log_2 n]}-1}^{n} \prod_{r=k}^{n} z_r \right| \right).
\end{aligned}
$$

The sums over k are sums of geometric progressions, and for each $l = 2, \ldots,$ $[\log_2 n]$, the absolute value of such a sum can be bounded in the following way

$$\left| z_{2^l - 1} + \ldots + z_{2^l - 1}^{2^l} \right| = \left| \frac{z_{2^l - 1} - z_{2^l - 1}^{2^l + 1}}{1 - z_{2^l - 1}} \right| \leq \frac{2}{\left| \frac{\log l}{2^l} \cdot \left(1 - e^{i2\pi m\alpha}\right) \right|}.$$

Then

$$\left| \frac{1}{n} \sum_{k=3}^{n} \prod_{r=k}^{n} \left(1 - a_r + a_r e^{i2\pi m\alpha}\right) \right| \leq \frac{2}{|1 - e^{i2\pi m\alpha}|} \cdot \frac{1}{n} \left(\sum_{l=2}^{[\log_2 n] - 1} \frac{2^l}{\log l} + \frac{2^{[\log_2 n]}}{\log[\log_2 n]} \right)$$

$$\leq \frac{2}{|1 - e^{i2\pi m\alpha}|} \cdot \sum_{l=2}^{[\log_2 n]} \frac{1}{2^{[\log_2 n] - l} \log l}.$$

The sum in the right side of the last inequality is a member of the convolution of the summable sequence $\left(1/2^k\right)_{k=0}^{\infty}$ and the sequence $(1/\log k)_{k=2}^{\infty}$ which converges to 0. Therefore, this convolution converges to 0 as $n \to \infty$. Thus, for the sequence $(S_k)_{k=0}^{\infty}$ the criterion (10) of the uniform distribution in probability is satisfied.

The following example shows that the uniform distribution of the sequence $(S_k)_{k=0}^{\infty}$ in probability is not implied by $*$-weak convergence of the sequence of distributions of S_k, $k = 0, 1, \ldots$, to the Lebesgue measure on $[0, 1)$, that is, in the terms of [6, 7], by the asymptotic uniformity of $(S_k)_{k=0}^{\infty}$ in distribution (mod 1).

Let $(k_l)_{l=1}^{\infty}$ be a lacunary sequence of natural numbers

$$k_{l+1} > b \left(k_l + 1\right), \; b > 1, \; k_1 = 1, \; l = 1, 2, \ldots \, .$$

Let $\xi_k = 0$ with probability 1 for $k \neq k_l$, and let ξ_{k_l}, $l = 1, 2, \ldots$ be uniform random variables on $[0, 1)$ which are independent in aggregate.

It is clear that the sequence of distributions of S_k, $k = 0, 1, \ldots$, $*$-weakly converges to the uniform distribution on $[0, 1)$. However, the condition (9) is not satisfied. Indeed, as $\varphi_r(2\pi m) = 1$ for $k_l < r < k_{l+1}$, $m = 1, 2, \ldots$, we have

$$\frac{1}{k_{l+1} - 1} \sum_{k=1}^{k_{l+1} - 1} \prod_{r=k}^{k_{l+1} - 1} \varphi_r(2\pi m) = \frac{k_{l+1} - k_l - 1}{k_{l+1} - 1}$$

$$> 1 - \frac{1}{b}.$$

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