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uniform distribution theory

ON TWO-DIMENSIONAL SEQUENCES COMPOSED BY ONE-DIMENSIONAL UNIFORMLY DISTRIBUTED SEQUENCES

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ABSTRACT. Let x_n and y_n, $n = 1, 2, \dots$, be sequences in the unit interval $[0, 1)$ and let $F(x, y)$ be a continuous function defined on $[0, 1]^2$. In this paper we consider limit points of sequence $\frac{1}{N}\sum_{n=1}^{N} F(x_n, y_n)$, $N = 1, 2, \dots$ A basic idea is to apply distribution functions $g(x, y)$ of two-dimensional sequence (x_n, y_n), $n = 1, 2, \dots$. It can be shown that every limit point has the form $\int_0^1\int_0^1 F(x, y)\mathrm{d}_x\mathrm{d}_y g(x, y)$. If, moreover, both sequences x_n and y_n are uniformly distributed, then distribution functions $g(x, y)$ are called copulas and we find extremes of $\int_0^1\int_0^1 F(x, y)\mathrm{d}_x\mathrm{d}_y g(x, y)$ assuming that the differential $\mathrm{d}_x\mathrm{d}_y F(x, y)$ has constant signum and also for $F(x, y) = f(x)y$ where $f(x)$ is a piecewise linear function.

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1. Introduction

This paper is motivated by [PS09] where the sequence $\frac{1}{N}\sum_{n=1}^{N}|x_n - y_n|$ was considered and it was proved that if x_n and y_n, $n = 1, 2, \dots$ are uniformly distributed (u.d.) in $[0, 1)$, then

$$\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}|x_n - y_n| \le \frac{1}{2}. \tag{1}$$

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Thus putting $y_n = x_{n+1}$, $n = 1, 2, \dots$ in the above inequality the authors found a new necessary condition

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |x_{n+1} - x_n| \le \frac{1}{2} \tag{2}$$

for u.d. of sequence x_n (for definitions consult, e.g., [KN06], [DT97] or [SP05]).

The aim of this paper is to study limit points of the sequence

$$\frac{1}{N} \sum_{n=1}^{N} F(x_n, y_n), \quad N = 1, 2, \dots \tag{3}$$

where $F(x, y)$ is arbitrary continuous function defined on $[0, 1]^2$ and for an arbitrary sequences x_n, y_n, $n = 1, 2, \dots$ in $[0, 1)$. We use distribution functions (d.f.s) $g(x, y)$ of a two-dimensional sequence (x_n, y_n), $n = 1, 2, \dots$ since every limit point of (3) is of the form

$$\int_0^1 \int_0^1 F(x, y) \mathrm{d}_x \mathrm{d}_y g(x, y). \tag{4}$$

Firstly, we study the lower and upper bounds of limit points of (3) assuming that the sequences x_n and y_n, $n = 1, 2, \dots$ are u.d. in $[0, 1)$. In this case, the d.f. $g(x, y)$ in (4) of a two-dimensional sequence (x_n, y_n) is called a copula.

In (3) it can be also used a sequence $(x_n, \Psi(x_n))$, $n = 1, 2, \dots$ where x_n is u.d. in $[0, 1)$ and $\Psi : [0, 1] \to [0, 1]$ is a so-called uniformly distribution preserving map. Thus, the limit points of (3) have the form $\int_0^1 F(x, \Psi(x))\mathrm{d}x$. In particular, for $F(x, y) = f(x)y$ we have

$$\int_0^1 f(x)\Psi(x)\mathrm{d}x. \tag{5}$$

A more general problem than (5) was introduced by S. Steinerberger [S09, p. 127]: He evaluates extremes of the integrals

$$\int_0^1 f_1(\Phi(x)) f_2(\Psi(x))\mathrm{d}x \tag{6}$$

where f, g are Riemann integrable functions and $\Phi, \Psi : [0, 1] \to [0, 1]$ are u.d.p. maps. He solved (6) by applying some Hardy-Littlewood inequality (see later).

Our paper is structured as follows:

In Section 3 we provide the maximum and the minimum of limit points of (3) under the assumptions that $\mathrm{d}_x\mathrm{d}_y F(x, y) > 0$ and that the sequences x_n and y_n are uniformly distributed in $[0, 1)$.

In Section 5 we extend the results of section 3 to the sequences x_n and y_n having a.d.f.s $g_1(x)$ and $g_2(x)$, respectively.

Section 6 provides a necessary condition for a copula $g(x,y)$ for which the integral $\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y)$ is extremal.

In Section 7 the maximal and minimal values of $\int_0^1 f_1(\Phi(x))f_2(\Psi(x))\mathrm{d}x$ are obtained over u.d.p. functions Φ,Ψ. As an application we find u.d.p. map $\Phi(x)$ which maximizes $\int_0^1 f(x)\Phi(x)\mathrm{d}x$ for piecewise linear function $f:[0,1]\to[0,1]$.

2. Basic definitions and results

• Denote the step d.f.

$$F_N(x,y)=\frac{\#\{n\le N;(x_n,y_n)\in[0,x)\times[0,y)\}}{N}, \tag{7}$$

and let $G((x_n,y_n))$ be a set of all possible limits $F_{N_k}(x,y)\to g(x,y)$ which exist for all points (x,y) of continuity of $g(x,y)$. These $g(x,y)$ are called d.f.s of sequence (x_n,y_n), $n=1,2,\dots$. Recall that if $G((x_n,y_n))$ is a singleton, i.e., $G((x_n,y_n))=\{g(x,y)\}$, then $g(x,y)$ is called an asymptotic d.f. (a.d.f.) of (x_n,y_n), $n=1,2,\dots$.

• If both sequences x_n and y_n, $n=1,2,\dots$ are u.d. in $[0,1)$, then the two-dimensional sequence (x_n,y_n) is not necessarily u.d. but every d.f. $g(x,y)\in G((x_n,y_n))$ satisfies

(i) $g(x,1)=x$ for $x\in[0,1]$ and

(ii) $g(1,y)=y$ for $y\in[0,1]$.

The d.f. $g(x,y)$ which satisfies (i) and (ii) is called a *copula*. For the fuller treatment of theory of copulas we refer the reader to [N99].

• A map $\Psi:[0,1]\to[0,1]$ is called *uniform distribution preserving* (u.d.p.) if for any u.d. sequence x_n, $n=1,2,\dots$ in $[0,1)$ the sequence $\Psi(x_n)$ is also u.d. Basic properties: A map $\Psi(x)$ is u.d.p. if and only if for every continuous $f:[0,1]\to\mathbb{R}$ we have $\int_0^1 f(\Psi(x))\mathrm{d}x=\int_0^1 f(x)\mathrm{d}x$. Every u.d.p. map is Jordan measurable. Other properties can be found in [SN10] and [SP05, 2.5.1].

• The Riemann-Stieltjes integral $\int_0^1\int_0^1 f(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y)$ is defined as the limit

$$\sum_{k=1}^{m}\sum_{l=1}^{n} f(\alpha_k,\beta_l)\big(g(x_k,y_l)+g(x_{k+1},y_{l+1})-g(x_k,y_{l+1})-g(x_{k+1},y_l)\big)$$
$$\to\int_0^1\int_0^1 f(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y) \tag{8}$$

if diameters of $[x_k, x_{k+1}] \times [y_l, y_{l+1}]$ tend to zero for partition $0 = x_0 < x_1 < \cdots < x_m = 1$ of x-axis, $0 = y_0 < y_1 < \cdots < y_n = 1$ of y-axis and for $(\alpha_k, \beta_l) \in [x_k, x_{k+1}] \times [y_l, y_{l+1}]$. This integral exists for continuous $f(x, y)$ and $g(x, y)$ with bounded variation, see W.I. Smirnov [S88, Par. 23]. Let $\square$ denote the rectangle $[x_k, x_{k+1}] \times [y_l, y_{l+1}]$ and denote

$$\square g(x, y) = g(x_k, y_l) + g(x_{k+1}, y_{l+1}) - g(x_k, y_{l+1}) - g(x_{k+1}, y_l).$$

If diameter $\square \to 0$, then we find the differential $\mathrm{d}_x\mathrm{d}_y g(x, y)$ as

$$\mathrm{d}_x\mathrm{d}_y g(x, y) = g(x, y) + g(x + \mathrm{d}x, y + \mathrm{d}y) - g(x, y + \mathrm{d}y) - g(x + \mathrm{d}x, y).$$

• $g(x, y)$ is a d.f. of a sequence (x_n, y_n), $n = 1, 2, \ldots$, $(x_n, y_n) \in [0, 1)^2$ if and only if the following three conditions are satisfied

(i) $\mathrm{d}_x\mathrm{d}_y g(x, y) \geq 0$ for every $(x, y) \in (0, 1)^2$.

(ii) $g(1, 1) = 1$, $g(x, 0) = 0$, $g(0, y) = 0$ for $x, y \in [0, 1]$.

(iii) $g(x, 1)$ and $g(1, y)$ are one-dimensional d.f.

We establish the following two basic expressions of limit points of (3).

Theorem 1. *For every continuous $F(x, y)$ the set of limit points of (3) coincides with*

$$\left\{ \int_0^1 \int_0^1 F(x, y)\mathrm{d}_x\mathrm{d}_y g(x, y); g(x, y) \in G((x_n, y_n)) \right\}. \tag{9}$$

P r o o f. By the Riemann-Stieltjes integration we have

$$\frac{1}{N}\sum_{n=1}^{N} F(x_n, y_n) = \int_0^1 \int_0^1 F(x, y)\mathrm{d}_x\mathrm{d}_y F_N(x, y). \tag{10}$$

a) Now, if $F_{N_k}(x, y) \to g(x, y)$ weakly as $k \to \infty$, then by the second Helly theorem (see [SP05, p. 4–5, Th. 4.1.0.11]) the equation (10) implies

$$\frac{1}{N_k}\sum_{n=1}^{N_k} F(x_n, y_n) \to \int_0^1 \int_0^1 F(x, y)\mathrm{d}_x\mathrm{d}_y g(x, y) \quad \text{as } k \to \infty. \tag{11}$$

b) If $\frac{1}{N_k}\sum_{n=1}^{N_k} F(x_n, y_n) \to A$ as $k \to \infty$, then by the first Helly theorem (see [SP05, p. 4–5, Th. 4.1.0.11]) there exists a subsequence N_k' of N_k such that for some d.f. $g(x, y)$ of (x_n, y_n) we have $F_{N_k'}(x, y) \to g(x, y)$ and then by the second Helly theorem we have $A = \int_0^1 \int_0^1 F(x, y)\mathrm{d}_x\mathrm{d}_y g(x, y)$.

From a) and b) there follows the desired property. $\square$

In following Theorem 2 we apply the equation:

$$\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y) = F(1,1) - \int_0^1 g(1,y)\mathrm{d}_y F(1,y)$$
$$- \int_0^1 g(x,1)\mathrm{d}_x F(x,1) + \int_0^1\int_0^1 g(x,y)\mathrm{d}_x\mathrm{d}_y F(x,y) \tag{12}$$

which we prove in following lemma.

Lemma 1. *Let $F(x,y)$ be a continuous function defined on $[0,1]^2$. Then for arbitrary N-term sequence $(x_1,y_1),\dots,(x_N,y_N)$ in $[0,1)^2$ with the step d.f. $F_N(x,y)$ we have*

$$\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y F_N(x,y) = F(1,1) - \int_0^1 F_N(1,y)\mathrm{d}_y F(1,y)$$
$$- \int_0^1 F_N(x,1)\mathrm{d}_x F(x,1) + \int_0^1\int_0^1 F_N(x,y)\mathrm{d}_x\mathrm{d}_y F(x,y). \tag{13}$$

Proof. Multiple both sides by N and employ an induction. For $N=1$ equation (13) follows directly from computation. Assume that for N the equation holds and add a new point (x_{N+1},y_{N+1}) to $(x_1,y_1),\dots,(x_N,y_N)$, exchange $F_N(x,y)$ by $F_{N+1}(x,y)$. We have $(N+1)F(1,1) = NF(1,1) + F(1,1)$ and

$$\begin{aligned}
(N+1)\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y F_{N+1}(x,y) =& N\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y F_N(x,y)\\
&+ F(x_{N+1},y_{N+1}),\\
-(N+1)\int_0^1 F_{N+1}(x,1)\mathrm{d}_x F(x,1) =& -N\int_0^1 F_N(x,1)\mathrm{d}_x F(x,1)\\
&- (F(1,1) - F(x_{N+1},1)),\\
-(N+1)\int_0^1 F_{N+1}(1,y)\mathrm{d}_y F(1,y) =& -N\int_0^1 F_N(1,y)\mathrm{d}_y F(1,y)\\
&- (F(1,1) - F(1,y_{N+1})),\\
(N+1)\int_0^1\int_0^1 F_{N+1}(x,y)\mathrm{d}_x\mathrm{d}_y F(x,y) =& N\int_0^1\int_0^1 F_N(x,y)\mathrm{d}_x\mathrm{d}_y F(x,y)\\
+(F(1,1) + F(x_{N+1},y_{N+1}) &- F(x_{N+1},1) - F(1,y_{N+1})).
\end{aligned}$$

Summing up

$$F(1,1) - (F(1,1) - F(x_{N+1},1)) - (F(1,1) - F(1,y_{N+1}))$$
$$+ (F(1,1) + F(x_{N+1},y_{N+1}) - F(x_{N+1},1) - F(1,y_{N+1}))$$

we obtain $F(x_{N+1},y_{N+1})$ and it is easily seen that (13) is valid also for $N+1$. □

If $F_N(x,y) \to g(x,y)$ as $N \to \infty$ in Lemma 1 we obtain (12) which directly implies Theorem 2.

Theorem 2. *Let both x_n and y_n be u.d. in $[0,1)$. Then the set of limit points of* (3) *can be expressed as*

$$F(1,1) - \int_0^1 y \mathrm{d}_y F(1,y) - \int_0^1 x \mathrm{d}_x F(x,1) + \left\{ \int_0^1 \int_0^1 g(x,y) \mathrm{d}_x \mathrm{d}_y F(x,y); g(x,y) \in G((x_n,y_n)) \right\}, \tag{14}$$

where + in (14) *is defined as $x + A = \{x + a; a \in A\}$.*

3. Extremes of $\lim_{k\to\infty} \frac{1}{N_k} \sum_{n=1}^{N_k} F(x_n, y_n)$

Theorem 3. *Let $F(x,y)$ be a Riemann integrable function defined on $[0,1]^2$ and assume that $\mathrm{d}_x \mathrm{d}_y F(x,y) > 0$ for every $(x,y) \in (0,1)^2$. Then for every u.d. sequences $x_n, y_n \in [0,1)$, $n = 1,2,\dots$ we have*

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} F(x_n,y_n) \le \int_0^1 F(x,x) \mathrm{d}x, \tag{15}$$

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} F(x_n,y_n) \ge \int_0^1 F(x,1-x) \mathrm{d}x. \tag{16}$$

Furthermore, for a sequence (x_n,y_n), $n = 1,2,\dots$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} F(x_n,y_n) = \int_0^1 F(x,x) \mathrm{d}x$$

if and only if (x_n,y_n) has the a.d.f. $g(x,y) = \min(x,y)$ and we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} F(x_n,y_n) = \int_0^1 F(x,1-x) \mathrm{d}x$$

if and only if $g(x,y) = \max(x+y-1,0)$.

If $\mathrm{d}_x \mathrm{d}_y F(x,y) < 0$, the right hand sides of (15) *and* (16) *are exchanged.*

Proof. We apply the Fréchet-Hoeffding bounds [N99, p. 9]

$$\max(x+y-1,0) \le g(x,y) \le \min(x,y) \tag{17}$$

which holds for every $(x,y)\in[0,1]^2$ and for every copula $g(x,y)$ in (14). The assumption $\mathrm{d}_x\mathrm{d}_yF(x,y)>0$ implies

$$\int_0^1\int_0^1 \max(x+y-1,0)\mathrm{d}_x\mathrm{d}_yF(x,y)\le\int_0^1\int_0^1 g(x,y)\mathrm{d}_x\mathrm{d}_yF(x,y)$$
$$\le\int_0^1\int_0^1 \min(x,y)\mathrm{d}_x\mathrm{d}_yF(x,y).$$

Since every copula is continuous, the left inequality is attained if and only if $g(x,y)=\max(x+y-1,0)$ and the right one if and only if $g(x,y)=\min(x,y)$.

Directly from the definition of a.d.f., for u.d. sequence x_n it follows that

(i) the sequence (x_n,x_n), $n=1,2,\dots$ has the a.d.f. $g(x,y)=\min(x,y)$ and

(ii) the sequence $(x_n,1-x_n)$, $n=1,2,\dots$ has the a.d.f. $g(x,y)=\max(x+y-1,0)$. From it follows

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}F(x_n,x_n)=\int_0^1 F(x,x)\mathrm{d}x=\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y\min(x,y),\tag{18}$$

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}F(x_n,1-x_n)=\int_0^1 F(x,1-x)\mathrm{d}x$$
$$=\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y\max(x+y-1,0).\tag{19}$$

And desired bounds follows from (18) and (19) using (17). □

4. Examples

Firstly, we prove the result (1) appeared in [PS09].

Example 1. Putting $F(x,y)=|x-y|$ we have $F(1,1)=0$, $F(1,x)=1-x$, $F(y,1)=1-y$, compute for $y>x$,

$\mathrm{d}_x\mathrm{d}_y|y-x|=(y+\mathrm{d}y-(x+\mathrm{d}x))=(y-x)-(y-(x+\mathrm{d}x))-(y+\mathrm{d}y-x)=0,$

and for $y=x$, $\mathrm{d}y=\mathrm{d}x$,

$\mathrm{d}_x\mathrm{d}_y|y-x|=|x+\mathrm{d}x-(x+\mathrm{d}x)|+|x-x|-|(x+\mathrm{d}x)-x|-|x-(x+\mathrm{d}x)|=-2\mathrm{d}x$

then we have

$$\int_0^1\int_0^1 |x-y|\mathrm{d}_x\mathrm{d}_yg(x,y)=\int_0^1 g(x,1)\mathrm{d}x+\int_0^1 g(1,y)\mathrm{d}y-2\int_0^1 g(x,x)\mathrm{d}x.\tag{20}$$

Thus for a copula $g(x,y)$, $g(x,1)=x$, $g(1,y)=y$ we have

$$\int_0^1\int_0^1 |x-y|\mathrm{d}_x\mathrm{d}_y g(x,y) = 1-2\int_0^1 g(x,x)\mathrm{d}x. \tag{21}$$

Finally, the lower bound in (17) for copulas $g(x,y)$ gives F. Pillichshammer's and S. Steinerberger's result [PS09] in the form

$$\int_0^1\int_0^1 |x-y|\mathrm{d}_x\mathrm{d}_y g(x,y) \le \int_0^1\int_0^1 |x-y|\mathrm{d}_x\mathrm{d}_y \max(x+y-1,0) = \frac{1}{2}. \tag{22}$$

F. Pillichshammer and S. Steinerberger compute in [PS09] also the limit $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}|x_{n+1}-x_n|$ for the van der Corpute sequence $x_n=\gamma_q(n)$. We give an alternative proof via d.f.s. in the following example.

Example 2. For the van der Corput sequence $x_n=\gamma_q(n)$, $n=0,1,\dots$ in base q we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}|x_{n+1}-x_n| = \frac{2(q-1)}{q^2}. \tag{23}$$

Proof. Every point $(\gamma_q(n),\gamma_q(n+1))$, $n=0,1,2,\dots$ lies on a line segment

$$Y = X-1+\frac{1}{q^k}+\frac{1}{q^{k+1}}, \quad X\in\left[1-\frac{1}{q^k},1-\frac{1}{q^{k+1}}\right] \tag{24}$$

for some $k=0,1,\dots$ and let T be the set of all points $(x,y)\in[0,1]^2$ which satisfy (24).

Since $\gamma_q(n)$ is u.d. the sequence $(\gamma_q(n),\gamma_q(n+1))$ has a.d.f. $g(x,y)$ which is a copula

$$g(x,y) = |\mathrm{Project}_x(([0,x)\times[0,y))\cap T)|, \tag{25}$$

where $\mathrm{Project}_x$ is a projection of two dimensional set to the x-axis. Thus $g(x,y)$ can be computed explicitly as

$$g(x,y)=\begin{cases} 0 & \text{if } (x,y)\in A,\\ 1-(1-y)-(1-x)=x+y-1 & \text{if } (x,y)\in B,\\ y-\frac{1}{q^i} & \text{if } (x,y)\in C_i,\\ x-1+\frac{1}{q^{i-1}} & \text{if } (x,y)\in D_i, \end{cases} \tag{26}$$

$i=1,2,...$

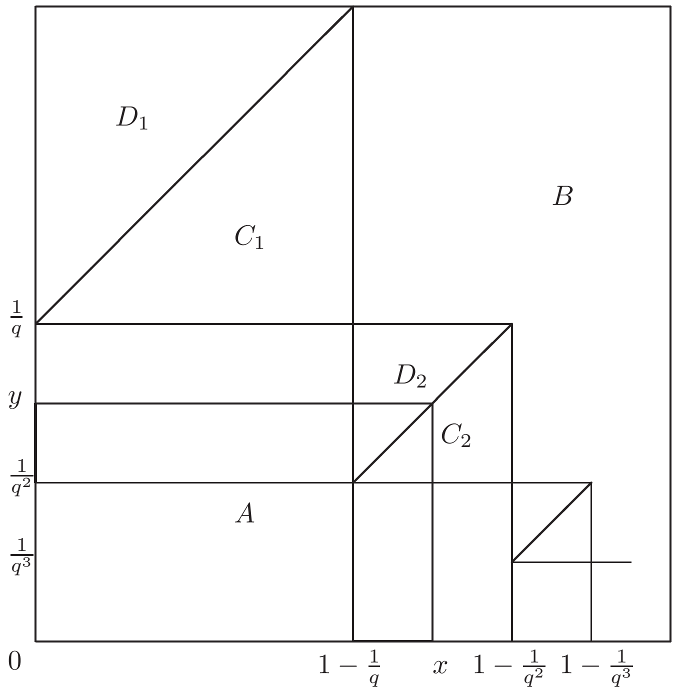

Figure 1: A.d.f. $g(x,y)$ of $(\gamma_q(n),\gamma_q(n+1)), n=1,2,...$

Hence

$$g(x,x)=\begin{cases}0, & \text{if } x\in\left[0,\frac{1}{q}\right],\\ x-\frac{1}{q}, & \text{if } x\in\left[\frac{1}{q},1-\frac{1}{q}\right],\\ 2x-1, & \text{if } x\in\left[1-\frac{1}{q},1\right]\end{cases}\tag{27}$$

and by (21)

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}|\gamma_q(n)-\gamma_q(n+1)|=1-2\int_0^1 g(x,x)\mathrm{d}x=\frac{2(q-1)}{q^2}.$$

□

5. Generalization

Now, we can generalize the result of Theorem 3 to the sequences x_n and y_n which are not necessarily u.d. but have a.d.f. $g_1(x)$ and $g_2(x)$, respectively. In the following theorem we use the inverse function $g^{-1}(x)$ of the d.f. $g(x)=g_1(x)$

or $g(x) = g_2(x)$ also for the case if the d.f. $g(x)$ is constant on the interval $I = (\alpha, \beta)$ with value c and to the left of α and to the right of β the d.f. $g(x)$ increases simultaneously, then we put

$$g^{-1}(c) = \beta \tag{28}$$

because in this case the $g^{-1}(z) < x \Leftrightarrow z < g(x)$ also holds for $z = c$.

Theorem 4. *Let $x_n \in [0,1)$ be a sequence with an a.d.f. $g_1(x)$ and $y_n \in [0,1)$ with an a.d.f. $g_2(x)$. Let us assume that $F(x,y)$ is a continuous function such that $\mathrm{d}_x\mathrm{d}_y F(x,y) > 0$ for every $(x,y) \in (0,1)^2$. Then we have*

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} F(x_n, y_n) \le \int_0^1 F(g_1^{-1}(x), g_2^{-1}(x))\mathrm{d}x, \tag{29}$$

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} F(x_n, y_n) \ge \int_0^1 F(g_1^{-1}(x), g_2^{-1}(1-x))\mathrm{d}x. \tag{30}$$

Furthermore, for the sequence (x_n, y_n), $n = 1, 2, \dots$ we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} F(x_n, y_n) = \int_0^1 F(g_1^{-1}(x), g_2^{-1}(x))\mathrm{d}x \tag{31}$$

if and only if (x_n, y_n) has the a.d.f. $g(x,y) = \min(g_1(x), g_2(y))$ and we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} F(x_n, y_n) = \int_0^1 F(g_1^{-1}(x), g_2^{-1}(1-x))\mathrm{d}x \tag{32}$$

if and only if $g(x,y) = \max(g_1(x) + g_2(y) - 1, 0)$.

If $\mathrm{d}_x\mathrm{d}_y F(x,y) < 0$ the right hand sides of (29) and (30) are exchanged.

Proof. Let $g(x,y)$ be a d.f. of a sequence (x_n, y_n), i.e., there exists a sequence $N_k \to \infty$ such that $F_{N_k}(x,y) \to g(x,y)$ and then

$$\lim_{k\to\infty} \frac{1}{N_k}\sum_{n=1}^{N_k} F(x_n, y_n) \to \int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y).$$

Furthermore, we have $g(x,1) = g_1(x)$ and $g(1,y) = g_2(y)$. Then by the Sklar theorem [N99, p.15, Th.2.3.3] for any such d.f. $g(x,y)$ there exists a copula $c(x,y)$ such that

$$g(x,y) = c(g_1(x), g_2(y))$$

for every $(x,y) \in [0,1]^2$. Applying the Fréchet-Hoeffding bounds (17) we find

$$\max(g_1(x) + g_2(y) - 1, 0) \le g(x,y) \le \min(g_1(x), g_2(y)). \tag{33}$$

Applying (12) we find

$$\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}F(x_n,y_n)\le\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y\min(g_1(x),g_2(y)),$$

$$\liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}F(x_n,y_n)\ge\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y\max(g_1(x)+g_2(y)-1,0).$$

Now, let z_n, $n=1,2,\dots$ be a u.d. sequence in $[0,1)$. Then we have
(i) $(g_1^{-1}(z_n),g_2^{-1}(z_n))$ has a.d.f. $g(x,y)=\min(g_1(x),g_2(y))$ and
(ii) $(g_1^{-1}(z_n),g_2^{-1}(1-z_n))$ has a.d.f. $g(x,y)=\max(g_1(x)+g_2(y)-1,0)$.
By the Helly theorem for the sequences (i) and (ii) we have (31) and (32)

$$\lim_{k\to\infty}\frac{1}{N_k}\sum_{n=1}^{N_k}F(g_1^{-1}(z_n),g_2^{-1}(z_n))\to\int_0^1 F(g_1^{-1}(x),g_2^{-1}(x))\mathrm{d}x$$
$$=\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y\min(g_1(x),g_2(y)),$$
$$\lim_{k\to\infty}\frac{1}{N_k}\sum_{n=1}^{N_k}F(g_1^{-1}(z_n),g_2^{-1}(1-z_n))\to\int_0^1 F(g_1^{-1}(x),g_2^{-1}(1-x))\mathrm{d}x$$
$$=\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y\max(g_1(x)+g_2(y)-1,0).$$

What is left to show is the proof of (i) and (ii). Assume that $g_1(x)$ and $g_2(x)$ are strictly increasing.

For (i) we note that $g_1^{-1}(z_n)<x\Leftrightarrow z_n<g_1(x)$ and $g_2^{-1}(z_n)<y\Leftrightarrow z_n<g_2(y)$ and thus $(g_1^{-1}(z_n),g_2^{-1}(z_n))\in[0,x)\times[0,y)\Leftrightarrow z_n\in[0,\min(g_1(x),g_2(y)))$.

For (ii) we note that $g_1^{-1}(z_n)<x\Leftrightarrow z_n<g_1(x)$ and $g_2^{-1}(1-z_n)<y\Leftrightarrow 1-z_n<g_2(y)$, $1-z_n<g_2(y)\Leftrightarrow 1-g_2(y)<z_n$ and thus $(g_1^{-1}(z_n),g_2^{-1}(1-z_n))\in[0,x)\times[0,y)\Leftrightarrow z_n\in(1-g_2(y),g_1(x))$. Then the density of such z_n is $\max(g_1(x)-(1-g_2(x)),0)$.

Finally, the uniqueness of extremal d.f. $g(x,y)$ follows from the existence of common point (x,y) of continuity for any two d.f.s $g(x,y)$. □

6. Extremes of $\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y)$

In Section 2 we saw that the extreme limit points of (3) $\frac{1}{N}\sum_{n=1}^{n}F(x_n,y_n)$ are the same as the extremes of (9) $\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y)$. In Section 3 we found the extremes of (9) for $F(x,y)$ with $\mathrm{d}_x\mathrm{d}_yF(x,y)$ of constant signum. In this section we study extremes of (9) for $F(x,y)$ without this condition. We only can find a criterion of necessary conditions of copula $g(x,y)$ which maximizes (9). The minimum can be studied similarly.

First of all, we reformulate Theorems 3 and 4 into following two theorems.

Theorem 5. *Let $F(x,y)$ be a Riemann integrable function defined on $[0,1]^2$. Assume that $\mathrm{d}_x\mathrm{d}_yF(x,y) > 0$ for $(x,y) \in (0,1)^2$. Then*

$$\max_{g(x,y)\text{-}copula} \int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y) = \int_0^1 F(x,x)\mathrm{d}x,$$
$$\min_{g(x,y)\text{-}copula} \int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y) = \int_0^1 F(x,1-x)\mathrm{d}x,$$

where the maximum is attained in $g(x,y) = \min(x,y)$ and the minimum in $g(x,y) = \max(x+y-1,0)$, uniquely.

Theorem 6. *Let us assume that $F(x,y)$ is a continuous function such that $\mathrm{d}_x\mathrm{d}_yF(x,y) > 0$ for every $(x,y) \in (0,1)^2$. Then for the extremes of integral $\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y)$ for $g(x,y)$ for which $g(x,1) = g_1(x)$ and $g(1,y) = g_2(y)$ we have*

$$\max_{g(x,y)} \int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y) = \int_0^1 F(g_1^{-1}(x), g_2^{-1}(x))\mathrm{d}x,$$
$$\min_{g(x,y)} \int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y) = \int_0^1 F(g_1^{-1}(x), g_2^{-1}(1-x))\mathrm{d}x,$$

where the maximum is attained in $g(x,y) = \min(g_1(x), g_2(y))$ and the minimum in $g(x,y) = \max(g_1(x)+g_2(y)-1,0)$, uniquely.

6.1. Criterion

Theorem 7. *Let us assume that a copula $g(x,y)$ maximizes the integral $\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y)$ and let $[X_1,X_2]\times[Y_1,Y_2]$ be an interval in $[0,1]^2$ such that the differential*

$$g(X_2,Y_2) + g(X_1,Y_1) - g(X_2,Y_1) - g(X_1,Y_2) > 0.$$

If for every interior point (x,y) of $[X_1,X_2]\times[Y_1,Y_2]$ the differential $\mathrm{d}_x\mathrm{d}_yF(x,y)$ has constant signum, then:

(i) *if $\mathrm{d}_x\mathrm{d}_yF(x,y) > 0$, then*

$$g(x,y) = \min(g(x,Y_2)+g(X_1,y)-g(X_1,Y_2),\, g(x,Y_1)+g(X_2,y)-g(X_2,Y_1)) \tag{34}$$

(ii) *if* $\mathrm{d}_x\mathrm{d}_y F(x,y) < 0$*, then*

$$g(x,y) = \max(g(x,Y_2) + g(X_2,y) - g(X_2,Y_2), g(x,Y_1) + g(X_1,y) - g(X_1,Y_1)) \tag{35}$$

for every $(x,y) \in [X_1,X_2] \times [Y_1,Y_2]$.

P r o o f. The basic idea of the proof is to extend the interval $[X_1,X_2] \times [Y_1,Y_2]$ with $F(x,y)$ linearly to $[0,1]^2$ with $F^*(x,y)$. Then it is possible to use Theorem 4 and find extremes of $\int_0^1\int_0^1 F^*(x,y)\mathrm{d}_x\mathrm{d}_y g^*(x,y)$. By inverse transformation we find desired extremal $g(x,y)$ on $[X_1,X_2] \times [Y_1,Y_2]$.

1^0. Let us start with a linear map $[X_1,X_2] \times [Y_1,Y_2] \to [0,1]^2$ defined by

$$\frac{x - X_1}{X_2 - X_1} = x', \quad \frac{y - Y_1}{Y_2 - Y_1} = y' \tag{36}$$

with inverse

$$x = x'(X_2 - X_1) + X_1, \quad y = y'(Y_2 - Y_1) + Y_1. \tag{37}$$

By this map we transform $F(x,y)$ into its image $F^*(x',y')$.

$$F^*(x',y') = [F(x,y)]_{\substack{x=x'(X_2-X_1)+X_1,\\ y=y'(Y_2-Y_1)+Y_1}} = F(x'(X_2 - X_1) + X_1, y'(Y_2 - Y_1) + Y_1) \tag{38}$$

for $(x',y') \in [0,1]^2$. We have

$$\mathrm{d}_{x'}\mathrm{d}_{y'} F^*(x',y') = \mathrm{d}_x\mathrm{d}_y F(x,y)(X_2 - X_1)(Y_2 - Y_1). \tag{39}$$

So the differential of $F(x,y)$ has the same signum as the differential of $F^*(x',y')$.

For the definition of $g^*(x',y')$ we use the auxiliary function $\tilde{g}(x,y)$ defined as follows: Let us assume that among $(x_1,y_1)\dots,(x_N,y_N)$ there are M-points in $[X_1,X_2] \times [Y_1,Y_2]$ with the local step d.f.

$$\tilde{F}_M(x,y) = \frac{1}{M}\#\{n \le N; (x_n,y_n) \in [X_1,x) \times [Y_1,y)\}$$

and assume $\tilde{F}_M(x,y) \to \tilde{g}(x,y)$ a.e. on $[X_1,X_2] \times [Y_1,Y_2]$.

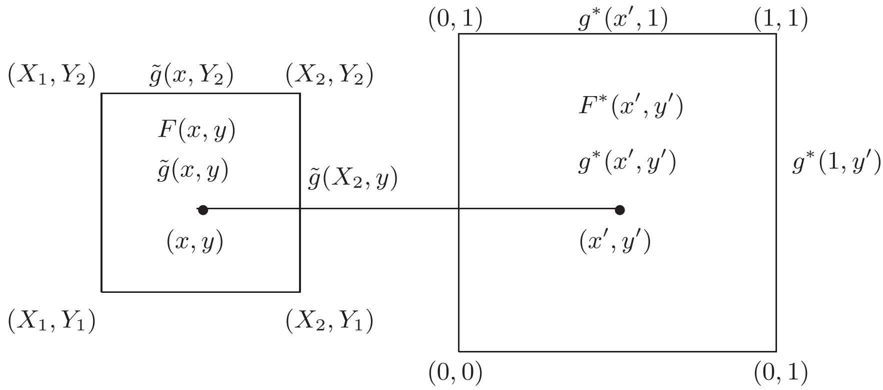

Figure 2: Linear map $(x, y) \to (x', y')$.

Since

$$\tilde{F}_M(x,y) = \frac{F_N(x,y) + F_N(X_1,Y_1) - F_N(x,Y_1) - F_N(X_1,y)}{F_N(X_2,Y_2) + F_N(X_1,Y_1) - F_N(X_2,Y_1) - F_N(X_1,Y_2)},$$

we have

$$\tilde{g}(x,y) = \frac{g(x,y) + g(X_1,Y_1) - g(x,Y_1) - g(X_1,y)}{g(X_2,Y_2) + g(X_1,Y_1) - g(X_2,Y_1) - g(X_1,Y_2)}. \tag{40}$$

Now, we put

$$g^*(x',y') = [\tilde{g}(x,y)]_{\substack{x=x'(X_2-X_1)+X_1,\\ y=y'(Y_2-Y_1)+Y_1}} = \tilde{g}(x'(X_2-X_1)+X_1, y'(Y_2-Y_1)+Y_1) \tag{41}$$

for $x', y' \in [0,1]^2$. Since

$$\frac{1}{M}\sum_{n=1}^{M} F(x_n,y_n) = \frac{1}{M}\sum_{n=1}^{M} F^*(x'_n,y'_n), \tag{42}$$

$$\frac{1}{M}\sum_{n=1}^{M} F(x_n,y_n) \to \iint_{[X_1,X_2]\times[Y_1,Y_2]} F(x,y)\mathrm{d}_x\mathrm{d}_y\tilde{g}(x,y), \tag{43}$$

$$\frac{1}{M}\sum_{n=1}^{M} F^*(x'_n,y'_n) \to \int_0^1\int_0^1 F^*(x',y')\mathrm{d}_{x'}\mathrm{d}_{y'}g^*(x',y') \tag{44}$$

we get that the integrals in (43) and (44) coincide. By (39) the differential of $F^*(x',y')$ has a constant signum on the open unit square. From Theorem 4 it follows that the integral (44) is maximal iff:

(i) $\mathrm{d}'_x\mathrm{d}'_y F^*(x',y') > 0$ implies $g^*(x',y') = \min(g^*(x',1), g^*(1,y'))$ and

(ii) $\mathrm{d}'_x\mathrm{d}'_y F^*(x',y') < 0$ implies $g^*(x',y') = \max(g^*(x',1) + g^*(1,y') - 1, 0)$.

From the maps $g^*(x',1) \longleftrightarrow \tilde{g}(x,Y_2)$, $g^*(1,y') \longleftrightarrow \tilde{g}(X_2,y)$ and from (42), it follows that the integral (43) is maximal if and only if:

(i) $\mathrm{d}_x\mathrm{d}_y F(x,y) > 0$ implies $\tilde{g}(x,y) = \min(\tilde{g}(x,Y_2), \tilde{g}(X_2,y))$ and

(ii) $\mathrm{d}_x\mathrm{d}_y F(x,y) < 0$ implies $\tilde{g}(x,y) = \max(\tilde{g}(x,Y_2) + \tilde{g}(X_2,y) - 1, 0)$.

Here from (40) we have

$$\begin{aligned} \tilde{g}(x,Y_2) &= \frac{g(x,Y_2) + g(X_1,Y_1) - g(x,Y_1) - g(X_1,Y_2)}{g(X_2,Y_2) + g(X_1,Y_1) - g(X_2,Y_1) - g(X_1,Y_2)}, \\ \tilde{g}(X_2,y) &= \frac{g(X_2,y) + g(X_1,Y_1) - g(X_2,Y_1) - g(X_1,y)}{g(X_2,Y_2) + g(X_1,Y_1) - g(X_2,Y_1) - g(X_1,Y_2)}. \end{aligned} \tag{45}$$

Using (45) in (i) and (ii) we find (34) and (35). □

For a fixed $F(x,y)$, the criterion in Theorem 7 can be fulfilled by several copulas which form local extremes of $\int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y)$. In order to find global extremes we need to choose optimal $g(x,Y_i)$ and $g(X_i,y)$ in Theorem 7. This problem is in general still unsolved. In the following theorem we give a solution for a special case.

Theorem 8. *Let us divide $[0,1]^2$ into two parts $[0,1]\times[Y,1]$ and $[0,1]\times[0,Y]$ and define continuous $F(x,y)$ by a composition*

$$F(x,y) = \begin{cases} F_1(x,y), & \text{if } (x,y) \in [0,1]\times[Y,1], \\ F_2(x,y), & \text{if } (x,y) \in [0,1]\times[0,Y], \end{cases}$$

where $\mathrm{d}_x\mathrm{d}_y F_1(x,y) > 0$ for interior points of $[0,1]\times[Y,1]$ and $\mathrm{d}_x\mathrm{d}_y F_1(x,y) < 0$ for interior points of $[0,1]\times[0,Y]$. Then we have

$$\begin{aligned} &\max_{g(x,y)\text{-copula}} \int_0^1\int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y) \\ &= \max_{h(x)} \left(\int_0^1 F_1(x, x-h(x)+Y)(1-h'(x))\mathrm{d}x + \int_0^1 F_2(x, Y-h(x))h'(x)\mathrm{d}x \right), \end{aligned} \tag{46}$$

where $h'(x)$ is the derivative of $h(x)$ and the maximum in (46) is over nondecreasing $h(x)$, $h(0)=0$, $h(1)=Y$ and $\max(x+Y-1,0) \le h(x) \le \min(x,Y)$.

P r o o f. As in the proof of Theorem 7 we start with the two linear maps (see Fig. 3):

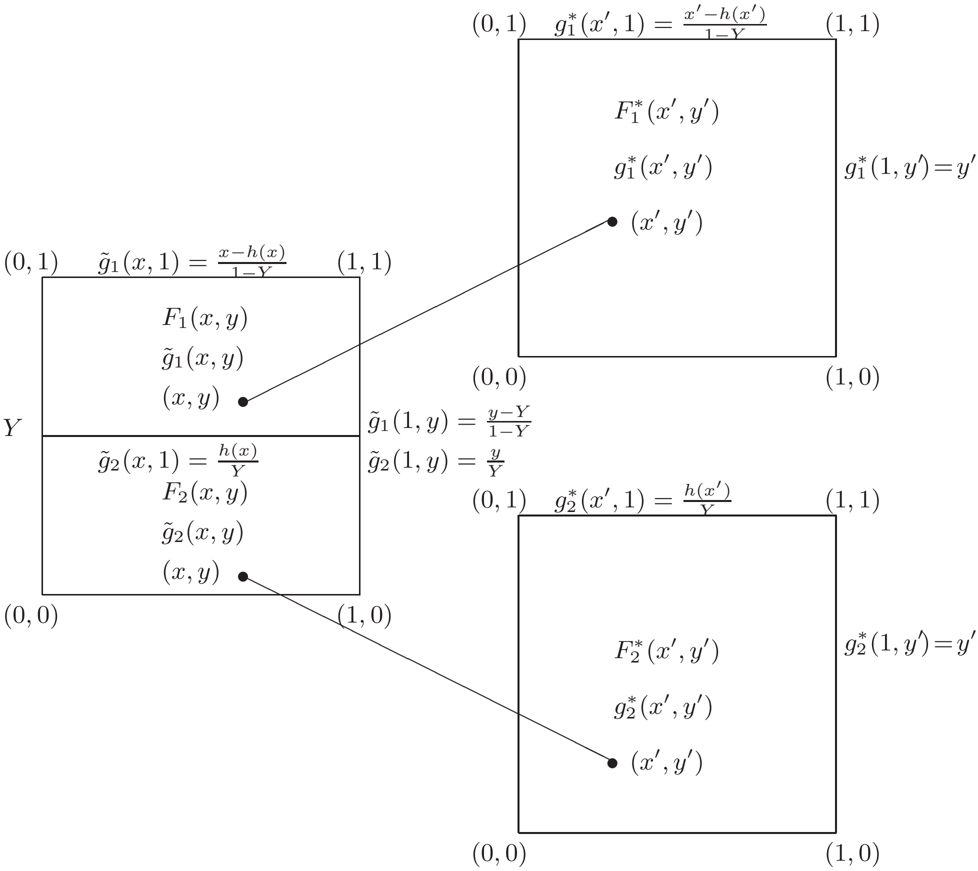

Figure 3: Linear maps $[0,1]\times[Y,1]\to[0,1]^2$ and $[0,1]\times[0,Y]\to[0,1]^2$.

$$[0,1]\times[Y,1]\to[0,1]^2,\ \text{where } x'=x,\quad y'=\frac{y-Y}{1-Y}, \tag{47}$$

$$[0,1]\times[0,Y]\to[0,1]^2,\ \text{where } x'=x,\quad y'=\frac{y}{Y}. \tag{48}$$

Then from (40) and from the properties $g(0,0)=g(0,y)=g(x,0)=0$, $g(1,1)=1$, $g(1,Y)=Y$ of copulas $g(x,y)$ it follows that

$$\tilde g_1(x,y)=\frac{g(x,y)-g(x,Y)}{1-Y},\quad \tilde g_2(x,y)=\frac{g(x,y)}{Y}. \tag{49}$$

Using the transformations (47) and (48) we find

$$g_1^*(x',y') = [\tilde{g}_1(x,y)]_{\substack{x=x',\\ y=y'(1-Y)+Y}} = \frac{g(x',y'(1-Y)+Y)-g(x',Y)}{1-Y} \tag{50}$$

$$g_2^*(x',y') = [\tilde{g}_2(x,y)]_{\substack{x=x',\\ y=y'Y}} = \frac{g(x',Y)}{Y}. \tag{51}$$

Put $g(x,Y)=h(x)$. Since $g(x,y)$ is a copula and for every copula $\max(x+y-1,0)\le g(x,y)\le\min(x,y)$, we have $h(0)=0$, $h(1)=Y$, $h(x)$ is nondecreasing and $\max(x+Y-1,0)\le h(x)\le\min(x,Y)$. From (50) and (51) we obtain

$$g_1^*(x',1) = \frac{x'-h(x')}{1-Y}, \quad g_1^*(1,y') = y', \tag{52}$$

$$g_2^*(x',1) = \frac{h(x')}{Y}, \quad g_2^*(1,y') = y'. \tag{53}$$

Then by Theorem 6 the distribution function $g_1^*(x',y')$ maximizes the integral $\int_0^1\int_0^1 F_1^*(x',y')\mathrm{d}_{x'}\mathrm{d}_{y'}g(x',y')$ and $g_2^*(x',y')$ maximizes the integral $\int_0^1\int_0^1 F_2^*(x',y')\mathrm{d}_{x'}\mathrm{d}_{y'}g(x',y')$ for fixed (52) and (53) iff

$$g_1^*(x',y') = \min\left(\frac{x'-h(x')}{1-Y}, y'\right), \tag{54}$$

$$g_2^*(x',y') = \max\left(\frac{h(x')}{Y}+y'-1, 0\right). \tag{55}$$

Now, from (49) it follows

$$g(x,y) = \begin{cases} \tilde{g}_1(x,y)(1-Y)+h(x), & \text{if } (x,y)\in[0,1]\times[Y,1],\\ \tilde{g}_2(x,y)Y, & \text{if } (x,y)\in[0,1]\times[0,Y]. \end{cases} \tag{56}$$

Using the inverse transformations of (47) and (48) we find

$$\tilde{g}_1(x,y) = [g_1^*(x',y')]_{\substack{x'=x,\\ y'=\frac{y-Y}{1-Y}}} = \min\left(\frac{x-h(x)}{1-Y}, \frac{y-Y}{1-Y}\right)$$

$$\tilde{g}_2(x,y) = [g_2^*(x',y')]_{\substack{x'=x,\\ y'=\frac{y}{Y}}} = \max\left(\frac{h(x)}{Y}+\frac{y}{Y}-1, 0\right). \tag{57}$$

By inserting (57) into (56) we obtain

$$g(x,y) = \begin{cases} \min(x-h(x), y-Y)+h(x), & \text{if } y\in[Y,1],\\ \max(h(x)+y-Y, 0), & \text{if } y\in[0,Y] \end{cases} \tag{58}$$

for every $x \in [0,1]$. It is easily seen that $g(x,y)$ has nonzero differential only on the curves $y = x - h(x) + Y$ and $y = Y - h(x)$ where

$$\mathrm{d}_x\mathrm{d}_y g(x,y) = \begin{cases} (1-h'(x))\mathrm{d}x, & \text{if } y = x - h(x) + Y, \\ h'(x)\mathrm{d}x, & \text{if } y = Y - h(x). \end{cases} \tag{59}$$

Finally, (59) implies

$$\int_0^1 \int_Y^1 F_1(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y) = \int_0^1 F_1(x, x-h(x)+Y)(1-h'(x))\mathrm{d}x$$

$$\int_0^1 \int_0^Y F_2(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y) = \int_0^1 F_2(x, Y-h(x))h'(x)\mathrm{d}x.$$

□

Remark 9. Rewrite (46) in the form $\int_0^1 G(x,h,h')\mathrm{d}x$. Then for the desired optimal $h(x)$ we can apply the Euler differential equation to the variation method: If the integral $\int_0^1 G(x,h,h')\mathrm{d}x$ attains on $h_0(x)$ a local extreme over all continuous $h(x)$ with continuous derivative $h'(x)$, $h(0) = h_0(0) = 0$, $h(1) = h_0(1) = 1$, then $h_0(x)$ satisfies

$$\frac{\partial G}{\partial h} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial G'}{\partial h} = 0. \tag{60}$$

For an application see the following example.

Example 3. Put $Y = \frac{1}{2}$ and define for $x \in [0,1]$

$$F(x,y) = \begin{cases} F_1(x,y) = 2\big(y - \frac{1}{2}\big)x, & \text{if } y \in \big[\frac{1}{2}, 1\big], \\ F_2(x,y) = 2\big(\frac{1}{2} - y\big)x, & \text{if } y \in \big[0, \frac{1}{2}\big]. \end{cases} \tag{61}$$

Then the integral in (46) has the form

$$\int_0^1 F_1(x, x-h(x)+Y)(1-h'(x))\mathrm{d}x + \int_0^1 F_2(x, Y-h(x))h'(x)\mathrm{d}x$$

$$= \frac{1}{6} + 2\int_0^1 (xh(x) - h^2(x))\mathrm{d}x. \tag{62}$$

By the Euler equation (60) we have

$$\frac{\partial(xh - h^2)}{\partial h} = x - 2h(x) = 0$$

which gives $h(x) = \frac{x}{2}$. Such $h(x)$ is nondecreasing, satisfies $\max(x + \frac{1}{2} - 1, 0) \le h(x) \le \min(x, \frac{1}{2})$ and thus the global extreme is

$$\max_{g(x,y)\text{-copula}} \int_0^1 \int_0^1 F(x,y)\mathrm{d}_x\mathrm{d}_y g(x,y) = \frac{1}{6} + 2\cdot\frac{1}{12} = \frac{1}{3}.$$

7. Extremes of $\int_0^1 f_1(\Phi(x))f_2(\Psi(x))\mathrm{d}x$

The problem presented in this section was also considered by S. Steinerberger in [S09] from where Theorem 10 is adapted. We are interested in finding extremes of (3) for $F(x,y) = f_1(x)f_2(x)$ and for u.d. sequences $x_n = \Phi(z_n)$ and $y_n = \Psi(z_n)$, where $\Phi(x)$ and $\Psi(x)$ are u.d.p. functions and z_n is a u.d. sequence.

Definition 1. Let $f : [0,1] \to \mathbb{R}$ be a Lebesgue measurable function, $g_f(x) = |f^{-1}([0,x))|$ be its d.f. and put $f^*(x) = g_f^{-1}(x)$. Here, if $g_f(x)$ is constant equal to c on interval (α,β) (maximal with respect to inclusion) then we put $g_f^{-1}(c) = \beta$ as in (28).

Theorem 10. *Let f_1 and f_2 be Riemann integrable functions on $[0,1]$. Let $\Phi(x)$ and $\Psi(x)$ be arbitrary u.d.p. transformations. Then*

$$\int_0^1 f_1^*(x)f_2^*(1-x)\mathrm{d}x \le \int_0^1 f_1(\Phi(x))f_2(\Psi(x))\mathrm{d}x \le \int_0^1 f_1^*(x)f_2^*(x)\mathrm{d}x \tag{63}$$

and these bounds are best possible. Also, every number within these bounds is attained by some u.d.p. functions $\Phi(x), \Psi(x)$.

In Steinerberger's proof the Hardy-Littlewood inequality in rearrangement theory was used [HL28], [HL30] (see [HLP34, Th. 378]):

$$\int_0^1 f_1(x)f_2(x)\mathrm{d}x \le \int_0^1 f_1^*(x)f_2^*(x)\mathrm{d}x. \tag{64}$$

From now on let $F(x,y) = f(x)\cdot y$ and we study the limit points of

$$\frac{1}{N}\sum_{n=1}^{N} f(x_n)\cdot y_n,$$

where x_n is a u.d. sequence and a u.d. sequence y_n is given by $y_n = \Phi(x_n)$, where $\Phi(x)$ is a u.d.p. map. This problem is equivalent to calculate

$$\max_{\Phi(x)-\text{u.d.p.}} \int_0^1 f(x)\Phi(x)\mathrm{d}x, \qquad \min_{\Phi(x)-\text{u.d.p.}} \int_0^1 f(x)\Phi(x)\mathrm{d}x. \tag{65}$$

Theorem 10 implies

Theorem 11. *For every Riemann integrable $f(x)$ we have*

$$\max_{\Phi(x)-u.d.p.} \int_0^1 f(x)\Phi(x)\mathrm{d}x = \int_0^1 f(x)g_f(f(x))\mathrm{d}x, \tag{66}$$

$$\min_{\Phi(x)-u.d.p.} \int_0^1 f(x)\Phi(x)\mathrm{d}x = \int_0^1 f(x)(1-g_f(f(x))\mathrm{d}x. \tag{67}$$

P r o o f.

$$\max_{\Phi(x)-\text{u.d.p.}} \int_0^1 f(x)\Phi(x)\mathrm{d}x = \int_0^1 f^*(x)x\mathrm{d}x = \int_0^1 g_f^{-1}(x)x\mathrm{d}x$$
$$= \int_0^1 g_f^{-1}(g_f(f(x)))g_f(f(x))\mathrm{d}x = \int_0^1 f(x)\Psi(x)\mathrm{d}x. \tag{68}$$

Analogously,

$$\min_{\Phi(x)-\text{u.d.p.}} \int_0^1 f(x)\Phi(x)\mathrm{d}x = \int_0^1 f^*(x)(1-x)\mathrm{d}x = \int_0^1 g_f^{-1}(x)(1-x)\mathrm{d}x$$
$$= \int_0^1 f(x)(1-\Psi(x))\mathrm{d}x. \tag{69}$$

Here $\Psi(x) = g_f(f(x))$ is a u.d.p. map since by [SP05, 2.3.7, pp. 2–28] $g_f(f(x_n))$ is u.d. for arbitrary u.d. sequence x_n, $n = 1, 2, \dots$. □

In the following part we will find an explicit form for the u.d.p. function $\Psi(x) = g_f(f(x))$ for pairwise linear functions $f : [0,1] \to [0,1]$.

Definition 2. A function $f : [0,1] \to [0,1]$ is called *piecewise linear* (p.l.) if there exists a system of ordinate intervals J_j, $j = 1, 2, \dots, k$, which are disjoint and whose union is the unit interval $[0,1]$ with the following property: For every J_j there exists a related system of abscissa intervals $I_{j,i}$, $i = 1, 2, \dots, l_j$ such that $f(x)|_{I_{j,i}}$ is an increasing or a decreasing diagonal of $I_{j,i} \times J_j$.

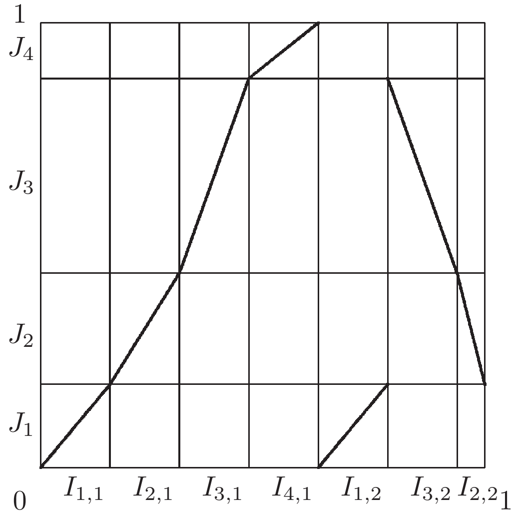

Figure 4: A piecewise linear function.

So every p.l. function can be defined by the sets J_j and $I_{j,i}$ and the signum of derivative on each $I_{j,i} \times J_j$, see, for example Fig. 4.

Theorem 12. *Let $f : [0,1] \to [0,1]$ be a p.l. function with the ordinate decomposition J_j, $j = 1, 2, \dots, k$ and the abscissa decomposition $I_{j,i}$, $i = 1, 2, \dots, l_j$. Define a p.l. function $\Psi(x)$ in the same abscissa decomposition $I_{j,i}$ but in a new ordinate decomposition J'_j with lengths*

$$|J'_j| = \sum_{i=1}^{l_j} |I_{j,i}|, \quad j = 1, 2, \dots, k, \tag{70}$$

and with the same ordering as J_j, $j = 1, 2, \dots, k$. Put the graph of $\Psi(x)|_{I_{j,i}}$ on $I_{j,i} \times J'_j$ as an increasing (decreasing) diagonal if and only if $f(x)|_{I_{j,i}}$ is an increasing (decreasing) diagonal. Note that if $f(x)$ is constant on the interval $I_{j,i}$, then J_i is a point and the graph $\Psi(x)|_{I_{j,i}}$ can be defined arbitrarily: either an increasing or a decreasing diagonal in $I_{j,i} \times J'_j$.

Then $\Psi(x)$ is a u.d.p. map and

$$\int_0^1 f(x)\Psi(x)\mathrm{d}x = \max_{\Phi(x)\text{-}u.d.p.} \int_0^1 f(x)\Phi(x)\mathrm{d}x.$$

For example

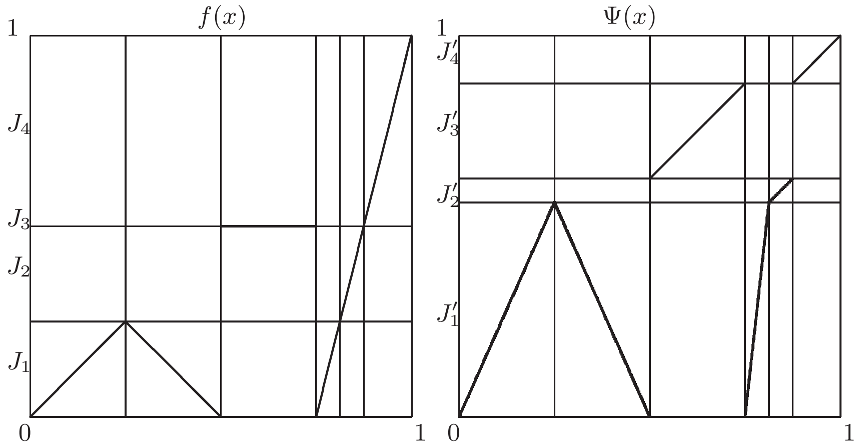

Figure 5: The best u.d.p. approximation.

S k e t c h o f p r o o f. According to Theorem 11 it suffices to prove that $\Psi(x)$ defined in Theorem 12 is equal to $g_f(f(x))$.

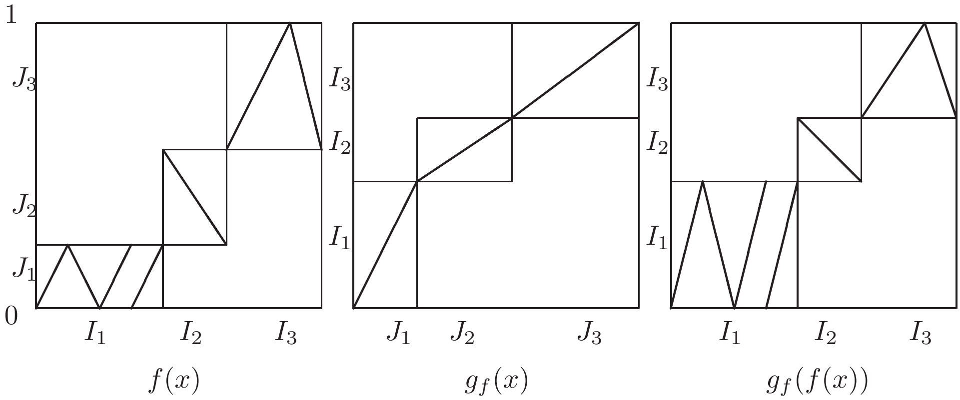

Figure 6: The equality between $\Psi(x)$ and $g_f(f(x))$.

Transform the p.l. function $f(x)$ in the following way: the intervals $I_{j,i}$ are rearranged such that for the same j the intervals $I_{j,i}$ are fused into I_j and these are ordered according to j from the left to the right. This can be done also backwards, so Fig. 6 shows that the function $g_f(f(x))$ is the same as the function $\Psi(x)$. Note that for such reorganized $f(x)$ the function $g_f(x)$ is still the same one. $\square$

In the original proof of Theorem 12 presented in the Strobl UDT2010 conference, there was used the following basic property of an extreme.

Theorem 13. *For an arbitrary Riemann integrable $f : [0,1] \to \mathbb{R}$ we have*

$$\int_0^1 f(x)\Psi(x)\mathrm{d}x = \max_{\Phi(x)\text{-}u.d.p.} \int_0^1 f(x)\Phi(x)\mathrm{d}x$$

if and only if

$$\int_0^1 (f(x) - \Psi(x))^2 \mathrm{d}x = \min_{\Phi(x)\text{-}u.d.p.} \int_0^1 (f(x) - \Phi(x))^2 \mathrm{d}x.$$

We will call such u.d.p. $\Psi(x)$ the best u.d.p. approximation of $f(x)$.

Proof. Let $\Psi_1(x), \Psi_2(x)$ be two u.d.p. functions and $f(x)$ be given. Then

$$\int_0^1 (f(x) - \Psi_1(x))^2 \mathrm{d}x < \int_0^1 (f(x) - \Psi_2(x))^2 \mathrm{d}x \Leftrightarrow$$
$$\int_0^1 f(x)\Psi_1(x)\mathrm{d}x > \int_0^1 f(x)\Psi_2(x)\mathrm{d}x.$$

Here the following property of u.d.p. functions is used

$$\int_0^1 \Psi_1^2(x)\mathrm{d}x = \int_0^1 \Psi_2^2(x)\mathrm{d}x = \int_0^1 x^2\mathrm{d}x = \frac{1}{3}.$$

□

Using Theorem 13, for some special cases, it can be proved that $\Psi(x)$ is the best u.d.p. approximation of a p.l. function $f(x)$. Here we add some consequences of Theorem 13, 12 and 11:

Corollary 1. *If $f(x)$ is a u.d.p. function, then $g_f(x) = x$ and*

$$\max_{\Phi(x)\text{-}u.d.p.} \int_0^1 f(x)\Phi(x)\mathrm{d}x = \int_0^1 f^2(x)\mathrm{d}x = \frac{1}{3}. \tag{71}$$

Corollary 2. *The best u.d.p. approximation of a p.l. function $f(x)$ is independent of the lengths of the ordinate intervals J_j. Therefore for a given u.d.p. function $\Psi(x)$ there are infinite many functions $f(x)$ for which $\Psi(x)$ is the best u.d.p. approximation, e.g., see Fig. 7.*

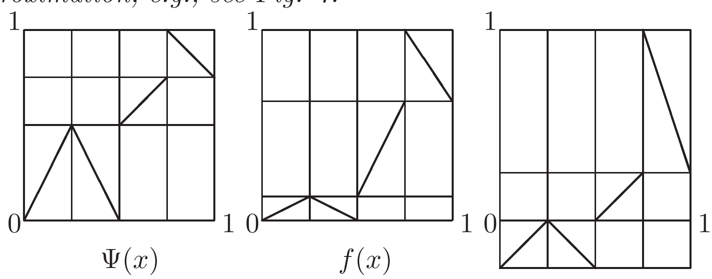

Figure 7: Functions with the same $\Psi(x)$.

Corollary 3. *For a u.d.p. function $\Psi(x)$ in Theorem 12, $\Psi(x)|_{I_{j,i}}$ can be expressed as*

$$\Psi(x) = \frac{|J'_j|}{|J_j|}f(x) + t'_{j-1} - t_{j-1}\frac{|J'_j|}{|J_j|},$$

where $|J'_j| = \sum_{i=1}^{l_j} |I_{j,i}|$ and t'_j is given recurrently as $t'_0 = 0$ and $t'_j = t'_{j-1} + |J'_j|$. So the integral $\int_0^1 f(x)\Psi(x)\mathrm{d}x$ can be calculated directly from $J_j = (t_{j-1}, t_j)$ and $J'_j = (t'_{j-1}, t'_j)$ as

$$\int_0^1 f(x)\Psi(x)\mathrm{d}x = \sum_{j=1}^{k} |J'_j| \left(t_{j-1}t'_{j-1} + \frac{1}{2}t_{j-1}|J'_j| + \frac{1}{2}t'_{j-1}|J_j| + \frac{1}{3}|J_j||J'_j| \right). \tag{72}$$

Corollary 4. *Let $f : [0,1] \to [0,1]$ be a Riemann integrable function. Then*

$$\max_{\Phi(x)-u.d.p.} \int_0^1 f(x)\Phi(x)\mathrm{d}x = \int_0^1 x g_f(x)\mathrm{d}g_f(x) = \frac{1}{2}\left(1 - \int_0^1 g_f^2(x)\mathrm{d}x\right). \quad (73)$$

Proof. Here as in Definition 1, $g_f(x) = |f^{-1}[0,x)|$ for $x \in [0,1]$. For every u.d. sequence x_n, $n = 1, 2, \dots$ the sequence $f(x_n)$ has a.d.f. $g_f(x)$ and then $g_f(f(x_n))$ is a u.d. sequence. By the Helly theorem we have two alternatives

$$\frac{1}{N}\sum_{n=1}^{N} f(x_n) g_f(f(x_n)) \to \int_0^1 x g_f(x)\mathrm{d}g_f(x) \quad (74)$$

$$\to \int_0^1 f(x) g_f(f(x))\mathrm{d}x. \quad (75)$$

Applying per-partes method of integration on (74) we find

$$\int_0^1 x g_f(x)\mathrm{d}g_f(x) = 1 - \int_0^1 g_f^2(x)\mathrm{d}x - \int_0^1 x g_f(x)\mathrm{d}g_f(x)$$

from which (73) follows. □

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