ON SEQUENCES INVOLVING PRIMES

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ABSTRACT. In this paper we give the set of distribution functions of $f(p_n)$ mod1 for a special class of functions $f(x)$. Also, we generalize a result on distribution functions shown by O. Strauch and O. Blažeková to a multi-dimensional case. Applying it, we prove that every uniform distributed sequence x_n mod1 and $\log p_n$ mod1 is statistically independent. By combining this and a result of G. Rauzy we derive that the sequences $p_n\theta + \log p_n$ and $p_n\theta + p_n/n$ are uniformly distributed mod1 with irrational θ.

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1. Introduction

For a real number x, let $[x]$ be the integral part of x, let $x \bmod 1 = \{x\} = x - [x]$ be the fractional part of x, and let $\|x\| = \min(\{x\}, 1 - \{x\})$. For $\mathbf{x} = (x_1, \ldots, x_s) \in \mathbb{R}^s$, let $\mathbf{x} \bmod 1 = (x_1, \ldots, x_s) \bmod 1 = (\{x_1\}, \ldots, \{x_s\})$. Let c_J be the characteristic function of the interval $J \subset [0, 1)^s$.

A function $g : [0, 1]^s \to [0, 1]$ is called a distribution function (abbreviated d.f.) if $g(\mathbf{x}) = 0$ for any $\mathbf{x}$ with a vanishing coordinate, $g(1, \ldots, 1) = 1$, and $g(\mathbf{x})$ is non-decreasing, i.e.

$$\Delta_{z_s-y_s}^{(s)} \cdots \Delta_{z_1-y_1}^{(1)} g(y_1, \ldots, y_s) = \Delta_{z_s-y_s}^{(s)} (\cdots (\Delta_{z_1-y_1}^{(1)} g(y_1, \ldots, y_s))) \geq 0$$

for any $0 \leq y_i < z_i \leq 1$ ($i = 1, 2, \ldots$), where $\Delta_{h_i}^{(i)} g(x_1, \ldots, y_i, \ldots, x_s) = g(x_1, \ldots, y_i + h_i, \ldots x_s) - g(x_1, \ldots, y_i, \ldots x_s)$ for $0 \leq y_i \leq y_i + h_i \leq 1$ (see [14, p.1-61]). It is easy to show that $\Delta_{z_s-y_s}^{(s)} \cdots \Delta_{z_1-y_1}^{(1)} g(y_1, \ldots, y_s) \leq 1$ for any $0 \leq y_i < z_i \leq 1$ ($i = 1, 2, \ldots$), and $g(x_1, \ldots, x_s)$ is non-decreasing with respect to x_i for $i = 1, \ldots, s$. The d.f. $g(1, \ldots, 1, x_{i_1}, 1, \ldots 1, x_{i_2}, 1, \ldots 1, x_{i_m}, 1, \ldots, 1)$ is called an m-dimensional face distribution function of g in variables $(x_{i_1}, x_{i_2}, \ldots, x_{i_m}) \in (0, 1)^m$, $m = 1, \ldots, s - 1$.

We identify two d.f.s $g(\mathbf{x})$ and $\tilde{g}(\mathbf{x})$ if

(i) $g(\mathbf{x}) = \tilde{g}(\mathbf{x})$ at every common point $\mathbf{x} \in (0,1)^s$ of continuity, and

(ii) $g(1,\ldots,1,x_{i_1},1,\ldots 1,x_{i_m},1,\ldots,1) = \tilde{g}(1,\ldots,1,x_{i_1},1,\ldots 1,x_{i_m},1,\ldots,1)$ for every common point $(x_{i_1},\ldots,x_{i_m}) \in (0,1)^m$ of continuity in every m-dimensional face distribution function of g and $\tilde{g}$, $m = 1,\ldots,s-1$ (see [14, p.1-61]).

For two s-dimensional sequences of real numbers $\mathbf{a}_n = (a_{n,1},\ldots,a_{n,s})$, $n = 1,2,\ldots$ and $\tilde{\mathbf{a}}_n = (\tilde{a}_{n,1},\ldots,\tilde{a}_{n,s})$, $n = 1,2,\ldots$, we define the empirical distribution by

$$F_N(\mathbf{x}) = \frac{1}{N}\sum_{n=1}^{N} c_{[0,x_1)\times\cdots\times[0,x_s)}(\{a_{n,1}\},\cdots,\{a_{n,s}\}),$$

$$\tilde{F}_N(\mathbf{x}) = \frac{1}{N}\sum_{n=1}^{N} c_{[0,x_1)\times\cdots\times[0,x_s)}(\{\tilde{a}_{n,1}\},\cdots,\{\tilde{a}_{n,s}\}),$$

for all $\mathbf{x} = (x_1,\ldots,x_s) \in [0,1]^s$.

If there exists an increasing sequence of positive integers N_k, $k = 1,2,\ldots$, such that

$$\lim_{k\to\infty} F_{N_k}(\mathbf{x}) = g(\mathbf{x})$$

at every continuity point $\mathbf{x} \in (0,1)^s$ of $g(\mathbf{x})$, and

$$\lim_{k\to\infty} F_{N_k}(1,\ldots,1,x_{i_1},1,\ldots 1,x_{i_m},1,\ldots,1)$$
$$= g(1,\ldots,1,x_{i_1},1,\ldots 1,x_{i_m},1,\ldots,1)$$

at every continuity point $(x_{i_1},\ldots,x_{i_m}) \in [0,1]^m$ of $g(1,\ldots,1,x_{i_1},1,\ldots 1,x_{i_m},1,\ldots,1)$ for $m = 1,\ldots,s-1$, then the function $g(\mathbf{x})$ is called the distribution function (abbreviated d.f.) of the sequence $\mathbf{a}_n \bmod 1$, $n = 1,2,\ldots$. We denote as $G(\mathbf{a}_n \bmod 1)$ the set of all d.f.s of the sequence $\mathbf{a}_n \bmod 1$, $n = 1,2,\ldots$ (see [14, p.1-61]). The sequence $\mathbf{a}_n$ is said to be uniformly distributed mod1 if

$$\lim_{N\to\infty} F_N(x_1,\ldots,x_s) = x_1\cdots x_s$$

for all $(x_1,\ldots,x_s) \in [0,1]^s$.

In [12] O. Strauch and O. Blažeková showed the following theorem.

THEOREM A ([12], Theorem 1). *Let x_n and y_n be two real sequences. Assume that all d.f.s in $G(x_n \bmod 1)$ are continuous at 0 and 1. Then the zero limit of fractional parts*

$$\lim_{n\to\infty}(x_n - y_n)\bmod 1 = 0$$

implies $G(x_n \bmod 1) = G(y_n \bmod 1)$.

R. Giuliano Antonini and O. Strauch [3] extended this theorem to the case of weighted distribution functions. In [3], the referee notes that if x_n, $y_n \in [0, 1)$, $n = 1, 2, \ldots$, then the assumption that all d.f.s in $G(x_n \bmod 1)$ are continuous at 0 and 1 can be omitted.

Let p_n denote the nth prime. Applying Theorem A, O. Strauch and O. Blažeková [12] proved that

$$G(p_n/n \bmod 1) = G(\log(n \log n) \bmod 1).$$

Furthermore, by two methods they proved

$$G(\log(n \log n) \bmod 1) = G(\log n \bmod 1). \tag{1.1}$$

It is well-known that

$$G(\log n \bmod 1) = \left\{ g_u(x) = \frac{e^x - 1}{e - 1} e^{-u} + (e^{\min(x,u)} - 1)e^{-u} : u \in [0, 1] \right\} \tag{1.2}$$

(see [6, pp.57-58], [14, 2.12]).

This paper is divided into three parts. The first part (Section 2) gives the set of d.f.s of $f(p_n) \bmod 1$ for a special class of functions $f(x)$ that is described in O. Strauch and O. Blažeková [12, Theorem 5].

In the second part (Section 3) we give a generalization of Theorem A to a multidimensional case. Applying it, we prove that every uniform distributed sequence $x_n \bmod 1$ and $\log p_n \bmod 1$ is statistically independent. By combining this and a result of G. Rauzy [8] (see Lemma B), we derive that the sequences $p_n\theta + \log p_n$ and $p_n\theta + p_n/n$ are uniformly distributed mod1.

In the third part (Section 4) we give the proof of Theorem 2.

2. Distribution of $f(p_n) \bmod 1$

Hadamard and de la Vallée Poussin proved the prime number theorem

$$\pi(x) = \operatorname{Li}(x) + O(xe^{-A\sqrt{\log x}}),$$

for some positive constant A (cf. P. Ribenboim [9, p.226]). By integration by parts

$$\operatorname{Li}(x) = \frac{x}{\log x - 1} - \frac{x}{(\log x)^2 (\log x - 1)} - \frac{2}{\log 2} - \frac{2}{(\log 2)^2} + 2 \int_2^x \frac{1}{(\log t)^3} dt.$$

Since

$$\int_2^x \frac{1}{(\log t)^3} dt = \int_2^{x^{1/2}} \frac{1}{(\log t)^3} dt + \int_{x^{1/2}}^x \frac{1}{(\log t)^3} dt \ll \frac{x}{(\log x)^3},$$

we have

$$\pi(x) = \frac{x}{\log x - 1} + O\left(\frac{x}{(\log x)^3}\right).$$

Hence

$$\frac{p_n}{n} = \log p_n - 1 + O\left(\frac{1}{\log p_n}\right). \tag{2.1}$$

On the other hand, it is known that

$$\frac{p_n}{n} = \log n + (\log\log n - 1) + o\left(\frac{\log\log n}{\log n}\right) \tag{2.2}$$

(see [9], [12]). From (2.1) and (2.2) it follows that

$$\log p_n = \log(n \log n) + o\left(\frac{\log\log n}{\log n}\right) + O\left(\frac{1}{\log p_n}\right).$$

Therefore we have

$$\lim_{n\to\infty} \|\log p_n - \log(n\log n)\| = 0. \tag{2.3}$$

Applying Theorem A with the continuity of $g(x) \in G(\log(n\log n) \bmod 1)$ at $x = 0$ and $x = 1$, we have

$$G(\log p_n \bmod 1) = G(\log(n\log n) \bmod 1).$$

Since $G(\log(n\log n) \bmod 1) = G(\log n \bmod 1)$ by (1.1),

$$G(\log p_n \bmod 1) = G(\log n \bmod 1).$$

The d.f.s of the sequence $\log p_n \bmod 1$ was studied by A. Wintner [16]. For a function $f(x)$ that satisfies conditions of the following theorem, O. Strauch and O. Blažeková showed some extension of the Koksma theorem [7] about the sequence $f(n)$ (see [12, Theorem 5]). They applied it to $f(x) = \log x$ and $f(x) = \log(x \log^{(i)} x)$, $i = 1, 2, \ldots$, where $\log^{(i)} x$ is the ith iterated logarithm $\log \cdots \log x$. We prove the following result for the sequence $f(p_n) \bmod 1$ by using the methods of [16] and [12, Theorem 5].

THEOREM 1. *Let the real-valued function $f(x)$ be strictly increasing for $x \geq 1$ and let $f^{-1}(x)$ be the inverse function of $f(x)$. Suppose that*
(i) $\lim_{k\to\infty} f^{-1}(k+1) - f^{-1}(k) = \infty$,
(ii) $\lim_{k\to\infty} \frac{f^{-1}(k+w_k)}{f^{-1}(k)} = \psi(u)$ *for every sequence* $w_k \in [0, 1]$

for which $\lim_{k\to\infty} w_k = u$, *where this limit defines the function* $\psi(u)$ *on* $[0,1]$,
(iii) $\psi(1) > 1$. *Then*

$$G(f(p_n) \bmod 1)$$
$$= \left\{ \tilde{g}_u(x) = \frac{\min(\psi(x), \psi(u)) - 1}{\psi(u)} + \frac{1}{\psi(u)} \cdot \frac{\psi(x) - 1}{\psi(1) - 1} : u \in [0,1] \right\}.$$

P r o o f. First we show that the sequence $\{f(p_n)\}$ is dense in $[0,1]$. For $0 \le a < b \le 1$, $a < \{f(p_n)\} \le b$ if and only if there exists $k \in \mathbb{Z}$ such that $f^{-1}(k+a) < p_n \le f^{-1}(k+b)$. For $\xi > 0$, let $\pi(\xi)$ be the number of primes p with $p \le \xi$. By the prime number theorem, we have

$$\pi(f^{-1}(k+b)) - \pi(f^{-1}(k+a))$$
$$= \frac{f^{-1}(k+b)}{\log f^{-1}(k+b)} - \frac{f^{-1}(k+a)}{\log f^{-1}(k+a)} + O\left(\frac{f^{-1}(k+b)}{\log^2 f^{-1}(k+b)}\right)$$
$$= \frac{f^{-1}(k+b)}{\log f^{-1}(k+b)} \left(1 - \frac{f^{-1}(k+a)}{f^{-1}(k+b)} \frac{\log f^{-1}(k+b)}{\log f^{-1}(k+a)}\right) + O\left(\frac{f^{-1}(k+b)}{\log^2 f^{-1}(k+b)}\right).$$

From (i) it follows that $\lim_{x\to\infty} f^{-1}(x) = \infty$, thus $\lim_{k\to\infty} \frac{f^{-1}(k+b)}{\log f^{-1}(k+b)} = \infty$. (ii) implies $\lim_{k\to\infty} \frac{f^{-1}(k+a)}{f^{-1}(k+b)} = \frac{\psi(a)}{\psi(b)} < 1$ and

$$\lim_{k\to\infty} \frac{\log f^{-1}(k+b)}{\log f^{-1}(k+a)} = \lim_{k\to\infty} \left(\frac{1}{\log f^{-1}(k+a)} \log \frac{f^{-1}(k+b)}{f^{-1}(k+a)}\right) + 1 = 1.$$

Therefore, $\pi(f^{-1}(k+b)) - \pi(f^{-1}(k+a)) \to \infty$ as $k \to \infty$. Hence, there exist k and n such that $f^{-1}(k+a) < p_n \le f^{-1}(k+b)$. So it was shown that the sequence $\{f(p_n)\}$ is dense in $[0,1]$.

Since the sequence $\{f(p_n)\}$ is dense in $[0,1]$, for $0 \le u \le 1$ there exists a sequence of positive integers n_j such that $u_j = \{f(p_{n_j})\} \to u$ as $j \to \infty$.
For $\xi > 0$, put $N(\xi) = \#\{n \ge 1 | f(p_n) < \xi\}$. For $0 \le x \le 1$ and $m \ge 1$, put $\phi(m; x) = \frac{1}{m} \#\{k \le m | \{f(p_k)\} < x\}$. Then we have

$$N(\xi)\phi(N(\xi); x) = \sum_{k=0}^{[\xi]-1} (N(k+x) - N(k)) + N(\min(\xi, [\xi]+x)) - N([\xi]).$$

The prime number theorem implies

$$N(\xi) = \#\{n | p_n < f^{-1}(\xi)\} = \pi(f^{-1}(\xi)) + c(\xi) \sim \frac{f^{-1}(\xi)}{\log f^{-1}(\xi)}, \qquad (2.4)$$

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where $c(\xi) = 0$ or 1.

Hence, we have

$$\phi(n_j - 1; x) = \phi(N(f(p_{n_j})); x)$$

$$= \frac{1}{N(f(p_{n_j}))} \sum_{k=0}^{h_j-1} (N(k+x) - N(k)) + \frac{N(h_j + \min(u_j, x))}{N(f(p_{n_j}))} - \frac{N(h_j)}{N(f(p_{n_j}))}$$

$$= \frac{\sum_{k=0}^{h_j-1}(N(k+x) - N(k))}{\sum_{k=0}^{h_j-1}(N(k+1) - N(k))} \cdot \frac{N(h_j) - N(0)}{N(h_j + u_j)}$$

$$+ \frac{N(h_j + \min(u_j, x))}{N(h_j + u_j)} - \frac{N(h_j)}{N(h_j + u_j)},$$

where $h_j = [f(p_{n_j})]$. Let $0 \le x \le 1$ be fixed. Since $N(n)$ is strictly increasing and tends to infinity, Cauchy-Stolz lemma implies

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n}(N(k+x) - N(k))}{\sum_{k=0}^{n}(N(k+1) - N(k))} = \lim_{k\to\infty} \frac{N(k+x) - N(k)}{N(k+1) - N(k)}$$

$$= \lim_{k\to\infty} \frac{N(k+x)/N(k) - 1}{N(k+1)/N(k) - 1}.$$

From (2.4), we obtain

$$\frac{N(k+x)}{N(k)} \sim \frac{\log f^{-1}(k)}{\log f^{-1}(k+x)} \cdot \frac{f^{-1}(k+x)}{f^{-1}(k)}.$$

Furthermore, (i) and (ii) yield

$$\lim_{k\to\infty} \frac{\log f^{-1}(k)}{\log f^{-1}(k+x)} = \lim_{k\to\infty} \left(\frac{1}{\log f^{-1}(k+x)} \log \frac{f^{-1}(k)}{f^{-1}(k+x)} \right) + 1 = 1.$$

Thus, we get

$$\lim_{k\to\infty} \frac{N(k+x)}{N(k)} = \psi(x),$$

and so,

$$\lim_{j\to\infty} \frac{\sum_{k=0}^{h_j-1}(N(k+x) - N(k))}{\sum_{k=0}^{h_j-1}(N(k+1) - N(k))} = \lim_{n\to\infty} \frac{\sum_{k=0}^{n}(N(k+x) - N(k))}{\sum_{k=0}^{n}(N(k+1) - N(k))} = \frac{\psi(x) - 1}{\psi(1) - 1}.$$

Using (ii) with $w_{h_j} = u_j$, $w_k = u$ otherwise, we obtain

$$\lim_{j\to\infty} \frac{f^{-1}(h_j + u_j)}{f^{-1}(h_j)} = \psi(u), \quad \lim_{j\to\infty} \frac{\log f^{-1}(h_j)}{\log f^{-1}(h_j + u_j)} = 1.$$

and hence

$$\lim_{j\to\infty} \frac{N(h_j)}{N(h_j + u_j)} = \lim_{j\to\infty} \frac{\log f^{-1}(h_j + u_j)}{\log f^{-1}(h_j)} \cdot \frac{f^{-1}(h_j)}{f^{-1}(h_j + u_j)} = \frac{1}{\psi(u)}.$$

Let $u < x < 1$ be fixed. Since $\min(u_j, x) = u_j$ for sufficiently large j, we obtain

$$\lim_{j\to\infty} \frac{N(h_j + \min(u_j, x))}{N(h_j + u_j)} = 1.$$

Let $0 \le x < u$ be fixed. For sufficiently large j, we have

$$\frac{N(h_j + \min(u_j, x))}{N(h_j + u_j)} = \frac{N(h_j + x)}{N(h_j + u_j)} \sim \frac{f^{-1}(h_j + x)/f^{-1}(h_j)}{f^{-1}(h_j + u_j)/f^{-1}(h_j)} \cdot \frac{\log f^{-1}(h_j + u_j)}{\log f^{-1}(h_j + x)}.$$

From (i) and (ii) it follows that

$$\lim_{j\to\infty} \frac{\log f^{-1}(h_j + u_j)}{\log f^{-1}(h_j + x)} = \lim_{j\to\infty} \left(\frac{1}{\log f^{-1}(h_j + x)} \log \frac{f^{-1}(h_j + u_j)}{f^{-1}(h_j + x)} \right) + 1 = 1.$$

Thus, we get

$$\lim_{j\to\infty} \frac{N(h_j + \min(u_j, x))}{N(h_j + u_j)} = \frac{\psi(x)}{\psi(u)}.$$

Therefore,

$$\lim_{j\to\infty} \frac{N(h_j + \min(u_j, x))}{N(h_j + u_j)} = \frac{\min(\psi(x), \psi(u))}{\psi(u)}.$$

Consequently,

$$\tilde{g}_u(x) = \lim_{j\to\infty} \phi(n_j - 1; x) = \frac{\min(\psi(x), \psi(u)) - 1}{\psi(u)} + \frac{1}{\psi(u)} \cdot \frac{\psi(x) - 1}{\psi(1) - 1}.$$

Conversely, if $g(x) \in G(f(p_n) \bmod 1)$, then there exists a sequence m_i such that $\lim_{i\to\infty} \phi(m_i; x) = g(x)$. The sequence $m_i + 1$ contains a subsequence n_j such that $\lim_{j\to\infty}\{f(p_{n_j})\} = u$ for some $0 \le u \le 1$. Therefore, we have $g(x) = \tilde{g}_u(x)$. □

As O. Strauch and O. Blažeková [12] applied Theorem 5 of [12] to $f(x) = \log x$ and $f(x) = \log(x \log^{(i)} x)$, $i = 1, 2, \ldots$, we can apply Theorem 1 to the same functions. In consequence, it is shown that the sequence $\log p_n$ and the sequences $\log(p_n \log^{(i)} p_n)$, $i = 1, 2, \ldots$ have the same distribution as one of the sequence $\log n$. This solves the Open problem 1.3 in [13]. S. Akiyama [1] proved that the sequence $\sum_{i=0}^{\ell-1} c_i \log p_{n+i}$ is not almost uniformly distributed modulo 1, where

c_i, $i = 0, 1, 2, \ldots$, are real numbers with $\sum_{i=0}^{\ell-1} c_i \neq 0$, i.e. $x \notin G(\sum_{i=0}^{\ell-1} c_i \log p_{n+i})$ (see also [14, 2.19.8]). Akiyama [1] used the following estimation

$$\sum_{i=0}^{\ell-1} c_i \log p_{n+i} - \left(\sum_{i=0}^{\ell-1} c_i \right) \log p_n = o(1).$$

Therefore by Theorem A (or Theorem 2),

$$G\left(\sum_{i=0}^{\ell-1} c_i \log p_{n+i} \bmod 1 \right) = G\left(\left(\sum_{i=0}^{\ell-1} c_i \right) \log p_n \bmod 1 \right).$$

On the other hand, by Theorem 1

$$G\left(c \log p_n \bmod 1 \right) = \left\{ \frac{e^{x/c} - 1}{e^{1/c} - 1} e^{-u/c} + (e^{\min(x/c, u/c)} - 1)e^{-u/c} : u \in [0, 1] \right\},$$

where $c = \sum_{i=0}^{\ell-1} c_i$. Hence, the sequence $\sum_{i=0}^{\ell-1} c_i \log p_{n+i}$ has the same distribution.

3. Distribution of the another type of sequence

We extend Theorem A to a multidimensional case as follows.

THEOREM 2. *Let $\mathbf{a}_n = (a_{n,1}, \ldots, a_{n,s})$, $\tilde{\mathbf{a}}_n = (\tilde{a}_{n,1}, \ldots, \tilde{a}_{n,s})$, $n = 1, 2, \ldots$ be s-dimensional sequences of real numbers. Suppose that for every d.f. $g(x_1, \ldots, x_s) \in G(\mathbf{a}_n \bmod 1)$, the face d.f. $g(1, \ldots, 1, x_i, 1, \ldots)$ is continuous at $x_i = 0, 1$ for $i = 1, \ldots, s$. If $\|a_{n,i} - \tilde{a}_{n,i}\| \to 0$ $(n \to \infty)$ for $i = 1, \ldots, s$, then $G(\mathbf{a}_n \bmod 1) = G(\tilde{\mathbf{a}}_n \bmod 1)$.*

The proof is described in Section 4. We recall definitions and results on statistical independence.

DEFINITION 1 (see [8, p.91]). *Let x_n and y_n be two infinite real sequences. Two sequences $x_n \bmod 1$ and $y_n \bmod 1$ are called statistically independent if*

$$\lim_{N \to \infty} \left(\frac{1}{N} \sum_{n=1}^{N} f(\{x_n\}) g(\{y_n\}) - \left(\sum_{n=1}^{N} f(\{x_n\}) \right) \left(\sum_{n=1}^{N} g(\{y_n\}) \right) \right) = 0$$

for every continuous real-valued functions f, g defined on $[0, 1]$.

LEMMA A (see [8, pp.97-98]). *Let x_n and y_n be two infinite real sequences. Two sequences x_n mod 1 and y_n mod 1 are statistically independent if and only if*

$$\lim_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N}e^{2\pi i(hx_n+ky_n)}-\left(\frac{1}{N}\sum_{n=1}^{N}e^{2\pi ihx_n}\right)\left(\frac{1}{N}\sum_{n=1}^{N}e^{2\pi iky_n}\right)\right)=0$$

for every integers $h\neq 0$, $k\neq 0$.

LEMMA 1. *Let x_n, $n=1,2,3\ldots$ be a real sequence. If x_n is uniformly distributed mod 1, then x_n mod 1 and $\log(n\log n)$ mod 1 are statistically independent.*

P r o o f. Suppose that x_n is uniformly distributed mod 1. By Weyl criterion we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}e^{2\pi ihx_n}=0. \tag{3.1}$$

Therefore by applying Lemma A it is sufficient to show that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}e^{2\pi i(hx_n+k\log(n\log n))}=0. \tag{3.2}$$

By Abel summation formula we obtain

$$\sum_{n=1}^{N}e^{2\pi i(hx_n+k\log(n\log n))}$$

$$=\sum_{n=1}^{N-1}\left(e^{2\pi ik\log(n\log n)}-e^{2\pi ik\log((n+1)\log(n+1))}\right)\sum_{m=1}^{n}e^{2\pi ihx_m}$$

$$+e^{2\pi ik\log(N\log N)}\sum_{n=1}^{N}e^{2\pi ihx_n}.$$

From

$$\left|e^{2\pi ik\log(n\log n)}-e^{2\pi ik\log((n+1)\log(n+1))}\right|\leq 2\pi|k|\frac{\log n+1}{n\log n}$$

and (3.1) we obtain (3.2). $\qquad\square$

THEOREM B (see [8, p.92], [4, Theorem 2]). *Let x_n and y_n be two infinite real sequences. Two sequences x_n mod1 and y_n mod1 are statistically independent if and only if for every $g\in G((x_n,y_n)\bmod 1)$*

$$g(x,y)=g(x,1)g(1,y)\qquad a.e.\ on\ [0,1]^2.$$

LEMMA 2. *Let x_n, $n=1,2,\ldots$ be a real sequence. If x_n is uniformly distributed mod1, then $G((x_n,\log p_n)\bmod 1)=G((x_n,\log(n\log n))\bmod 1)=\{xg_u(y):u\in[0,1]\}$.*

P r o o f. Let $g \in G((x_n, \log(n \log n)) \bmod 1)$. From Lemma 1, it follows that x_n mod1 and $\log(n \log n)$ mod1 are statistically independent. Applying Theorem B, we have

$$g(x, y) = g(x, 1)g(1, y) \text{ a.e. on } [0, 1]^2$$

Since $g(x, 1) \in G(x_n \bmod 1) = \{x\}$, we have $g(x, 1) = x$ at every continuity point x of $g(x, 1)$. We have $g(x, 1) = x$ for all $x \in [0, 1]$. Since $g(1, y) \in G(\log(n \log n) \bmod 1)$, by (1.2) there exists $u \in [0, 1]$ such that $g(1, y) = g_u(y)$ at every continuity point y of $g(1, y)$. Since $g_u(y)$ is continuous, we obtain $g(1, y) = f_u(y)$ for every $y \in [0, 1]$. It follows that

$$g(x, y) = xg_u(y) \text{ for every } (x, y) \in [0, 1]^2.$$

By this and (2.3), Theorem 2 yields
$G((x_n, \log p_n) \bmod 1) = G((x_n, \log(n \log n)) \bmod 1)$. $\qquad \square$

THEOREM 3. *Let x_n, $n = 1, 2, \ldots$ be a real sequence. If x_n is uniformly distributed* mod 1*, then x_n mod 1 and $\log p_n$ mod 1 is statistically independent.*

P r o o f. Let $g \in G((x_n, \log p_n) \bmod 1)$. By Lemma 2 there exist $\tilde{g} \in G((x_n, \log(n \log n)) \bmod 1)$ such that $g(x, y) = \tilde{g}(x, y)$ at every continuity point $(x, y) \in [0, 1]^2$. Since $\tilde{g}(x, y)$ is continuous on $[0, 1]^2$,

$$g(x, y) = \tilde{g}(x, y) \tag{3.3}$$

for every $(x, y) \in [0, 1]^2$. By Lemma 1 and Theorem B we have
$\tilde{g}(x, y) = \tilde{g}(x, 1)\tilde{g}(1, y)$ almost everywhere. Hence from (3.3) it follows that

$$g(x, y) = g(x, 1)g(1, y)$$

almost everywhere. This and Theorem B complete the proof of Theorem 3. $\quad \square$

LEMMA B (see [8, p.96], [14])**.** *Let x_n and y_n be sequences of real numbers such that x_n mod 1 and y_n mod 1 are statistically independent, and x_n is uniformly distributed* mod 1*. Then the sequence $x_n + y_n$ is uniformly distributed* mod 1*.*

It is well known that $p_n\theta$ is uniformly distributed mod 1, for an irrational number θ (see [15, Chapter 11]). By this result, Theorem 3 and Lemma B we obtain the following result.

THEOREM 4. *If θ is irrational, then $p_n\theta + \log p_n$ is uniformly distributed* mod 1.

Furthermore, we have the following result in same way as the proof of Theorem 4. If θ is irrational, then $p_n(\theta + 1/n)$ is uniformly distributed mod 1.

4. Proof of Theorem 2

In order to prove Theorem 2 we need some properties of monotone functions of several variables in [2].

DEFINITION 2 (see [2]). *Let $F(x_1, \ldots, x_s)$ be a real valued function defined on the interval $J = [0,1]^s$. The function F is said to be monotone if $F(x_1, \ldots, x_s)$ is non-decreasing with respect to x_i for $1 \leq i \leq s$, and if $\Delta_{h_k}^{(k)}(\Delta_{h_j}^{(j)} F(x_1, \ldots, x_s)) \geq 0$ for $0 \leq x_i \leq 1$ $(i = 1, \ldots, s)$, $1 \leq j < k \leq s$, $h_j \geq 0$, $h_k \geq 0$, $x_j + h_j \leq 1$, $x_k + h_k \leq 1$.*

LEMMA C (see [2], Theorem 1). *If $F(x_1, \ldots, x_s)$ be a monotone function on the interval J, then there is a set R consisting of the points of a countable collection $(s-1)$-space $x_i = \alpha_i^{(j)}$, $i = 1, 2, \ldots, s$, $j = 1, 2, \ldots$, such that F is continuous at all points of $J - R$.*

REMARK 1. *Since $F(1, \ldots, 1, x_i, 1, \ldots, 1)$ is non-decreasing with respect to x_i, it has a countable number of discontinuities $x_i = \alpha_{ij}$, $i = 1, \ldots, s$, $j = 1, 2, \ldots$. The set R is the union of the sets in $[0,1]^s$: $x_i = \alpha_{ij}$, $i = 1, \ldots, s$, $j = 1, 2, \ldots$.*

LEMMA D (see [2], Theorem 2). *Let $F_n(x_1, \ldots, x_s)$ be a sequence of monotone functions on J with $|F_n(x_1, \ldots, x_s)| \leq A$ $(n = 1, 2, \ldots)$ for some $A > 0$. Then, there exists a sequence of positive integers $n_1 < n_2 < \cdots$ and a non-decreasing function $F(x_1, \ldots, x_s)$ such that*

$$\lim_{k \to \infty} F_{n_k}(x_1, x_2, \ldots, x_s) = F(x_1, x_2, \ldots, x_s) \quad \text{on } J.$$

LEMMA 3. *Let g be a distribution function. If there exist y_{i,n_i} and z_{i,n_i} $(n_i = 1, 2, \ldots)$ for $i = 1, 2, \ldots, s$ such that $0 < y_{i,n_i} < z_{i,n_i} < 1$ $(i = 1, 2, \ldots, s)$, $\lim_{n_i \to \infty} y_{i,n_i} = 0$, $\lim_{n_i \to \infty} z_{i,n_i} = 1$, and*

$$\lim_{\substack{n_i \to \infty \\ i=1 \cdots s}} \Delta_{z_{s,n_s} - y_{s,n_s}}^{(s)} \cdots \Delta_{z_{1,n_1} - y_{1,n_1}}^{(1)} g(y_{1,n_1}, \cdots, y_{s,n_s}) :=$$

$$\lim_{n_s \to \infty} \cdots \lim_{n_1 \to \infty} \Delta_{z_{s,n_s} - y_{s,n_s}}^{(s)} \cdots \Delta_{z_{1,n_1} - y_{1,n_1}}^{(1)} g(y_{1,n_1}, \cdots, y_{s,n_s}) = 1.$$

then

$$\lim_{\substack{n_i \to \infty \\ i=1 \cdots s}} g(z_{1,n_1}, \ldots, z_{s,n_s}) = 1 \quad \text{and} \quad \lim_{\substack{n_i \to \infty \\ i=1 \cdots s}} g(t_{1,n_1}, \ldots, t_{s,n_s}) = 0,$$

where $t_{j,n_j} = y_{j,n_j}$ for some j and either $t_{i,n_i} = y_{i,n_i}$ or $t_{i,n_i} = z_{i,n_i}$ for $i = 1, \ldots, s$.

P r o o f. By induction on s we shall prove this lemma.

For $s = 1$, from $\lim_{n_1 \to \infty} (g(z_{1,n_1}) - g(y_{1,n_1})) = 1$ and $0 \leq g(y_{1,n_1}) \leq g(z_{1,n_1}) \leq 1$ we have

$$\lim_{n_1 \to \infty} g(z_{1,n_1}) = 1, \ \lim_{n_1 \to \infty} g(y_{1,n_1}) = 0.$$

For $s > 1$ we have

$$\lim_{\substack{n_i \to \infty \\ i=1\cdots s}} \Delta^{(s)}_{z_{s,n_s} - y_{s,n_s}} \cdots \Delta^{(1)}_{z_{1,n_1} - y_{1,n_1}} g(y_{1,n_1}, \cdots, y_{s,n_s})$$

$$= \lim_{\substack{n_i \to \infty \\ i=1\cdots s-1}} \Delta^{(s-1)}_{z_{s-1,n_{s-1}} - y_{s-1,n_{s-1}}} \cdots \Delta^{(1)}_{z_{1,n_1} - y_{1,n_1}}$$

$$\lim_{n_s \to \infty} g(y_{1,n_1}, \cdots, y_{s-1,n_{s-1}}, z_{s,n_s}) -$$

$$\lim_{\substack{n_i \to \infty \\ i=1\cdots s-1}} \Delta^{(s-1)}_{z_{s-1,n_{s-1}} - y_{s-1,n_{s-1}}} \cdots \Delta^{(1)}_{z_{1,n_1} - y_{1,n_1}}$$

$$\lim_{n_s \to \infty} g(y_{1,n_1}, \cdots, y_{s-1,n_{s-1}}, y_{s,n_s}) = 1.$$

Set

$$g_0(y_1, \cdots, y_{s-1}) := \lim_{n_s \to \infty} g(y_1, \cdots, y_{s-1}, y_{s,n_s}),$$

$$g_1(y_1, \cdots, y_{s-1}) := \lim_{n_s \to \infty} g(y_1, \cdots, y_{s-1}, z_{s,n_s}).$$

It is shown that $g_k(y_1, \cdots, y_{s-1})$ is a d.f. for $k = 0, 1$. Hence we obtain

$$\lim_{\substack{n_i \to \infty \\ i=1\cdots s-1}} \Delta^{(s-1)}_{z_{s-1,n_{s-1}} - y_{s-1,n_{s-1}}} \cdots \Delta^{(1)}_{z_{1,n_1} - y_{1,n_1}} g_1(y_{1,n_1}, \cdots, y_{s-1,n_{s-1}}) = 1.$$

From this and the induction hypothesis we obtain

$$\lim_{\substack{n_i \to \infty \\ i=1\cdots s-1}} \lim_{n_s \to \infty} g(z_{1,n_1}, \cdots, z_{s-1,n_{s-1}}, z_{s,n_s}) = \lim_{\substack{n_i \to \infty \\ i=1\cdots s}} g_1(z_{1,n_1}, \ldots, z_{s-1,n_{s-1}}) = 1,$$

$$\lim_{\substack{n_i \to \infty \\ i=1\cdots s-1}} \lim_{n_s \to \infty} g(t_{1,n_1}, \cdots, t_{n_{s-1,s-1}}, z_{s,n_s}) = \lim_{\substack{n_i \to \infty \\ i=1\cdots s}} g_1(t_{1,n_1}, \ldots, t_{s-1,n_{s-1}}) = 0,$$

where either $t_{i,n_i} = y_{i,n_i}$ or $t_{i,n_i} = z_{i,n_i}$ for $1 \leq i \leq s-1$ and $t_{j,n_j} = y_{j,n_j}$ for some j. Since $g(x_1, \ldots, x_{s-1}, x_s)$ is non-decreasing with respect to x_s,

$$\lim_{\substack{n_i \to \infty \\ i=1\cdots s-1}} \lim_{n_s \to \infty} g(t_{1,n_1}, \cdots, t_{s-1,n_{s-1}}, y_{s,n_s})$$

$$\leq \lim_{\substack{n_i \to \infty \\ i=1\cdots s-1}} \lim_{n_s \to \infty} g(t_{1,n_1}, \cdots, t_{s-1,n_{s-1}}, z_{s,n_s}) = 0.$$

Thus

$$\lim_{\substack{n_i \to \infty \\ i=1\cdots s}} g(t_{1,n_1}, \cdots, t_{s-1,n_{s-1}}, y_{s,n_s}) = 0.$$

$\square$

We prove the two dimensional case of Theorem 2. The proof is analogous to the proof of Theorem 1 in [12].

Since either $\|x_n - \tilde{x}_n\| = \{x_n - \tilde{x}_n\} = (x_n - [x_n - \tilde{x}_n]) - \tilde{x}_n$ or $\|x_n - \tilde{x}_n\| = 1 - \{x_n - \tilde{x}_n\} = \tilde{x}_n - (x_n - [x_n - \tilde{x}_n] - 1)$, we can assume that $|a_{n,i} - \tilde{a}_{n,i}| \to 0$ for $i = 1, 2$.

Let $g(\mathbf{x}) \in G(\mathbf{a}_n \bmod 1)$. There exists the sequence N_k' such that

$$\lim_{k \to \infty} F_{N_k'}(\mathbf{x}) = g(\mathbf{x})$$

at every continuity point $\mathbf{x} \in (0,1)^2$ of $g(\mathbf{x})$.
By Lemma D there exists the subsequence N_{k_j}' of the sequence N_k' and $\tilde{g}(\mathbf{x}) \in G(\tilde{\mathbf{x}}_n \bmod 1)$ such that

$$\lim_{j \to \infty} \tilde{F}_{N_{k_j}'}(\mathbf{x}) = \tilde{g}(\mathbf{x})$$

for all $\mathbf{x} \in [0,1]^2$.
Thus there exists N_j such that

$$\lim_{j \to \infty} F_{N_j}(\mathbf{x}) = g(\mathbf{x})$$

at every continuity point $\mathbf{x} \in (0,1)^2$ of $g(\mathbf{x})$, and

$$\lim_{j \to \infty} \tilde{F}_{N_j}(\mathbf{x}) = \tilde{g}(\mathbf{x})$$

for all $\mathbf{x} \in [0,1]^2$.

Applying the second Helly theorem, we have

$$\lim_{j \to \infty} \int_{[0,1]^2} e^{2\pi i(h_1 x_1 + h_2 x_2)} dF_{N_j}(x_1, x_2) = \int_{[0,1]^2} e^{2\pi i(h_1 x_1 + h_2 x_2)} dg(x_1, x_2),$$

$$\lim_{j \to \infty} \int_{[0,1]^2} e^{2\pi i(h_1 x_1 + h_2 x_2)} d\tilde{F}_{N_j}(x_1, x_2) = \int_{[0,1]^2} e^{2\pi i(h_1 x_1 + h_2 x_2)} d\tilde{g}(x_1, x_2)$$

for $h_i = 0, \pm 1, \pm 2, \dots$ $(i = 1, 2)$.
Hence

$$\lim_{j \to \infty} \frac{1}{N_j} \sum_{n=1}^{N_j} e^{2\pi i(h_1 a_{n,1} + h_m a_{n,2})} = \int_{[0,1]^2} e^{2\pi i(h_1 x_1 + h_2 x_2)} dg(x_1, x_2),$$

$$\lim_{j \to \infty} \frac{1}{N_j} \sum_{n=1}^{N_j} e^{2\pi i(h_1 \tilde{a}_{n,1} + h_2 \tilde{a}_{n,2})} = \int_{[0,1]^2} e^{2\pi i(h_1 x_1 + h_2 x_2)} d\tilde{g}(x_1, x_2)$$

for $h_i = 0, \pm 1, \pm 2, \ldots$ $(i = 1, 2)$.

Since

$$\left| \frac{1}{N_j} \sum_{n=1}^{N_j} e^{2\pi i(h_1 a_{n,1} + h_2 a_{n,2})} - \sum_{n=1}^{N_j} e^{2\pi i(h_1 \tilde{a}_{n,1} + h_2 \tilde{a}_{n,2})} \right|$$

$$\leq \frac{1}{N_j} \sum_{n=1}^{N_j} |e^{2\pi i(h_1(a_{n,1} - \tilde{a}_{n,1}) + h_2(a_{n,2} - \tilde{a}_{n,2}))} - 1| \to 0$$

as $j \to \infty$, we have

$$\lim_{j \to \infty} \frac{1}{N_j} \sum_{n=1}^{N_j} e^{2\pi i(h_1 a_{n,1} + h_2 a_{n,2})} = \lim_{j \to \infty} \frac{1}{N_j} \sum_{n=1}^{N_j} e^{2\pi i(h_1 \tilde{a}_{n,1} + h_2 \tilde{a}_{n,2})}.$$

Therefore we obtain

$$\int_{[0,1]^2} e^{2\pi i(h_1 x_1 + h_2 x_2)} dg(x_1, x_2) = \int_{[0,1]^2} e^{2\pi i(h_1 x_1 + h_2 x_2)} d\tilde{g}(x_1, x_2)$$

for $h_i = 0, \pm 1, \pm 2, \ldots$ $(i = 1, 2)$.

By the Weierstrass approximation theorem, we have for any complex-valued continuous function f on $[0,1]^2$ with period 1 in each variable

$$\int_{[0,1]^2} f(x_1, x_2) dg(x_1, x_2) = \int_{[0,1]^2} f(x_1, x_2) d\tilde{g}_1(x_1, x_2). \tag{4.1}$$

For $0 \leq y < z \leq 1$ and sufficiently small $\delta > 0$, we define the function $f_{yz}^{(\delta)}$ by $f_{yz}^{(\delta)}(x) = 0$ $(0 \leq x \leq y)$, $f_{yz}^{(\delta)}(x) = x/\delta - y/\delta$ $(y < x < y + \delta)$, $f_{yz}^{(\delta)}(x) = 1$ $(y + \delta \leq x \leq z - \delta)$, $f_{yz}^{(\delta)}(x) = -x/\delta + z/\delta$ $(z - \delta < x < z)$, $f_{yz}^{(\delta)}(x) = 0$ $(z \leq x \leq 1)$.

Since $g(x_1, 1)$ is non-decreasing with respect to x_1, it has a countable number of dis-continuities $\alpha_{1j} \in [0, 1]$, $j = 1, 2, \ldots$. Similarly $\tilde{g}(x_1, 1)$ has a countable number of dis-continuities $\tilde{\alpha}_{1,j} \in [0, 1]$, $j = 1, 2, \ldots$. Since $g(1, x_2)$ is non-decreasing with respect to x_2, it has a countable number of dis-continuities $\alpha_{2j} \in [0, 1]$, $j = 1, 2, \ldots$. Similarly $\tilde{g}(1, x_2)$ has a countable number of dis-continuities $\tilde{\alpha}_{2j} \in [0, 1]$, $j = 1, 2, \ldots$.

Let $R_1 = (\cup_{j=1}^{\infty}\{(\alpha_{1j}, x_2) : 0 \leq x_2 \leq 1\}) \cup (\cup_{j=1}^{\infty}\{(x_1, \alpha_{2j}) : 0 \leq x_1 \leq 1\})$, $R_2 = (\cup_{j=1}^{\infty}\{(\tilde{\alpha}_{1j}, x_2) : 0 \leq x_2 \leq 1\}) \cup (\cup_{j=1}^{\infty}\{(x_1, \tilde{\alpha}_{1j}) : 0 \leq x_1 \leq 1\})$, and $R = R_1 \cup R_2$.

Let (y_1, y_2), (z_1, z_2) be in $[0,1]^2 - R$ with $0 < y_i < z_i < 1$ for $i = 1, 2$. Set

$$f_\delta(x_1, x_2) = f_{y_1 z_1}^{(\delta)}(x_1) \cdot f_{y_2 z_2}^{(\delta)}(x_2).$$

By (4.1) we obtain

$$\int_{[0,1]^2} f_\delta(x_1, x_2) dg(x_1, x_2) = \int_{[0,1]^2} f_\delta(x_1, x_2) d\tilde{g}(x_1, x_2). \qquad (4.2)$$

The left-hand side of (4.2) is written in the form

$$\int_{[0,1]^2} f_\delta(x_1, x_2) dg(x_1, x_2) = \sum_{J_1, J_2} \int_{J_1 \times J_2} f_\delta(x_1, x_2) dg(x_1, x_2)$$

where $\sum_{J_1, J_2}$ stands for the sum over all intervals
$J_i \in \{[y_i, y_i + \delta], [y_i + \delta, z_i - \delta], [z_i - \delta, z_i]\}$, $i = 1, 2$.
If $J_i \neq [y_i + \delta, z_i - \delta]$ for some i, then

$$\lim_{\delta \to 0} \int_{J_1 \times J_2} f_\delta(x_1, x_2) dg(x_1, x_2) = 0,$$

for in the case of $J_1 = [y_1, y_1 + \delta]$ we have

$$0 \leq \int_{J_1 \times J_2} f_\delta(x_1, x_2) dg(x_1, x_2) \leq \int_{J_1 \times [0,1]} dg(x_1, x_2) = g(y_1 + \delta, 1) - g(y_1, 1) \to 0$$

as $\delta \to 0$, since g is continuous at $(y_1, 1)$ by Lemma C.
Furthermore, since g is continuous at (x_1, x_2) with $x_i \in \{y_i, z_i\}$, $i = 1, 2$, we obtain

$$\lim_{\delta \to 0} \int_{[y_1 + \delta, z_1 - \delta] \times [y_2 + \delta, z_2 - \delta]} f_\delta(x_1, x_2) dg(x_1, x_2) = \Delta^{(2)}_{z_2 - y_2} \Delta^{(1)}_{z_1 - y_1} g(y_1, y_2),$$

hence

$$\lim_{\delta \to 0} \int_{[0,1]^2} f_\delta(x_1, x_2) dg(x_1, x_2) = \Delta^{(2)}_{z_2 - y_2} \Delta^{(1)}_{z_1 - y_1} g(y_1, y_2).$$

Similarly we obtain

$$\lim_{\delta \to 0} \int_{[0,1]^2} f_\delta(x_1, x_2) d\tilde{g}(x_1, x_2) = \Delta^{(2)}_{z_2 - y_2} \Delta^{(1)}_{z_1 - y_1} \tilde{g}(y_1, y_2).$$

Therefore from (4.2) it follows that

$$\Delta^{(2)}_{z_2 - y_2} \Delta^{(1)}_{z_1 - y_1} g(y_1, y_2) = \Delta^{(2)}_{z_2 - y_2} \Delta^{(1)}_{z_1 - y_1} \tilde{g}(y_1, y_2). \qquad (4.3)$$

Let $(y_{n_1}^{(1)}, y_{n_2}^{(2)})$, $(z_{n_1}^{(1)}, z_{n_2}^{(2)})$ be in $[0,1]^2 - R$ for $n_i = 1, 2, \ldots (i = 1, 2)$ with
$0 < y_{i,n_i}^{(i)} < z_{i,n_i}^{(i)} < 1$, $\lim_{n_i \to \infty} y_{n_i}^{(i)} = 0$, and $\lim_{n_i \to \infty} z_{n_i}^{(i)} = 1$ for $i = 1, 2$.
From Lemma C, $g(1, 1) = 1$, and $g(\mathbf{x}) = 0$ for any $\mathbf{x}$ with a vanishing coordinate we obtain

$$\lim_{\substack{n_i \to \infty \\ i=1,2}} \Delta^{(2)}_{z_{n_2}^{(2)} - y_{n_2}^{(2)}} \Delta^{(1)}_{z_{n_1}^{(1)} - y_{n_1}^{(1)}} g(y_{n_1}^{(1)}, y_{n_2}^{(2)}) = 1.$$

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From (4.3) we have

$$\lim_{\substack{n_i \to \infty \\ i=1,2}} \Delta^{(2)}_{z^{(2)}_{n_2} - y^{(2)}_{n_2}} \Delta^{(1)}_{z^{(1)}_{n_1} - y^{(1)}_{n_1}} \tilde{g}(y^{(1)}_{n_1}, y^{(2)}_{n_2}) = 1.$$

Using Lemma 3, we obtain

$$\lim_{\substack{n_i \to \infty \\ i=1,2}} \tilde{g}(z^{(1)}_{n_1}, z^{(2)}_{n_2}) = 1, \quad \lim_{\substack{n_i \to \infty \\ i=1,2}} \tilde{g}(t^{(1)}_{n_1}, t^{(2)}_{n_2}) = 0,$$

where $t^{(j)}_{n_j} = y^{(j)}_{n_j}$ for some j and either $t^{(i)}_{n_i} = y^{(i)}_{n_i}$ or $t^{(i)}_{n_i} = z^{(i)}_{n_i}$ for $i = 1, 2$. Since $\tilde{g}(x_1, x_2)$ is non-decreasing with respect to each variable, we have

$$\lim_{\substack{n_i \to \infty \\ i=1,2}} \tilde{g}(t^{(1)}_{n_1}, t^{(2)}_{n_2}) = 0, \tag{4.4}$$

where $t^{(j)}_{n_j} = y^{(j)}_{n_j}$ for some j and either $t^{(i)}_{n_i} = y^{(i)}_{n_i}$ or $t^{(i)}_{n_i} = z_i$ for $i = 1, 2$.

From (4.3) we obtain

$$\Delta^{(2)}_{z_2 - y^{(2)}_{n_2}} \Delta^{(1)}_{z_1 - y^{(1)}_{n_1}} g(y^{(1)}_{n_1}, y^{(2)}_{n_2}) = \Delta^{(2)}_{z_2 - y^{(2)}_{n_2}} \Delta^{(1)}_{z_1 - y^{(1)}_{n_1}} \tilde{g}(y^{(1)}_{n_1}, y^{(2)}_{n_2}),$$

so

$$\tilde{g}(z_1, z_2) - \tilde{g}(z_1, y^{(2)}_{n_2}) - \tilde{g}(y^{(1)}_{n_1}, z_2) + \tilde{g}(y^{(1)}_{n_1}, y^{(2)}_{n_2})$$
$$= g(z_1, z_2) - g(z_1, y^{(2)}_{n_2}) - g(y^{(1)}_{n_1}, z_2) + g(y^{(1)}_{n_1}, y^{(2)}_{n_2}).$$

By using (4.4) and continuity of g at $(z_1, 0)$, $(0, z_2)$. we get

$$\tilde{g}(z_1, z_2) = g(z_1, z_2).$$

Letting $z_1 \to 1$, we get

$$\lim_{z_1 \to 1} \tilde{g}(z_1, z_2) = g(1, z_2), \tag{4.5}$$

because g is continuous at $(1, z_2)$.

Set $z_1 = z^{(1)}_{n_1}, y_2 = y^{(2)}_{n_2}$ in (4.3), then by letting $n_1 \to \infty, n_2 \to \infty$ we have

$$\lim_{n_1 \to \infty} \tilde{g}(z^{(1)}_{n_1}, z_2) - \tilde{g}(y_1, z_2) = g(1, z_2) - g(y_1, z_2).$$

Using (4.5), we have

$$\tilde{g}(y_1, z_2) = g(y_1, z_2).$$

In the same way, we can show that $\tilde{g}(x_1, x_2) = g(x_1, x_2)$ for $x_i \in \{y_i, z_i\}$ ($i = 1, 2$). Therefore we obtain

$$G(\mathbf{a}_n \bmod 1) \subset G(\tilde{\mathbf{a}}_n \bmod 1). \tag{4.6}$$

If we assume that $\tilde{g}(x_1, 1)$ and $\tilde{g}(1, x_2)$ is continuous at $x_i = 0, 1$ for $i = 1, 2$, then (4.6) is shown in the same way, hence

$$G(\tilde{\mathbf{a}}_n \bmod 1) \subset G(\mathbf{a}_n \bmod 1).$$

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Thus we have

$$G(\mathbf{a}_n \bmod 1) = G(\tilde{\mathbf{a}}_n \bmod 1).$$

This proof can be extended to the general s-dimensional case.

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