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SOME APPLICATIONS OF W. RUDIN'S INEQUALITY TO PROBLEMS OF COMBINATORIAL NUMBER THEORY

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ABSTRACT. In the paper we obtain some new applications of well-known W. Rudin's theorem concerning lacunary series to problems of combinatorial number theory. We generalize a result of M.-C. Chang on $L_2(\Lambda)$-norm of Fourier coefficients of a set (here Λ is a dissociated set), and prove a dual version of the theorem. Our main instrument is computing of eigenvalues of some operators.

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1. Introduction

A well–known theorem of W. Rudin concerning lacunary series (see [22, 23]) states for any complex function with condition that support of its Fourier transform belongs to a set "without arithmetical structure" all L_p-norms, $p \geq 2$ are equivalent to L_2-norm. Let us formulate the last result more precisely. Let $\mathbf{G} = (\mathbf{G}, +)$ be an Abelian group with additive group operation $+$. By $\widehat{\mathbf{G}}$ denote the Pontryagin dual of $\mathbf{G}$. In other words $\widehat{\mathbf{G}}$ is the group of homomorphisms ξ from $\mathbf{G}$ to $\mathbf{R}/\mathbf{Z}$, $\xi : x \to \xi \cdot x$. A set $\Lambda \subseteq \widehat{\mathbf{G}}$, $\Lambda = \{\lambda_1, \ldots, \lambda_{|\Lambda|}\}$ is called *dissociated* if any identity of the form

$$\sum_{i=1}^{|\Lambda|} \varepsilon_i \lambda_i = 0\,,$$

where $\varepsilon_i \in \{0, \pm 1\}$ implies that all ε_i are equal zero.

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Theorem 1.1. *Let $\mathbf{G}$ be a finite Abelian group. There exists an absolute constant $C > 0$ such that for any dissociated set $\Lambda \subseteq \mathbf{G}$, any complex numbers $a_n \in \mathbb{C}$, and all positive integers $p \geq 2$ the following inequality holds*

$$\frac{1}{|\mathbf{G}|} \sum_{x \in \mathbf{G}} \left| \sum_{\xi \in \Lambda} a_\xi e^{2\pi i \xi \cdot x} \right|^p \leq \ (C\sqrt{p})^p \left(\sum_{\xi \in \Lambda} |a_\xi|^2 \right)^{p/2} . \tag{1}$$

Theorem 1.1 is widely used in combinatorial number theory, mainly in solution of so-called “inverse” problems (see books [19, 34]). Thus, using Rudin’s theorem M.-C. Chang proved the following result (see [6, 12, 9]).

Theorem 1.2. *Let $\mathbf{G}$ be a finite Abelian group, and $\delta \in (0,1]$ be a real number. Let also $\Lambda \subseteq \mathbf{G}$ be a dissociated set, $S \subseteq \mathbf{G}$ be an arbitrary set, $|S| = \delta|\mathbf{G}|$. Then*

$$\sum_{\xi \in \Lambda} \left| \sum_{x \in S} e^{2\pi i \xi \cdot x} \right|^2 \leq \ C_1 |S|^2 \log(1/\delta) \,, \tag{2}$$

where $C_1 > 0$ is an absolute constant.

Theorem 1.2 was repeatedly used in problems concerning large exponential sums (see papers [6, 8, 9, 25, 24] and others). On sets of large exponential sums see [7], [10]—[12], [34], [35]—[1], [27]—[30]. In paper [3] Rudin’s theorem was used for finding arithmetic progressions in sumsets, and for studying properties of sets with small doubling (see [26], [32]). In the paper we obtain new applications of beautiful Theorem 1.2 to problems of combinatorial number theory. Firstly, we prove a sharp generalization of Chang’s theorem.

Theorem 1.3. *Let $\mathbf{G}$ be a finite Abelian group, $\delta \in (0,1]$ be a real number, and l be a positive integer, $l \geq 2$. Suppose that $\Lambda \subseteq \mathbf{G}$ is a dissociated set, and $S \subseteq \mathbf{G}$ is an arbitrary set, $|S| = \delta|\mathbf{G}|$. Then*

$$\sum_{\xi \in \Lambda} \left| \sum_{x \in S} e^{2\pi i \xi \cdot x} \right|^{l+1} \leq \ C_2 |S| \left(\sum_{\xi \neq 0} \left| \sum_{x \in S} e^{2\pi i \xi \cdot x} \right|^{2l} \right)^{1/2} \cdot \ \log^{1/2}(1/\delta) \,, \tag{3}$$

where $C_2 > 0$ is an absolute constant.

Secondly, we obtain a “dual” version of Theorem 1.3. For two complex functions $f, g \ : \ \mathbf{G} \to \mathbb{C}$ define its convolution by the formula

$$(f * g)(x) := \sum_{y \in \mathbf{G}} f(y) g(x - y).$$

Theorem 1.4. *Let* $\mathbf{G}$ *be a finite Abelian group, and* l *be a positive integer,* $l \ge 2$. *Suppose that* $\Lambda \subseteq \mathbf{G}$ *is a dissociated set, and* $S_1, \dots, S_l \subseteq \mathbf{G}$ *are arbitrary sets. Then*

$$\sum_{x \in \Lambda} (S_1 * S_2 * \dots * S_l)^2(x) \;\le\; C_3 \frac{|S_l|}{N} \left(\sum_\xi \prod_{j=1}^{l-1} \left| \sum_{x \in S_j} e^{2\pi i \xi \cdot x} \right|^2 \right) \cdot \, \log N \,, \qquad (4)$$

where $C_3 > 0$ *is an absolute constant.*

Structure of the paper is the following. In section 1 we introduce operators (matrices) T^φ_ψ and S^φ_ψ. This family of matrices is our main instrument of investigation. We study spectrums of matrices T^φ_ψ and S^φ_ψ and its eigenvectors. In Section 3 we reformulated Rudin's theorem on the language of eigenvalues of operators $\mathrm{T}^\varphi_\Lambda$, where Λ is a dissociated set and φ is a function (see Proposition 3.1) and derive Theorems 1.3, 1.4. Also we show that Theorem 1.3 is sharp and for some choice of parameters Theorem 1.4 is also sharp.

Let us say a few words about the notation. If $S \subseteq \mathbf{G}$ is a set then we will write $S(x)$ for the characteristic function. In other words $S(x) = 1$ if $x \in S$ and zero otherwise. By log denote logarithm base two. Signs $\ll$ and $\gg$ are usual Vinogradov's symbols. If n is a positive integer then we will write $[n]$ for the segment $\{1, 2, \dots, n\}$.

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2. Operators T^φ_ψ and S^φ_ψ

We need in Fourier analysis in our proof. Let $\mathbf{G}$ be a finite Abelian group, $N := |\mathbf{G}|$. It is well–known that in the case the dual group $\widehat{\mathbf{G}}$ is isomorphic to $\mathbf{G}$. Let also f be a function from $\mathbf{G}$ to $\mathbb{C}$. By $(\Phi f)(\xi) = \widehat{f}(\xi)$ denote the Fourier transformation of f

$$(\Phi f)(\xi) = \widehat{f}(\xi) = \sum_{x \in \mathbf{G}} f(x) e(-\xi \cdot x) \,, \qquad (5)$$

where $e(x) = e^{2\pi i x}$. We will use the following basic facts

$$\|f\|_2^2 := \sum_{x \in \mathbf{G}} |f(x)|^2 = \frac{1}{N} \sum_{\xi \in \widehat{\mathbf{G}}} |\widehat{f}(\xi)|^2 = \frac{1}{N} \|\widehat{f}\|_2^2 \,. \qquad (6)$$

$$\langle f, g \rangle := \sum_{x \in \mathbf{G}} f(x) \overline{g(x)} = \frac{1}{N} \sum_{\xi \in \widehat{\mathbf{G}}} \widehat{f}(\xi) \overline{\widehat{g}(\xi)} = \frac{1}{N} \langle \widehat{f}, \widehat{g} \rangle \,. \qquad (7)$$

$$\sum_{y\in\mathbf{G}}\left|\sum_{x\in\mathbf{G}} f(x)g(y-x)\right|^2 = \frac{1}{N}\sum_{\xi\in\widehat{\mathbf{G}}} |\widehat{f}(\xi)|^2|\widehat{g}(\xi)|^2\,. \tag{8}$$

$$f(x) = \frac{1}{N}\sum_{\xi\in\widehat{\mathbf{G}}} \widehat{f}(\xi)e(\xi\cdot x)\,. \tag{9}$$

If

$$(f*g)(x) := \sum_{y\in\mathbf{G}} f(y)g(x-y),$$

then

$$\widehat{f*g} = \widehat{f}\widehat{g} \quad \text{and} \quad (\widehat{fg})(x) = \frac{1}{N}(\widehat{f}*\widehat{g})(x)\,. \tag{10}$$

By $*_{l-1}$ denote the result of using convolution l times. Let also

$$(f*_0 f)(x) := f(x).$$

We will write $\sum_s$ instead of $\sum_{s\in\mathbf{G}}$ and $\sum_\xi$ instead of $\sum_{\xi\in\widehat{\mathbf{G}}}$ for brevity.

Definition 2.1. Let $\varphi,\psi:\mathbf{G}\to\mathbb{C}$ be two functions. By $\mathrm{T}^{\varphi}_{\psi}$ denote the following operator on the space of functions $\mathbf{G}^{\mathbb{C}}$

$$\left(\mathrm{T}^{\varphi}_{\psi}f\right)(x) = \psi(x)\left(\widehat{\varphi^c}*f\right)(x) = \psi(x)\widehat{\varphi^c\widehat{f^c}}(x)\,, \tag{11}$$

where f is an arbitrary complex function on $\mathbf{G}$ and f^c is the function $f^c(x) = f(-x)$. Also we need in "more symmetric" (see identity (19) below) operator $\mathrm{S}^{\varphi}_{\psi}$

$$\left(\mathrm{S}^{\varphi}_{\psi}f\right)(x) = \psi(x)(\widehat{\varphi^c}*\overline{\psi}f)(x) = \psi(x)(\widehat{\varphi^c\widehat{\overline{\psi^c}f^c}})(x)\,. \tag{12}$$

In particular if $\psi\equiv 1$, then $\mathrm{T}^{\varphi}_{\psi}$ is the convolution operator by $\widehat{\varphi^c}$ and if $\varphi\equiv 1$, then $\mathrm{T}^{\varphi}_{\psi}$ is the operator of multiplication by the function ψ.

Let us express the operators $\mathrm{T}^{\varphi}_{\psi}$, $\mathrm{S}^{\varphi}_{\psi}$ as composition of more simple operators. Let

$$(\mathrm{C}f)(x) = f(-x) = f^c(x)$$

and for any complex function $\rho:\mathbf{G}\to\mathbb{C}$ let

$$(\mathrm{P}_\rho f)(x) = \rho(x)f(x)\,.$$

Clearly, $\mathrm{C}^2 = \mathrm{I}$ is the identity operator and for any two functions ρ_1, ρ_2 the following holds $\mathrm{P}_{\rho_1}\mathrm{P}_{\rho_2} = \mathrm{P}_{\rho_1\rho_2}$. We have

$$\mathrm{C}\Phi = \Phi\mathrm{C}\,,\quad \mathrm{C}\mathrm{P}_\rho = \mathrm{P}_{\rho^c}\mathrm{C}\,,\quad \Phi^2 = N\cdot\mathrm{C}\,. \tag{13}$$

The last formulas imply

$$\mathrm{T}_\psi^\varphi = \mathrm{P}_\psi \Phi \mathrm{P}_{\varphi^c} \Phi \mathrm{C} = \mathrm{C}\mathrm{P}_{\psi^c} \Phi \mathrm{P}_\varphi \Phi = \mathrm{P}_\psi \Phi \mathrm{C} \mathrm{P}_\varphi \Phi \tag{14}$$

and

$$\mathrm{S}_\psi^\varphi = \mathrm{P}_\psi \Phi \mathrm{P}_{\varphi^c} \Phi \mathrm{P}_{\overline{\psi}^c} \mathrm{C} = \mathrm{C}\mathrm{P}_{\psi^c} \Phi \mathrm{P}_\varphi \Phi \mathrm{P}_{\overline{\psi}} = \mathrm{P}_\psi \Phi \mathrm{C} \mathrm{P}_\varphi \Phi \mathrm{P}_{\overline{\psi}} . \tag{15}$$

We will need it in more formulas.

We have

$$\mathrm{C}(a * b) = \mathrm{C}a * \mathrm{C}b \,, \qquad \langle a, \overline{b} * c\rangle = \langle b, \overline{a} * \mathrm{C}c\rangle \tag{16}$$

for any functions a, b and c. Let $\mathcal{C}$ be the conjugation operator. Clearly, $\mathcal{C}^2 = \mathrm{I}$. Besides

$$\mathcal{C}\mathrm{C} = \mathrm{C}\mathcal{C}\,, \quad \mathcal{C}\Phi = \Phi\mathcal{C}\mathrm{C} = \Phi\mathrm{C}\mathcal{C}\,, \quad \Phi\mathcal{C} = \mathcal{C}\Phi\mathrm{C} = \mathcal{C}\mathrm{C}\Phi \,. \tag{17}$$

Now let us find operators $\left(\mathrm{T}_\psi^\varphi\right)^*$ and $\left(\mathrm{S}_\psi^\varphi\right)^*$. First of all, note that formula (7) is equivalent to identity $\Phi^* = \Phi\mathrm{C} = \mathrm{C}\Phi$. Secondly, we have $\mathrm{P}_\rho^* = \mathrm{P}_{\overline{\rho}}$ and $\mathrm{C}^* = \mathrm{C}$. From this and identities (13), we get

$$\left(\mathrm{T}_\psi^\varphi\right)^* = \mathrm{C}\Phi \mathrm{P}_{\overline{\varphi}} \Phi \mathrm{P}_{\overline{\psi}} = \Phi \mathrm{P}_{\overline{\varphi^c}} \Phi \mathrm{P}_{\overline{\psi^c}} \mathrm{C} \,, \tag{18}$$

and

$$\left(\mathrm{S}_\psi^\varphi\right)^* = \mathrm{C}\mathrm{P}_{\psi^c} \Phi \mathrm{P}_{\overline{\varphi}} \Phi \mathrm{P}_{\overline{\psi}} = \mathrm{P}_\psi \Phi \mathrm{P}_{\overline{\varphi^c}} \Phi \mathrm{P}_{\overline{\psi^c}} \mathrm{C} = \mathrm{S}_\psi^{\overline{\varphi}} \,. \tag{19}$$

In particular, an operator S_ψ^φ is hermitian, provided by φ is a real function. Finally, note that

$$\left\langle \mathrm{T}_\psi^\varphi u, v \right\rangle = \sum_x \psi(x)\overline{v}(x)(\widehat{\varphi^c} * u)(x) = \sum_x \varphi(x)\widehat{u}(x)\overline{(\widehat{\overline{\psi} v})(x)} \tag{20}$$

and

$$\left\langle \mathrm{S}_\psi^\varphi u, v \right\rangle = \sum_x \psi(x)\overline{v}(x)(\widehat{\varphi^c} * \overline{\psi} u)(x) = \sum_x \varphi(x)(\widehat{\overline{\psi} u})(x)\overline{(\widehat{\overline{\psi} v})(x)} \,. \tag{21}$$

To obtain the last formulas we have used identity (7).

We need in some statements from linear algebra. Let M be an arbitrary matrix $n \times n$, $M = (m_{ij})_{i,j=1}^n$. By $\mathrm{Spec}\,(M)$ (or $\mathrm{Spec}\, M$) denote the multiset of n eigenvalues of M. For example $\mathrm{Spec}\,\mathrm{P}_\rho = \{\rho(x_1), \ldots, \rho(x_N)\}$, where $\{x_j\}_{j=1}^N = \mathbf{G}$ and

$$\delta_c(x) = \begin{cases} 1 & \text{if } x = c, \\ 0, & \text{otherwise.} \end{cases}$$

are orthonormal eigenfunctions of operator P_ρ. Lemma below can be find in [16, p. 104].

Lemma 2.2. *For any matrices M_1 and M_2 (possibly singular), we have*

$$\mathrm{Spec}\,(M_1 M_2) = \mathrm{Spec}\,(M_2 M_1)\,. \tag{22}$$

Denote by

$$\mu_j(M), \quad j = 1, \dots, n$$

the eigenvalues of M. Let us remind that a matrix M is called *symmetric* (or hermitian) if

$$M^* = \ M, \quad \text{where} \quad M^* = (m^*_{ij})^n_{i,j=1},\ m^*_{ij} = \overline{m}_{ji}.$$

It is well-known that all eigenvalues of such matrices are real. Suppose that the eigenvalues are arranged in order of magnitude

$$\mu_1(M) \ge \mu_2(M) \ge \dots \ge \mu_n(M)\,. \tag{23}$$

Courant-Fischer's theorem (see, e.g., [16, p. 115], gives us the characterization of eigenvalues of symmetric matrices.

Theorem 2.3. *Let M be a symmetric matrix $n{\times}n$. Suppose that its eigenvalues μ_j are arranged like in* (23)*. Then for any*

$$r = 0, 1, \dots, n-1,$$

we have

$$\mu_{r+1}(M) = \min_{v_1, \dots, v_r} \ \max_{f\,:\,\langle f, v_j\rangle = 0,\, \|f\|_2 = 1} \langle Mf, f\rangle\,,$$

where the minimum is taken over all systems of orthogonal vectors $v_1, \dots, v_r$. Besides, for any vectors u and v the following holds

$$|\langle Mu, v\rangle| \le |\mu_1(M)| \cdot \|u\|_2 \|v\|_2\,.$$

The case

$$\mathbf{G} = \mathbb{Z}/p\mathbb{Z} := \mathbb{Z}_p,$$

where p is a prime number is important for applications. Below, we will obtain some statements for the situation. In the proof we will need in Chebotarëv's theorem (see, e.g., [21]).

Theorem 2.4. *Let $\mathbf{G} = \mathbb{Z}_p$, and p be a prime number. Then any submatrix of the operator Φ is nonsingular.*

By $\mathrm{supp}\, f$ denote the support of an arbitrary function $f : \mathbf{G} \to \mathbb{C}$ that is the set

$$\{x \in \mathbf{G} \ : \ f(x) \ne 0\}.$$

Theorem 2.4 implies the following result (see, e.g., [34] or [33]).

Theorem 2.5. *Let $\mathbf{G} = \mathbb{Z}_p$, and p be a prime number. Let also $f : \mathbf{G} \to \mathbb{C}$ be an arbitrary non–zero complex function. Then*

$$|\operatorname{supp} f| + |\operatorname{supp} \widehat{f}| \geq p + 1 \,. \tag{24}$$

By $\operatorname{Span}\{f_j\}$ denote the linear hull of a system of vectors (functions) $\{f_j\}$.

Corollary 2.6. *Let $\mathbf{G} = \mathbb{Z}_p$, and p be a prime number. Suppose that $S = \{s_1, \ldots, s_{|S|}\} \subseteq \mathbb{Z}_p$ is an arbitrary nonempty set. Let also l be a positive integer, $\{f_j\}$ be a family of functions,* $\operatorname{supp} f_j \subseteq S$, *and* $\dim(\operatorname{Span}\{f_j\}) = l$. *Then there exist l linearly independent functions $g_i \in \operatorname{Span}\{f_j\}$, $i \in [l]$ such that $|\operatorname{supp} \widehat{g_i}| \geq p - |S| + l$. If $l = |S|$, then $g_i = \delta_{s_i}$, $i \in [|S|]$.*

P r o o f. Choosing an appropriate basis, we find linearly independent functions $g_i \in \operatorname{Span}\{f_j\}$, $i \in [l]$ such that $|\operatorname{supp} g_i| \leq |S| - l + 1$, and if $l = |S|$, then $g_i = \delta_{s_i}$, $i \in [|S|]$. Using Theorem 2.5, we get $|\operatorname{supp} \widehat{g_i}| \geq p - |S| + l$, $i \in [l]$. This completes the proof. □

Proposition 2.7. *We have*

1) $\operatorname{Spec}\left(\mathrm{T}^{\psi}_{\varphi}\right) = \operatorname{Spec}\left(\mathrm{T}^{\varphi}_{\psi^c}\right) = \operatorname{Spec}\left(\mathrm{T}^{\varphi^c}_{\psi}\right)$.

2) $\operatorname{Spec}\left(\mathrm{T}^{\varphi}_{\psi}\left(\mathrm{T}^{\varphi}_{\psi}\right)^*\right) = N \cdot \operatorname{Spec}\left(\mathrm{T}^{|\varphi|^2}_{|\psi|^2}\right)$ *and* $\mathrm{T}^{\varphi}_{\psi}\left(\mathrm{T}^{\varphi}_{\psi}\right)^* = N \cdot \mathrm{S}^{|\varphi|^2}_{\psi}$.

3) $\operatorname{Spec}\left(\mathrm{S}^{|\varphi|^2}_{\psi}\right) = \operatorname{Spec}\left(\mathrm{S}^{|\psi^c|^2}_{\varphi}\right) = \operatorname{Spec}\left(\mathrm{S}^{|\psi|^2}_{\varphi^c}\right)$.

4) $\operatorname{Spec}\left(\mathrm{S}^{\varphi}_{\psi}\right) = \operatorname{Spec}\left(\mathrm{T}^{\varphi}_{|\psi|^2}\right)$.

P r o o f. By (14), we have

$$\mathrm{T}^{\psi}_{\varphi} = \left(\mathrm{P}_{\varphi}\Phi\right)\left(\mathrm{P}_{\psi^c}\Phi\mathrm{C}\right) \tag{25}$$

and

$$(\mathrm{P}_{\psi^c}\Phi\mathrm{C})(\mathrm{P}_{\varphi}\Phi) = \mathrm{P}_{\psi^c}\Phi\mathrm{P}_{\varphi^c}\Phi\mathrm{C} = \mathrm{T}^{\varphi}_{\psi^c} \,.$$

By Lemma 2.2, we get

$$\operatorname{Spec}\left(\mathrm{T}^{\psi}_{\varphi}\right) = \operatorname{Spec}\left(\mathrm{T}^{\varphi}_{\psi^c}\right).$$

Similarly, using the second identity from (14) instead of (25) and calculations from (15), we obtain

$$\operatorname{Spec}\left(\mathrm{T}^{\psi}_{\varphi}\right) = \operatorname{Spec}\left(\mathrm{T}^{\varphi^c}_{\psi}\right).$$

Let us prove the second part of our proposition. From (13), (14), (15) and (18), we have

$$\mathrm{T}^{\varphi}_{\psi}\left(\mathrm{T}^{\varphi}_{\psi}\right)^* = \mathrm{P}_{\psi}\Phi\mathrm{P}_{\varphi^c}\Phi\mathrm{C}\Phi\mathrm{P}_{\overline{\varphi^c}}\Phi\mathrm{P}_{\overline{\psi^c}}\mathrm{C} = N \cdot \left(\mathrm{P}_{\psi}\Phi\right)\left(\mathrm{P}_{|\varphi^c|^2}\Phi\mathrm{P}_{\overline{\psi^c}}\mathrm{C}\right) = N \cdot \mathrm{S}^{|\varphi|^2}_{\psi}.$$

Further, by (14)

$$\left(\mathrm{P}_{|\varphi^c|^2}\Phi\mathrm{P}_{\overline{\psi^c}}\mathrm{C}\right)\left(\mathrm{P}_\psi\Phi\right)=\mathrm{P}_{|\varphi^c|^2}\Phi\mathrm{P}_{|\psi^c|^2}\Phi\mathrm{C}$$

and by Lemma 2.2 again and also by the first part, we get

$$\mathrm{Spec}\left(\mathrm{T}^{\varphi}_{\psi}\left(\mathrm{T}^{\varphi}_{\psi}\right)^*\right)=N\cdot\mathrm{Spec}\left(\mathrm{T}^{|\psi|^2}_{|\varphi^c|^2}\right)=N\cdot\mathrm{Spec}\left(\mathrm{T}^{|\varphi|^2}_{|\psi|^2}\right).$$

The third part is a corollary of 1) and 2).

We need to check the last part of Proposition 2.7. We have

$$\mathrm{S}^{\varphi}_{\psi}=\left(\mathrm{P}_\psi\Phi\right)\left(\mathrm{P}_{\varphi^c}\Phi\mathrm{P}_{\overline{\psi}^c}\mathrm{C}\right).$$

Hence the multiset $\mathrm{Spec}\left(\mathrm{S}^{\varphi}_{\psi}\right)$ is coincident with the spectrum of operator

$$\mathrm{P}_{\varphi^c}\Phi\mathrm{P}_{\overline{\psi}^c}\mathrm{C}\mathrm{P}_\psi\Phi=\mathrm{P}_{\varphi^c}\Phi\mathrm{P}_{|\psi^c|^2}\Phi\mathrm{C}=\mathrm{T}^{|\psi|^2}_{\varphi^c}.$$

Using 1), we get $\mathrm{Spec}\left(\mathrm{S}^{\varphi}_{\psi}\right)=\mathrm{Spec}\left(\mathrm{T}^{\varphi}_{|\psi|^2}\right)$. This completes the proof. $\square$

Note that $\mathrm{Spec}\left(\mathrm{S}^{\varphi}_{\psi}\right)\neq\mathrm{Spec}\left(\mathrm{S}^{\psi}_{\varphi}\right)$, in general.

Now let us consider a class of operators $\mathrm{T}^{\varphi}_{\psi}$ of special type. Let

$$S=\left\{s_1,\dots,s_{|S|}\right\}\subseteq\mathbf{G}$$

be an arbitrary set. We are interested in the class of operators $\mathrm{T}^{\varphi}_{\psi}$ of the form T^{φ}_{S}, where $\psi(x)=S(x)$ is the characteristic function of our set S.

Let

$$L(S)=\{f\ :\ \mathrm{supp}\,f\subseteq S\}\quad\text{and}\quad L(\overline{S})=\{f\ :\ \mathrm{supp}\,f\subseteq\mathbf{G}\setminus S\}.$$

Clearly, the linear space $f:\mathbf{G}\to\mathbb{C}$ is a direct sum of subspaces $L(S)$ and $L(\overline{S})$. Obviously,

$$\mathrm{dim}L(S)=|S|,\quad\mathrm{dim}L(\overline{S})=N-|S|\quad\text{and}\quad\mathrm{T}^{\varphi}_{S}\big(L(S)\big)\subseteq L(S).$$

Definition 2.8. By $\overline{\mathrm{T}}^{\varphi}_{S}$ denote the restriction of operator T^{φ}_{S} onto the space $L(S)$. Let also $\overline{\mathrm{S}}^{\varphi}_{S}$ be the restriction of S^{φ}_{S} onto $L(S)$.

Clearly,

$$\overline{\mathrm{T}}^{\varphi}_{S}=\overline{\mathrm{S}}^{\varphi}_{S}\quad\text{but}\quad\mathrm{T}^{\varphi}_{S}\neq\mathrm{S}^{\varphi}_{S},$$

in general. Nevertheless, we will prove (see proposition below) that

$$\mathrm{Spec}\left(\mathrm{T}^{\varphi}_{S}\right)=\mathrm{Spec}\left(\mathrm{S}^{\varphi}_{S}\right).$$

The matrix of operator $\overline{\mathrm{T}}^{\varphi}_{S}$ is $\left(\overline{\mathrm{T}}^{\varphi}_{S}\right)_{ij}=\widehat{\varphi}(s_i-s_j)$. It is easy to see that $\overline{\mathrm{T}}^{\varphi}_{S}$ is a symmetric operator, provided by φ is a real function.

Proposition 2.9. *We have*

1) $\mathrm{Spec}\left(\mathrm{T}_S^\varphi\right) = \mathrm{Spec}\left(\mathrm{S}_S^\varphi\right) = \mathrm{Spec}\left(\overline{\mathrm{T}}_S^\varphi\right) \bigcup (0,\dots,0)$, *where* 0 *is taken* $N-|S|$ *times.*
2) *Let* $\varphi(x)$ *be a nonnegative function. Then the operator* $\overline{\mathrm{T}}_S^\varphi$ *is nonnegative definite.*
3) *Let* $\mathbf{G} = \mathbb{Z}_p$, p *be a prime number, and* $S \subseteq \mathbf{G}$ *be an arbitrary nonempty set. Suppose also that* $\varphi(x)$ *is a nonnegative function. The operator* $\overline{\mathrm{T}}_S^\varphi$ *is positively definite iff there exist at least* $|S|$ *elements* $x \in \mathbf{G}$ *such that* $\varphi(x) \neq 0$.

P r o o f. The fourth part of Proposition 2.7 implies the equality

$$\mathrm{Spec}\left(\mathrm{T}_S^\varphi\right) = \mathrm{Spec}\left(\mathrm{S}_S^\varphi\right).$$

Further, the operator S_S^φ has $N-|S|$ linearly independent eigenfunctions corresponding zero, namely $\delta_{\overline{s}}(x)$, $\overline{s} \notin S$. The last functions are linearly independent with eigenfunctions of the restriction of the operator S_S^φ onto $L(S)$. Besides

$$\mathrm{Spec}\left(\overline{\mathrm{S}}_S^\varphi\right) = \mathrm{Spec}\left(\overline{\mathrm{T}}_S^\varphi\right) \subseteq \mathrm{Spec}\left(\mathrm{T}_S^\varphi\right)$$

and we have proved the first part of the Proposition.

Further, let M be the matrix such that

$$M_{st} = \left\{\varphi^{1/2}(t)e(-st)\right\}, \qquad s \in S,\ t \in \mathbf{G}.$$

Then

$$MM^* = \overline{\mathrm{T}}_S^\varphi$$

and the operator $\overline{\mathrm{T}}_S^\varphi$ is nonnegative definite. Let now $\mathbf{G} = \mathbb{Z}_p, p$ be a prime number. If there are $x_1,\dots,x_l \in \mathbf{G}$, $l \geq |S|$ such that $\varphi(x_j) \neq 0$, $j \in [l]$, then choose a square submatrix of M, say M', which corresponds to elements $x_1,\dots,x_{|S|}$. By Theorem 2.4, we have $\det M' \neq 0$. Using Binet-Cauchy's formula, we get

$$\det \overline{\mathrm{T}}_S^\varphi \geq \left(\det M'\right)^2 > 0.$$

Hence all eigenvalues of the matrix $\overline{\mathrm{T}}_S^\varphi$ are positive. Remember $\overline{\mathrm{T}}_S^\varphi$ is a symmetric operator, we obtain that it is positively definite. This completes the proof. □

Example 2.10. Let $S_1, S_2 \subseteq \mathbb{Z}_p$ be arbitrary sets such that $|S_1| \leq |S_2|$. Then by the last proposition, we see that the operator $\overline{\mathrm{T}}_{S_1}^{S_2}$ is positively definite. Note that $0 \in \mathrm{Spec}\left(\overline{\mathrm{T}}_{S_2}^{S_1}\right)$ with multiplicity $|S_2|-|S_1|$. Hence the operator $\overline{\mathrm{T}}_{S_2}^{S_1}$ is singular.

Formula (20) can be rewritten for operator $\overline{\mathrm{T}}^{\varphi}_S$ as

$$\left\langle \overline{\mathrm{T}}^{\varphi}_S u, v \right\rangle = \sum_x (\widehat{\varphi^c} * u)(x)\overline{v}(x) = \sum_x \varphi(x)\widehat{u}(x)\overline{\widehat{v}(x)} = \left\langle \overline{\mathrm{S}}^{\varphi}_S u, v \right\rangle , \qquad (26)$$

where u, v are arbitrary functions such that $\operatorname{supp} u, \operatorname{supp} v \subseteq S$. Besides

$$\operatorname{tr}\left(\overline{\mathrm{T}}^{\varphi}_S\right) = \operatorname{tr}\left(\overline{\mathrm{S}}^{\varphi}_S\right) = |S|\widehat{\varphi}(0) = \sum_{j=1}^{|S|} \mu_j\left(\overline{\mathrm{T}}^{\varphi}_S\right) = \sum_{j=1}^{N} \mu_j\left(\mathrm{T}^{\varphi}_S\right). \qquad (27)$$

If φ is a real function, then as was noted before $\overline{\mathrm{T}}^{\varphi}_S$ is a symmetric matrix. In particular, it is a normal matrix. Using (16), and identities

$$(S * S^c)(z) = (S * S^c)(-z), \quad \widehat{S^c}(\xi) = \overline{\widehat{S}(\xi)},$$

we get

$$\begin{aligned} \operatorname{tr}\left(\overline{\mathrm{T}}^{\varphi}_S\left(\overline{\mathrm{T}}^{\varphi}_S\right)^*\right) &= \sum_z |\widehat{\varphi}(z)|^2 (S * S^c)(z) \\ &= \sum_z \big(\varphi * \varphi^c\big)(z)|\widehat{S}(z)|^2 = \sum_{j=1}^{|S|} \mu_j^2\left(\overline{\mathrm{T}}^{\varphi}_S\right) = \sum_{j=1}^{N} \mu_j^2\left(\mathrm{T}^{\varphi}_S\right). \qquad (28) \end{aligned}$$

The identities from the first part of Proposition 2.7 show that there is a duality between operators $\mathrm{T}^{\varphi}_{\psi}$ and $\mathrm{T}^{\psi}_{\varphi}$. Having this in mind one can suppose that it should exists a dual version of operator $\overline{\mathrm{T}}^{\varphi}_S$. In the rest of the section we will define the dual version which we will call $\overline{\mathrm{T}}^{S}_{\varphi}$. Also we will study eigenvalues and eigenfunctions of operators T^{φ}_S and T^{S}_{φ} in the case $\mathbf{G} = \mathbb{Z}_p$, where p is a prime number.

Let us make a general remark. Let $f(x)$ be an eigenfunction of operator $\mathrm{T}^{\varphi}_{\psi}$ corresponding an eigenvalue μ. In other words $f(x)$ is a non–zero function and $\mathrm{T}^{\varphi}_{\psi} f = \mu f$. Consider operator $(\mathrm{M}_{\varphi} f)(x) := \varphi(x)\widehat{f}(x)$ and define the function $F := \mathrm{M}_{\varphi} f$. Using the third identity from (14), we get $\mathrm{T}^{\psi^c}_{\varphi} F = \mu F$. Thus, if $F(x)$ is a non-zero function, then F is an eigenfunction of operator $\mathrm{T}^{\psi^c}_{\varphi}$ corresponding to μ.

Recall that a matrix called *simple* if there exist a basis of its eigenvectors.

Proposition 2.11. *Let φ, ψ be arbitrary functions.*

1) *Suppose that $\mathrm{T}^{\varphi}_{\psi}$ is a simple matrix. Then $\mathrm{T}^{\varphi^c}_{\psi^c}$ is also a simple matrix.*
2) *Let $\mathrm{T}^{\varphi}_{\psi}$ be a simple matrix and φ, ψ are real functions. Then all matrices $\mathrm{T}^{\varphi^c}_{\psi}$, $\mathrm{T}^{\varphi}_{\psi^c}$, $\mathrm{T}^{\varphi^c}_{\psi^c}$ are simple.*
3) *Suppose that $\mathrm{T}^{\varphi}_{\psi}$ is a simple matrix and* $\operatorname{supp} \varphi = \mathbf{G}$. *Then $\mathrm{T}^{\psi}_{\varphi^c}$ is also a simple matrix.*

P r o o f. Let f be an eigenfunction of operator T_ψ^φ corresponding an eigenvalue μ. Then

$$\psi(x)(\widehat{\varphi^c} * f)(x) = \mu f(x)\,. \tag{29}$$

Apply the operator C to the last identity and use formula 16, we get 1). Let us prove the second part. Apply the operator $\mathcal{C}$ to (29) and use formula (17), we obtain that $\mathrm{T}_\psi^{\varphi^c}$ is a simple matrix. The simplicity of another two matrices can be derived from the first part.

Further, the third identity from (13) gets $\Phi^4 = N^2 \cdot \mathrm{I}$. Hence Φ is a non–singular operator. Let $\{f_j\},\ j \in [N]$ be a basis of eigenfunctions of the operator T_ψ^φ. Since T_ψ^φ is a simple matrix, then these functions are linearly independent. For each $j \in [N]$ consider the function $F_j = \mathrm{M}_\varphi f_j$. Using the condition $\operatorname{supp}\varphi = \mathbf{G}$ and non-singularity of the operator Φ, we get that these functions are linearly independent and non-zero, in particular. This completes the proof. $\square$

Now let us consider the case $\mathbf{G} = \mathbb{Z}_p$.

Proposition 2.12. *Let $\mathbf{G} = \mathbb{Z}_p$, p be a prime number, and $S \subseteq \mathbf{G}$ be an arbitrary set.*

1) *Let ψ be a nonnegative function. If there are at least $|S|$ elements $x \in \mathbf{G}$ such that $\psi(x) \neq 0$, then T_ψ^S is a simple matrix.*
2) *Let φ be a nonnegative function. If for all $x \in \mathbf{G}$, we have $\varphi(x) \neq 0$, then T_S^φ is a simple matrix.*

P r o o f. Using formula (11) and the identity $\widehat{e(ax)} = p \cdot \delta_a(x)$ is is easy to see that the functions $e(\overline{s}x)$, $\overline{s} \notin S$ are eigenfunctions of the operator T_ψ^S corresponding to zero. Clearly, these functions are linearly independent. Further, by Proposition 2.7, we have

$$\operatorname{Spec}\left(\mathrm{T}_\psi^S\right) = \operatorname{Spec}\left(\mathrm{T}_{S^c}^\psi\right) = \operatorname{Spec}\left(\mathrm{T}_S^{\psi^c}\right).$$

Using the third part of Proposition 2.9, we see T_ψ^S has $|S|$ positive eigenvalues. Hence, the correspondent eigenfunctions are linearly independent with $e(\overline{s}x)$, $\overline{s} \notin S$. Let f_j, $j \in [|S|]$ be eigenfunctions of operator $\overline{\mathrm{T}}_{S^c}^\psi$. Clearly, $\operatorname{supp} f_j \subseteq S^c$, $j \in [|S|]$. Let $F_j = M_\psi f_j$, $j \in [|S|]$. We assert that these functions are linearly independent and non-zero, in particular. Indeed, Theorem 2.5 and condition $|\operatorname{supp}\psi| \geq |S^c| = |S|$ imply that the operator M_ψ is invertible onto $L(S^c)$. Using the argument as before, we get F_j are linearly independent functions of the operator T_ψ^S corresponding positive eigenvalues. Hence T_ψ^S is a simple matrix.

Let us prove the second part of Proposition 2.8. For each $\overline{s} \notin S$ consider the equation $M_\varphi f_{\overline{s}} = e(-\overline{s}x)$. Here $f_{\overline{s}}$ is an unknown function. Since $\varphi(x) \neq 0$ for all $x \in \mathbf{G}$, it follows that the equation is solvable. Using formula (11) it is easy to see that $\mathrm{T}^{\varphi}_S f_{\overline{s}} = 0$ for all $\overline{s} \notin S$. Besides, the functions $f_{\overline{s}}$, $\overline{s} \notin S$ are linearly independent. Indeed, if

$$\sum_{\overline{s} \notin S} c_{\overline{s}} f_{\overline{s}} \equiv 0, \qquad \text{then} \qquad \sum_{\overline{s} \notin S} c_{\overline{s}} \widehat{f_{\overline{s}}} \equiv 0.$$

Remember the definition of the functions $f_{\overline{s}}$, we have

$$\sum_{\overline{s} \notin S} c_{\overline{s}} \varphi^{-1}(x) e(-\overline{s}x) \equiv 0.$$

Using Theorem 2.4 for the matrix Φ, we obtain that all coefficients $c_{\overline{s}}$ equal zero. Hence, the all the functions $f_{\overline{s}}$, $\overline{s} \notin S$ are linearly independent. We have $|\operatorname{supp} \varphi| = p \geq |S|$. By assumption $\varphi(x)$ is a non-negative function. Using the third part of Proposition 2.9, we get all eigenvalues of the operator $\overline{\mathrm{T}}^{\varphi}_S$ are positive. Whence, the functions $f_{\overline{s}}$, $\overline{s} \notin S$ are linearly independent with the eigenfunctions of $\overline{\mathrm{T}}^{\varphi}_S$. Thus, T^{φ}_S is a simple matrix. This completes the proof. $\square$

Finally, let us define the operator $\overline{\mathrm{T}}^{S}_{\psi}$. Let $\psi(x)$ be an arbitrary complex function. Consider the linear space

$$L^*(S) = \{ f \ : \ f = \psi(x) a(x), \operatorname{supp} \widehat{a} \subseteq S \} \quad \text{and, analogously,} \quad L^*(\overline{S}).$$

Since

$$\mathrm{T}^{S}_{\psi} f = \psi(x) (\widehat{S^c \widehat{f^c}})(x),$$

it follows that the space $L^*(S)$ is invariant under the action of the operator T^{S}_{ψ}. By definition, the restriction T^{S}_{ψ} onto $L^*(S)$ is the operator $\overline{\mathrm{T}}^{S}_{\psi}$. Analogously, the space $L^*(S)$ is invariant under the action of the operator S^{S}_{ψ}, and we can define $\overline{\mathrm{S}}^{S}_{\psi}$ as the restriction of the operator S^{S}_{ψ} onto the space. It is easy to see that

$$L^*(S) = \mathrm{M}_\psi L(S) \quad \text{and} \quad L^*\big(\overline{S}\big) = \mathrm{M}_\psi L\big(\overline{S}\big).$$

Thus, in particular, we have

$$\operatorname{Spec} \overline{\mathrm{T}}^{S}_{\psi} = \operatorname{Spec} \overline{\mathrm{T}}^{\psi}_{S} = \operatorname{Spec} \overline{\mathrm{S}}^{\psi}_{S},$$

provided by the operator M_ψ is invertible onto $L(S)$ (the last situation happens if $\operatorname{supp} \psi = \mathbf{G}$, for example, or $|\operatorname{supp} \psi| \geq |S|$, if $\mathbf{G} = \mathbb{Z}_p$). It is not difficult to see (from the fourth part of Proposition 2.7, say) that

$$\operatorname{Spec} \overline{\mathrm{T}}^{S}_{\psi} \neq \operatorname{Spec} \overline{\mathrm{S}}^{S}_{\psi},$$

in general.

Let us define the dimension of the space $L^*(S)$.

Statement 2.13. Let p be a prime number, $\mathbf{G} = \mathbb{Z}_p$, $S \subseteq \mathbf{G}$ be a nonempty set, and ψ be an arbitrary complex function. Then $\dim\big(L^*(S)\big) = \min\{|S|, |\operatorname{supp}\psi|\}$.

P r o o f. Since $\dim\big(L(S)\big) = |S|$ it follows that

$$\dim\big(L^*(S)\big) \le \min\big\{|S|, |\operatorname{supp}\psi|\big\} := m.$$

Let $\operatorname{supp}\psi = \{a_1, \ldots, a_t\}$. Consider the case $t \le |S|$ (we can use similar arguments at the opposite situation). Apply Corollary 2.6 to the system of linearly independent eigenfunctions of the operator $\overline{\mathrm{T}}^{\psi}_{S}$. To prove the inequality $\dim(L^*(S)) \ge m = t$ it is sufficiently to solve the equation

$$\sum_{s \in S} c_s e(-sx) = \delta_{a_i}(x) \quad \text{for each} \ \ i \in [t].$$

Here coefficients c_s are unknowns. Clearly, the last equation is solvable, because by Theorem 2.4 the matrix $\{e(-sx)\}_{s\in S,\, x \in \operatorname{supp}\psi}$ has full rank. This completes the proof. $\square$

Example 2.14. Let $S_1, S_2 \subseteq \mathbb{Z}_p$ be arbitrary nonempty sets such that

$$|S_1| \le |S_2|.$$

Then the operator $\overline{\mathrm{T}}^{S_2}_{S_1}$ can be defined as the restriction of $\mathrm{T}^{S_2}_{S_1}$ onto $L(S_1)$ and it can be defined as the restriction of $\mathrm{T}^{S_2}_{S_1}$ onto $L^*(S_2)$. It is easy to see that all these definitions define the same operator.

3. Some generalizations of M.–C. Chang's inequality

Properties of the operators $\mathrm{T}^{\varphi}_{\psi}$, $\overline{\mathrm{T}}^{\varphi}_{S}$, which were considered at the previous section allow us to prove a theorem of M.-C. Chang [6] from the theory of large exponential sums. Moreover, we will show that the theorem and its generalizations of different types can be derived from Proposition 3.1, which is essentially a reformulation of Rudin's inequality (1).

For a real function $\varphi(x)$ let $\varphi_0(x) = \varphi(x)$, if $x \ne 0$ and zero otherwise.

Proposition 3.1. *Let $\mathbf{G}$ be a finite Abelian group, $\varphi(x)$ be a real non-zero function, and $\Lambda \subseteq \mathbf{G}$ be a dissociated set. Then*

$$\Big|\mu_1\Big(\overline{\mathrm{T}}^{\varphi}_{\Lambda}\Big)\Big| \ll \|\varphi\|_1 \Big(\log\big(N\|\varphi\|_\infty \|\varphi\|_1^{-1}\big) + 1\Big). \tag{30}$$

Let now $|\varphi(0)| = \|\varphi\|_\infty$. *If* $|\varphi(0)| \ge \|\varphi_0\|_1$, *then*

$$\left|\mu_1\left(\overline{\mathrm{T}}^{\varphi}_{\Lambda}\right)\right| \ll |\varphi(0)| \log N\,. \tag{31}$$

If $|\varphi(0)| \le \|\varphi_0\|_1$, *then*

$$\left|\mu_1\left(\overline{\mathrm{T}}^{\varphi}_{\Lambda}\right)\right| \ll \|\varphi_0\|_1\left(\log\left(N\|\varphi\|_\infty\|\varphi_0\|_1^{-1}\right) + 1\right). \tag{32}$$

P r o o f. By assumption $\varphi(x)$ is a real function, so $\mu_1\left(\overline{\mathrm{T}}^{\varphi}_{\Lambda}\right)$ is a real number. Let w be an arbitrary function, $\operatorname{supp} w \subseteq \Lambda$, $\|w\|_2 = 1$. Using (26), we have

$$\sigma := \max_{\|w\|_2=1} \left\langle \overline{\mathrm{T}}^{\varphi}_{\Lambda} w, w \right\rangle = \sum_x |\widehat{w}(x)|^2 \varphi(x)\,.$$

Let $k > 0$ be an integer parameter. By Rudin's inequality and Hölder's inequality, we get

$$\begin{aligned} |\sigma|^k &\le \sum_x |\widehat{w}(x)|^{2k} \cdot \left(\sum_x |\varphi(x)|^{k/(k-1)}\right)^{k-1} = \sigma_1 \cdot \sigma_2 \\ &\le \left(2C^2\right)^k N \|w\|_2^{2k} k^k \|\varphi\|_\infty \|\varphi\|_1^{k-1} \\ &= \left(2C^2\right)^k k^k \|\varphi\|_1^k \cdot N \|\varphi\|_\infty \|\varphi\|_1^{-1}\,, \end{aligned} \tag{33}$$

where $C > 0$ is the absolute constant from (1). Putting

$$k = \left[\log\left(N\|\varphi\|_\infty\|\varphi\|_1^{-1}\right)\right] + 1$$

and using Propositions 2.7, 2.9, and Theorem 2.3, we obtain

$$\mu_1\left(\mathrm{T}^{\Lambda}_{\varphi}\right) = \mu_1\left(\overline{\mathrm{T}}^{\varphi}_{\Lambda}\right) \le \sigma \ll \|\varphi\|_1 \left(\log\left(N\|\varphi\|_\infty\|\varphi\|_1^{-1}\right) + 1\right)\,.$$

We need to check (31), (32). Return to (33). We have

$$\sigma_2^{1/(k-1)} = \sum_x |\varphi(x)|^{k/(k-1)} \le |\varphi(0)|^{k/(k-1)} + \|\varphi_0\|_\infty^{1/(k-1)} \|\varphi_0\|_1\,.$$

By assumption $|\varphi(0)| = \|\varphi\|_\infty$. If $|\varphi(0)| \ge \|\varphi_0\|_1$, then $\sigma_2 \le 2^{k-1}|\varphi(0)|^k$. Put $k = [\log N] + 1$ and use the arguments as before, we get (31). If $|\varphi(0)| \le \|\varphi_0\|_1$, then $\sigma_2 \le 2^{k-1}\|\varphi\|_\infty\|\varphi_0\|_1^{k-1}$. Substitute the last inequality into (33) and put $k = \left[\log(N\|\varphi\|_\infty\|\varphi_0\|_1^{-1})\right] + 1$, we obtain (32). This completes the proof. $\square$

Example 3.2. Suppose that $\varphi(x) = S(x)$, where S is an arbitrary subset of the group $\mathbf{G}$. Then inequality (30) implies

$$\left|\mu_1\left(\overline{\mathrm{T}}^{\Lambda}_{S}\right)\right| \ll |S| \log 1/\delta\,. \tag{34}$$

Let us obtain some applications of Rudin's inequality. First of all derive Chang's Theorem from Proposition 3.1.

Theorem 3.3. *Let $\delta \in (0,1]$ be a real number, $\Lambda \subseteq \mathbf{G}$ be a dissociated set, and $S \subseteq \mathbf{G}$ be an arbitrary set, $|S| = \delta N$. Then for any function f, $\operatorname{supp} f \subseteq S$, we have*

$$\sum_{x\in\Lambda} |\widehat{f}(x)|^2 \ll |S| \cdot \|f\|_2^2 \log(1/\delta)\,. \tag{35}$$

In particular,

$$\sum_{x\in\Lambda} |\widehat{S}(x)|^2 \ll |S|^2 \log(1/\delta)\,. \tag{36}$$

P r o o f. We give even two proofs. Put $\varphi(x)$ equals $S(x)$. We have (34). By (27), we get $\mu_1\big(\overline{\mathrm{T}}^{\Lambda}_S\big) > 0$. Using Courant-Fischer's Theorem and identity (26), we obtain

$$\sum_{x\in\Lambda} |\widehat{f}(x)|^2 = \left\langle \overline{\mathrm{T}}^{\Lambda}_S f, f\right\rangle \le \mu_1\left(\overline{\mathrm{T}}^{\Lambda}_S\right)\|f\|_2^2 \ll |S| \cdot \|f\|_2^2 \log(1/\delta)$$

as required.

Let us give another proof. Using Theorem 2.3, formulas (6), (10), (11), (16), (34), the second part of the Proposition 2.7 and Proposition 3.1, we get

$$\left\langle \mathrm{T}^S_{\Lambda^c}\widehat{f^c}, \mathrm{T}^S_{\Lambda^c}\widehat{f^c}\right\rangle = \sum_{x\in\Lambda} |(\widehat{S} * \widehat{f})(x)|^2 = N^2 \sum_{x\in\Lambda} |\widehat{f}(x)|^2 \le \mu_1\left(\mathrm{T}^S_{\Lambda^c}(\mathrm{T}^S_{\Lambda^c})^*\right) \cdot \|\widehat{f^c}\|_2^2$$
$$= N \cdot \mu_1\left(\overline{\mathrm{T}}^S_{\Lambda^c}\right) \cdot N\|f\|_2^2 \ll |S| \cdot \|f\|_2^2 \log(1/\delta) N^2\,.$$

This completes the proof. □

Note 3.4. We can analogously obtain a generalization of Chang's Theorem, belonging J. Bourgain [5] (see detailed proof in [30]).

Our approach allows to prove new formulas. For example, using (6), the bound $\mu_1\left(\overline{\mathrm{T}}^{\Lambda}_S\right) \ll |S| \log 1/\delta$, formulas (7), (10) and Courant-Fischer's theorem, we get

$$\left\langle \left(\overline{\mathrm{T}}^{\Lambda}_S\right)^2 S, S\right\rangle = \sum_z S(z)\left(\widehat{\Lambda}^c * S(\widehat{\Lambda}^c * S)\right)(z) = \sum_{z\in\Lambda} \overline{\widehat{S}(z)}\big(\widehat{S} * (\widehat{S}\Lambda)\big)(z)$$
$$\le \mu_1^2\left(\overline{\mathrm{T}}^{\Lambda}_S\right) \cdot \|S\|_2^2 \ll |S|^3 \log^2(1/\delta)\,.$$

Hence for any dissociated set Λ, we have

$$\left|\sum_{z\in\Lambda} \overline{\widehat{S}(z)}(\widehat{S} * \widehat{S}\Lambda)(z)\right| \ll |S|^3 \log^2(1/\delta)\,.$$

Note 3.5. Using (27), (28), we can obtain some information about all eigenvalues $\mu_j\big(\overline{\mathrm{T}}^S_\Lambda\big) = \mu_j$, $j \in [|\Lambda|]$, not only $\mu_1\big(\overline{\mathrm{T}}^S_\Lambda\big)$. First of all, by Proposition 2.9 all these eigenvalues are nonnegative (in the case $\mathbf{G} = \mathbb{Z}_p$, p is a prime number, we have $\mu_j > 0$ for any j). Secondly, from (27), we have $\sum_{j=1}^{|\Lambda|} \mu_j = |\Lambda||S|$. Thirdly, by formula (28), we get

$$\sum_{j=1}^{|\Lambda|} \mu_j^2 = \sum_z |\widehat{S}(z)|^2 (\Lambda * \Lambda^c)(z) = |S|^2|\Lambda| + \sum_{z \in \Lambda \dotplus \Lambda} |\widehat{S}(z)|^2 \,,$$

where $\Lambda \dotplus \Lambda^c = \{\lambda_1 - \lambda_2 \ :\ \lambda_1, \lambda_2 \in \Lambda,\ \lambda_1 \neq \lambda_2\}$. Finally, using an inequality of J. Bourgain (see [5] or [30])

$$\sum_{z \in \Lambda \dotplus \Lambda} |\widehat{S}(z)|^2 \ll |S|^2 \log^2(1/\delta) \,,$$

we obtain

$$\sum_{j=1}^{|\Lambda|} (\mu_j - |S|)^2 \ll |S|^2 \log^2(1/\delta) \,.$$

In particular, the last formula implies that there are at most $O(1)$ numbers μ_j such that $\mu_j \gg |S| \log(1/\delta)$.

Now let us obtain Theorems 1.3 and 1.4.

Proof of Theorem 1.3. Let $f(x) = S(x) - \delta$. Clearly, $\widehat{f}(0) = 0$ and $\widehat{f}(x) = \widehat{S}(x)$, $x \neq 0$. Define the function $F(x)$ by the formula $\widehat{F}(x) = |\widehat{S}(x)|$. If $l = 2$, then put $u(x) = (F * f)(x)$. If $l = 2k$, $k \geq 2$, then let

$$u(x) = (F * (f *_{k-1} f)) * (f^c *_{k-2} f^c)(x).$$

Finally, if $l = 2k+1$, $k \geq 1$, then put $u(x) = (f *_k f) * (f^c *_{k-1} f^c)(x)$. Let also $v(x) = S(x)$. By assumption Λ is a dissociated set. Hence $0 \notin \Lambda$. Using formula (20) and inclusion $\operatorname{supp} v \subseteq S$, it is easy to see that for any $l \geq 2$, we have

$$\sigma := \left\langle \mathrm{T}^\Lambda_S u, v \right\rangle = \sum_{x \in \Lambda} |\widehat{S}(x)|^{l+1} \,. \tag{37}$$

Using the Cauchy-Schwarz inequality, the second part of Proposition 2.7 and the bound $\mu_1(\mathrm{T}^\Lambda_S) \ll |S| \log(1/\delta)$, we get

$$\sigma^2 \leq \langle \mathrm{T}^\Lambda_S u, \mathrm{T}^\Lambda_S u \rangle \cdot \langle v, v \rangle \leq N \mu_1\left(\mathrm{T}^\Lambda_S\right) \|u\|_2^2 \|v\|_2^2 \ll |S|^2 \log(1/\delta) \cdot \sum_{x \neq 0} |\widehat{S}(x)|^{2l}$$

because of $|\widehat{u}(x)|^2 = |\widehat{S}(x)|^{2l}$ and the Parseval identity. Substitute the last identity into (37), we get

$$\sum_{x\in\Lambda} |\widehat{S}(x)|^{l+1} \ll |S| \log^{1/2}(1/\delta) \cdot \left(\sum_{x\neq 0} |\widehat{S}(x)|^{2l} \right)^{1/2}$$

as required. This completes the proof. $\square$

Note 3.6. Let $\delta, \alpha \in (0,1]$ be real parameters, $\alpha \le \delta$, and $\Lambda \subseteq \mathbb{Z}_N$ be an arbitrary dissociated set, such that $|\Lambda| \gg (\delta/\alpha)^2 \log(1/\delta)$. Developing the approach from papers [10, 11], in article [29] (see Theorem 2.8) the following set S was constructed. Suppose that there are some restrictions onto parameters δ, α, and $|\Lambda| \ll (\delta/\alpha)^2 \log(1/\delta)$. Then exists a set S, $S \subseteq \mathbb{Z}_N$, $|S| = \delta N$ such that

1) For all $x \neq 0$, we have $|\widehat{S}(x)| \ll \alpha N$.
2) For each $x \in \Lambda \bigsqcup (-\Lambda)$, the following holds $|\widehat{S}(x)| \gg \alpha N$.
2) For any $x \neq 0$, $x \in \Lambda \bigsqcup (-\Lambda)$, we have $|\widehat{S}(x)| \ll \varepsilon N$, where $\varepsilon = \alpha^2 \delta^{-1}$.

The set S gives us an example showing that inequality (3) of Theorem 1.3 is sharp. Indeed, by 2) and the inequality $|\Lambda| \gg (\delta/\alpha)^2 \log(1/\delta)$, we obtain

$$\sum_{x\in\Lambda} |\widehat{S}(x)|^{l+1} \gg |\Lambda| (\alpha N)^{l+1} \gg (\alpha N)^l \delta^2 \alpha^{-1} \log(1/\delta) . \tag{38}$$

From the other hand, using Parseval's identity and 1), 3), we get

$$\begin{aligned} \sum_{x\neq 0} |\widehat{S}(x)|^{2l} &= \sum_{x\in\Lambda\bigsqcup(-\Lambda)} |\widehat{S}(x)|^{2l} + \sum_{x\notin\Lambda\bigsqcup(-\Lambda),\, x\neq 0} |\widehat{S}(x)|^{2l} \\ &\ll |\Lambda|(\alpha N)^{2l} + (\varepsilon N)^{2l-2} \delta N^2 \ll |\Lambda|(\alpha N)^{2l} . \end{aligned}$$

To prove the last inequality we have supposed that $\alpha \ll \delta \cdot \delta^{1/(2l-2)}$. Thus

$$\begin{aligned} |S| \log^{1/2}(1/\delta) \cdot \left(\sum_{x\neq 0} |\widehat{S}(x)|^{2l} \right)^{1/2} &\ll \delta N \log^{1/2}(1/\delta) |\Lambda|^{1/2} (\alpha N)^l \\ &\ll (\alpha N)^l \delta^2 \alpha^{-1} \log(1/\delta) \end{aligned} \tag{39}$$

and we see that assuming the condition $\alpha \ll \delta \cdot \delta^{1/(2l-2)}$, we have lower bound (38) is coincident (up to constants) to upper bound (39).

Now let us prove Theorem 1.4.We obtain even a tiny stronger result.

Theorem 3.7. *Let* $\mathbf{G}$ *be a finite Abelian group, and* l *be a positive integer,* $l \ge 2$. *Let also* $\Lambda \subseteq \mathbf{G}$ *be dissociated set, and* $S_1, \dots, S_l \subseteq \mathbf{G}$ *be arbitrary sets. Then*

$$\sum_{x\in\Lambda} \big(S_1 * S_2 * \dots * S_l\big)^2(x) \ll \frac{|S_l|}{N} \left(\sum_x \prod_{j=1}^{l-1} |\widehat{S}_j(x)|^2 \right) \cdot \log N\,. \qquad (40)$$

If

$$\prod_{j=1}^{l-1} |S_j|^2 \ge \sum_{x\neq 0} \prod_{j=1}^{l-1} |\widehat{S}_j(x)|^2$$

then

$$\sum_{x\in\Lambda} \big(S_1 * S_2 * \dots * S_l\big)^2(x) \ll \frac{|S_l|}{N} \left(\prod_{j=1}^{l-1} |S_j|^2 \right) \cdot \log N\,. \qquad (41)$$

If

$$\prod_{j=1}^{l-1} |S_j|^2 \le \sum_{x\neq 0} \prod_{j=1}^{l-1} |\widehat{S}_j(x)|^2$$

then

$$\sum_{x\in\Lambda} \big(S_1 * S_2 * \dots * S_l\big)^2(x) \ll \frac{|S_l|}{N} \left(\sum_{x\neq 0} \prod_{j=1}^{l-1} |\widehat{S}_j(x)|^2 \right) \cdot \log N\,. \qquad (42)$$

Suppose that

$$S_1 = S_2 = \dots = S_{l-1} = S, \qquad |S| \le N/2$$

and

$$|S|^{2l-2} \le \sum_{x\neq 0} |\widehat{S}(x)|^{2l-2}.$$

Then

$$\sum_{x\in\Lambda} \big((S *_{l-2} S) * S_l\big)^2(x) \ll \frac{l|S_l|}{N} \left(\sum_{x\neq 0} |\widehat{S}(x)|^{2l-2} \right) \cdot \log|S|\,. \qquad (43)$$

Finally, always

$$\sum_{x\in\Lambda} \big(S_1 * S_2\big)^2(x) \ll |S_1||S_2| \cdot \log\big(\min\{|S_1|, |S_2|\}\big)\,. \qquad (44)$$

Proof of Theorem 3.7. Let $f(x) = S_l(x)$, $\varphi(x) = \prod_{j=1}^{l-1} \widehat{S}^c_j(x)$. Using the arguments similar to the second variant of the proof of Theorem 3, we get

$$\begin{aligned}\left\langle \mathrm{T}^{\varphi}_{\Lambda^c} f^c, \mathrm{T}^{\varphi}_{\Lambda^c} f^c \right\rangle &= \sum_{x\in\Lambda} |(\widehat{\varphi} * f)(x)|^2 = N^2 \sum_{x\in\Lambda} (S_1 * S_2 * \cdots * S_l)^2(x) \\ &\le \mu_1\big(\mathrm{T}^{\varphi}_{\Lambda^c} (\mathrm{T}^{\varphi}_{\Lambda^c})^*\big) \cdot \|f^c\|_2^2 \\ &= N \cdot \mu_1\Big(\overline{\mathrm{T}}^{|\varphi|^2}_{\Lambda^c}\Big) \cdot \|f\|_2^2 \ll N|S_l|\mu_1\big(\overline{\mathrm{T}}^{\varphi^*}_{\Lambda^c}\big)\,,\end{aligned}$$

where

$$\varphi^*(x) = |\varphi(x)|^2 = \prod_{j=1}^{l-1} |\widehat{S}^c_j(x)|^2.$$

We have

$$\|\varphi^*\|_\infty = \varphi^*(0) = \prod_{j=1}^{l-1} |S_j|^2 \quad \text{and} \quad \|\varphi^*\|_1 \ge \prod_{j=1}^{l-1} |S_j|^2.$$

Proposition 3.1 gives us some inequalities for the quantity $\mu_1(\overline{\mathrm{T}}^{\varphi^*}_{\Lambda^c})$. Applying (30), (31), we get (40) and (41), correspondingly. If

$$\prod_{j=1}^{l-1} |S_j|^2 \le \sum_{x\ne 0} \prod_{j=1}^{l-1} |\widehat{S}_j(x)|^2,$$

then

$$\|\varphi^*_0\|_1 \ge |\varphi^*(0)| = \|\varphi^*\|_\infty$$

and inequality (32) implies (42). In the case $l = 2$ the quantity $\|\varphi^*\|_1$ can be computed. Indeed, by Parseval's identity, we obtain $\|\varphi^*\|_1 = |S_1|N$. Whence, inequality (44) holds. Finally, we need to check (43). By assumption

$$|S|^{2l-2} \le \sum_{x\ne 0} |\widehat{S}(x)|^{2l-2}.$$

Thus we can use inequality (32) of Proposition 3.1 to estimate $\mu_1\big(\overline{\mathrm{T}}^{\varphi^*}_{\Lambda^c}\big)$. By assumption $|S| \le N/2$. The last bound and Parseval's identity imply

$$\sum_{x\ne 0} |\widehat{S}(x)|^2 = N|S| - |S|^2 \ge 2^{-1} N|S|.$$

Using Hölder's inequality, we obtain

$$\|\varphi^*_0\|_1 \ge 2^{-(l-1)} N|S|^{l-1}$$

and inequality (43). This concludes the proof. □

Corollary 3.8. *Let* $\mathbf{G}$ *be a finite Abelian group,* r *be a positive integer. Let also* $\Lambda \subseteq \mathbf{G}$ *be a dissociated set, and* $S_1, S_2 \subseteq \mathbf{G}$ *be arbitrary sets. Suppose that for any* $x \in \Lambda$ *the following holds* $(S_1 * S_2)(x) \geq r$*. Then*

$$|\Lambda| \ll r^{-2}|S_1||S_2| \cdot \log\bigl(\min\bigl\{|S_1|, |S_2|\bigr\}\bigr) \,. \tag{45}$$

Note 3.9. Let $\mathbf{G} = (\mathbb{Z}/p\mathbb{Z})^n$, p be a prime number (usefulness of considering such groups was discussed in survey [13]), and $S_1 = S_2 = P$, where P be a linear subspace of $\mathbf{G}$. Clearly, any dissociated set $\Lambda \subseteq P$ has the cardinality at most $\log|P|$ and there are dissociated sets $\Lambda \subseteq P$ such that $|\Lambda| \gg \log|P|$. From the other hand from (44), we have

$$r = |P| \quad \text{and} \quad \sum_{x\in\Lambda} (S_1 * S_2)^2(x) = |\Lambda||P|^2 \ll |P|^2 \log|P|.$$

Hence $|\Lambda| \ll \log|P|$. Thus, at least in the situation when the parameter r is large inequality (44) and consequently Corollary 3.8 are sufficiently sharp.

If r is a small number, then there is a different bound for the cardinality of Λ. We thank to S. Yekhanin and K. Talwar for pointed us the fact. For simplicity, let $\mathbf{G} = (\mathbb{Z}/2\mathbb{Z})^n$, and $S_1 = S_2 = S$. Take $p = cr^{-1/2}\log^{1/2}|S|$, where $c > 0$ is an absolute constant, and suppose that $r \gg \log|S|$. Let us choose a random subset $S' \subseteq S$ such that any element x belongs to S' with probability p. Clearly, the expectation of the cardinality of the set S' equals $p|S|$. By assumption for any $x \in \Lambda$, we have $(S*S)(x) \geq r$. Hence choosing the constant c, we get $\Lambda \subseteq S' + S'$ with positive probability (probability of the event that Λ is not in $S'+S'$ does not exceed $|\Lambda|(1-p^2)^r \leq |S|^2(1-p^2)^r$). It is easy to see (the proof was also suggested to us by S. Yekhanin and can be found, e.g., in [15]) that $|\Lambda| \ll |S'|$. Hence we have with positive probability that $|\Lambda| \ll |S'| \ll p|S| \ll |S|r^{-1/2}\log^{1/2}|S|$. If r is small, then the last bound is better than (45).

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