

# A DICHOTOMY BETWEEN UNIFORM DISTRIBUTIONS OF THE STERN-BROCOT AND THE FAREY SEQUENCE

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ABSTRACT. We employ infinite ergodic theory to show that the even Stern-Brocot sequence and the Farey sequence are uniformly distributed with respect to certain canonical weightings. As a corollary we derive the precise asymptotic for the Lebesgue measure of continued fraction sum-level sets as well as connections to asymptotic behaviours of geometrically and arithmetically restricted Poincaré series. Moreover, we give relations of our main results to elementary observations for the Stern-Brocot tree.

*Communicated by Cor Kraaikamp*

*Dedicated to the memory of Gérard Rauzy*

## 1. Introduction and statements of result

In this paper we consider weighted uniform distributions for the following two canonical sequences: the *Farey sequence*  $(\mathcal{F}_n)_{n \in \mathbb{N}}$  which is given by

$$\mathcal{F}_n := \{p/q : 0 < p \leq q \leq n, \gcd(p, q) = 1\},$$

and the *even Stern-Brocot sequence*  $(\mathcal{S}_n)_{n \in \mathbb{N}}$  which is given by

$$\mathcal{S}_n := \{s_{n,2k}/t_{n,2k} : k = 1, \dots, 2^{n-1}\},$$

where the integers  $s_{n,k}$  and  $t_{n,k}$  are defined recursively by

$$s_{0,1} := 0 \quad \text{and} \quad s_{0,2} := t_{0,1} := t_{0,2} := 1;$$

$$s_{n+1,2k-1} := s_{n,k} \quad \text{and} \quad t_{n+1,2k-1} := t_{n,k}, \quad \text{for } k = 1, \dots, 2^n + 1;$$

$$s_{n+1,2k} := s_{n,k} + s_{n,k+1} \quad \text{and} \quad t_{n+1,2k} := t_{n,k} + t_{n,k+1}, \quad \text{for } k = 1, \dots, 2^n.$$

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2010 Mathematics Subject Classification: Primary 10K05; Secondary 11B57.

Keywords: Uniform distribution, Stern-Brocot sequence, Farey sequence, infinite ergodic theory.

The following theorem states the main results of this paper, where  $\delta_x$  denotes the Dirac distribution at  $x \in [0, 1]$ ,  $*$ -lim the weak limit of measures, and  $\lambda$  the Lebesgue measure on  $[0, 1]$ . Note that, throughout, all appearing fractions will always be assumed to be reduced.

**THEOREM 1.1.** *For the even Stern-Brocot sequence we have that*

$$*\text{-}\lim_{n \rightarrow \infty} \log(n^2) \sum_{p/q \in \mathcal{S}_n} q^{-2} \delta_{p/q} = \lambda, \quad (1)$$

and for the Farey sequence we have that

$$*\text{-}\lim_{n \rightarrow \infty} \frac{\zeta(2)}{\log n} \sum_{p/q \in \mathcal{F}_n} q^{-2} \delta_{p/q} = \lambda. \quad (2)$$

In fact, for the derivation of the assertion in (1) we will show that the following more general measure theoretical result holds. In here,  $T : [0, 1] \rightarrow [0, 1]$  denotes the *Farey map* defined by

$$T(x) := \begin{cases} x/(1-x) & \text{for } x \in [0, 1/2] \\ (1-x)/x & \text{for } x \in (1/2, 1]. \end{cases}$$

**THEOREM 1.2.** *For each rational number  $v/w \in (0, 1]$  we have that*

$$*\text{-}\lim_{n \rightarrow \infty} \log(n^{vw}) \sum_{p/q \in T^{-n}\{v/w\}} q^{-2} \delta_{p/q} = \lambda. \quad (3)$$

In a nutshell, the proofs of these results are obtained as follows. The convergence in (2) is derived by combining *Toeplitz's Lemma* and a classical result by Landau [Lan24] and Mikolás [M48] with a well-know estimate involving *Euler's totient function*  $\varphi : \mathbb{N} \rightarrow \mathbb{N}_0$ , given by  $\varphi(n) := \text{card}\{1 \leq m \leq n : \text{gcd}(m, n) = 1\}$ . Whereas, the proof of Theorem 1.2, and consequently, since  $\mathcal{S}_n = T^{-(n-1)}(1/2)$ , the proof of (1) is obtained from the following slightly more technical result, for which we employ certain techniques from recent results in infinite ergodic theory.

**PROPOSITION 1.3.** *For each interval  $[\alpha, \beta] \subset (0, 1]$  we have that*

$$*\text{-}\lim_{n \rightarrow \infty} \left( \frac{\log n}{\log(\beta/\alpha)} \cdot \lambda|_{T^{-n}([\alpha, \beta])} \right) = \lambda.$$

Note that Proposition 1.3 has the following immediate number theoretical implication, which has been the main result of [KS12a] where we confirmed the conjecture of Fiala and Kleban in [FK10] (see also Remark 2.1 following the proof of Proposition 1.3). Therefore, Proposition 1.3 gives an alternative proof of this

conjecture. In order to state this implication, recall that the regular continued fraction expansion of a number  $x \in (0, 1]$  is given by

$$x =: [x_1, x_2, \dots] := \frac{1}{x_1 + \frac{1}{x_2 + \dots}},$$

where  $x_i \in \mathbb{N}$ , for all  $i \in \mathbb{N}$ . Also, we write  $a_n \sim b_n$  if  $\lim_{n \rightarrow \infty} a_n/b_n = 1$ .

**COROLLARY 1.4.** *We have that*

$$\lambda \left( \left\{ [x_1, x_2, \dots] : \sum_{i=1}^k x_i = n, k \in \mathbb{N} \right\} \right) \sim \frac{1}{\log_2 n}.$$

Further immediate consequences of the results in Theorem 1.1 and Theorem 1.2 are given in the following two corollaries.

**COROLLARY 1.5.** *We have that*

$$*\lim_{n \rightarrow \infty} \frac{\zeta(2)}{n} \sum_{\substack{p/q \in [0,1] \\ \log q \leq n}} q^{-2} \delta_{p/q} = \lambda \quad \text{and} \quad *\lim_{n \rightarrow \infty} \frac{\log(n^2)}{n} \sum_{\substack{p/q = [x_1, \dots, x_k] \\ \sum_{i=1}^k x_i \leq n}} q^{-2} \delta_{p/q} = \lambda.$$

The latter dichotomy can also be expressed as a dichotomy between *partial geometric Poincaré sums* and *partial algebraic Poincaré sums* for the modular group  $\Gamma := PSL_2(\mathbb{Z})$ . For results of this type on the algebraic growth rates of Poincaré series for more general Kleinian groups we refer to [KS12b]. In the following corollary,  $d$  refers to the hyperbolic metric in the upper plane model of hyperbolic space and  $\|\cdot\|$  denotes the word length in  $\Gamma$  with respect to the two generators  $z \mapsto z + 1$  and  $z \mapsto -1/z$  of the modular group  $\Gamma$ . Also, we write  $a_n \asymp b_n$  if  $a_n/b_n$  is uniformly bounded away from zero and infinity.

**COROLLARY 1.6.** *We have that*

$$\sum_{\substack{\gamma \in \Gamma \\ d(0, \gamma(0)) \leq n}} e^{-d(0, \gamma(0))} \asymp n \quad \text{and} \quad \sum_{\substack{\gamma \in \Gamma \\ \|\gamma\| \leq n}} e^{-d(0, \gamma(0))} \asymp \frac{n}{\log n}.$$

*Remark 1.7.* (i) Note that the results in Theorem 1.1 complement well-known results on weak convergence of empirical measures with constant weight 1 for the sequences  $(\mathcal{F}_n)$  and  $(\mathcal{S}_n)$ . More precisely, in [M48] (see also [CP88, D90, K53, K54, Lan24]) it was shown that  $(\mathcal{F}_n)$  is uniformly distributed, that is,

$$*\lim_{n \rightarrow \infty} \frac{1}{\text{card}(\mathcal{F}_n)} \sum_{p/q \in \mathcal{F}_n} \delta_{p/q} = \lambda. \tag{4}$$

On the other hand, it is known that the Stern-Brocot sequence is not uniformly distributed. In fact, an immediate consequence of the results in [KS08] is that

$$\ast\text{-}\lim_{n \rightarrow \infty} \frac{1}{\text{card}(\mathcal{S}_n)} \sum_{p/q \in \mathcal{S}_n} \delta_{p/q} = m_T,$$

where  $m_T$  refers to the *measure of maximal entropy* for the Farey map  $T$ . Here, the reader might like to recall that the distribution function of  $m_T$  is equal to the *Minkowski question mark function* (see e.g. [KS08]) and hence, the two measures  $m_T$  and  $\lambda$  are mutually singular. In fact, a numerical calculation has shown that the Hausdorff dimension  $\dim_H(m_T) := \inf \{\dim_H(X) : m_T(X) = 1\}$  of the measure  $m_T$  is approximable equal to 0.875 (see e.g. [KS08, Lag92, TU95]).

(ii) In order to tie the results in Theorem 1.1 (1) and Theorem 1.2 to elementary number theory and, in particular, to give a clarification of the significance of  $v/w$  in Theorem 1.2, we mention the following observation concerning the even Stern-Brocot tree. For each reduced fraction  $v/w \in (0, 1)$  and for all  $n \in \mathbb{N}_0$ , we have

$$\sum_{p/q \in T^{-n}(v/w)} \frac{1}{pq} = \frac{1}{vw}. \quad (5)$$

To see this first in an elementary way, note that we have  $p/q \in \mathcal{S}_n$  if and only if  $T^{-1}(p/q) = \{p/(p+q), q/(p+q)\} \subset \mathcal{S}_{n+1}$ . Furthermore, with  $\kappa : \bigcup_{n \in \mathbb{N}} \mathcal{S}_n \rightarrow \mathbb{R}$  given by  $\kappa(p/q) := 1/(pq)$ , one immediately verifies that

$$\kappa(p/(p+q)) + \kappa(q/(p+q)) = \kappa(p/q).$$

The proof now follows by induction. Note that for the special case  $v/w = 1/2$  one can use the fact that  $\mathcal{S}_n = T^{-(n-1)}(1/2)$ , and then (5) becomes

$$\sum_{p/q \in \mathcal{S}_n} \frac{2}{pq} = 1, \text{ for all } n \in \mathbb{N},$$

which has also been observed by the Canadian music theorist Pierre Lamothe (see [G11, p. 146]).

Alternatively, the equality in (5) can also be deduced immediately from the well-known fixed point equation for the Perron-Frobenius operator  $\mathcal{L}$  associated with the Farey map  $T$  (see Section 2.1 for the definition). For this let  $h$  denote the eigenfunction of  $\mathcal{L}$  associated with the eigenvalue 1. It is well known that  $h$  is given by  $h(x) := 1/x$ , which consequently gives that

$$\sum_{y \in T^{-n}(x)} |(T^n)'(y)|^{-1} h(y) = h(x), \text{ for all } x \in (0, 1) \text{ and } n \in \mathbb{N}_0.$$

Since  $|(T^n)'(p/q)| = q^2/w^2$  for all  $p/q \in T^{-n}(v/w)$ , we have that

$$\begin{aligned} \sum_{p/q \in T^{-n}(v/w)} \frac{w^2}{pq} &= \sum_{p/q \in T^{-n}(v/w)} \frac{w^2}{q^2} \cdot \frac{q}{p} = \sum_{y \in T^{-n}(v/w)} |(T^n)'(y)|^{-1} h(y) \\ &= h\left(\frac{v}{w}\right) = \frac{w}{v} \end{aligned}$$

and hence, the statement in (5) follows.

Finally, let us apply Theorem 1.2 to obtain yet another proof of the statement in (5), and this proof will implicitly use dual aspects of the Perron-Frobenius operator. More precisely, by applying Theorem 1.2 twice, we obtain the following, which immediately implies (5). For each  $n \in \mathbb{N}_0$  and for every reduced fraction  $v/w \in (0, 1)$ , we have

$$\begin{aligned} \sum_{p/q \in T^{-n}(v/w)} \frac{1}{pq} \cdot \lambda &= \sum_{p/q \in T^{-n}(v/w)} \ast\text{-}\lim_{k \rightarrow \infty} \log k \sum_{r/s \in T^{-k}(p/q)} s^{-2} \delta_{r/s} \\ &= \frac{1}{vw} \ast\text{-}\lim_{k \rightarrow \infty} \log(k^{vw}) \sum_{p/q \in T^{-(n+k)}(v/w)} q^{-2} \delta_{p/q} = \frac{1}{vw} \cdot \lambda. \end{aligned}$$

## 2. Proofs of Theorem 1.1, 1.2 and Proposition 1.3

### 2.1. Proof of Proposition 1.3

As already mentioned in the introduction, the proof of Proposition 1.3 will make use of results from infinite ergodic theory. Therefore, let us first recall a few basic facts and results from infinite ergodic theory for the Farey map. (For an overview, further definitions and details concerning infinite ergodic theory in general, the reader is referred to [A97]). First, note that the *Perron-Frobenius operator*  $\mathcal{L} : L_1(\mu) \rightarrow L_1(\mu)$  associated with the Farey map is given by

$$\mathcal{L}(f) = |u'_0| \cdot (f \circ u_0) + |u'_1| \cdot (f \circ u_1), \text{ for all } f \in L_1(\mu),$$

where  $\mu$  denotes the infinite,  $\sigma$ -finite and  $T$ -invariant measure absolutely continuous with respect to the Lebesgue measure  $\lambda$  and where  $u_0$  and  $u_1$  refer to the inverse branches of  $T$ , which are given for  $x \in [0, 1]$  by

$$u_0(x) = x/(1+x) \text{ and } u_1(x) = 1/(1+x).$$

In fact, with  $\varphi_0 : [0, 1] \rightarrow [0, 1]$  defined by  $\varphi_0(x) := x$ , the measure  $\mu$  is explicitly given by

$$d\lambda = \varphi_0 d\mu.$$

Moreover, recall that the *transfer operator*  $\widehat{T} : L_1(\mu) \rightarrow L_1(\mu)$  associated with the Farey map is the positive linear operator which is given by

$$\mu\left(\mathbb{1}_C \cdot \widehat{T}(f)\right) = \mu\left(\mathbb{1}_{T^{-1}(C)} \cdot f\right), \text{ for all } f \in L_1(\mu), C \in \mathcal{A},$$

where  $\mathcal{A}$  refers to the Borel  $\sigma$ -algebra of  $[0, 1]$  and  $\mathbb{1}_C$  denotes the characteristic function of  $C$ . One immediately verifies that the two operators  $\widehat{T}$  and  $\mathcal{L}$  are related through

$$\widehat{T}(f) = \varphi_0 \cdot \mathcal{L}(f/\varphi_0), \text{ for all } f \in L_1(\mu).$$

It is then well known that the *Farey system*  $([0, 1], T, \mathcal{A}, \mu)$  is a conservative, ergodic and measure preserving dynamical systems. (Recall that *conservative and ergodic* means that for all function  $f$  element of

$$L_1^+(\mu) := \{f \in L_1(\mu) : f \geq 0 \text{ and } \mu(f \cdot \mathbb{1}_{[0,1]}) > 0\},$$

we have  $\mu$ -almost everywhere that  $\sum_{n \geq 0} \widehat{T}^n(f) = \infty$ ; also, *measure preserving* means that  $\widehat{T}(\mathbb{1}_{[0,1]}) = \mathbb{1}_{[0,1]}$ .

Now, the crucial notion for proving Proposition 1.3 is provided by the following concept of a uniformly returning set which was introduced in [KS107].

*A set  $C \in \mathcal{A}$  with  $0 < \mu(C) < \infty$  is called uniformly returning for some  $f \in L_1^+(\mu)$ , if there exists a positive increasing sequence  $(w_n)_{n \in \mathbb{N}}$  of positive reals such that  $\mu$ -almost everywhere and uniformly in  $C$  we have*

$$\lim_{n \rightarrow \infty} w_n \widehat{T}^n(f) = \mu(f).$$

In [KS107, Lemma 3.3] it was shown that for the Farey system we have that every interval contained in  $[1/2, 1]$  is uniformly returning, for each function

$$f \in \mathcal{D} := \left\{g \in C^2([0, 1]) : g, g' \geq 0; g'' \leq 0; \int g \, d\mu = 1\right\}.$$

(In fact, very recently it has been shown in [K11] that every closed interval in  $(0, 1]$  is uniformly returning for each  $f \in \mathcal{D}$ ). Moreover, in [KS107, Section 3.1] it was shown that in the situation of the Farey system the sequence  $(w_n)_{n \in \mathbb{N}}$  can be chosen to be equal to  $(\log n)_{n \in \mathbb{N}}$ . (For further examples of one dimensional dynamical systems which allow uniformly returning sets for some appropriate functions we refer to [T00]). We are now in the position to give the proof of Proposition 1.3.

**Proof of Proposition 1.3.** Consider the function  $\varphi_t$  given by  $\varphi_t : x \mapsto x \cdot \exp(tx)$ . The first aim is to show that for all  $t \in [-1, 1]$  we have

$$\widehat{T}\varphi_t \in \mathcal{D}.$$

Indeed, for  $t \in [-1, 0]$  this is an immediate consequence of the facts that  $\varphi_t$  is increasing and concave and that  $\widehat{T}(\mathcal{D}) \subset \mathcal{D}$ . For  $t \in (0, 1]$ , a straight forward computation shows that the first derivative of  $\widehat{T}\varphi_t$  at  $x \in [0, 1]$  is given by

$$\left(\widehat{T}\varphi_t\right)'(x) = \frac{\varphi_t'\left(\frac{x}{x+1}\right) - x\varphi_t'\left(\frac{1}{x+1}\right)}{(x+1)^3} + \frac{\varphi_t\left(\frac{1}{x+1}\right) - \varphi_t\left(\frac{x}{x+1}\right)}{(x+1)^2}.$$

For the second derivative we then obtain

$$\begin{aligned} \left(\widehat{T}\varphi_t\right)''(x) &= \frac{(-2xt - 6x + 2t + xt^2 + 2x^3 - 4tx^2 - 4) \exp\left(\frac{tx}{x+1}\right)}{(x+1)^6} \\ &\quad + \frac{(2tx - 6x - 2t + xt^2 + 2x^3 + 4tx^2 - 4) \exp\left(\frac{t}{x+1}\right)}{(x+1)^6}. \end{aligned}$$

This immediately implies that  $\left(\widehat{T}\varphi_t\right)'' \leq 0$ , for all  $t \in (0, 1]$ . Therefore,  $\left(\widehat{T}\varphi_t\right)'$  is decreasing on  $[0, 1]$  with  $\left(\widehat{T}\varphi_t\right)'(1) = 0$ , which shows that on  $[0, 1]$  we have that  $\left(\widehat{T}\varphi_t\right)' \geq 0$ . Hence, we can apply [KS108, Lemma 3.2], which then implies that  $\widehat{T}\varphi_t \in \mathcal{D}$ , for all  $t \in [-1, 1]$ .

We proceed by noting that [KS108, Lemma 3.3] guarantees that every interval contained in  $[1/2, 1]$  is a uniformly returning set for  $\varphi_t$ , for each  $t \in [-1, 1]$ . In order to complete the proof of the proposition, we employ the method of moments as follows. The aim is to show that for each  $[\alpha, \beta] \subset (0, 1]$  and for each  $t \in [-1, 1]$ , we have for the moment generating function at  $t$  that

$$\lim_{n \rightarrow \infty} \int \exp(tx) \cdot \frac{\log n}{\mu([\alpha, \beta])} \cdot \mathbb{1}_{T^{-n}([\alpha, \beta])}(x) \, d\lambda(x) = \int \exp(tx) \, d\lambda(x).$$

To see this, we argue by induction as follows. For  $[\alpha, \beta] \subset [1/2, 1]$ , we have that

$$\begin{aligned} &\lim_{n \rightarrow \infty} \int \exp(tx) \cdot \frac{\log n}{\mu([\alpha, \beta])} \cdot \mathbb{1}_{T^{-n}([\alpha, \beta])}(x) \, d\lambda(x) \\ &= \lim_{n \rightarrow \infty} \frac{\log n}{\mu([\alpha, \beta])} \cdot \mu(\varphi_t \cdot \mathbb{1}_{T^{-n}([\alpha, \beta])}) = \lim_{n \rightarrow \infty} \frac{\log n}{\mu([\alpha, \beta])} \cdot \mu\left(\widehat{T}^n \varphi_t \cdot \mathbb{1}_{[\alpha, \beta]}\right) \\ &= \mu(\varphi_t) = \int \exp(tx) \, d\lambda(x). \end{aligned}$$

Next, suppose that the assertion holds for any interval which is contained in the set  $\mathcal{E}_n := \bigcup_{k=0}^{n-1} T^{-k}([1/2, 1])$ , and consider an interval  $[\alpha, \beta] \subset T^{-n}([1/2, 1]) \setminus \mathcal{E}_n$ .

Since  $T([\alpha, \beta]) \subset \mathcal{E}_n$ , we then have

$$\begin{aligned}
 & \lim_{m \rightarrow \infty} \int \exp(tx) \cdot \frac{\log m}{\mu([\alpha, \beta])} \cdot \mathbb{1}_{T^{-m}([\alpha, \beta])}(x) \, d\lambda(x) \\
 &= \lim_{m \rightarrow \infty} \frac{\log m}{\mu([\alpha, \beta])} \cdot \mu \left( \widehat{T}^m \varphi_t \cdot \left( \mathbb{1}_{T^{-1}(T([\alpha, \beta]))} - \mathbb{1}_{T^{-1}(T([\alpha, \beta]) \cap \mathcal{E}_n)} \right) \right) \\
 &= \lim_{m \rightarrow \infty} \frac{\log m}{\mu([\alpha, \beta])} \cdot \left( \mu \left( \widehat{T}^{m+1} \varphi_t \cdot \mathbb{1}_{T([\alpha, \beta])} \right) - \mu \left( \widehat{T}^m \varphi_t \cdot \mathbb{1}_{T^{-1}(T([\alpha, \beta]) \cap \mathcal{E}_n)} \right) \right) \\
 &= \frac{\mu(\varphi_t)}{\mu([\alpha, \beta])} \left( \mu(T([\alpha, \beta])) - \mu(T^{-1}(T([\alpha, \beta]) \cap \mathcal{E}_n)) \right) = \int \exp(tx) \, d\lambda(x).
 \end{aligned}$$

This finishes the proof of Proposition 1.3.  $\square$

*Remark 2.1.* We remark that the assertion in Corollary 1.2 is an immediate consequence of Proposition 1.3. Indeed, by choosing  $[\alpha, \beta] = [1/2, 1]$  and observing that (see [KS12a, Lemma 2.1])

$$T^{-(n-1)}([1/2, 1]) = \left\{ [x_1, x_2, \dots] : \sum_{i=1}^k x_i = n, k \in \mathbb{N} \right\},$$

it follows that

$$\lambda \left( \left\{ [x_1, x_2, \dots] : \sum_{i=1}^k x_i = n, k \in \mathbb{N} \right\} \right) \sim \frac{\log 2}{\log n}.$$

## 2.2. Proof of Theorem 1.2

The following two lemmata will be required in the proof of Theorem 1.2.

**LEMMA 2.2.**

$$\sum_{p/q \in \mathcal{S}_n} q^{-2} \asymp \frac{1}{\log n}.$$

*Proof.* First, note that there is a 1–1 correspondence between the sequence  $(\mathcal{S}_n)$  and the set of connected components of  $T^{-(n-1)}([1/2, 1])$ . That is, if  $p/q = [a_1, \dots, a_n] \in \mathcal{S}_n$ , where  $a_n > 1$ , then one of these connected component is given by

$$\begin{aligned}
 C_n(p/q) &:= \{ [x_1, x_2, \dots] : x_i = a_i \text{ for } 1 \leq i \leq n \} \\
 &\cup \{ [x_1, x_2, \dots] : x_i = a_i \text{ for } 1 \leq i \leq n-1, x_n = a_n - 1, x_{n+1} = 1 \}.
 \end{aligned}$$

Using standard Diophantine estimates, we find that  $\lambda(C_n(p/q)) \asymp 1/q^2$ . Hence, an application of Proposition 1.3, in particular Corollary 1.4, gives the assertion of the lemma.  $\square$

For the next lemma note that the sequence  $(\mathcal{S}_n)$  can also be expressed in terms of the inverse branches  $u_0$  and  $u_1$  of the Farey map  $T$ . Namely, one immediately verifies that the orbit of the unit interval under the free semi-group  $\Phi$  generated by  $u_0$  and  $u_1$  is in 1–1 correspondence to the set of all Stern-Brocot intervals

$$\left\{ \left[ \frac{s_{n,k}}{t_{n,k}}, \frac{s_{n,k+1}}{t_{n,k+1}} \right] : n \in \mathbb{N}_0; k = 1, \dots, 2^n \right\}.$$

Note that for each rational number  $v/w \in (0, 1]$  we have that

$$\{T^{-n} \{v/w\} : n \in \mathbb{N}\} = \{\gamma(v/w) : \gamma \in \Phi\}.$$

Moreover, note that the  $\Phi$ -orbit of 1 is equal to the set of rational numbers in  $(0, 1)$ . More precisely, we have that if  $\gamma \in \Phi$  then  $\gamma(1) = v/w$ , for some  $v, w \in \mathbb{N}$  such that  $v < w$  and  $\gcd(v, w) = 1$ , and for the modulus of the derivative of  $\gamma$  at 1 we have that  $|\gamma'(1)| = w^{-2}$ .

In the following we let  $\mathcal{U}_\varepsilon(x)$  denote the interval centred at  $x \in \mathbb{R}$  of Euclidean diameter  $\text{diam}(\mathcal{U}_\varepsilon(x))$  equal to  $\varepsilon > 0$ .

**LEMMA 2.3.** *For each  $g \in \Phi$  there exists  $\Delta : (0, 1] \rightarrow \mathbb{R}_+$  with  $\lim_{s \rightarrow 0} \Delta(s) = 0$  such that for each  $h \in \Phi$  and  $\varepsilon > 0$  sufficiently small, we have*

$$|\text{diam}(h(\mathcal{U}_\varepsilon(g(1)))) - \varepsilon |(h'(g(1)))| < \varepsilon |(hg)'(1)| \Delta(\varepsilon).$$

*Proof.* By the bounded distortion property, we have for each  $z \in (0, 1)$  that there exists  $\Delta_z : (0, 1] \rightarrow \mathbb{R}_+$  with  $\lim_{s \rightarrow 0} \Delta_z(s) = 0$  such that, for each  $\varepsilon > 0$  sufficiently small,

$$\sup_{\substack{x, y \in \mathcal{U}_\varepsilon(z) \\ \gamma \in \Phi}} \left| \frac{|\gamma'(x)|}{|\gamma'(y)|} - 1 \right| < \Delta_z(\varepsilon).$$

This implies that for fixed  $g \in \Phi$  we have, for each  $h \in \Phi$  and  $\varepsilon > 0$  sufficiently small,

$$\left| \frac{\text{diam}(h(\mathcal{U}_\varepsilon(g(1))))}{\varepsilon |h'(g(1))|} - 1 \right| < \Delta_{g(1)}(\varepsilon).$$

From this we deduce that

$$|\text{diam}(h(\mathcal{U}_\varepsilon(g(1)))) - \varepsilon |h'(g(1))| < \varepsilon \frac{|(hg)'(1)|}{|g'(1)|} \Delta_{g(1)}(\varepsilon) = \varepsilon |(hg)'(1)| \Delta(\varepsilon).$$

This finishes the proof. □

**Proof of Theorem 1.2.** Let  $g \in \Phi$  be given and define, for  $\varepsilon > 0$  sufficiently small,

$$\mathcal{U}_{g,\varepsilon,n} := T^{-(n-1)}(\mathcal{U}_\varepsilon(g(1))).$$

With  $u_{g,\varepsilon} := 1/\mu(\mathcal{U}_\varepsilon(g(1))) = 1/\log((g(1) + \varepsilon/2)/(g(1) - \varepsilon/2))$ , consider the measure  $\nu_{g,\varepsilon,n}$  which is given, for each  $n \in \mathbb{N}$ , by

$$\nu_{g,\varepsilon,n} = u_{g,\varepsilon} \log n \cdot \lambda|_{\mathcal{U}_{g,\varepsilon,n}}.$$

By Proposition 1.3, we then have that  $*\text{-}\lim_{n \rightarrow \infty} \nu_{g,\varepsilon,n} = \lambda$ . Also, let us define the atomic measure  $\rho_{g,\varepsilon,n}$ , for each  $n \in \mathbb{N}$ , by

$$\rho_{g,\varepsilon,n} := u_{g,\varepsilon} \log n \sum_{f(1) \in T^{-(n-1)}(g(1))} \varepsilon \frac{|f'(1)|}{|g'(1)|} \cdot \delta_{f(1)}.$$

Then observe that

$$\lim_{\varepsilon \searrow 0} \varepsilon u_{g,\varepsilon} = \lim_{\varepsilon \searrow 0} \frac{\varepsilon}{\log \frac{g(1) + \varepsilon/2}{g(1) - \varepsilon/2}} = \lim_{\varepsilon \searrow 0} \frac{\varepsilon}{\varepsilon/(g(1) - \varepsilon/2)} = g(1),$$

and consider the measures  $\rho_{g,n}$  defined, for each  $n \in \mathbb{N}$ , by

$$\rho_{g,n} := g(1) \log n \sum_{f(1) \in T^{-(n-1)}(g(1))} \frac{|f'(1)|}{|g'(1)|} \cdot \delta_{f(1)}.$$

Using Lemma 2.2 and Lemma 2.3, we now obtain the following for all  $x \in [0, 1]$ , where  $F_{g,\varepsilon,n}^{(\nu)}$ ,  $F_{g,\varepsilon,n}^{(\rho)}$  and  $F_{g,n}^{(\rho)}$  denote the distribution functions of the measures  $\nu_{g,\varepsilon,n}$ ,  $\rho_{g,\varepsilon,n}$  and  $\rho_{g,n}$ , and where we write  $a_n \ll b_n$  if  $a_n/b_n$  is uniformly bounded from above:

$$\begin{aligned} \left| F_{g,\varepsilon,n}^{(\nu)}(x) - F_{g,n}^{(\rho)}(x) \right| &\leq \left| F_{g,\varepsilon,n}^{(\nu)}(x) - F_{g,\varepsilon,n}^{(\rho)}(x) \right| + \left| F_{g,\varepsilon,n}^{(\rho)}(x) - F_{g,n}^{(\rho)}(x) \right| \\ &\ll u_{g,\varepsilon} \log n \sum_{hg(1) \in T^{-(n-1)}(g(1))} \left| \text{diam}(h(\mathcal{U}_\varepsilon(g(1)))) - \varepsilon \frac{|(hg)'(1)|}{|g'(1)|} \right| \\ &\quad + \frac{\varepsilon u_{g,\varepsilon} \log n}{n^2} + |g(1) - \varepsilon u_{g,\varepsilon}| \log n \sum_{f(1) \in T^{-(n-1)}(g(1))} |f'(1)| \\ &\ll (\varepsilon u_{g,\varepsilon} \Delta(\varepsilon) + |g(1) - \varepsilon u_{g,\varepsilon}|) \log n \sum_{f(1) \in T^{-(n-1)}(g(1))} |f'(1)| \\ &\ll |g(1) - \varepsilon u_{g,\varepsilon}| + g(1) \Delta(\varepsilon). \end{aligned}$$

This holds for each  $\varepsilon > 0$  arbitrarily small and hence, we obtain that

$$*\text{-}\lim_{n \rightarrow \infty} \rho_{g,n} = \lambda.$$

The proof of Theorem 1.2 now follows, if we insert in the definition of  $\rho_{g,n}$  the fact that  $g(1)$  can be written in form of a reduced fraction  $v/w$  and that then  $|g'(1)| = w^{-2}$ , as well as similarly, that  $f(1)$  can be written in form of a reduced fraction  $p/q$  and that then  $|f'(1)| = q^{-2}$ .  $\square$

### 2.3. Proof of Theorem 1.1 (2)

Proof of Theorem 1.1 (2). For  $n \in \mathbb{N}$  define

$$\mathcal{F}_n^* := \{m/n : 0 < m \leq n, \gcd(m, n) = 1\}$$

and let  $\psi(n) := \text{card}(\mathcal{F}_n)$ . Recall the classical estimate  $\psi(n) \sim n^2 / (2\zeta(2))$  and observe that the statement in (4) then implies that for each continuous function  $f : [0, 1] \rightarrow \mathbb{R}_{\geq 0}$ , we have

$$\chi_n := \frac{2\zeta(2)}{n^2} \sum_{r \in \mathcal{F}_n} f(r) \rightarrow \lambda(f), \text{ for } n \text{ tending to infinity.}$$

An application of Toeplitz's Lemma then gives that

$$\lim_{n \rightarrow \infty} \frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} \chi_k = \lambda(f).$$

By setting  $f_n := \sum_{p/n \in \mathcal{F}_n^*} f(p/n)$ , we next observe that for each  $n \geq 2$ , we have

$$\frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} \chi_k = \frac{2\zeta(2)}{\log n} \sum_{k=1}^n \frac{1}{k^3} \sum_{m=1}^k f_m = \frac{2\zeta(2)}{\log n} \sum_{m=1}^n \sum_{k=m}^n \frac{1}{k^3} f_m.$$

By comparing the sum  $\sum_{k=m}^n k^{-3}$  with the corresponding integral  $\int_m^n x^{-3} dx$ , we obtain

$$\begin{aligned} \frac{\zeta(2)}{\log n} \sum_{k=1}^n \frac{f_k}{k^2} - \frac{\zeta(2)}{n^2 \log n} \sum_{k=1}^n f_k &\leq \frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} \chi_k \\ &\leq \frac{\zeta(2)}{\log n} \sum_{k=1}^n \frac{f_k}{k^2} - \frac{\zeta(2)}{n^2 \log n} \sum_{k=1}^n f_k + \frac{\zeta(2)}{\log n} \sum_{k=1}^n \frac{f_k}{k^3}. \end{aligned}$$

Finally, note that we clearly have that

$$\frac{\zeta(2)}{n^2 \log n} \sum_{k=1}^n f_k \sim \frac{\lambda(f)}{2 \log n}$$

and that

$$\frac{\zeta(2)}{\log n} \sum_{k=1}^n \frac{\varphi(k)}{k^3} \frac{f_k}{\varphi(k)} \leq \frac{\zeta(2)}{\log n} \sum_{k=1}^n \frac{\varphi(k)}{k^3} \|f\|_{\infty} \sim \frac{\|f\|_{\infty} (\zeta(2))^2}{\zeta(3) \log n},$$

where  $\varphi(n) := \text{card}(\mathcal{F}_n^*)$  denotes Euler's totient function. Hence, it now follows that

$$\lim_{n \rightarrow \infty} \frac{\zeta(2)}{\log n} \sum_{q=1}^n \frac{1}{q^2} \sum_{p/q \in \mathcal{F}_q^*} f\left(\frac{p}{q}\right) = \lim_{n \rightarrow \infty} \frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} \chi_k = \lambda(f).$$

This finishes the proof of (2) in Theorem 1.1. □

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A DICHOTOMY BETWEEN THE STERN-BROCOT AND THE FAREY SEQUENCE

Received August 9, 2011  
Accepted September 19, 2011

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