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uniform distribution theory

ON MOVING AVERAGES AND CONTINUED FRACTIONS

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ABSTRACT. We use the moving average ergodic theorem to derive various results concerning moving averages of continued fractions previously known only for non-moving averages and then derived using the pointwise ergodic theorem.

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1. Introduction

We begin by introducing some notation. Let Z be a collection of points in $\mathbf{Z} \times \mathbf{N}$ and let

$$Z^h = \big\{(n,k) : (n,k) \in Z \ \text{ and } \ k \geq h\big\},$$

$$Z^h_\alpha = \big\{(z,s) \in \mathbf{Z}^2 : |z-y| < \alpha(s-r) \text{ for some } (y,r) \in Z^h\big\}$$

and

$$Z^h_\alpha(\lambda) = \big\{n : (n,\lambda) \in Z^h_\alpha\big\} \qquad (\lambda \in \mathbf{N}).$$

Geometrically we can think of Z^1_α as the lattice points contained in the union of all solid cones with aperture α and vertex contained in $Z^1 = Z$. We say a sequence of pairs of natural numbers $(n_l, k_l)_{l=1}^\infty$ is *Stoltz* if there exists a collection of points Z in $\mathbf{Z} \times \mathbf{N}$, and a function $h = h(t)$ tending to infinity with t such that

$$(n_l, k_l)_{l=t}^\infty \in Z^{h(t)}$$

and there exist h_0, α_0 and $A > 0$ such that for all integers $\lambda > 0$ we have

$$\big|Z^{h_0}_{\alpha_0}(\lambda)\big| \leq A\lambda\,.$$

This elaborate condition is interesting because of the following theorem [BJR].

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Theorem 1. *Let (X, β, μ, T) denote a dynamical system, with set X, a σ-algebra of its subsets β, a measure μ defined on the measurable space (X, β) such that $\mu(X) = 1$ and a measurable, measure preserving map T from X to itself. Suppose f is in $L^1(X, \beta, \mu)$ and that the sequence of pairs of natural numbers $(n_l, k_l)_{l=1}^{\infty}$ is Stoltz, then if (X, β, μ, T) is ergodic,*

$$m_f(x) = \lim_{l\to\infty} \frac{1}{k_l} \sum_{i=1}^{k_l} f(T^{n_l+i}x),$$

exists almost everywhere with respect to Lebesgue measure.

Note that if

$$m_{l,f}(x) = \frac{1}{k_l} \sum_{i=1}^{k_l} f(T^{n_l+i}x),$$

then

$$m_{l,f}(Tx) - m_{l,f}(x) = k_l^{-1}\Big(f\big(T^{n_l+k_l+1}\big) - f\big(T^{n_l+1}x\big)\Big).$$

This means that

$$m_f(Tx) = m_f(x), \quad \mu \text{ almost everywhere.}$$

A dynamical system (X, β, μ, T) is called ergodic if given any $A \in \beta$ we have

$$T^{-1}A := \{x \in X : Tx \in A\} = A\,,$$

the set A has either full or null measure. A standard fact in ergodic theory is that if (X, β, μ, T) is ergodic and $m_f(Tx) = m_f(x)$ almost everywhere, $m_f(x) = \int_X f d\mu$, μ almost everywhere [CFS]. The term Stoltz is used here because the condition on $(k_l, n_l)_{l=1}^{\infty}$ is analogous to the condition required in the classical non-radial limit theorem for harmonic functions also called a Stoltz condition, which suggested the above theorem to the authors of [BJR]. Averages, where $k_l = 1$ for all l will be called non-moving. Moving averages satisfying the above hypothesis can be constructed by taking, for instance,

$$n_l = 2^{2^l} \quad \text{and} \quad k_l = 2^{2^{l-1}}.$$

In this paper we apply this theorem to derive some new results about the moving averages of continued fraction expansions of almost all real numbers with respect to Lebesgue measure. The details of our results are described in the next section. The corresponding results for non-moving averages are known and some are classical.

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2. Description and derivation of results

2.1. Averages of partial quotients

In this section we study the metrical theory of regular continued fractions. Let x be an irrational number in $(0,1)$. The familiar expansion of x as a regular continued fraction is denoted

$$\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{1}{a_4+\cdots}}}}, \tag{2.1}$$

or more compactly as

$$x \;=\; [0;\, a_1, a_2, \cdots],$$

where a_n is in $\mathbf{N}$ $(n \in \mathbf{N})$. Here and henceforth in this paper for a real number y the symbol $[y]$ denotes the greatest integer not greater than y and $\{y\}$ denotes its fractional part, that is $\{y\} = y - [y]$. Truncation of (2.1) yields the regular convergents $(p_n/q_n)_{n=0}^{\infty}$. The terms $(a_i)_{i=0}^{\infty}$ are called the partial quotients of x. Define the regular continued fraction map $T : [0,1) \to [0,1)$ by $Tx = \left\{\frac{1}{x}\right\}$ if $x \neq 0$ and $T(0) = 0$. Define the function $a(x)$ on $[0,1]$ by $a(x) = \left[\frac{1}{x}\right]$ if $x \neq 0$ and $a(0) = \infty$. Then $T(x) = [0; a_2, a_3, \cdots]$ and $a_{n+1}(x) = a\big(T^n(x)\big)$. As C. F. Gauss observed, the map T preserves the measure g defined on $[0,1)$ by

$$g(A) = \frac{1}{\log 2}\int_A \frac{dx}{1+x},$$

for each set A contained in the Borel σ-algebra β of $[0,1)$. It is known [CFS] that the dynamical system $([0,1), \beta, g, T)$ is ergodic. We proved the following theorem, the non-moving version of which appears in [RN].

Theorem 2.1. *Suppose the real valued function defined on the non-negative reals F is continuous and increasing. For each natural number n and arbitrary non-negative real numbers $b_1, \cdots, b_n$ we define the mean*

$$M_{F,n}(b_1, \cdots, b_n) = F^{-1}\left[\frac{F(b_1) + \cdots + F(b_n)}{n}\right].$$

Suppose $(n_l, k_l)_{l=1}^{\infty}$ is Stoltz. Then

$$\lim_{l\to\infty} M_{F,l}\big(a_{k_l+1}(x), \cdots, a_{n_l+k_l}(x)\big) = F^{-1}\left[\frac{1}{\log 2}\int_0^1 \frac{F\big(a_1(t)\big)}{t+1}dt\right].$$

almost everywhere with respect to Lebesgue measure.

Proof. If
$$\frac{1}{\log 2}\int_0^1 \frac{F\big(a_1(x)\big)}{x+1}dx < \infty,$$
then the result is an immediate consequence of Theorem 1. We otherwise argue as follows. Let
$$f_M(x) = \begin{cases} F(a_1(x)) & \text{if } \big|F\big(a_1(x)\big)\big| \le M, \\ M & \text{if } \big|F\big(a_1(x)\big)\big| > M. \end{cases}$$

This means that for each $M \geq |F(1)|$ and almost every x with respect to Lebesgue measure
$$\liminf_{l\to\infty} \frac{1}{k_l}\sum_{i=1}^{k_l} F\big(a_{n_l+i}(x)\big) \geq \liminf_{l\to\infty} \frac{1}{k_l}\sum_{i=1}^{k_l} f_M\big(T^{n_l+i}x\big),$$
$$\frac{1}{\log 2}\int_0^1 \frac{f_M(x)}{x+1}\,dx.$$
Letting M tend to ∞ completes the proof. □

This result has a number of interesting arithmetic consequences some of which are detailed as corollaries now and which follow for appropriate choice of F.

Corollary 2.2. *If $(n_l, k_l)_{l=1}^{\infty}$ is Stoltz, then*
$$\lim_{l\to\infty} \frac{1}{k_l}\sum_{i=1}^{k_l} a_{n_l+i}(x) = \infty,$$
almost everywhere with respect to Lebesgue measure.

Corollary 2.3. *If $(n_l, k_l)_{l=1}^{\infty}$ is Stoltz, then*
$$\lim_{l\to\infty} \big(a_{n_l+1}(x)\cdots a_{n_l+k_l}(x)\big)^{\frac{1}{k_l}} = \Pi_{k=1}^{\infty}\left(1+\frac{1}{k(k+2)}\right),$$
almost everywhere with respect to Lebesgue measure.

Corollary 2.4. *Suppose $(n_l, k_l)_{l=1}^{\infty}$ is Stoltz, and let $P_l(x,q)$ $(l = 1, 2, \cdots)$ denote the number of $a_{n_l+1}(x), \cdots, a_{n_l+k_l}(x)$ such that $a_{n_l+i}(x) = q$. Then*
$$\lim_{l\to\infty} \frac{P_l(x,q)}{k_l} = \frac{1}{\log 2}\log\frac{(q+1)^2}{q(q+2)},$$
almost everywhere with respect to Lebesgue measure.

2.2. Average behaviour of approximation constants $\big(\theta_n(x)\big)_{n=1}^{\infty}$

There are interesting properties of the regular continued fraction expansion that do not follow from a study of the dynamical system $\big([0,1),\beta,g,T\big)$ alone. In particular we need the following theorem proved in [INT].

Theorem 2.5. *Let $\Omega = \big([0,1)\setminus\mathbf{Q}\big)\times[0,1)$. Now let β be the Borel σ algebra of subsets of Ω and let μ be the probability measure on the measurable space (Ω,β) with density $(\log 2)^{-1}(1+xy)^{-1}$. Define the map*

$$\mathcal{T}(x,y) = \left(Tx, \frac{1}{\left[\frac{1}{x}\right]+y}\right), \qquad (x,y)\in\Omega.$$

Then the dynamical system $(\Omega,\beta,\mu,\mathcal{T})$ is measurable, measure preserving and also ergodic.

The transformation $\mathcal{T}$ is in fact a concrete form of the natural extension of T. It is useful at this point to set $T_n = T^n(x)$ for $(n\geq 0)$ and $V_n = \frac{q_{n-1}}{q_n}$ also for $(n\geq 0)$. Then $T_n = [0; a_{n+1}, a_{n+2}\cdots]$ and a simple calculation shows that $V_n = [0; a_n,\cdots,a_1]$. Moreover, the sequence $(T_n,V_n)_{n=0}^{\infty}$ is contained in Ω. Notice that

$$\mathcal{T}^n(x,y) = \Big(T^n x, [0;\ a_n,\cdots,a_2,a_1+y]\Big) \qquad (0\leq y\leq 1;\ n=1,2,\cdots)$$

and hence in particular

$$\mathcal{T}^n(x,0) = (T_n,V_n). \tag{2.2}$$

First we consider the well known diophantine inequality [HW]

$$\left|x-\frac{p_n}{q_n}\right| \leq \frac{1}{q_n^2}. \tag{2.3}$$

This motivates the definition of the function $\theta_n(x)$ $(n=1,2,\cdots)$ as the quantity that satisfies

$$\left|x-\frac{p_n}{q_n}\right| = \frac{\theta_n(x)}{q_n^2}.$$

Of course, this means that $0\leq\theta_n(x)\leq 1$. It is noted in [K] that

$$\theta_n(x) = \frac{1}{\left(\frac{1}{T^n x}+\frac{q_{n-1}}{q_n}\right)}. \tag{2.4}$$

Also in [J] it is shown that for each irrational number x the sequence

$$\big(\theta_n(x),\theta_{n+1}(x)\big)_{n=1}^{\infty}$$

is contained in the triangle with verticies $(0,0),(1,0)$ and $(0,1)$. We have the following theorem.

Theorem 2.6. *Suppose* $f_1(r,s) = \frac{1}{\log 2}\frac{1}{\sqrt{1-4rs}}$ *for* (r,s) *such that* $0 \le r+s < 1$, *and* $f_1(r,s) = 0$ *otherwise. Then if the sequence of pairs of natural numbers* $(n_l, k_l)_{l=1}^{\infty}$ *is Stoltz*

$$\lim_{l\to\infty} \frac{1}{k_l}\Big|\big\{1 \le l \le \ k_l : \theta_{n_l+i}(x) < a;\ \theta_{n_l+i+1}(x) < b\big\}\Big| = \int_0^a\int_0^b f_1(r,s)drds,$$

almost everywhere with respect to Lebesgue measure.

Proof. Set Y_a to be the curve $(u+v)^{-1} = a$ and Z_b to be the curve $(u^{-1}+v^{-1})^{-1} = b$ in the (u,v) plane with $0 < a \le 1$ and $0 < b \le 1$. The curves Y_a and Z_b do not intersect in $[0,1]\times[0,1]$ when $a+b > 1$ because otherwise $(1-u)(1-v) \le\ 0$ which is clearly not possible. On the other hand, if $a+b \le 1$, the curves Y_a and Z_b intersect at one point whose coordinates are

$$\left(\frac{1-\sqrt{1-4ab}}{2a},\ \frac{1-\sqrt{1-4ab}}{2b}\right). \tag{2.5}$$

Now, because,

$$\theta_n(x) = \left(T^{n+1}x + \frac{q_{n+1}}{q_n}\right)^{-1},$$

as the reader will easily verify, $\theta_n(x) < a$ if and only if the point $\left(T^{n+1}x, \frac{q_n}{q_{n+1}}\right)$ lies in the (u,v) plane under the curve Y_a. Similarly, as the reader will also easily verify,

$$\theta_n(x) = \left(\frac{1}{T^n x} + \frac{q_{n-1}}{q_n}\right)^{-1}$$

thus the inequality $\theta_{n+1}(x) < b$ is satisfied if and only if this point lies above the curve Z_b. Now set $W(a,b)$ to be the region in $[0,1]\times[0,1]$ bounded by the curves Y_a, Z_b, the u axis and the v axis. Suppose for a given $\epsilon > 0$ that

$$U(a,b,\epsilon) \subset W(a,b) \subset V(a,b,\epsilon),$$

where $W(a,b)$ is an ϵ neighbourhood of $U(a,b,\epsilon)$ and $V(a,b,\epsilon)$ is an ϵ neighbourhood of $W(a,b)$. Then there is a natural number $N_0 = N_0(\epsilon)$ such that if $N > N_0$ for all y in $[0,1]$,

$$\mathcal{T}(x,y) \in\ V(a,b,\epsilon),$$

implies

$$\mathcal{T}(x,0) \in\ W(a,b),$$

and

$$\mathcal{T}(x,0) \in\ W(a,b),$$

$$\mathcal{T}(x,y) \in\ U(a,b,\epsilon).$$

From this it follows that

$$\begin{aligned}
&\lim_{l\to\infty} \frac{1}{k_l} \left|\left\{1 \le i \le k_l \ : \mathcal{T}^{n_l+i}(x,y) \in V(a,b,\epsilon)\right\}\right| \\
&\le \liminf_{l\to\infty} \frac{1}{k_l} \left|\left\{1 \le i \le k_l \ : \mathcal{T}^{n_l+i}(x,0) \in W(a,b)\right\}\right| \\
&\le \limsup_{l\to\infty} \frac{1}{k_l} \left|\left\{1 \le i \le k_l \ : \mathcal{T}^{n_l+i}(x,0) \in W(a,b)\right\}\right| \\
&\le \lim_{l\to\infty} \frac{1}{k_l} \left|\left\{1 \le i \le k_l : \mathcal{T}^{n_l+i}(x,y) \in U(a,b,\epsilon)\right\}\right|.
\end{aligned}$$

Thus using Theorem 2.1 all limits exist for almost all x with respect to Lebesgue measure, and as ϵ can be taken arbitrarily small equals $\omega(W(a,b))$. Now

$$\begin{aligned}
\omega\big(W(a,b)\big) &= \frac{1}{\log 2} \iint_{W(a,b)} \frac{du dv}{(1+uv)^2} \\
&= \frac{1}{\log 2}(1-\sqrt{1-4ab}) - \log\frac{1}{2}(1+\sqrt{1-4ab}).
\end{aligned}$$

Observe that

$$\frac{\partial^2 \omega\big(W(a,b)\big)}{\partial a \partial b} = \frac{1}{\log 2}\frac{1}{\sqrt{1-4ab}},$$

which completes the proof of Theorem 2.6 □

Theorem 2.7. *Suppose the sequence of pairs of natural numbers $(n_l,k_l)_{l=1}^{\infty}$ is Stoltz. Let the function $F_1 : [0,1] \to [0,1]$ be defined by*

$$F_1(z) = \frac{z}{\log z} \ \text{on}\ \left[0,\frac{1}{2}\right] \quad \text{and} \quad F_1(z) = \frac{1}{\log 2}(1 - z + \log 2z) \ \text{on}\ \left[\frac{1}{2},1\right].$$

Then

$$\lim_{l\to\infty} \frac{1}{k_l} \left|\left\{1 \le i \le k_l : \theta_{n_l+i}(x) \le z\right\}\right| = F_1(z), \tag{2.6}$$

almost everywhere with respect to Lebesgue measure.

In the stationary case this result was conjectured by H. W. Lenstra Jr. and proved in [BJW].

P r o o f o f T h e o r e m 2.7. Denote by $\Omega(c)$ with $c \ge 1$ that part of Ω on or above the hyperbola $\frac{1}{x} + y = c$. By (2.3) and (2.4) the statement

$$\theta_n(x) \le z \qquad \text{for} \ \ z \in [0,1]$$

is equivalent to the statement

$$\mathcal{T}^n(x,0) \in \Omega\left(\frac{1}{z}\right).$$

It is also readily verified that there exists an integer $n_0(\epsilon)$ such that for all n greater than $n_0(\epsilon)$ and all y in $[0,1]$ if

$$\mathcal{T}^n(x,y) \in \Omega\left(\frac{1}{z}+\epsilon\right),$$

then

$$\mathcal{T}^n(x,0) \in \Omega\left(\frac{1}{z}\right).$$

Also if

$$\mathcal{T}^n(x,0) \in \Omega\left(\frac{1}{z}\right),$$

then

$$\mathcal{T}^n(x,y) \in \Omega\left(\frac{1}{z}-\epsilon\right).$$

From this it follows that for almost all (x,y) with respect to the measure μ we have

$$\begin{aligned}
&\lim_{l\to\infty} \frac{1}{k_l}\left|\left\{1 \le i \le k_l : \mathcal{T}^{n_l+i}(x,y) \in \Omega\left(\frac{1}{z}+\epsilon\right)\right\}\right| \\
&\le \liminf_{l\to\infty} \frac{1}{k_l}\left|\left\{1 \le i \le k_l : \mathcal{T}^{n_l+i}(x,0) \in \Omega\left(\frac{1}{z}\right)\right\}\right| \\
&\le \limsup_{l\to\infty} \frac{1}{k_l}\left|\left\{1 \le i \le k_l : \mathcal{T}^{n_l+i}(x,0) \in \Omega\left(\frac{1}{z}\right)\right\}\right| \\
&\le \lim_{l\to\infty} \frac{1}{k_l}\left|\left\{1 \le i \le k_l : \mathcal{T}^{n_l+i}(x,y) \in \Omega\left(\frac{1}{z}-\epsilon\right)\right\}\right|.
\end{aligned}$$

Using the fact that $(n_l,k_l)_{l=1}^{\infty}$ is Stoltz, both limits exist and are

$$\mu\left(\Omega\left(\frac{1}{z}+\epsilon\right)\right) \quad \text{and} \quad \mu\left(\Omega\left(\frac{1}{z}-\epsilon\right)\right),$$

respectively. Since ϵ is arbitrary the limit (2.5) exists and is equal to $\mu(\Omega(\frac{1}{z}))$ for almost all x with respect to Lebesgue measure. We straightforwardly verify that

$$\mu\left(\Omega\left(\frac{1}{z}\right)\right) = F(z).$$

□

Corollary 2.8. *Suppose the sequence of pairs of natural numbers* $(n_l, k_l)_{l=1}^{\infty}$ *is Stoltz. Then*

$$\lim_{l\to\infty} \frac{1}{k_l} \sum_{i=1}^{k_l} \theta_{n_l+i}(x) = \frac{1}{4\log 2},$$

almost everywhere with respect to Lebesgue measure.

Corollary 2.9. *For* t *in* $[0,1]$ *let* $f_2(t) = \frac{1}{2\log 2}\big((1+t)\log(1+t+(1-t)\log(1-t)\big)$. *Then if the sequence of pairs of natural numbers* $(n_l, k_l)_{n=1}^{\infty}$ *is Stoltz*

$$\lim_{l\to\infty} \frac{1}{k_l} \left|\{1 \le i \le \ k_l : \theta_{n_l+i}(x) + \theta_{n_l+i+1}(x) < t\}\right| = f_2(t),$$

almost everywhere with respect to Lebesgue measure.

Proof. Set $\Delta_1(t)$ to be the triangle with vertices at $(0,0)$, $(t,0)$ and $(0,t)$ for a particular t in $(0,1]$. By Theorem 2.6 we have

$$\lim_{l\to\infty} \frac{1}{k_l} \Big|\{1 \le i \le k_l : \theta_{n_l+i}(x) + \theta_{n_l+i+1}(x) < t\}\Big| = \frac{1}{\log 2} \iint_{\Delta_1(t)} \frac{drds}{\sqrt{1-4rs}}.$$

By the change of variables $x = r + s$ and $y = r - s$ we find that

$$\begin{aligned}
\frac{1}{\log 2} \iint_{\Delta_1(t)} \frac{drds}{\sqrt{1-4rs}} &= \frac{1}{2\log 2} \int_0^t \left(\int_{-x}^{x} \frac{dy}{\sqrt{1+y^2-x^2}} \right) dx \\
&= \frac{1}{2\log 2} \int_0^t \left[\log\left(y + \sqrt{1+y^2-x^2}\right)\right]_{-x}^{x} dx \\
&= \frac{1}{2\log 2} \int_0^t \log \frac{1+x}{1-x} dx,
\end{aligned}$$

as required. □

Corollary 2.10. *For* t *in* $[0,1]$ *let*

$$f_2(t) = \frac{1}{\log 2} \left(\frac{1}{2}\pi t - 2t \arctan + \log\big(1+t^2\big)\right).$$

Then if the sequence of pairs of natural numbers $(n_l, k_l)_{l=1}^{\infty}$ *is Stoltz*

$$\lim_{l\to\infty} \frac{1}{k_l} \left|\left\{1 \le i \le k_l : \left|\theta_{n_l+i}(x) - \theta_{n_l+i+1}(x)\right| \le t\right\}\right| = f_2(t),$$

almost everywhere with respect to Lebesgue measure.

Proof. Set $\Delta_2(t)$ to be the triangle with vertices at $(t,0)$ $(1,0)$ and $\big(\frac{1}{2}(1+t), \frac{1}{2}(1-t)\big)$ for a particular t in $(0,1]$. By Theorem 2.6 we have

$$\lim_{l\to\infty} \frac{1}{N} \Big|\{1 \le i \le k_l : \left|\theta_{n_l+i}(x) - \theta_{n_l+i+1}(x)\right| < t\}\Big| = 1 - \frac{2}{\log 2} \iint_{\Delta_2(t)} \frac{drds}{\sqrt{1-4rs}}.$$

By the change of variables $x = r + s$ and $y = r - s$ we find that

$$\begin{aligned} 1 - \frac{2}{\log 2} \iint_{\Delta_2(t)} \frac{drds}{\sqrt{1-4rs}} &= 1 - \frac{1}{\log 2} \int_t^1 \left(\int_y^1 \frac{dx}{\sqrt{1+y^2-x^2}} \right) dy \\ &= 1 - \frac{1}{\log 2} \int_t^1 \left[\arcsin \frac{x}{\sqrt{1+y^2}} \right]_{x=y}^{x=1} dy \\ &= 1 - \frac{1}{\log 2} \int_t^1 \left(\frac{\pi}{2} - 2 \arctan y \right) dy \\ &= \frac{1}{\log 2} \int_0^t \left(\frac{\pi}{2} - 2 \arctan y \right) dy, \end{aligned}$$

as required. □

Corollary 2.11. *For t in $[0,1]$ let*

$$f_3(t) = \sqrt{1-4t} - \frac{1}{\log 2} \sqrt{1-4t} \log\left(1 + \sqrt{1-4t}\right) - \frac{1}{2\log 2}\left(1 - \sqrt{1-4t}\right) \log t.$$

If the sequence of pairs of integers $(n_l, k_l)_{l=1}^{\infty}$ is Stoltz, then

$$\lim_{l\to\infty} \frac{1}{k_l} \left| \left\{ 1 \le i \le k_l : \theta_{n_l+i}(x)\theta_{n_l+i+1}(x) \le t \right\} \right| = f_3(t),$$

almost everywhere with respect to Lebesgue measure.

2.3. The distribution of other sequences attached to the regular continued fraction expansion

We now consider some sequences other than $(\theta_n(x))_{n=1}^{\infty}$.

Theorem 2.12. *Suppose z is in $[0,1]$ and for irrational x in $(0,1)$ set $Q_n(x) = \frac{q_{n-1}(x)}{q_n(x)}$ for each positive integer n. Suppose also that the sequence of pairs of natural numbers $(n_l, k_l)_{l=1}^{\infty}$ is Stoltz. Then*

$$\lim_{l\to\infty} \frac{1}{k_l} \left| \left\{ 1 \le i \le k_l : Q_{n_l+i}(x) \le z \right\} \right| = F_2(z) = \frac{\log(1+z)}{\log 2}$$

almost everywhere with respect to Lebesgue measure.

Proof. Using the fact that $(n_l, k_l)_{l=1}^{\infty}$ is Stoltz, we see that

$$\lim_{l\to\infty} \frac{1}{k_l} \left| \left\{ 1 \le i \le k_l : T^{n_l+i}(x) \le z \right\} \right| = \frac{1}{\log 2} \int_0^z \frac{dx}{1+x} = \frac{\log(1+z)}{\log 2}.$$

Now note that if for a set E in β if $\overline{E}$ denotes $\{(x,y) : (y,x) \in E\}$, then $\mu(E) = \mu(\overline{E})$ and so $(Q_n(x))_{n=1}^{\infty}$ is distributed identically to $(T^n x)_{n=1}^{\infty}$ and the theorem follows as a consequence. □

Theorem 2.13. *For irrational x in $(0,1)$ set*

$$r_n(x) = \frac{\left|x - \frac{p_n}{q_n}\right|}{\left|x - \frac{p_{n-1}}{q_{n-1}}\right|} \qquad (n = 1, 2, \cdots).$$

Further, for z in $[0,1]$ let

$$F_3(z) = \frac{1}{\log 2}\left(\log(1+z) - \frac{z}{1+z}\log z\right). \tag{2.7}$$

Suppose also that the sequence of pairs of natural numbers $(n_l, k_l)_{l=1}^{\infty}$ is Stoltz. Then

$$\lim_{l\to\infty} \frac{1}{k_l}\left|\left\{1 \le i \le k_l : r_{n_l+i}(x) \le z\right\}\right| = F_3(z),$$

almost everywhere with respect to Lebesgue measure.

Proof. It follows from the fact that

$$x - \frac{p_n}{q_n} = \frac{(-1)^n T^n x}{q_n(q_n + q_{n-1}T^n x)} \tag{2.8}$$

[BJW] and the fact that

$$\frac{1}{T^{n-1}x} = a_n + T^n x \quad \text{and that} \quad r_n(x) = \frac{q_n}{q_{n-1}} T^n x. \tag{2.9}$$

Arguing as in the proof of Theorem 2.7 we see that F_3 exists for almost all x and that for z in $[0,1]$ the value of $F_3(z)$ is equal to the μ measure of the part of Ω under the curve $xy = z$. A simple calculation shows that F_3 is given by (2.7) as specified. □

Corollary 2.14. *Suppose the sequence of pairs of natural numbers $(n_l, k_l)_{l=1}^{\infty}$ is Stoltz. Then*

$$\lim_{l\to\infty} \frac{1}{k_l}\sum_{i=1}^{k_l} r_{n_l+i}(x) = \frac{\pi^2}{12\log 2},$$

almost everywhere with respect to Lebesgue measure.

Proof. The limit is $\int_0^1 z dF_3(z)$. □

Another well known inequality is the following

$$\left|x - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}} \qquad (n = 1, 2, \cdots)$$

which motivates the following result.

Theorem 2.15. *For each irrational number x in $(0,1)$ define the function $d_n(x)$ for each natural number n by the identity*

$$\left|x-\frac{p_n}{q_n}\right|=\frac{d_n(x)}{q_nq_{n-1}}.$$

Suppose the sequence of pairs of natural numbers $(n_l,k_l)_{n=1}^{\infty}$ is Stoltz. Suppose also F_4 is defined on $[0,1]$ as

$$F_4(z)=0 \text{ if } z \text{ is in } \left[0,\frac{1}{2}\right]$$

and

$$F_4(z)=\frac{1}{\log 2}\big(2\log z+(1-z)\log(1-z)+\log z\big)$$

if z is in $\left[\frac{1}{2},1\right]$. Then

$$\lim_{l\to\infty}\frac{1}{k_l}\left|\left\{1\le i\le k_l : d_{n_l+i}(x)\le z\right\}\right|=F_4(z),$$

almost everywhere with respect to Lebesgue measure.

Proof. From (2.8) and (2.9) we readily see that

$$d_n(x)=\frac{1}{1+\frac{q_n}{q_{n-1}}T^{n+1}x}\qquad (n=1,2,\cdots).$$

Hence $F_4(z)$ equals the μ measure of the part of Ω above the curve $xy=\frac{1}{z}-1$. Note that for $z\le\frac{1}{2}$ this is an empty set. $\square$

Finally, in this section we consider the inequality

$$\left|x-\frac{p_n}{q_n}\right|<\frac{1}{2q_nq_{n-1}}\qquad (n=1,2,\cdots).$$

This is sharper than (2.2) whenever $a_n=1$. That is for almost all x with frequency $2-\frac{\log 3}{\log 2}$. See [K] for details. This motivates the following result.

Theorem 2.16. *For each irrational number x in $(0,1)$ define the function $D_n(x)$ for each natural number n by the identity*

$$\left|x-\frac{p_n}{q_n}\right|=\frac{D_n(x)}{q_nq_{n-1}}\qquad (n=1,2,\cdots).$$

Suppose the sequence of pairs of natural numbers $(n_l, k_l)_{l=1}^{\infty}$ *is Stoltz. Suppose* F_5 *is defined on* $[0,1]$ *by*

$$F_5(z) = \frac{1}{\log 2}\left(\log z - \frac{z}{2}\log z - \frac{2-z}{2}\log(2-z)\right)$$

if z *is in* $[0,1]$. *Then*

$$\lim_{l\to\infty} \frac{1}{k_l}|\{1 \le i \le k_l : D_{n_l+i}(x) \ \le \ z\}| \ = \ F_5(z),$$

almost everywhere with respect to Lebesgue measure.

Proof. It is not difficult to verify that

$$D_n(x) \ = \ 2\frac{1}{\left(\frac{q_n}{q_{n-1}}\frac{1}{T^n x} + 1\right)} \qquad (n \ = \ 1, 2, \cdots)$$

As earlier in the proof of Theorem 2.7 $F_5(z)$ denotes the μ measure of the part of Ω under the hyperbola $xy = \frac{z}{2-z}$ when z is in $[0,1]$. This completes the proof of the theorem. □

Corollary 2.17. *Suppose the sequence of pairs of natural numbers* $(n_l, k_l)_{l=1}^{\infty}$ *is Stoltz. Then*

$$\lim_{l\to\infty} \frac{1}{k_l}\sum_{i=1}^{k_l} D_{n_l+i}(x) = 1 - \frac{1}{2\log 2},$$

almost everywhere with respect to Lebesgue measure.

Proof. The limit is $\int_0^1 z dF_5(z) = 1 - \frac{1}{2\log 2}$. □

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