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ON THE DISCREPANCY OF SOME GENERALIZED KAKUTANI'S SEQUENCES OF PARTITIONS

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Abstract. In this paper we study a class of generalized Kakutani's sequences of partitions of $[0, 1]$, constructed by using the technique of successive $\rho-$refinements. Our main focus is to derive bounds for the discrepancy of these sequences. The approach that we use is based on a tree representation of the sequence of partitions which is precisely the parsing tree generated by Khodak's coding algorithm. With the help of this technique we derive (partly up to a logarithmic factor) optimal upper bound in the so-called rational case. The upper bounds in the irrational case that we obtain are weaker, since they heavily depend on Diophantine approximation properties of a certain irrational number. Finally, we present an application of these results to a class of fractals.

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1. Introduction

In this paper we will study uniformly distributed sequences of partitions of $[0, 1]$, a concept which has been introduced in 1976 by Kakutani, [13].

Definition 1.1. *Let (π_n) be a sequence of partitions of $[0, 1]$ represented by $\pi_n = \{[t_{i-1}^{(n)}, t_i^{(n)}] : 1 \leq i \leq k(n)\}$, where $0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_{k(n)}^{(n)} = 1$. The sequence (π_n) is said to be uniformly distributed (u.d.) if for any continuous*

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function f on $[0,1]$ we have

$$\lim_{n\to\infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} f\left(t_i^{(n)}\right) = \int_0^1 f(t)\ dt.$$

Equivalently, (π_n) is u.d. if the sequence of discrepancies

$$D_n = \sup_{0\le a<b\le 1} \left| \frac{1}{k(n)} \sum_{i=1}^{k(n)} \chi_{[a,b[}\left(t_i^{(n)}\right) - (b-a) \right| \tag{1}$$

tends to 0 as $n \to \infty$ (for more details on the theory of uniform distribution see [9] or [17]; χ_M denotes the characteristic function of the set M).

Kakutani's sequence of partitions is defined in the following way. Let $\alpha \in]0,1[$ be given and start with the unit interval $[0,1]$. In the first step this interval is divided into two intervals $[0,\alpha], [\alpha,1]$ of lengths α and $1-\alpha$, respectively. In the second step the largest interval is partitioned into two subintervals of lengths proportional to α and $1-\alpha$, respectively. For example, if $\alpha = \frac{1}{3}$ then the interval $\left[\frac{1}{3},1\right]$ is split into $\left[\frac{1}{3},\frac{5}{9}\right], \left[\frac{5}{9},1\right]$. In this way one proceeds further. Note that one always considers all intervals of maximal length at once.

Definition 1.2. *If $\alpha \in]0,1[$ and $\pi = \{[t_{i-1},t_i] : 1 \le i \le k\}$ is any interval partition of $[0,1]$, then Kakutani's α-refinement of π (which will be denoted by $\alpha\pi$) is obtained by splitting all the intervals of π having maximal length in two parts, proportional to α and $1-\alpha$ respectively.*

Consequently, if we denote by κ_n Kakutani's sequence of partitions we have that it can be written as $\kappa_n = \alpha^n\omega$, where $\omega = \{[0,1]\}$. Kakutani proved that for every $\alpha \in]0,1[$ the sequence of partitions (κ_n) of $[0,1]$ is u.d. (cf. [13]).

In a recent paper [19], Kakutani's splitting procedure has been generalized by splitting the longest intervals of a partition π into a finite number of parts homothetically to a given finite interval partition ρ of $[0,1]$. The resulting interval partition $\rho\pi$ is called a *ρ-refinement* of π. As for the α-refinements (which correspond to $\rho-$refinements with $\rho = \{[0,\alpha],[\alpha,1]\}$) the following result holds (cf. [19]).

Theorem 1.3. *The sequence $(\rho^n\omega)$ of successive ρ-refinements of the trivial partition $\omega = \{[0,1]\}$ is u.d..*

A natural problem posed in [19], which is interesting for possible applications, is to estimate the behaviour of the discrepancy as n tends to infinity. The only known discrepancy bounds for sequences of this kind have been obtained in [4] by Carbone, who considered the so-called LS-sequences that evolve from partitions

ρ with L subintervals of $[0,1]$ of length α and S subintervals of length α^2 (where α is given by the equation $L\alpha + S\alpha^2 = 1$).

In this paper, we analyze this problem with a new approach based on a parsing tree (related to Khodak's coding algorithm [15]) which represents the successive ρ-refinements. In particular, we will use refinements of the results obtained in [8] about Khodak's algorithm to give an estimate of the discrepancy for a class of sequences of partitions constructed by successive $\rho-$refinements. Suppose that ρ is a partition of $[0,1]$ consisting of m subintervals of lengths $p_1, \ldots, p_m$. In the so-called rational case (which means that all fractions $(\log p_i)/(\log p_j)$ are rational, see Definition 2.1) we will provide very precise bounds for the discrepancy. Note that LS-sequences are included into the rational case, therefore we generalize the results of [4]. However, we are also able to cover several irrational cases (which means that at least one of the fractions $(\log p_i)/(\log p_j)$ is irrational).

Let us give a brief outline of the structure of the paper. In Section 2 we introduce Khodak's algorithm and analyze the correspondence between subintervals of $[0,1]$ and nodes of the parsing tree. Moreover, we extend an asymptotic result from [8]. In Section 3 we present our main results in the rational case. In particular, we obtain an upper bound of the form

$$D_n = O\left((\log k(n))^d k(n)^{-\eta}\right)$$

where η is a real positive constant and $d \geq 0$ an integer (both values are explicit). Furthermore, this upper bound is the best possible (despite of a logarithmic factor in a special case). In Section 4 we discuss some instances of the irrational case for $m = 2$. They are much more involved than in the rational case. Finally, in Section 5 we give some examples and applications including LS-sequences and u.d. sequences of partitions on a class of fractals. Some auxiliary results which are used in Section 2 are collected in Section 6.

2. ρ-refinements and Khodak's algorithm

From now on, consider a partition ρ of $[0,1]$ consisting of m intervals of lengths $p_1, \ldots, p_m$ and the sequence of ρ-refinements of the trivial partition $\omega = \{[0,1]\}$.

Our goal is to construct recursively an m-ary tree T which represents the process of successive $\rho-$refinements of ω. An m-ary tree is an ordered rooted tree, where each node has either m ordered successors (we call such a node *internal node*) or it is a leaf with no successors (which we call also *external node*). The numbers $1, \ldots, m$ induce a natural labelling on the nodes. Suppose that the unique path from the root to a node x at level l is encoded by the sequence $(j_1, j_2, \ldots, j_l)$, $j_i \in \{1, \ldots, m\}$, then we set $P(x) = p_{j_1} p_{j_2} \cdots p_{j_l}$ and this is the label of the node x. This can be also considered as the probability of reaching the node x with a random walk that starts at the root and moves away from it according to the probabilities $p_1, \ldots, p_m$. For completeness the root a is labelled with $P(a) = 1$. If T is a finite m-ary tree then the labels of the external nodes sum up to 1 (this fact follows easily by induction). Hence, the shape of an m-ary tree (together with $p_1, \ldots, p_m$) gives rise to a probability distribution.

The start of our iteration is a tree that only consists of the root which is then an external node (with probability 1). In the first step the root is replaced by an internal node together with m ordered successing leaves that are given the probability distribution $p_1, \ldots, p_m$. At each further iteration we select all leaves y with largest label $P(y)$ and grow m children out of each of them. Actually this construction corresponds to the ρ-refinements procedure of the sequence $(\rho^n \omega)$. At level n the leaves of the tree correspond to the intervals of $\rho^n \omega$ and the labels of the leaves to the lengths of these intervals. This procedure exactly leads to the same parsing tree of Tunstall's code [8] (the words $(j_1, j_2, \ldots, j_l)$ which encodes the paths from the root to the leaves are the phrases of the dictionary). It is important to note that at each iteration we can have different leaves of the same highest probability, but Tunstall's algorithm selects (randomly) only one of these leaves and grows m children out of it.

There is a second way to describe this tree evolution process, namely by Khodak's algorithm [15]. Fix a number $r \in]0, p_{min}[$, where $p_{min} = \min\{p_1, \ldots, p_m\}$, and consider all nodes x in an infinite m-ary tree with $P(x) \geq r$. Let us denote these nodes by $\mathcal{I}(r)$. Of course, if $P(x) \geq r$ then all nodes x' on the path from the root to x satisfy $P(x') \geq r$, too. Hence, the nodes of $\mathcal{I}(r)$ constitute a finite subtree and will be called the *internal nodes* of Khodak's construction. Finally, we append to the internal nodes all successor nodes y which, by construction, satisfy $p_{min} r \leq P(y) < r$ and we denote them by $\mathcal{E}(r)$. These nodes are called the *external nodes* of Khodak's construction. We denote by $M_r = |\mathcal{E}(r)|$ the number of external nodes. Obviously in correspondence to r we have got a finite m-ary tree $\mathcal{T}(r) = \mathcal{I}(r) \cup \mathcal{E}(r)$ and it is clear that this tree grows when r decreases. For certain values r, the external nodes y of largest value $P(y) = r$ turn into internal nodes and all their successors become new external nodes. Actually, the tree $\mathcal{T}(r)$ grows in correspondence to a decreasing sequence of values (r_j). Indeed, when $r \in]r_j, r_{j-1}]$ the tree remains the same, i.e $\mathcal{T}(r) = \mathcal{T}(r_{j-1})$.

The parsing tree resulting from Khodak's algorithm is exactly the same as the tree constructed by Tunstall's algorithm. However, we have to observe that in Khodak's construction all leaves with the same highest probability are selected to generate the children at once, while in Tunstall's algorithm they are selected one by one in an arbitrary order. Now, in the procedure of successive $\rho-$refinements, at each step we select the intervals having maximal lenght at once and we split them at the same time. So, Khodak's algorithm and $\rho-$refinements procedure not only are exactly represented by the same tree but they also have a common structure which allows to create a useful correspondence between them.

In fact, if we fix a step j in the $\rho-$refinements procedure, then the tree associated to the partition $\rho^j\omega$ is exactly $\mathcal{T}(r_j)$. Therefore, we will mainly consider the values of the sequence (r_j) for which the tree constructed by Khodak's algorithm actually grows. Note that we have to start with the value $r_1 = 1$ which corresponds to $\rho\omega$. The number of external nodes in $\mathcal{E}(r_j)$ equals the number of points defining the partition $\rho^j\omega$, i.e. $M_{r_j} = k(j)$. Of course, if $r \in]r_j, r_{j-1}]$ then $M_r = M_{r_{j-1}} = k(j-1)$.

From here on we denote by $\mathscr{E}_{r_j}$ the family of all intervals of the partition $\rho^j\omega$ corresponding to the leaves belonging to $\mathcal{E}(r_j)$ and the order of the intervals in $\mathscr{E}_{r_j}$ corresponds to the left-to-right order of the external nodes in $\mathcal{E}(r_j)$. We will call all the intervals belonging to each $\mathscr{E}_r$ for $r \in]0,1]$ *elementary intervals.*

In the following we denote by H the entropy of the probability distribution $p_1, \dots, p_m$, which is defined as

$$H = p_1 \log\left(\frac{1}{p_1}\right) + \cdots + p_m \log\left(\frac{1}{p_m}\right).$$

Definition 2.1. *We say that* $\log\left(\frac{1}{p_1}\right), \dots, \log\left(\frac{1}{p_m}\right)$ *are* rationally related *if there exists a positive real number* Λ *such that* $\log\left(\frac{1}{p_1}\right), \dots, \log\left(\frac{1}{p_m}\right)$ *are integer multiples of* Λ*, that is*

$$\log\left(\frac{1}{p_j}\right) = n_j\Lambda, \quad \textit{with } n_j \in \mathbb{Z} \textit{ for } j = 1, \dots, m.^1 \tag{2}$$

Equivalently, all fractions $(\log p_i)/(\log p_j)$ *are rational. W.l.o.g. we can assume that* Λ *is as large as possible which is equivalent to assume* $\gcd(n_1, \dots, n_m) = 1$.

Similarly we say that $\log\left(\frac{1}{p_1}\right), \dots, \log\left(\frac{1}{p_m}\right)$ *are* irrationally related *if they are not rationally related.*

One of the main results from [8] provides asymptotic information on the number M_r of external nodes in Khodak's construction. Actually these relations can be used to prove Theorem 1.3. However, in order to derive bounds for the discrepancy of the sequence $(\rho^n\omega)$ we need more precise information on the error terms. Therefore we have extended the analysis of [8].

[1]Actually this means that the $\log(p_i)$ are simply commensurable, but we chose the present notation in accordance with the paper [8].

Theorem 2.2. *Let M_r be the number of the external nodes corresponding to the parameter r in Khodak's construction, that is, the number of nodes in $\mathcal{E}(r)$.*

- *If $\log\left(\frac{1}{p_1}\right), \ldots, \log\left(\frac{1}{p_m}\right)$ are rationally related, let $\Lambda > 0$ be the largest real number such that (2) holds. Then there exist a real number $\eta > 0$ and an integer $d \geq 0$ such that*
$$M_r = \frac{(m-1)}{rH} Q_1\left(\log\left(\frac{1}{r}\right)\right) + O\left((\log r)^d r^{-(1-\eta)}\right), \tag{3}$$
where
$$Q_1(x) = \frac{\Lambda}{1-e^{-\Lambda}} e^{-\Lambda\left\{\frac{x}{\Lambda}\right\}}$$
and $\{y\}$ is the fractional part of the real number y. Furthermore, the error term is optimal.
- *If $\log\left(\frac{1}{p_1}\right), \ldots, \log\left(\frac{1}{p_m}\right)$ are irrationally related, then*
$$M_r = \frac{(m-1)}{rH} + o\left(\frac{1}{r}\right). \tag{4}$$
In particular, if $m = 2$ and $\gamma = (\log p_1)/(\log p_2)$ is badly approximable then
$$M_r = \frac{(m-1)}{rH}\left(1 + O\left(\frac{(\log\log(1/r))^{1/4}}{(\log(1/r))^{1/4}}\right)\right). \tag{5}$$
If p_1 and p_2 are algebraic then there exists an effectively computable constant $\kappa > 0$ with
$$M_r = \frac{(m-1)}{rH}\left(1 + O\left(\frac{(\log\log(1/r))^{\kappa}}{(\log(1/r))^{\kappa}}\right)\right). \tag{6}$$

P r o o f.
Set $v = \frac{1}{r}$ and denote by $A(v)$ the number of internal nodes (root node included) in Khodak's construction with parameter $r = 1/v$, that is,
$$A(v) = \sum_{x:P(x) \geq \frac{1}{v}} 1.$$
Hence, the number of external nodes generated at the step corresponding to the parameter r is
$$M_r = (m-1)A(v) + 1. \tag{7}$$
The key relation is that $A(v)$ satisfies the following recurrence (see [8, Lemma 2])
$$A(v) = \begin{cases} 0 & \text{if } v < 1, \\ 1 + \sum_{j=1}^{m} A(p_j v) & \text{if } v \geq 1. \end{cases} \tag{8}$$

For the asymptotic analysis of $A(v)$ (and consequently that of M_r) we distinguish between the rational and the irrational case.

Rational case
If the $\log(1/p_j)$ are rationally related then $A(v)$ is constant for $v \in [e^{\Lambda n}, e^{\Lambda(n+1)}[$ (for every integer n). Hence, it suffices to study the behaviour of the sequence $G(n) = A(e^{\Lambda n})$ which satisfies the recurrence

$$G(n) = \begin{cases} 0 & \text{if } n < 0 \\ 1 + \sum_{j=1}^{m} G(n - n_j) & \text{if } n \geq 0 \end{cases}$$

where $n_j = \frac{\log\left(\frac{1}{p_j}\right)}{\Lambda}$ for $j = 1, \ldots, m$. The generating function $g(z) = \sum_{n \geq 0} G(n) z^n$ is then given by

$$g(z) = \frac{1}{(1-z)f(z)}, \tag{9}$$

where $f(z) = 1 - z^{n_1} - \cdots - z^{n_m}$. By Definition 2.1, it follows that $e^{-\Lambda}$ is a positive real root of f. Moreover, it is proved in [5, p. 271, Lemma 8] that if we denote by $\omega_1, \ldots, \omega_h$ all the other different roots (with multiplicities μ_i) of f then $|\omega_i| > e^{-\Lambda}$ for $i = 1, \ldots, h$. (Here we use the assumption that $n_1, \ldots, n_m$ are coprime). Hence, it follows that

$$G(n) = \frac{\Lambda e^{\Lambda n}}{H(1 - e^{-\Lambda})} + \sum_{i=1}^{h} P_i(n) \omega_i^{-n} - \frac{1}{m-1}, \tag{10}$$

where the polynomials P_i are non-zero and their degrees correspond to the algebraic multiplicities of ω_i. This follows from the representation in (9) which implies that all zeros of $f(z)$ have to contribute to $G(n)$ (according to their multiplicities). Obviously relation (10) implies (3) of Theorem 2.2 for some $\eta > 0$. Note that in view of (7) the constant term $-1/(m-1)$ disappears when we translate the asymptotics of $G(n)$ to M_r.

Next we study the error term (without the constant term $-1/(m-1)$) in more detail. W.l.o.g. we can assume that $\omega_1, \ldots, \omega_k$ (with $k \leq h$) are those roots of $f(z)$ with smallest modulus

$$|\omega_i| = e^{-\Lambda(1-\eta)} \tag{11}$$

(for some $\eta > 0$) such that $P_i \neq 0$ and the degrees of P_i are maximal and all equal to $d \geq 0$, for $1 \leq i \leq k$. This means that the difference between $G(n)$ and the asymptotic leading term is bounded by

$$\delta(n) = \left| G(n) - \frac{\Lambda e^{\Lambda n}}{H(1-e^{-\Lambda})} + \frac{1}{m-1} \right| \le C n^d e^{\Lambda(1-\eta)n}$$

for some constant $C > 0$. More precisely $\delta(n)$ can be written as

$$\delta(n) = \left| n^d \sum_{i=1}^{k} \widetilde{c}_i \, \omega_i^{-n} \right| + O\left(n^{d-1} e^{\Lambda(1-\eta)n} \right).$$

Note that the leading term of $\delta(n)$ is certainly non-zero (as a sequence). If it were zero then the sequence $G(n)$ would satisfy a linear recurrence with a characteristic polynomial of degree smaller than the degree of $(1-z)f(z)$. However, this contradicts the representation (9) of the generating function $g(z)$. Furthermore, since all roots of $f(z)$ are either real or appear in conjugate pairs of complex numbers we can rewrite the sum $n^d \sum\limits_{i=1}^{k} \widetilde{c}_i \, \omega_i^{-n}$ as

$$n^d e^{\Lambda(1-\eta)n} \sum_{i=1}^{k'} c_i' \cos(2\pi\theta_i n + \alpha_i).$$

By Lemma 6.1 there exists a $\delta > 0$ and infinitely many n such that

$$\left| \sum_{i=1}^{k'} c_i' \cos(2\pi\theta_i n + \alpha_i) \right| \ge \delta.$$

This shows that

$$\delta(n) \ge C' n^d e^{\Lambda(1-\eta)n}$$

for infinitely many n and for some constant $C' > 0$. This means that the error term in (3) is optimal.

Irrational case

The analysis in the irrational case is much more involved. Instead of using power series we use the Mellin transform

$$A^*(s) = \int_0^\infty A(v) v^{s-1} \, dv.$$

By using the fact that the Mellin transform of $A(av)$ is $a^{-s} A^*(s)$, a simple analysis of recurrence (8) reveals that the Mellin transform $A^*(s)$ of $A(v)$ is given by

$$A^*(s) = \frac{-1}{s\left(1 - p_1^{-s} - \cdots - p_m^{-s}\right)}, \qquad \Re(s) < -1.$$

In order to find asymptotics of $A(v)$ as $v \to \infty$ one can directly use the Tauberian theorem (for the Mellin transform) by Wiener-Ikehara [16, Theorem 4.1]. For this purpose we have to check that $s_0 = -1$ is the only (polar) singularity on the line $\Re(s) = -1$ and that $(s+1)A^*(s)$ can be analytically extended to a region that contains the line $\Re(s) = -1$. However, in the irrational case this follows by a lemma of Schachinger [18, p. 454, Lemma 4]. In particular, one finds $A(v) \sim v/H$ and so (4) but this simple procedure does not provide any information about the error term.

In order to make our presentation as simple as possible we will restrict ourselves to the case $m = 2$ and we will also assume certain conditions on the Diophantine properties of the irrational number

$$\gamma = \frac{\log p_1}{\log p_2}.$$

We use the simplified notation $p = p_1$ and $q = p_2$.

The main idea to obtain error terms for $A(v)$ is to use the formula for the inverse Mellin transform

$$A(v) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\sigma-iT}^{\sigma+iT} A^*(s) v^{-s} ds, \qquad \sigma < -1, \tag{12}$$

and to shift the line of integration to the right. Of course, all polar singularities of $A^*(s)$ (which are given by the solutions of the equation $p^{-s} + q^{-s} = 1$ and by $s = 0$) give rise to a polar singularity of $A(v)$. Unfortunately, the order of magnitude of $A^*(s)$ is $O(1/s)$. Hence the integral in (12) is not absolutely convergent. It is therefore convenient to *smooth* the problem and to study the function

$$A_1(v) = \int_0^v A(w)\, dw$$

which is given by

$$A_1(v) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} A^*(s) \frac{v^{-s+1}}{1-s}\, ds = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{v^{-s+1}}{s(s-1)(1-p^{-s}-q^{-s})}\, ds,$$

with $\sigma < -1$. By [18, p. 454, Lemma 4] we know that all zeros of the equation $p^{-s} + q^{-s} = 1$ which are different from -1 satisfy $-1 < \Re(s) \le \sigma_0$ for some σ_0. Furthermore, there is $\tau > 0$ such that in each box of the form

$$B_k = \{s \in \mathbb{C} : -1 < \Re(s) \le \sigma_0,\ (2k-1)\tau \le \Im(s) < (2k+1)\tau\}, \quad k \in \mathbb{Z} \setminus \{0\},$$

there is precisely one zero of $p^{-s} + q^{-s} = 1$ that we denote by s_k. Hence, by shifting the line of integration to the right (namely to $\Re(s) = \sigma_1$ for some

$\sigma_1 > \max\{\sigma_0+1,1\})$ and by collecting all residues we obtain

$$A_1(v) = \frac{v^2}{2H} + \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{v^{1-s_k}}{s_k(s_k-1)H(s_k)} - v - \frac{1}{1-p^{-1}-q^{-1}} + \frac{1}{2\pi i}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty} \frac{v^{-s+1}}{s(s-1)(1-p^{-s}-q^{-s})}\,ds,$$

where $H(s) = p^{-s}\log(1/p) + q^{-s}\log(1/q)$. Clearly the integral can be estimated by

$$\frac{1}{2\pi i}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty} \frac{v^{-s+1}}{s(s-1)(1-p^{-s}-q^{-s})}\,ds = O\left(v^{-\sigma_1+1}\right).$$

Hence we just have to deal with the sum of residues $\sum v^{1-s_k}/(s_k(s_k-1)H(s_k))$.

First it is an easy exercise to show that there exists $\delta > 0$ such that $|H(s_k)| \ge \delta$ for all $k \in \mathbb{Z}\setminus\{0\}$. W.l.o.g. we can assume that $q < p$ and so $\frac{\log q}{\log p} < 1$. Since s_k is a zero of $p^{-s} + q^{-s} = 1$ we have that

$$\begin{aligned} H(s_k) &= p^{-s_k}\log\left(\frac{1}{p}\right) + q^{-s_k}\log\left(\frac{1}{q}\right) \\ &= (1-q^{-s_k})\log\left(\frac{1}{p}\right) + q^{-s_k}\log\left(\frac{1}{q}\right) \\ &= \log\left(\frac{1}{p}\right)\left(1 - q^{-s_k}\left(1-\frac{\log q}{\log p}\right)\right). \end{aligned}$$

So it follows that

$$|H(s_k)| = \log\left(\frac{1}{p}\right)\cdot\left|1 - q^{-s_k}\left(1-\frac{\log q}{\log p}\right)\right| \ge \log\left(\frac{1}{p}\right)\cdot\left|1 - |q^{-s_k}|\cdot\left|1-\frac{\log q}{\log p}\right|\right|.$$

Therefore, since $\left(1-\frac{\log q}{\log p}\right) < 1$ and $\Re(s_k) < 1$, there exists $c_0 > 0$ such that

$$|H(s_k)| \ge c_0 \log\left(\frac{1}{p}\right) = \delta.$$

Thus, we do not have to care about the factor $1/H(s_k)$.

Next assume that γ is a badly approximable irrational number which means that γ has a bounded continued fraction representation. Here Lemma 6.2 shows that all zeros $s_k \ne -1$ of the equation $p^{-s} + q^{-s} = 1$ satisfy $\Re(s_k) > -1 + c/\Im(s_k)^2$ for some constant $c > 0$. Hence it follows that $\Re(s_k) > -1 + c_1/k^2$ for some constant $c_1 > 0$ and we can estimate the sum of residues by

$$\left|\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{v^{1-s_k}}{s_k(s_k-1)H(s_k)}\right| \le \left|\sum_{0<|k|\le K} \frac{v^{1-s_k}}{s_k(s_k-1)H(s_k)}\right| + \left|\sum_{|k|>K} \frac{v^{1-s_k}}{s_k(s_k-1)H(s_k)}\right|$$

$$\leq C_1 v^{2-c_1/K^2} \sum_{0<|k|\leq K} \frac{1}{k^2} + C_2 v^2 \sum_{|k|>K} \frac{1}{k^2}$$
$$\leq C_3 v^2 \left(v^{-c_1/K^2} + \frac{1}{K} \right)$$

where C_1, C_2, C_3 are appropriate positive constants.

Thus, by choosing $K = \sqrt{c_1 (\log v)/(\log\log v)}$, we obtain the upper bound

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{v^{1-s_k}}{s_k(s_k-1)H(s_k)} = O\left(v^2 \frac{\sqrt{\log\log v}}{\sqrt{\log v}} \right)$$

and consequently

$$A_1(v) = \frac{v^2}{2H} \left(1 + O\left(\frac{\sqrt{\log\log v}}{\sqrt{\log v}} \right) \right).$$

Finally, by an application of Lemma 6.5, the previous relation implies

$$A(v) = \frac{v}{H} \left(1 + O\left(\frac{(\log\log v)^{1/4}}{(\log v)^{1/4}} \right) \right).$$

Similarly we can deal with the case when p and q are algebraic. In this case, all the solutions of the equation $p^{-s} + q^{-s} = 1$ (that are different from -1) satisfy $\Re(s_k) > -1 + \frac{D}{\Im(s_k)^{2C}}$ for some positive constants C, D (see Lemma 6.3). Then with the same procedure as above we get

$$\left| \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{v^{1-s_k}}{s_k(s_k-1)H(s_k)} \right| \leq C_4 v^2 \left(v^{-c_2 K^{-2C}} + \frac{1}{K} \right).$$

Hence, if we choose $K = (c_2(\log v)/(\log\log v))^{1/(2C)}$ we obtain (after a second application of Lemma 6.5) that

$$A(v) = \frac{v}{H} \left(1 + O\left(\frac{(\log\log v)^{\kappa}}{(\log v)^{\kappa}} \right) \right),$$

where $\kappa = \frac{1}{4C}$.

This completes the proof of Theorem 2.2.

□

3. Discrepancy bounds in the rational case

In this section, we are going to consider a partition ρ of $[0,1]$ consisting of m intervals of lengths $p_1,\ldots,p_m$ such that $\log\left(\frac{1}{p_1}\right),\ldots,\log\left(\frac{1}{p_m}\right)$ are rationally related.

By Theorem 2.2 we know that M_r is asymptotically given by

$$M_{r_n} = \frac{c'}{r_n} + O\left((\log r_n)^d r_n^{-(1-\eta)}\right), \quad r = r_n = e^{-\Lambda n}, \tag{13}$$

for some $\eta > 0$ and some integer $d \geq 0$, where $c' = (m-1)\Lambda/(H(1-e^{-\Lambda}))$ and the error term is optimal. Recall also that $k(n) = M_{r_n}$ which gives an asymptotic expansion for $k(n)$ of the form

$$k(n) \sim \frac{(m-1)\Lambda}{H(e^{\Lambda}-1)}\, e^{\Lambda n}.$$

Theorem 3.1. *Suppose that the lengths of the intervals of a partition ρ are $p_1,\ldots,p_m$ and assume that $\log\left(\frac{1}{p_1}\right),\ldots,\log\left(\frac{1}{p_m}\right)$ are rationally related. Furthermore, let $\eta > 0$ and $d \geq 0$ be given as in Theorem 2.2.*

Then the discrepancy of the sequence of partitions $(\rho^n\omega)$ is bounded by

$$D_n = \begin{cases} O\left((\log k(n))^d k(n)^{-\eta}\right) & \text{if } 0 < \eta < 1, \\ O\left((\log k(n))^{d+1} k(n)^{-1}\right) & \text{if } \eta = 1, \\ O\left(k(n)^{-1}\right) & \text{if } \eta > 1. \end{cases} \tag{14}$$

Moreover, there exists a $\delta > 0$ and infinitely many n such that

$$D_n \geq \begin{cases} \delta\,(\log k(n))^d k(n)^{-\eta} & \text{if } 0 < \eta \leq 1, \\ \delta\, k(n)^{-1} & \text{if } \eta > 1. \end{cases} \tag{15}$$

P r o o f.
For notational convenience we set

$$\Delta_n = \sup_{0<y\leq 1} \left| \sum_{i=1}^{k(n)} \chi_{[0,y[}\left(t_i^{(n)}\right) - k(n)y \right|$$

where $t_i^{(n)}$ are the points defining the partition $\rho^n\omega$. Then we have

$$D_n \leq \frac{2\Delta_n}{k(n)}.$$

Fix a step in the algorithm corresponding to a certain parameter r of the form $r = e^{-n\Lambda}$ for some integer $n \geq 0$, and consider an interval $A = [0,y[\subset [0,1]$. We want to estimate the number of elementary intervals belonging to $\mathscr{E}_r$ which are contained in A. For this purpose, let us fix another parameter $\overline{r}$, of the form $\overline{r} = e^{-\overline{n}\Lambda}$ with an integer $0 \leq \overline{n} < n$, corresponding to a previous step in Khodak's construction. At this previous step, we have $M_{\overline{r}}$ intervals I_j generated by the construction. If we denote by $l(I_j)$ the length of the interval I_j, then we have that

$$p_{min}\overline{r} \leq l(I_j) < \overline{r}, \quad \text{for} \quad j = 1, \ldots, M_{\overline{r}}, \tag{16}$$

since the lengths of $I_j \in \mathscr{E}_{\overline{r}}$ correspond to the values $P(y)$ of the external nodes y in $\mathcal{E}(\overline{r})$.

Suppose that precisely the first h of these intervals I_j are contained in A, so $U = I_1 \cup \ldots \cup I_h \subset A$. Now, we want to estimate the number of elementary intervals in $\mathscr{E}_r$ contained in each I_j. Khodak's construction shows that this equals precisely the number of external nodes in the subtree of the node x that is related to the interval I_j. An important feature of Khodak's construction is that subtrees of $\mathcal{T}(r)$ rooted at an internal node $x \in \mathcal{I}(r)$ are parts of a self-similar infinite tree and therefore they are constructed in the same way as the whole tree. So, one just has to replace r by $\frac{r}{P(x)}$. Hence, by using this remark in (13), the number N_{I_j} of subintervals of I_j (corresponding to the value r) equals

$$N_{I_j} = M_{\frac{r}{l(I_j)}} = \frac{c'}{r} l(I_j) + O\left(|\log r|^d \frac{l(I_j)^{1-\eta}}{r^{1-\eta}} \right).$$

Therefore, we have that the number N_U of elementary intervals in $\mathscr{E}_r$ contained in U is

$$N_U = N_{I_1} + \ldots + N_{I_h} = \frac{c'}{r}(l(I_1) + \ldots + l(I_h)) + O\left(\frac{|\log r|^d}{r^{1-\eta}} \sum_{j=1}^{h} l(I_j)^{1-\eta} \right).$$

By using (16) and the fact that $h \leq M_{\overline{r}} = O(1/\overline{r})$ we obtain

$$N_U = \frac{c'}{r}(l(I_1) + \ldots + l(I_h)) + O\left(|\log r|^d \frac{\overline{r}^{(-\eta)}}{r^{(1-\eta)}} \right).$$

The total number of intervals equals $M_r = c'/r + O(|\log r|^d r^{-1+\eta})$, so it follows

$$N_U - M_r l(U) = O\left(|\log r|^d \frac{\overline{r}^{(-\eta)}}{r^{(1-\eta)}} \right) + O\left(\frac{|\log r|^d}{r^{1-\eta}} \right) = O\left(|\log r|^d \frac{\overline{r}^{(-\eta)}}{r^{(1-\eta)}} \right).$$

Since $N_A - M_r l(A) = (N_U - M_r l(U)) + (N_{A \setminus U} - M_r l(A \setminus U))$, it remains to study the difference

$$N_{A \setminus U} - M_r l(A \setminus U) = \left(N_{A \setminus U} - M_{r/l(I_{h+1})} \frac{l(A \setminus U)}{l(I_{h+1})} \right) +$$

$$+\left(M_{r/l(I_{h+1})}\frac{l(A\setminus U)}{l(I_{h+1})}-M_r l(A\setminus U)\right).$$

The second part of the sum can be directly estimated by

$$\left|M_{r/l(I_{h+1})}\frac{l(A\setminus U)}{l(I_{h+1})}-M_r l(A\setminus U)\right|=O\left(|\log r|^d\,\frac{\overline{r}^{(1-\eta)}}{r^{(1-\eta)}}\right),$$

whereas the first part is bounded by

$$\left|N_{A\setminus U}-M_{r/l(I_{h+1})}\frac{l(A\setminus U)}{l(I_{h+1})}\right|\le \Delta_{n-\overline{n}}.$$

Summing up and taking the supremum over all sets $A=[0,y[$ we obtain the recurrence relation

$$\Delta_n\le\Delta_{n-\overline{n}}+O\left(|\log r|^d\,\frac{\overline{r}^{(-\eta)}}{r^{(1-\eta)}}\right).$$

We now set $\overline{n}=1$ and recall that $r=e^{-\Lambda n}$ (and also $\overline{r}=e^{-\Lambda\overline{n}}=e^{-\Lambda}$). Thus, we get

$$\Delta_n\le\Delta_{n-1}+O\left(n^d e^{\Lambda n(1-\eta)}\right).$$

We distinguish between three cases.

(1) $0<\eta<1$. In this case we get

$$\Delta_n=O\left(\sum_{k\le n}k^d e^{\Lambda k(1-\eta)}\right)=O\left(n^d e^{\Lambda n(1-\eta)}\right)$$

which implies $D_n=O\left((\log k(n))^d k(n)^{-\eta}\right)$.

(2) $\eta=1$. In this case we get $\Delta_n=O(n^{d+1})$ and consequently

$$D_n=O\left((\log k(n))^{d+1}k(n)^{-1}\right).$$

(3) $\eta>1$. Here we have

$$\Delta_n=O\left(\sum_{k\le n}k^d e^{-\Lambda k(\eta-1)}\right)=O\left(1\right)$$

which rewrites as $D_n=O\left(k(n)^{-1}\right)$.

This completes the proof of the upper bounds in (14).
In order to give a lower bound of the discrepancy it is sufficient to handle the case $0<\eta\le 1$. If $\eta>1$ we just use the trivial lower bound $D_n\ge 1/k(n)$

which meets the upper bound. For the remaining case $0 < \eta \leq 1$ we consider the interval $A = [0, p_1[$. We also recall that we can write M_r (for $r = r_n = e^{-\Lambda n}$) as

$$M_r = c'\, e^{\Lambda n} + \delta_n,$$

where δ_n has a representation of the form

$$\delta_n = n^d e^{\Lambda n(1-\eta)} \sum_{i=1}^{k} c_i \cos(2\pi\theta_i n + \alpha_i) + O\left(n^{d-1} e^{\Lambda n(1-\eta)}\right).$$

Recall that the leading part of δ_n is a non-zero sequence, that is,

$$\mu_n = \sum_{i=1}^{k} c_i \cos(2\pi\theta_i n + \alpha_i)$$

is a non-zero sequence.

Now we obtain (as above)

$$\begin{aligned} N_A - M_r l(A) &= M_{r/p_1} - M_r p_1 \\ &= \delta_{n-n_1} - p_1 \delta_n \\ &= n^d e^{\Lambda n(1-\eta)} p_1 \left(p_1^{-\eta} \mu_{n-n_1} - \mu_n\right) + O\left(n^{d-1} e^{\Lambda n(1-\eta)}\right). \end{aligned}$$

Suppose that $\mu'_n = p_1^{-\eta} \mu_{n-n_1} - \mu_n = 0$ for all n. Since $p_1^{-\eta} > 1$ and μ_n is a non-zero sequence it would follow that μ_n is an unbounded sequence. This is of course impossible. Hence, the sequence μ'_n is non-zero.

Consequently we can apply Lemma 6.1 to the function

$$\begin{aligned} f(n) &= p_1^{-\eta} \sum_{i=1}^{k} c_i \cos(2\pi\theta_i (n - n_1) + \alpha_i) - \sum_{i=1}^{k} c_i \cos(2\pi\theta_i n + \alpha_i) \\ &= \sum_{i=1}^{k} c'_i \cos(2\pi\theta_i n + \alpha'_i) \end{aligned}$$

(for proper constants c'_i and α'_i) and obtain that there exists a $\delta > 0$ and infinitely many n such that

$$|N_A - M_r l(A)| \geq \delta n^d e^{\Lambda n(1-\eta)}.$$

Consequently

$$D_n \geq \frac{1}{M_r} |N_A - M_r l(A)| \geq \delta' n^d e^{-\Lambda n \eta}$$

for some $\delta' > 0$. This completes the proof of the lower bounds in (15).

□

4. Discrepancy bounds in the irrational case

As mentioned above, the case when $\log\left(\frac{1}{p_1}\right), \ldots, \log\left(\frac{1}{p_m}\right)$ are irrationally related is much more difficult to handle since the error term in the asymptotic expansion for M_r is not explicit in general (see (4) in Theorem 2.2). Nevertheless, we can provide upper bounds in some cases of interest.

Suppose that $m = 2$ and set $p = p_1$, $q = p_2$ and $\gamma = (\log p)/(\log q)$.

Proposition 4.1. *Suppose that $\gamma = (\log p)/(\log q)$ is irrational. Then the number of intervals $k(n)$ of the partition $\rho^n\omega$ is asymptotically given by*

$$k(n) \sim \frac{pq(m-1)}{H} \exp\left(\sqrt{2n \log\left(\frac{1}{p}\right) \cdot \log\left(\frac{1}{q}\right)}\right).$$

P r o o f.
Let r be the parameter in Khodak's construction that corresponds to the $n-$th step, then $M_r = k(n)$. By (4) in Theorem 2.2 we have that

$$k(n) \sim \frac{m-1}{H} \cdot \frac{1}{r}. \tag{17}$$

Note that there is a one-to-one correspondence between the probability of each node x, that is $P(x) = p^k q^l$, and the non-negative integral lattice points (k, ℓ). So the number n of steps corresponding to the value r is approximatively given by the cardinality of the set

$$\begin{aligned}\{x \in \mathcal{T}(r) : P(x) \geq r\} &= \{(k,l) \in \mathbb{Z}^2_{\geq 0} : p^k q^l \geq r\} \\ &= \left\{(k,l) \in \mathbb{Z}^2_{\geq 0} : k \log\left(\frac{1}{p}\right) + l \log\left(\frac{1}{q}\right) \leq \log\left(\frac{1}{r}\right)\right\}.\end{aligned}$$

Hence, by using the fact that γ is irrational it follows that

$$\begin{aligned} n &= \#\{x \in \mathcal{T}(r) : P(x) \geq r\} \\ &= \frac{1}{2}\left(\frac{\log\left(\frac{1}{r}\right)}{\log\left(\frac{1}{p}\right)} + 1\right) \cdot \left(\frac{\log\left(\frac{1}{r}\right)}{\log\left(\frac{1}{q}\right)} + 1\right) + o\left(\log\left(\frac{1}{r}\right)\right),\end{aligned}$$

which implies

$$\frac{1}{r} \sim pq \exp\left(\sqrt{2n \log\left(\frac{1}{p}\right) \cdot \log\left(\frac{1}{q}\right)}\right).$$

The conclusion follows by using this relation in (17).

□

In Theorem 2.2 we have considered the case when γ is badly approximable and the case when p and q are algebraic. By using these results we can show the following theorem for the discrepancy in the irrational case.

Theorem 4.2. *Suppose that the lenghts of the intervals of a partition ρ of $[0,1]$ are p and $q = 1-p$ and let $\gamma = \frac{\log p}{\log q}$. If $\gamma \notin \mathbb{Q}$ and it is badly approximable, then the discrepancy of $(\rho^n\omega)$ is bounded by*

$$D_n = O\left(\left(\frac{\log\log(k(n))}{\log(k(n))}\right)^{\frac{1}{4}}\right), \quad \text{as } n \to \infty.$$

Furthermore, if p and q are algebraic and $\gamma \notin \mathbb{Q}$ then

$$D_n = O\left(\left(\frac{\log\log(k(n))}{\log(k(n))}\right)^{\kappa}\right), \quad \text{as } n \to \infty,$$

where $\kappa > 0$ is an effectively computable constant (see Theorem 2.2).

Note that the upper bounds for the discrepancy we obtained are worse than $k(n)^{-\beta}$ for any $\beta > 0$. Actually, it seems that we cannot do really better in the irrational case. This is due to the fact that $\liminf_{k\neq 0} \Re(s_k) = -1$ where s_k, $k \neq 0$, runs through all the zeros of the equation $p^{-s} + q^{-s} = 1$ different from $s_0 = -1$. Indeed, the continued fraction expansion of $\gamma = (\log p)/(\log q)$ could be used to obtain more explicit upper bounds. However, since they are all rather poor it is probably not worth working them out in detail. The case $m > 2$ is even more involved, compare with the discussion of [10].

Proof.
We use a procedure similar to that of the proof of Theorem 3.1, but now we use the asymptotic expansion

$$M_r = \frac{c''}{r} + O\left(\frac{1}{r}\left(\frac{\log\log\frac{1}{r}}{\log\frac{1}{r}}\right)^{\xi}\right)$$

where $c'' = (m-1)/H$. Moreover, we have that $\xi = \frac{1}{4}$ when γ is badly approximable, while $\xi = \kappa$ when p, q are algebraic (see (5) and (6) in Theorem 2.2).

First it follows that

$$N_U - M_r l(U) = \frac{c''}{r}(l(I_1) + \ldots + l(I_h)) + O\left(\frac{h\overline{r}}{r}\left(\frac{\log\log\frac{\overline{r}}{r}}{\log\frac{\overline{r}}{r}}\right)^{\xi}\right)$$
$$- \frac{c''}{r}(l(I_1) + \ldots + l(I_h)) + O\left(\frac{1}{r}\left(\frac{\log\log\frac{1}{r}}{\log\frac{1}{r}}\right)^{\xi}\right) =$$

$$= O\left(\frac{1}{r}\left(\frac{\log\log\frac{1}{r}}{\log\frac{1}{r}}\right)^{\xi}\right).$$

For the interval $A \setminus U$ we use the (trivial) bounds $N_{A\setminus U} \leq M_{r/l(I_{h+1})} = O(\overline{r}/r)$ and $l(I_{h+1}) = O(\overline{r})$ to end up with the upper bound

$$D_n = O\left(\left(\frac{\log\log\frac{\overline{r}}{r}}{\log\frac{\overline{r}}{r}}\right)^{\xi}\right) + O\left(\overline{r}\right).$$

Hence, by choosing

$$\overline{r} = \left(\frac{\log\log\frac{1}{r}}{\log\frac{1}{r}}\right)^{\xi}$$

we finally obtain

$$D_n = O\left(\left(\frac{\log\log\frac{1}{r}}{\log\frac{1}{r}}\right)^{\xi}\right).$$

This completes the proof of Theorem 4.2.

□

5. Applications

5.1. LS-sequences

We recall that LS-sequences of partitions are iterated ρ-refinements of the trivial partition $\omega = \{[0,1]\}$, where ρ consists of L subintervals of $[0,1]$ of length α and S subintervals of length α^2 with $0 < \alpha < 1$ is given by the equation $L\alpha + S\alpha^2 = 1$.

For instance, if $L = S = 1$ then $\alpha = \frac{\sqrt{5}-1}{2}$ and we obtain the so-called Kakutani-Fibonacci sequence. Here we have $p_1 = \alpha$ and $p_2 = 1 - \alpha = \alpha^2$ and consequently

$$\log\left(\frac{1}{\alpha}\right) = n_1\Lambda \quad \text{and} \quad \log\left(\frac{1}{\alpha^2}\right) = n_2\Lambda,$$

for $\Lambda = -\log\alpha$, $n_1 = 1$ and $n_2 = 2$. By following the lines of the proof of Theorem 2.2 and in particular (11) we can explicitly get the value of η. In fact, since the roots of the equation $1 - z - z^2 = 0$ are given by $z_1 = \frac{\sqrt{5}-1}{2} = \alpha = e^{-\Lambda}$

and $z_2 = \frac{-\sqrt{5}-1}{2}$ it follows that $d = 0$ and

$$\eta = 1 + \frac{\log|z_2|}{\Lambda} = 1 + \frac{\log\left|\frac{-\sqrt{5}-1}{2}\right|}{-\log\left(\frac{\sqrt{5}-1}{2}\right)} = 2.$$

According to Theorem 3.1, this shows that the discrepancy is of the order of $1/k(n)$ (and therefore it is optimal).

In the general case set $m = L + S$. Of course we are in the rational case since $p_i = \alpha$ or $p_i = \alpha^2$ for $i = 1, \ldots, m$. More precisely, according to Definition 2.1, we have $\Lambda = \log(1/\alpha)$ and $n_i \in \{1, 2\}$ corresponding to $p_i = \alpha^{n_i}$. The zeros of the equation

$$1 - Lz - Sz^2 = 0$$

are given by $z_1 = \frac{-L+\sqrt{L^2+4S}}{2S} = \alpha$ and $z_2 = \frac{-L-\sqrt{L^2+4S}}{2S}$. Hence,

$$\eta = 1 + \frac{\log\left|\frac{-L-\sqrt{L^2+4S}}{2S}\right|}{\Lambda} = 1 + \frac{\log\left(\frac{L+\sqrt{L^2+4S}}{2S}\right)}{\Lambda}.$$

Consequently, we have $\eta < 1$ if and only if $\frac{L+\sqrt{L^2+4S}}{2S} < 1$ or if $S > L + 1$. Similarly we have $\eta = 1$ if and only if $S = L + 1$ and $\eta > 1$ if and only if $S < L + 1$. This is in perfect accordance with the results of Carbone [4]. The discrepancy bounds are (of course) also of the same kind.

5.2. Sequences Related to Pisot Numbers

A Pisot number β is an algebraic integer (larger than 1) with the property that all its conjugates have modulus smaller than 1. A prominent example of Pisot numbers are the real roots of a polynomial of the form

$$z^k - a_1 z^{k-1} - a_2 z^{k-2} - \cdots - a_k = 0, \tag{18}$$

where a_j are positive integers with $a_1 \geq a_2 \geq \cdots \geq a_k$, see [3]. In this case the polynomial in (18) is also irreducible over the rationals.

Suppose now that ρ is a partition of $m = a_1 + a_2 + \cdots + a_k$ intervals, where a_j intervals have length α^j, $1 \leq j \leq k$, with $\alpha = 1/\beta$ and β is the Pisot number related to the polynomial (18). Note that we have

$$a_1\alpha + a_2\alpha^2 + \cdots + a_k\alpha^k = 1.$$

Since all conjugates of α have now modulus larger than 1, it follows that $\eta > 1$. This means that the order of magnitude of the discrepancy is optimal, namely $1/k(n)$. LS-sequences are a special instance for $k = 2$, $a_1 = L$ and $a_2 = S$ with $L \geq S$.

5.3. Multiple Zeros

In the Pisot case all complex zeros of the polynomial are simple, since the polynomial is irreducible over the rationals. However, this is not necessarily true in less restrictive cases than Pisot numbers. For example, consider one interval of length $\alpha = 1/5$, 16 intervals of lengths $\alpha^2 = 1/25$ and 20 intervals of lengths $\alpha^3 = 1/125$. Since $\alpha + 16\alpha^2 + 20\alpha^3 = 1$ we have a proper partition ρ of $[0, 1]$. Here the roots of the polynomial $z + 16z^2 + 20z^3 = 1$ are $z_1 = \alpha = 1/5$ and $z_2 = z_3 = -1/2$ (which is a double root). Hence, we obtain $\eta = 1 - (\log 2)/(\log 5) = 0.56932\ldots < 1$ and $d = 1$. Consequently, the discrepancy is bounded by

$$D_n = O((\log k(n))\, k(n)^{-\eta}),$$

and this upper bound is optimal.

5.4. The rational case on fractals

The same procedure of ρ-refinements could be also used to construct u.d. sequences of partitions on fractals generated by an iterated function system (IFS) satisfying the Open Set Condition (OSC). This class of fractals has been already considered in [12], where the authors introduced a general algorithm to produce u.d. sequences of partitions and of points on fractals generated by an IFS consisting of similarities which have the same ratio and which satisfy the OSC.

Now we can extend these results eliminating the restriction that the similarities have the same ratio. In fact, we will describe an analogue of the method of successive $\rho-$refinements which allows to produce sequences of partitions on this class of fractals. Actually, we will introduce a new correspondence between nodes of the tree associated to Khodak's algorithm and the subsets belonging to the partitions generated on the fractal.

Let $h \in \mathbb{N}$ and let $\varphi = \{\varphi_1, \ldots, \varphi_m\}$ be a system of m similarities defined on $\mathbb{R}^h$ having ratios $c_1, \ldots, c_m \in]0, 1[$ respectively and satisfying the OSC. Let F be the attractor of φ, that is, $F = \bigcup_{i=1}^{m} \varphi_i(F)$, let S be its Hausdorff dimension, and denote by $\mathcal{H}^S$ the Hausdorff measure of dimension S. Moreover, we will consider the normalized S-dimensional Hausdorff measure P on the fractal F, which is given by

$$P(A) = \frac{\mathcal{H}^S(A)}{\mathcal{H}^S(F)} \quad \text{for any Borel set } A \subset F$$

(recall that P is a regular probability measure).

Start with a tree having a root node of probability 1, which corresponds to the fractal F, and m leaves corresponding to the m images of F through the m

similarities, i.e. $\varphi_1(F), \dots, \varphi_m(F)$. The probability of each node is given by the probability P of the corresponding subset, that is, $p_i = P(\varphi_i(F)) = c_i^S$. At each iteration we select the leaves having the highest probability and grow m children out of each of them. On the fractal, this corresponds to apply successively the m similarities only to those subsets having the highest probability at this certain step. By iterating this procedure, we obtain a parsing tree associated to the sequence of partitions on the fractal F, which is the same tree as that generated by Khodak's algorithm.

Let us denote by (π_n) the sequence of partitions of F constructed by this technique, i.e.

$$\pi_n = \left\{\varphi_{j_{k(n)}}\varphi_{j_{(k(n)-1)}} \cdots\ \varphi_{j_1}(F) : j_1, \dots, j_{k(n)} \in \{1, \dots, m\}\right\}. \tag{19}$$

where $k(n)$ is the number of sets constructed at the $n-$th step .

Let us denote by $\mathscr{E}_n$ the collection of the $k(n)$ sets E_i^n belonging to the partition π_n and by $\mathscr{E}$ the union of the families $\mathscr{E}_n$, by varying n. The sets of the class $\mathscr{E}$ are called *elementary sets.*

In [12], it is proved that the class $\mathscr{E}$ is determining and consists of P-continuity sets. Now, similarly to (1) we can define the elementary discrepancy of the partition π_n as follows

$$D_n^{\mathscr{E}} = \sup_{E\in\mathscr{E}} \left| \frac{1}{k(n)} \sum_{i=1}^{k(n)} \chi_E\left(t_i^{(n)}\right) - P(E)\right|,$$

where we denote by $t_i^{(n)}$ a point chosen in $E_i^n \in \mathscr{E}_n$. By using a procedure similar to the one used in the proof of Theorem 3.1 we can prove the following theorem.

Theorem 5.1. *Let* (π_n) *be the sequence of partitions of* F *given in (19). Assume that* $\log\left(\frac{1}{p_1}\right), \dots, \log\left(\frac{1}{p_m}\right)$ *are rationally related. Then we have the following bounds for the elementary discrepancy of* (π_n)

$$D_n^{\mathscr{E}} = \begin{cases} O\left((\log k(n))^d k(n)^{-\eta}\right) & \text{if } 0 < \eta \le 1, \\ O\left(k(n)^{-1}\right) & \text{if } \eta > 1. \end{cases} \tag{20}$$

Furthermore, both upper bounds are best possible.

Proof.
Fix a step in the algorithm corresponding to a certain parameter r of the form $r = e^{-n\Lambda}$ for some integer $n \ge 0$. First we observe that the number $N_E^{(n)}$ of elementary sets in $\mathscr{E}_n$ which are contained in a fixed elementary set E is given by $M_{r/P(E)}$. This fact implies that

$$N_E^{(n)} = \frac{c'}{r} P(E) + O\left(|\log r|^d\, r^{-1+\eta} P(E)^{1-\eta}\right). \tag{21}$$

This proves (20) directly for $\eta \le 1$ and also shows that this bound is optimal.

If $\eta > 1$ then we argue recursively. The set E is either contained in the collection $\mathscr{E}_1 = \{\varphi_1(F), \ldots, \varphi_m(F)\}$, which means that we can use (21) for $P(E) \in \{p_1, \ldots, p_m\}$, or it is part of $E_j = \varphi_j(F)$ for some j. In the latter case we can rewrite $N_E^{(n)} - k(n)P(E)$ as

$$N_E^{(n)} - k(n)P(E) = \left(N_E^{(n)} - k(n-1)\frac{P(E)}{P(E_j)} \right) + \left(k(n-1)\frac{P(E)}{P(E_j)} - k(n)P(E) \right),$$

which leads to a recurrence of the form

$$\Delta_n^{\mathscr{E}} = \sup_{E \in \mathscr{E}} \left| N_E^{(n)} - k(n)P(E) \right| \le \Delta_{n-1}^{\mathscr{E}} + O\left(n^d e^{\Lambda n(1-\eta)} \right).$$

Here we have

$$\Delta_n^{\mathscr{E}} = O\left(\sum_{k \le n} k^d e^{-\Lambda k(\eta-1)} \right) = O(1)$$

and consequently $D_n^{\mathscr{E}} \le 2\Delta_n^{\mathscr{E}}/k(n) = O(1/k(n))$ (which is also optimal).

□

In particular it follows that the sequence of partitions (π_n) is u.d. with respect to P. Actually, this remains true in the irrational case, too. However, we can only derive effective upper bounds for the discrepancy in very specific cases.

There are few papers devoted to u.d. sequences on fractals and to estimates of their discrepancy (see [6, 7, 11]). The various types of discrepancy considered depend very much on the geometric features of the fractal. At the moment the choice of elementary discrepancy seems to be the only one which allows to overcome the problem of the peculiar complexity of each fractal and to give explicit results for all the fractals generated by IFS and satisfying the OSC. So a still open problem is finding a definition of discrepancy, different from the elementary one, which is general at least for the fractals of this class. An attempt in this direction has been proposed by Albrecher, Matoušek and Tichy in [1], but it concerns the average discrepancy.

6. Auxiliary Results

In this section we collect some auxiliary results that are used in the proof of Theorem 2.2 (see Section 2).

6.1. Trigonometric Sums

Lemma 6.1. *Let $f(n) = \sum_{i=1}^{k} c_i \cos(2\pi\theta_i n + \alpha_i)$, $c_i, \alpha_i, \theta_i \in \mathbb{R}$ be defined for non-negative integers n and suppose that f is not identically zero. Then there exists $\delta > 0$ such that $|f(n)| \geq \delta$ for infinitely many non-negative integers n.*

Proof.
We have to distinguish two cases.

Case 1: $\theta_1, \dots, \theta_k$ are rationally related.
There exist $\Lambda \in \mathbb{R} \setminus \{0\}$ and $k_i \in \mathbb{Z}$ such that $\theta_i = \Lambda k_i$. In this case, we can rewrite the function as follows

$$f(n) = \sum_{i=1}^{k} c_i \cos(2\pi\Lambda n k_i + \alpha_i) = \sum_{i=1}^{k} c_i \cos(2\pi\{\Lambda n\} k_i + \alpha_i),$$

where $\{x\}$ denotes the fractional part of x.

Hence, $f(n) = g(\{\Lambda n\})$ where $g(x) = \sum_{i=1}^{k} c_i \cos(2\pi k_i x + \alpha_i)$ is a periodic non-zero function of period 1.

Case 1.1: If $\Lambda \in \mathbb{Q}$, then $\Lambda = \frac{p}{q}$ for some coprime integers $p, q \in \mathbb{Z}$ and the sequence $f(n)$ attains periodically the set of values

$$g\left(\left\{\frac{pn}{q}\right\}\right), \quad n = 0, \dots, q-1.$$

Since they are not all equal to zero there exists $\delta > 0$ such that $|f(n)| = |g(\{\Lambda n\})| \geq \delta$ for infinitely many n. In particular we can use a linear subsequence $qn + r$ for which $|f(qn + r)| \geq \delta$.

Case 1.2: If $\Lambda \notin \mathbb{Q}$, then the sequence $(\{\Lambda n\})$ is u.d. and consequently dense in the interval $[0, 1]$. Hence, there (again) exists $\delta > 0$ such that $|f(n)| = |g(\{\Lambda n\})| \geq \delta$ for infinitely many n.

Case 2: $\theta_1, \dots, \theta_k$ are irrationally related.
Here we divide the θ_i in groups which are rationally related. Assume that we have s groups $\{\theta_i : i \in I_j\}$, $j = 1, \dots, s$, and in each group we write

$$\theta_i = \Lambda_j k_i, \quad i \in I_j$$

with $k_i \in \mathbb{Z}$ and some $\Lambda_j \in \mathbb{R} \setminus \{0\}$.

In this case, we distinguish between three different sub-cases.

Case 2.1: $1, \Lambda_1, \dots, \Lambda_s$ are linearly independent over $\mathbb{Q}$ and as a consequence $\Lambda_1, \dots, \Lambda_s \notin \mathbb{Q}$. We set $f_j(x) = \sum_{i \in I_j} c_i \cos(2\pi x k_i + \alpha_i)$ (where

we assume w.l.o.g. that f_j is non-zero) and $g(x_1, \ldots, x_s) = \sum_{j=1}^{s} f_j(x_j)$.
Then

$$f(n) = \sum_{j=1}^{s} f_j(\{n\Lambda_j\}) = g\left(\{n\Lambda_1\}, \ldots, \{n\Lambda_s\}\right).$$

By Kronecker's Theorem, the sequence $(\{n\Lambda_1\}, \ldots, \{n\Lambda_s\})$ is dense in the cube $[0,1]^s$. Thus, it follows (as above) that there exists $\delta > 0$ such that $|f(n)| \geq \delta$ for infinitely many n.
Note that by the same reasoning it follows that for every $\varepsilon > 0$ we have $|f(n)| \leq \varepsilon$ for infinitely many n. (Here we also use that fact that f has zero mean.) This observation will be used in Case 2.3.

Case 2.2: $1, \Lambda_1, .., \Lambda_s$ are linearly dependent over $\mathbb{Q}$ and $\Lambda_1, .., \Lambda_s \notin \mathbb{Q}$.
In this case there exist $q, p_1, \ldots, p_s \in \mathbb{Z}$ with $q = p_1\Lambda_1 + \ldots + p_s\Lambda_s$. Suppose (w.l.o.g.) that $p_1 > 0$ and consider the subsequence of integers $(p_1 n)$

$$\begin{aligned} f(p_1 n) &= \sum_{j=1}^{s} f_j(n\Lambda_j p_1) \\ &= f_1(n(q - \Lambda_2 p_2 - \cdots - \Lambda_s p_s)) + \sum_{j=2}^{s} f_j(n\Lambda_j p_1). \end{aligned}$$

By using the addition theorem for cosine and rewriting the sum accordingly we obtain a representation of the form

$$f(p_1 n) = \sum_{j=2}^{s} \tilde{f}_j(n\Lambda_j p_j),$$

where $\tilde{f}_j$ are certain trigonometric polynomials.
This means that we have eliminated Λ_1. In this way we can proceed further. When we have that $1, p_2\Lambda_2, \ldots, p_s\Lambda_s$ are linearly independent over $\mathbb{Q}$, then we argue as in Case 2.1. However, if $1, p_2\Lambda_2, \ldots, p_s\Lambda_s$ are linearly dependent over $\mathbb{Q}$ then we repeat the elimination procedure etc. Note that this elimination procedure terminates, since we assumed that in this case $\Lambda_1, \ldots, \Lambda_s \notin \mathbb{Q}$. Hence, we always end up in Case 2.1.

Case 2.3: $\Lambda_1, \ldots, \Lambda_s$ are not all irrational.
Here we represent $f(n) = h_1(n) + h_2(n)$, where

$$h_1(n) = \sum_{j \in \{j : \Lambda_j \in \mathbb{Q}\}} f_j(n) \quad \text{and} \quad h_2(n) = \sum_{j \in \{j : \Lambda_j \notin \mathbb{Q}\}} f_j(n).$$

If h_1 is non-zero then we can argue as in Case 1.1. All appearing θ_i are rational and consequently there exists a linear subsequence $(qn+r)$ such that $|h_1(qn+r)| \geq \frac{\delta}{2}$ for some $\delta > 0$. Next we reduce the sum $h_2(qn+r)$ to a sum of the form that is discussed in Case 2.1 (possibly we have to eliminate several terms as discussed in Case 2.2). It follows that there exist infinitely many n such that $|h_2(qn+r)| \leq \delta/2$. Hence we have $|f(n)| \geq \delta$ for infinitely many n.
If h_1 is zero for all non-negative integers we just have to consider h_2. But this case is precisely that of Case 2.2.

□

6.2. Zerofree Regions

The purpose of the next two lemmas is to discuss zero-free regions of the equation

$$1 - p^{-s} - q^{-s} = 0 \tag{22}$$

where p, q are positive numbers with $p + q = 1$. It is clear that $s = -1$ is a solution and that all solutions have to satisfy $\Re(s) \geq -1$. (Otherwise, we would have $|p^{-s}| + |q^{-s}| < 1$.) Furthermore, it is easy to verify that there are no solutions (other than $s = -1$) on the line $\Re(s) = -1$ if and only if the ratio $\gamma = (\log p)/(\log q)$ is irrational, compare also with [18]. It is also known that there exist $\sigma_0, \tau > 0$ such that in each box of the form

$$B_k = \{s \in \mathbb{C} : -1 \leq \Re(s) \leq \sigma_0,\ (2k-1)\tau \leq \Im(s) < (2k+1)\tau\}, \quad k \in \mathbb{Z} \setminus \{0\},$$

there is precisely one zero of (22), and there are no other zeros.

However, the position of the zeros in B_k is by no means clear. Nevertheless, with the help of the continued fraction expansion of γ it is possible to construct infinitely many zeros s of (22) with $\Re(s) < -1 + \varepsilon$ (for every $\varepsilon > 0$). Therefore it is natural to ask for zero-free regions of this equation. Actually one has to assume some Diophantine conditions on γ to get precise information.

Lemma 6.2. *If γ is badly approximable then for every solution $s \neq -1$ of (22) we have that*

$$\Re(s) > \frac{c}{(\Im(s))^2} - 1$$

for some positive constant c.

Proof.
We recall that an irrational number γ is badly approximable if its continued fraction expansion $\gamma = [a_0; a_1; \ldots]$ is bounded, that is, there exists a positive

constant D such that $\max_{j\ge 1}(a_j) \le D$. Equivalently we have the property that there exists a constant $d > 0$ such that

$$\left|\gamma - \frac{k}{l}\right| \ge \frac{d}{l^2} \tag{23}$$

for all non-zero integers k, l (see [14]).

In order to make the presentation of the proof more transparent we make a shift by 1 and consider the equation

$$p^{1-s} + q^{1-s} = 1 \tag{24}$$

and show that all non-zero solutions satisfy $\Re(s) > c/\Im(s)^2$ for some positive constant c depending on γ.

Suppose that $s = \sigma + i\tau$ is a zero of (24) with $\sigma > 0$. Furthermore, we assume that $\sigma \le \varepsilon$, where ε is a sufficiently small constant (that will be fixed in a moment). Since $p + q = 1$ and $|p^{1-s}| = p^{1-\sigma} = p(1 + O(\varepsilon)) > p$ and $|q^{1-s}| = q^{1-\sigma} = q(1+O(\varepsilon)) > q$ we can only have a solution if the arguments of p^{1-s} and q^{1-s} are small. (Actually they have to be of order $O(\sqrt{\varepsilon})$ if ε is chosen sufficiently small). W.l.o.g. we write

$$\arg(p^{1-s}) = \tau\log(1/p) = 2\pi k + \eta_1 \quad \text{and} \quad \arg(q^{1-s}) = \tau\log(1/q) = 2\pi l - \eta_2$$

for some integers k, l and certain positive numbers η_1, η_2 (which are of order $O(\sqrt{\varepsilon})$). More precisely, making a local expansion in (24) we obtain

$$\eta_2 = \frac{p}{q}\eta_1 + O(\eta_1^2) \quad \text{and} \quad \sigma = \frac{p}{2qH}\eta_1^2 + O(\eta_1^4).$$

Furthermore we have

$$\begin{aligned}\gamma &= \frac{\tau\log\frac{1}{p}}{\tau\log\frac{1}{q}}\\ &= \frac{2\pi k + \eta_1}{2\pi l - \eta_2}\\ &= \frac{k}{l} + \frac{1}{2\pi}\left(\frac{1}{l} + \frac{kp}{l^2 q}\right)\eta_1\left(1 + O\left(\frac{\eta_1}{l}\right)\right).\end{aligned}$$

This means that k/l is close to γ and by applying (23) it follows that

$$\eta_1 \ge \frac{d'}{|l|}$$

for some constant $d' > 0$. Consequently we obtain $\sigma \ge d''/l^2$ (for some constant $d'' > 0$) which translates directly to $\sigma > c/\tau^2$ for some positive constant c.

□

Next we consider the case of algebraic number p and q with the property that $\log(p)/\log(q)$ is irrational.

Lemma 6.3. *If $p, q \in]0,1[$ are positive algebraic numbers with $p+q=1$ and the property that $\log(p)/\log(q)$ is irrational, then for every solution $s \neq -1$ of (22) we have*

$$\Re(s) > \frac{D}{(\Im(s))^{2C}} - 1$$

with effectively computable positive constants C, D.

The classical theorem of Gelfond-Schneider says that if $\gamma = \log(p)/\log(q)$ is irrational for algebraic numbers p and q then γ is transcendental. Baker's Theorem [2] gives also effective bounds for Diophantine approximation of γ that will be used in the subsequent proof of Lemma 6.3. (Recall that the height of an algebraic number is the maximum of the absolute values of the relatively prime integer coefficients in its minimal defining polynomial, while its degree is the degree of this polynomial.)

Theorem 6.4 (Baker's Theorem [2])**.** *Let $\gamma_1, \ldots, \gamma_n$ be non-zero algebraic numbers with degrees at most d and heights at most A. Further, $\beta_0, \beta_1, \ldots, \beta_n$ are algebraic numbers with degree at most d and heights at most B (≥ 2). Then for*

$$\Lambda = \beta_0 + \beta_1 \log \gamma_1 + \ldots + \beta_n \log \gamma_n$$

we have either $\Lambda = 0$ or $|\Lambda| \geq B^{-C}$, where C is an effectively computable number depending only on n, d, and A.

Proof. (Lemma 6.3)
We apply Theorem 6.4 for the algebraic number $\gamma_1 = p$ and $\gamma_2 = q$ and the integers $\beta_0 = 0$, $\beta_1 = l$, and $\beta_2 = -k$. Then $B = \max\{|k|, |l|\}$. W.l.o.g. we may assume that $p > q$ which ensures that we only have to consider cases with $|k| \leq |l|$. Thus

$$|l \log p - k \log q| > B^{-C}$$

and consequently

$$\left| \frac{\log p}{\log q} - \frac{k}{l} \right| > \left(\frac{1}{\log q} \right) \frac{B^{-C}}{l} > \left(\frac{1}{\log q} \right) \frac{1}{l^{1+C}}, \tag{25}$$

where C is effectively computable.

By using (25) instead of (23) in the proof of Lemma 6.2 we complete the proof of Lemma 6.3.

□

6.3. Differentiating Asymptotic Expansions

Lemma 6.5. *Suppose that $f(v)$ is a non-negative increasing function for $v \geq 0$. Assume that*

$$F(v) = \int_0^v f(w)dw$$

has the asymptotic expansion

$$F(v) = \frac{v^{\lambda+1}}{(\lambda+1)} (1 + O(g(v))) \quad \text{as} \quad v \to \infty,$$

where $\lambda > -1$ and $g(v)$ is a decreasing function that tends to zero as $v \to \infty$. Then

$$f(v) = v^{\lambda} \left(1 + O\left(g(v)^{\frac{1}{2}}\right)\right) \quad \text{as} \quad v \to \infty.$$

P r o o f.
By the assumption we have that there exist $v_0, c > 0$ such that for all $v \geq v_0$ we have

$$\left| F(v) - \frac{v^{\lambda+1}}{(\lambda+1)} \right| \leq c|g(v)| \frac{v^{\lambda+1}}{(\lambda+1)}.$$

Now, set $h = |g(v)|^{\frac{1}{2}} v$. By monotonicity, for $v \geq v_0$ we get

$$\frac{F(v+h) - F(v)}{h} = \frac{1}{h} \int_v^{v+h} f(w)dw \geq \frac{1}{h} \int_v^{v+h} f(v)dw = f(v)$$

and so

$$\begin{aligned}
f(v) &\leq \frac{F(v+h) - F(v)}{h} \\
&\leq \frac{1}{h}\left(\frac{(v+h)^{\lambda+1}}{\lambda+1} - \frac{v^{\lambda+1}}{\lambda+1}\right) + \frac{1}{h}\left(c|g(v+h)| \frac{(v+h)^{\lambda+1}}{(\lambda+1)} + c|g(v)| \frac{v^{\lambda+1}}{(\lambda+1)}\right) \\
&\leq \frac{1}{h(\lambda+1)} \left(v^{\lambda+1} + (\lambda+1)v^{\lambda}h + O(v^{\lambda-1}h^2) - v^{\lambda+1}\right) + O\left(|g(v)| \frac{v^{\lambda+1}}{h}\right) \\
&= v^{\lambda} + O\big(v^{\lambda-1}h\big) + O\left(|g(v)| \frac{v^{\lambda+1}}{h}\right) \\
&= v^{\lambda} + O\left(v^{\lambda-1}|g(v)|^{\frac{1}{2}} v\right) + O\left(|g(v)| \frac{v^{\lambda+1}}{|g(v)|^{\frac{1}{2}} v}\right) \\
&= v^{\lambda} + O\left(v^{\lambda}|g(v)|^{\frac{1}{2}}\right).
\end{aligned}$$

Similarly we derive a corresponding lower bound from the inequality

$$f(v) \geq (F(v) - F(v-h))/h.$$

□

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