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uniform distribution theory

J.J. SYLVESTER'S TWO CONVEX SETS THEOREM AND G.-L. LESAGE'S THEORY OF GRAVITY

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ABSTRACT. Given two convex sets K_1 and K_2 in the plane, J.J. Sylvester computes the measure $m(K_1, K_2)$ of the family of straight lines which meet both K_1 and K_2. As their distance $d = d(K_1, K_2)$ increases to infinity

$$m(K_1, K_2) = h(K_1)h(K_2)/d + O(1/d^2)$$

for some $h(K_1) \geq 0$ and $h(K_2) \geq 0$, suggesting Newton's law of attraction in the plane. We discuss the analogy in the spirit of G. -L. Lesage.

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dédié à Gérard Rauzy
mathématicien profond et pertinent,
à l'ami chaleureux disparu.

1. Introduction, Georges-Louis Lesage and gravitation

Georges-Louis Lesage (1724 – 1803), a Swiss mathematician and physicist, thought to explain Newton's $1/d^2$ law of gravitation in the following way. Space, he assumed, is filled with small invisible particles (*particules ultramondaines* as he called them) which move in straight lines and hit material bodies uniformly from all directions so that any isolated body in space will stay at rest if initially at rest. Suppose now that two bodies K_1 and K_2 face each other. Each one of them shadows off almost all particles in between them so that mostly particles pushing K_1 and K_2 together are left. Attraction reduces therefore to pushing

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forces! Lesage justified $1/d^2$ law considering elementary geometry of solid angles in space.

For many reasons, this mechanistic theory very much in the spirit of Descartes, cannot hold. It does not take into account the masses, it depends on the shape of the bodies, it ignores those particles which could bounce off from one body to the other, and furthermore, the "attraction" would vary with the absolute speed of the bodies in the Aether. Lesage was able to answer all these criticisms by adding to his model ad hoc unconvincing hypotheses. He is today almost completely forgotten though H. Poincaré [11], R. Feynman [6] and J.-M. Lévy-Leblond [10] do mention him with new arguments against the model. See also H. Chabot [4] and the Lesage site [9]. No one, and we in particular, would hope to rehabilitate his theory. Yet it seems so clever, simple and beautiful from a mathematical point of view that we feel it deserves further discussion.

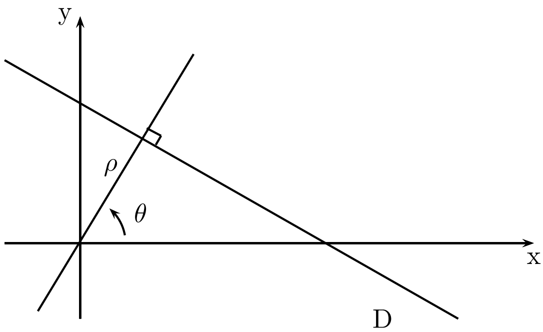
J.J. Sylvester's geometric probability results [13] which dates from 1890 could be seen (our interpretation) as a toy-model for Lesage's approach. Having died in 1803 long before Sylvester's birth in 1814, he clearly could know nothing about Sylvester! For an introduction to geometric probability we refer to M.W. Crofton [5], L.A. Santaló [12], D.A. Klain and G.-C. Rota [7], R.V. Ambartzumian [1], S. Buchin [2], R. Langevin [8], etc.

Guided by simplicity, we shall only develop the computations in the plane so that the $1/d^2$ law reads $1/d$.

2. Geometric Probability and Sylvester's theorem

Buffon's needle problem and its solution [3] date from the 18^{th} century giving birth to geometric probability. It is only toward the end of the 19^{th} century that the field was seriously considered and attracted many a mathematician. Let us recall one of the fundamental results in the plane.

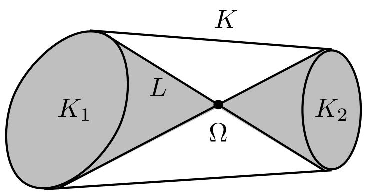
A straight line D is completely determined by its algebraic distance $\rho \in \mathbb{R}$ to the origin and by the angle $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ of the $x-axis$ with the vector orthogonal to D.



Clearly $(-\rho, \theta+\pi) = (\rho, \theta)$, so that the space of straight lines can be identified with a Möbius manifold M. The natural measure on M is the Lebesgue measure $d\rho d\theta$. An important result states that the measure of the set of lines which meet a given segment Γ is $2|\Gamma|$, twice the length of Γ.

Let K_1 and K_2 be two disjoint smooth compact convex sets in the plane. Let K be the convex hull of $K_1 \cup K_2$. We define the "twisted hull" L as follows.

Two of the four common tangent lines to K_1 and K_2 meet in the interior point Ω. Let L_1 (resp. L_2) be the convex hull of $K_1 \cup \{\Omega\}$ (resp. $K_2 \cup \{\Omega\}$). The twisted hull is $L = L_1 \cup L_2$. In case of polygonal convex sets, the definition is similar.



For any set A we let ∂A denote its boundary and $|\partial A|$ the length of the boundary. We are now in a position to state Sylvester's theorem [13], [12], [1] and [2]. We do not reproduce its proof here.

Theorem 1. *(Sylvester)*

The measure of the set of lines which meet K_1 (resp. K_2) without intersecting K_2 (resp. K_1) is

$$m_1 = |\partial K| + |\partial K_1| - |\partial L|$$

resp.

$$m_2 = |\partial K| + |\partial K_2| - |\partial L|.$$

The measure of the set of lines which meet both K_1 and K_2 is

$$m = |\partial L| - |\partial K|.$$

We shall only use the last part of the theorem.

If $K_1 \cap K_2 \neq \emptyset$, the results still hold provided $|\partial L|$ is replaced by $|\partial K_1| + |\partial K_2|$.

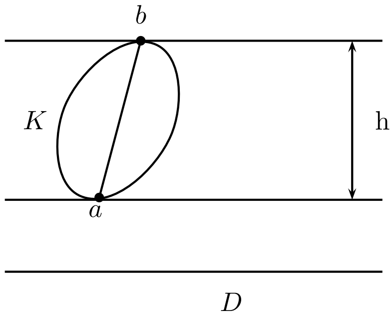
In the case $K_1 \cap K_2 = \emptyset$, could it be that the introduction of the twisted hull is related to the Möbius structure of the family of lines, even though Sylvester's proof does not use it?

3. Lesage particles

We now present our hypothesis. We identify Lesage particles with their trajectories, i.e., with straight lines. In the left hand side of the figure below, the straight line represents two particles falling on K_1 respectively at points a and b.

On the right hand the line represents two particles, one striking K_1 at c and one striking K_2 at d and therefore pushing K_1 and K_2 together. The measure of the family of lines hitting both K_1 and K_2, namely $|\partial L| - |\partial K|$ is the excess of particles pushing K_1 and K_2 together.

Let us introduce a few notations and definitions before stating our results. Given a compact convex set and a direction D, we consider the *height* h of K relative to the direction D i.e., the projection of K on a line orthogonal to D. We agree to call the segment $[a\,b]$ a diameter of K relative to D. It may happen that the diameter relative to a direction is not uniquely determined, but this is of no importance.



Let K_1 and K_2 be two compact convex sets. Define the distance between K_1 and K_2

$$d(K_1, K_2) = \inf_{a_1 \in K_1, a_2 \in K_2} ||a_1 - a_2||$$

where $||\dots||$ is the usual Euclidian norm. There exist two points $a_1^0 \in K_1$ and $a_2^0 \in K_2$ for which $d(K_1, K_2) = ||a_1^0 - a_2^0||$. We will note $\underline{d} = \underline{a_1^0 a_2^0}$ the vector distance between K_1 and K_2. The h_1 of K_1 and h_2 of K_2 relative to the direction of $\underline{d}$ will play a central role in the following theorem.

In Theorem 2 below, when we assume that d increases we mean the following. Let $\underline{d}^0$ the distance vector between K_1 and K_2. Fix K_1 and translate K_2 by a vector $\underline{d}$ supported by $\underline{d}^0$ such that $d = ||\underline{d}|| \to +\infty$.

Theorem 2.

Let K_1 and K_2 be two disjoint convex polygons each with finitely many edges. As their distance d increases to infinity, the measure of the set of straight lines meeting both K_1 and K_2 is of the form

$$m = \frac{h_1 h_2}{d} + O(1/d^2)$$

where $h_1 \geq 0$ and $h_2 \geq 0$ are the respective heights of K_1 and K_2 relative to their vector distance $\underline{d}$.

Remark 1.

We expect Theorem 2 to be valid for arbitrary compact convex sets K_1 and K_2, but we were unable to present a clearcut argument.

Theorem 2 demonstrates very clearly that Lesage's model based on Sylvester's theorem which depends on the orientation of the bodies cannot explain Newton's law which is independent of orientation. In particular if K_1 and K_2 are straight segments supported by a same straight line, then $m(K_1, K_2) = 0$ in complete contradiction with Newton's law.

If K_1 and K_2 are two disjoints discs we get a more precise results which we shall proof first.

Theorem 3.

Let K_1 and K_2 be two disjoint discs with radii R_1 and R_2. If $\delta > R_1 + R_2$ is the distance of their centers, the measure of the family of those lines which meet both K_1 and K_2 is

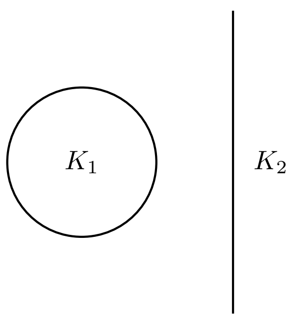
$$\begin{aligned} m &= 2(R_1+R_2)\arcsin\frac{R_1+R_2}{\delta} - 2(R_1-R_2)\arcsin\frac{R_1-R_2}{\delta} \\ &+ 2\sqrt{\delta^2-(R_1+R_2)^2} - 2\sqrt{\delta^2-(R_1-R_2)^2}. \end{aligned}$$

Remark 2.

If $R_1/\delta \to 0$ and $R_2/\delta \to 0$ it is easily verified that

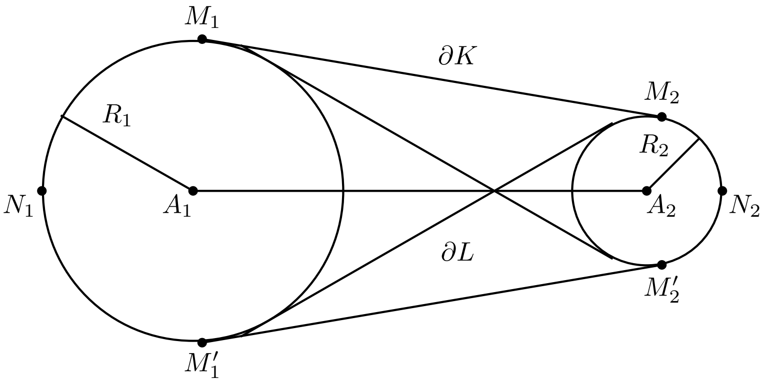
$$m = 4R_1R_2/\delta + O(1/\delta^2).$$

which is consistent with Theorem 2 since $\delta = d - (R_1 + R_2)$ and $h_1 = 2R_1$, $h_2 = 2R_2$.
If R_1 and d are kept fixed and $R_2 \to +\infty$, $\delta \to +\infty$ with $R_2/\delta \to 1$, then K_2 becomes a half-plane. According to Theorem 3, m tend to $2\pi R_1$ which is no surprise. Indeed except for those lines which are parallel to ∂K_2 and whose measure therefore vanishes, lines which meet both K_1 and K_2 are precisely those which intersect K_1 alone.



4. Proof of Theorem 3

We compute $m = |\partial L| - |\partial K|$.



Now

$$\partial K = \widehat{M_1 N_1 M_1'} + M_1' M_2' + \widehat{M_2' N_2 M_2} + M_2 M_1.$$

Elementary geometry leads to

$$|\partial K| = \pi(R_1 + R_2) + 2(R_1 - R_2) \arcsin \frac{R_1 - R_2}{\delta} + 2\sqrt{\delta^2 - (R_1 - R_2)^2}.$$

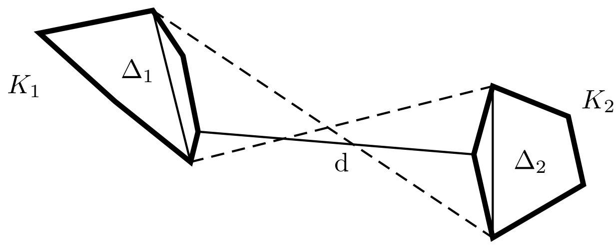
Similarly

$$|\partial L| = \pi(R_1 + R_2) + 2(R_1 + R_2) \arcsin \frac{R_1 + R_2}{\delta} + 2\sqrt{\delta^2 - (R_1 + R_2)^2},$$

which establishes the theorem.

5. Proof of Theorem 2

Let K_1 and K_2 be two disjoint polygonal compact convex sets with finitely many sides, and let $d = d(K_1, K_2)$ be their distance which we assume to be large compared to $|\partial K_1|$ and $|\partial K_2|$. Define h_1 and h_2 as in Section 3.



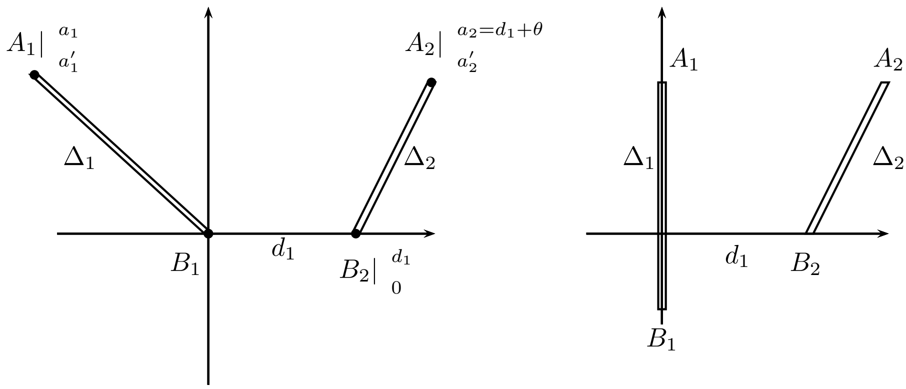
A_1 and B_1 on ∂K_1 are the endpoints of the diameter of K_1 associated with the direction of the vector distance $\underline{d}$. Put $\Delta_1 = A_1 B_1$. Similarly we consider $\Delta_2 = A_2 B_2$ for the set K_2.

Let $m = m(K_1, K_2)$ be the measure of the set of straight lines which meet both K_1 and K_2 and let $m(\Delta_1, \Delta_2)$ be the measure of the set of straight lines which intersect both Δ_1 and Δ_2. K_1 and K_2 are both polygonal with finitely many sides, therefore when d is large enough, one has $m = m(\Delta_1, \Delta_2)$. Hence we are led to compute $m(\Delta_1, \Delta_2)$.

Denote by d_1 the vector which measures the distance between Δ_1 and Δ_2. Let Q be the intersection point of the two straight lines supporting Δ_1 and Δ_2. There are essentially two cases to study according to whether $Q \in \Delta_1 \cup \Delta_2$ for all d_1, or $Q \notin \Delta_1 \cup \Delta_2$ for d_1 large enough.

In the first case either h_1 or h_2 is zero, and all other terms of $|\partial L| - |\partial K|$ associated with Δ_1 and Δ_2 are also $O(1/d_1^2)$. Therefore $m(\Delta_1, \Delta_2) = O(1/d_1^2)$ and since $d_1 \sim d$, $m = O(1/d^2)$. This establishes the theorem in that case since $h_1 h_2 = 0 + O(1/d^2)$.

We now study the second case which is subdivided in two sub-cases according to the following figures.



Since the computations are very similar we shall only treat the first sub-case and choose the origin to be B_1. Then

$$\begin{aligned} m(\Delta_1, \Delta_2) &= |\partial L| - |\partial K| \\ &= B_1A_2 + A_1B_2 - B_1B_2 - A_1A_2. \end{aligned}$$

Now

$$\begin{aligned} B_1A_2 &= (a_2^2 + {a_2'}^2)^{1/2} = ((d_1+\theta)^2 + {a_2'}^2)^{1/2} \\ &= d_1\left[1 + \frac{2\theta d_1 + \theta^2 + (a_2')^2}{2d_1^2} - \frac{4\theta d_1^2}{8d_1^4}\right] + O(1/d_1^2) \\ &= d_1 + \frac{2\theta d_1 + {a_2'}^2}{2d_1} + O(1/d_1^2). \end{aligned}$$

$$\begin{aligned} A_1B_2 &= ((d_1 - a_1)^2 + {a_1'}^2)^{1/2} \\ &= d_1\left[1 + \frac{-2a_1d_1 + a_1^2 + (a_1')^2}{2d_1^2} - \frac{4a_1^2d_1^2}{8d_1^4}\right] + O(1/d_1^2) \\ &= d_1 + \frac{-2a_1d_1 + {a_1'}^2}{2d_1} + O(1/d_1^2). \end{aligned}$$

Then $B_1B_2 = d_1$ and

$$\begin{aligned} A_1A_2 &= ((d_1+\theta-a_1)^2+(a_2'-a_1')^2)^{1/2} \\ &= d_1\left[1+\frac{2(\theta-a_1)d_1+(\theta-a_1)^2+(a_2'-a_1')^2}{2d_1^2}\right. \\ &\qquad\left.-\frac{4(\theta-a_1)^2d_1^2}{8d_1^4}\right]+O(1/d_1^2) \\ &= d_1+\frac{1}{2d_1}\left[2\theta d_1-2a_1d_1+{a_2'}^2+{a_1'}^2-2a_1'a_2'\right]+O(1/d_1^2). \end{aligned}$$

Finally since $d_1 \sim d$

$$m(\Delta_1,\Delta_2)=\frac{a_1'a_2'}{d}+O(1/d^2)=\frac{h_1h_2}{d}+O(1/d^2).$$

6. Conclusion

The 3-body problem is well known to be deep and difficult in the context of Newton gravity. Likewise, given three convex sets in the plane, computing the measures of the sets of straight lines meeting respectively 0, 1, 2 or 3 of these sets is also complex as realized by Sylvester [13] and Ambartzumian [1].

As we are about to conclude our paper, we should mention that Lesage's theory is valid in a somewhat different context. According to Casimir's effect in quantum physics, two parallel metal plates in empty space attract each other, or rather are pushed together. The reason is that empty space is structured by electromagnetic waves. Between the plates, the wave-lengths are necessarily divisors of the distance between the plates, whereas outside, the wave-lengths are arbitrary. Therefore these waves are much more numerous and push the plates one onto the other.

A similar effect can actually be observed in a completely trivial situation. Imagine two parallel logs of wood floating close together on the surface of the sea. There are less waves between the logs than beyond them and so as before the logs are eventually brought together. It is an amusing fact that Lesage thought of particles when at his time and later during the whole of the 19th century wave theory permeated almost all physics (light, sound, etc,...). Continuum was the master word. Lesage's philosophy with his particles was, to some extent, 200 years in advance...

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