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A NOTE ON THE EXTREME DISCREPANCY OF THE HAMMERSLEY NET IN BASE 2

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ABSTRACT. In this note, we study lower bounds on the extreme discrepancy of the Hammersley net in base 2. The Hammersley net in base 2 can be interpreted as a finite two-dimensional analogue of the well known (one-dimensional) van der Corput sequence in base 2. For the van der Corput sequence it is known that its star discrepancy equals its extreme discrepancy. In this paper, we prove the rather surprising fact that the same does not hold for the Hammersley net, by giving lower bounds on its extreme discrepancy. We furthermore state a few remarks on upper bounds and conclude with a conjecture.

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1. Introduction and notation

In this paper, we study a special, but very well known type of a finite two-dimensional point set in the unit square $[0,1)^2$, namely the Hammersley point set in base 2. By "point set", here and in the following, we do not mean a proper set but a finite or infinite sequence of points, i.e., points may occur repeatedly.

The Hammersley point set is in some sense a finite two-dimensional analogue of the so-called van der Corput sequence in base 2. The van der Corput sequence is defined as follows.

Definition 1. Let $n \geq 0$ be an integer and let $n_0 + n_1 2 + n_2 2^2 + \cdots$, $n_i \in \{0,1\}$ for $i \geq 0$, be its unique base 2 representation. The radical inverse function (to the base 2) of n is defined by $\phi(n) = \sum_{i=0}^{\infty} \frac{n_i}{2^{i+1}}$. The van der Corput sequence V in base 2 is defined as the sequence

$$(x_n)_{n\geq 0} = \big(\phi(n)\big)_{n\geq 0}.$$

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Note that the van der Corput sequence is a digital $(0,1)$-sequence over $\mathbb{Z}_2$ in the sense of Niederreiter, see [13, 14] for further details, and [5] for a recent overview of related results. If one wants to have a (finite) two-dimensional version of the van der Corput sequence, it is near at hand to study the well known Hammersley point set, which is defined as follows.

Definition 2. Let an integer $m \geq 1$ be given. The Hammersley point set in base 2 with 2^m points in $[0,1)^2$ is the finite sequence $(\boldsymbol{x}_n)_{n=0}^{2^m-1}$, where

$$\boldsymbol{x}_n = \left(\frac{n}{2^m}, \phi(n)\right).$$

We denote this point set by H_m.

Note that H_m is *the* prototype of a digital $(0,m,2)$-net over $\mathbb{Z}_2$, see again [13, 14], which is why it is frequently referred to as Hammersley net. Alternatively, H_m is sometimes also called Roth net, but we shall use the term Hammersley net in the following.

In our paper, we study the discrepancy of the Hammersley point set, in particular we deal with its extreme and its star discrepancy. For defining these two types of discrepancy, we need a discrepancy function, which measures the relative difference between the number of points in a given subinterval of the s-dimensional unit cube $[0,1)^s$ and its volume. For the following definitions, we denote by $\mathbf{0}$ the zero vector, and for real vectors

$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_s), \qquad \boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_s) \in [0,1)^s,$$

we denote by $[\boldsymbol{\alpha}, \boldsymbol{\beta})$ the cartesian product of $[\alpha_1, \beta_1), [\alpha_2, \beta_2), \ldots, [\alpha_s, \beta_s)$.

We define a local discrepancy function as follows.

Definition 3. Let $s \geq 1$. For $\boldsymbol{\alpha}, \boldsymbol{\beta} \in [0,1]^s$ and a point set $P = (\boldsymbol{p}_n)_{n=0}^{N-1} \subseteq [0,1)^s$, let $\Delta(P, \boldsymbol{\alpha}, \boldsymbol{\beta})$ denote the discrepancy function of P evaluated at $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, which is defined as

$$\Delta(P, \boldsymbol{\alpha}, \boldsymbol{\beta}) = A_N\big(P, [\boldsymbol{\alpha}, \boldsymbol{\beta})\big) - N\lambda_s\big([\boldsymbol{\alpha}, \boldsymbol{\beta})\big),$$

where $A_N(P, [\boldsymbol{\alpha}, \boldsymbol{\beta}))$ denotes the number of points of P in $[\boldsymbol{\alpha}, \boldsymbol{\beta})$, and λ_s denotes the s-dimensional Lebesgue measure.

Using the discrepancy function, we now define the star discrepancy and the extreme discrepancy.

Definition 4. Let $P = (\boldsymbol{p}_n)_{n=0}^{N-1}$ be a point set of N points in the s-dimensional unit cube, $[0,1)^s$. The star discrepancy of P is defined as

$$D_N^*(P) := \sup_{\boldsymbol{\beta} \in [0,1]^s} \left| \frac{\Delta(P, \mathbf{0}, \boldsymbol{\beta})}{N} \right|.$$

For an infinite sequence, D_N^* denotes the discrepancy of the first N points of the sequence.

Definition 5. Let $P = (\boldsymbol{p}_n)_{n=0}^{N-1}$ be a point set of N points in the s-dimensional unit cube, $[0,1)^s$. The extreme discrepancy of P is defined as

$$D_N(P) := \sup_{\boldsymbol{\alpha},\boldsymbol{\beta}\in[0,1]^s} \left| \frac{\Delta(P,\boldsymbol{\alpha},\boldsymbol{\beta})}{N} \right|.$$

Again, for an infinite sequence, D_N denotes the discrepancy of the first N points of the sequence.

Note that the discrepancy of a point set gives information on its quality of uniform distribution. The lower the discrepancy of a point set, the more evenly its points are distributed in the unit cube. On the other hand, high discrepancy indicates a relatively uneven distribution of points. Since the van der Corput sequence and the Hammersley point set are very well known examples of point sets in $[0,1)$ and $[0,1)^2$, respectively, much is known about their distribution properties. In particular, their star discrepancies have been studied frequently, see, e.g., [1]–[4], [6]–[12], and the references therein. It is known (see [2, 8, 12]) that the local discrepancy function of the Hammersley net is always non-negative, and the star discrepancy of H_m for $m \geq 2$ can be calculated explicitly by

$$2^m D_{2^m}^*(H_m) = \frac{m}{3} + \frac{13}{9} - (-1)^m \frac{4}{9 \cdot 2^m}. \tag{1}$$

Furthermore, it has been shown in [4] that H_m has essentially the worst star discrepancy among all $(0,m,2)$-nets in base 2.

For the van der Corput sequence, it is known that its local discrepancy is also never negative, and its star discrepancy satisfies the upper bound

$$ND_N^*(V) \leq \frac{\log_2 N}{3} + 1 \tag{2}$$

for all $N \geq 1$, where $\log_2$ is the logarithm to the base 2. In this note, we will mostly be interested in the "leading term" in discrepancy bounds, i.e., in the case of Equations (1) and (2), we are most interested in the coefficient that goes with the logarithm of the number of points. Note that from Equations (1) and (2) it follows that, for both the Hammersley net as well as the first points of the van der Corput sequence, the leading term in the star discrepancy is at most

$$\frac{\log_2 K}{3},$$

where K is the number of points involved. Note also that, due to results in [9], it is known that the van der Corput sequence has the worst star discrepancy among all $(0,1)$-sequences in base 2.

Regarding the one-dimensional sequence V, it has been shown by Béjian and Faure (see [1]) that its star discrepancy always equals its extreme discrepancy, i.e.,

$$ND_N^*(V) = ND_N(V) \tag{3}$$

for all $N \geq 1$. So also for $ND_N(V)$ the leading term in the discrepancy bound is $(\log_2 N)/3$.

As there are many similarities between V and H_m, their discrepancies, and their roles among $(0,1)$-sequences and $(0,m,2)$-nets, respectively, one would suspect that an analogue of Equation (3) also holds for H_m. In this paper, we show the rather surprising fact that this is *not* the case. The rest of this note is organized as follows. In Section 2, we show a lower bound on the extreme discrepancy of H_m, thereby proving that the analogue of (3) does not hold for H_m. In Section 3, we discuss possible issues with an upper bound on $D_{2^m}(H_m)$ and conclude with a conjecture.

2. Lower bounds on the extreme discrepancy of the Hammersley net

In this section, we establish useful lower bounds on the extreme discrepancy D_{2^m} of H_m. To this end, we need some notation.

For a given real number $\mu \in [0,1)$ and $m \geq 1$, we say that μ is m-bit, if μ is of the form

$$\mu = \frac{\mu_1}{2} + \frac{\mu_2}{2^2} + \frac{\mu_3}{2^3} + \cdots + \frac{\mu_m}{2^m},$$

with $\mu_1, \mu_2, \ldots, \mu_m \in \{0,1\}$. Furthermore, we denote the base 2 representation of $\mu \in [0,1)$ as

$$\mu = 0.\mu_1\mu_2\ldots\mu_m.$$

Whenever we speak of an m-bit number we mean, in the following, that at most the first m digits in its base 2 expansion are different from zero.

Our lower bound on the extreme discrepancy of H_m is obtained by explicitly giving intervals where the local discrepancy function attains a certain value. We need to distinguish three cases, depending on the remainder when m is divided by 3.

We first show the following result.

Proposition 1. *Let $k \geq 1$ and let $m = 3k$. Furthermore, let $\alpha^{(1)}, \alpha^{(2)}, \beta^{(1)}, \beta^{(2)}$ have the following base 2 representation:*

$$\alpha^{(1)} = \beta^{(1)} = 0.\underbrace{001001001\ldots001001}_{m\text{ bits}},$$

$$\alpha^{(2)} = \beta^{(2)} = 0.\underbrace{110110110\ldots110111}_{m\text{ bits}}.$$

Then we have

$$\Delta\Big(H_m, \big(\alpha^{(1)}, \beta^{(1)}\big), \big(\alpha^{(2)}, \beta^{(2)}\big)\Big) = \frac{10m}{21} + \frac{4}{49} - \frac{2^{2-m}}{49}.$$

Proof. Let k, m, and $\alpha^{(1)}, \alpha^{(2)}, \beta^{(1)}, \beta^{(2)}$ be as above. First of all, we note that

$$\begin{aligned}
&\Delta\Big(H_m, \big(\alpha^{(1)}, \beta^{(1)}\big), \big(\alpha^{(2)}, \beta^{(2)}\big)\Big) \\
&= \Delta\left(H_m, \mathbf{0}, \left(\alpha^{(2)}, \beta^{(2)}\right)\right) - \Delta\left(H_m, \mathbf{0}, \left(\alpha^{(1)}, \beta^{(2)}\right)\right) \\
&- \Delta\left(H_m, \mathbf{0}, \left(\alpha^{(2)}, \beta^{(1)}\right)\right) + \Delta\left(H_m, \mathbf{0}, \left(\alpha^{(1)}, \beta^{(1)}\right)\right). \qquad (4)
\end{aligned}$$

We make use of a formula for the discrepancy function Δ for m-bit α and β, established in [12],

$$\Delta\big(H_m, \mathbf{0}, (\alpha, \beta)\big) = \sum_{u=0}^{m-1} \|2^u \beta\| \left(\alpha_{m-u} \oplus \alpha_{m+1-j(u,\alpha,\beta,m)}\right).$$

Here $\|\cdot\|$ denotes the distance to the nearest integer, i.e., $\|x\| = \min\{x - \lfloor x \rfloor, 1 - (x - \lfloor x \rfloor)\}$, $\oplus$ means addition modulo 2, and α_i, β_i are the i-th digits of α and β, respectively, in their base 2 representation (we set $\alpha_{m+k} = 0$, $k \geq 1$). Furthermore, the function $j(u, \alpha, \beta, m)$ is defined as follows:

$$j(u, \alpha, \beta, m) = \begin{cases} 0 & \text{if } u = 0, \\ 0 & \text{if } \alpha_{m+1-j} = \beta_j,\ 1 \leq j \leq u, \\ \max\{1 \leq j \leq u : \alpha_{m+1-j} \neq \beta_j\} & \text{otherwise.} \end{cases}$$

Let us first study $\Delta\big(H_m, \mathbf{0}, \big(\alpha^{(2)}, \beta^{(2)}\big)\big)$. It is easily checked that we have

$$j\big(0, \alpha^{(2)}, \beta^{(2)}, m\big) = j\big(1, \alpha^{(2)}, \beta^{(2)}, m\big) = j\big(2, \alpha^{(2)}, \beta^{(2)}, m\big) = 0$$

in this case. Furthermore, one obtains

$$j\big(3l, \alpha^{(2)}, \beta^{(2)}, m\big) = 3l,\ 1 \leq l \leq k - 1$$

and

$$j\big(3l + 1, \alpha^{(2)}, \beta^{(2)}, m\big) = j\big(3l + 2, \alpha^{(2)}, \beta^{(2)}, m\big) = 3l + 1,\ 1 \leq l \leq k - 1.$$

Considering the form of $\alpha^{(2)}$, then one obtains immediately

$$\alpha^{(2)}_{m-u} \oplus \alpha^{(2)}_{m+1-j\left(u,\alpha^{(2)},\beta^{(2)},m\right)} = 1, \ \forall u \in \{0,\ldots,m-1\}.$$

Hence,

$$\Delta\left(H_m, \mathbf{0}, \left(\alpha^{(2)}, \beta^{(2)}\right)\right) = \sum_{u=0}^{m-1} \left\| 2^u \beta^{(2)} \right\|.$$

Now note that

$$\beta^{(2)} = \sum_{l=0}^{k-2}\left(\frac{1}{2^{3l+1}} + \frac{1}{2^{3l+2}}\right) + \frac{1}{2^{3k-2}} + \frac{1}{2^{3k-1}} + \frac{1}{2^{3k}}.$$

Hence, for $u \in \{0,1,\ldots,m-1\}$, $u \equiv 0 \pmod 3$,

$$\left\|2^u\beta^{(2)}\right\| = \left\| \sum_{l=0}^{k-u/3-2}\left(\frac{1}{2^{3l+1}} + \frac{1}{2^{3l+2}}\right) + \frac{1}{2^{3(k-u/3)-2}} + \frac{1}{2^{3(k-u/3)-1}} + \frac{1}{2^{3(k-u/3)}} \right\|.$$

As a consequence,

$$\sum_{\substack{u=0 \\ u\equiv 0 \pmod 3}}^{m-1} \left\|2^u\beta^{(2)}\right\| = \sum_{r=1}^{k}\left(1 - \left(\sum_{l=0}^{r-2}\left(\frac{1}{2^{3l+1}} + \frac{1}{2^{3l+2}}\right) + \frac{1}{2^{3r-2}} + \frac{1}{2^{3r-1}} + \frac{1}{2^{3r}}\right)\right).$$

Similarly, it is easy to see that

$$\sum_{\substack{u=0 \\ u\equiv 1 \pmod 3}}^{m-1} \left\|2^u\beta^{(2)}\right\| = \sum_{r=2}^{k}\left(1 - \left(\frac{1}{2} + \sum_{l=1}^{r-2}\left(\frac{1}{2^{3l}} + \frac{1}{2^{3l+1}}\right) + \frac{1}{2^{3r-3}} + \frac{1}{2^{3r-2}} + \frac{1}{2^{3r-1}}\right)\right) + \frac{1}{4}.$$

Furthermore, we obtain

$$\sum_{\substack{u=0 \\ u\equiv 2 \pmod 3}}^{m-1} \left\|2^u\beta^{(2)}\right\| = \sum_{r=2}^{k}\left(\sum_{l=1}^{r-2}\left(\frac{1}{2^{3l-1}} + \frac{1}{2^{3l}}\right) + \frac{1}{2^{3r-4}} + \frac{1}{2^{3r-3}} + \frac{1}{2^{3r-2}}\right) + \frac{1}{2}.$$

This yields

$$\Delta\left(H_m, \mathbf{0}, \left(\alpha^{(2)}, \beta^{(2)}\right)\right) = \sum_{u=0}^{m-1}\left\|2^u\beta^{(2)}\right\| = \frac{2m}{7} + \frac{1}{49} - \frac{2^{-m}}{49}.$$

Let us now study $\Delta\Big(H_m, \mathbf{0}, \big(\alpha^{(1)}, \beta^{(2)}\big)\Big)$. In this case, it is easy to check that

$$\alpha_{m-u} \oplus \alpha_{m+1-j(u,\alpha^{(1)},\beta^{(2)},m)} = 1$$

if and only if $u \equiv 0 \pmod 3$. Hence,

$$\Delta\Big(H_m, \mathbf{0}, \big(\alpha^{(1)}, \beta^{(2)}\big)\Big) = \sum_{\substack{u=0 \\ u \equiv 0 \pmod 3}}^{m-1} \left\| 2^u \beta^{(2)} \right\|.$$

From above, it is easily derived that

$$\Delta\Big(H_m, \mathbf{0}, \big(\alpha^{(1)}, \beta^{(2)}\big)\Big) = \frac{m}{21} - \frac{1}{49} + \frac{2^{-m}}{49}.$$

Furthermore, it turns out that

$$\Delta\Big(H_m, \mathbf{0}, \big(\alpha^{(2)}, \beta^{(1)}\big)\Big) = \sum_{\substack{u=0 \\ u \equiv 0 \pmod 3}}^{m-1} \left\| 2^u \beta^{(1)} \right\|.$$

Since $\beta^{(2)} = 1 - \beta^{(1)}$ and since $\|x\| = \|1-x\|$, it follows that

$$\Delta\Big(H_m, \mathbf{0}, \big(\alpha^{(2)}, \beta^{(1)}\big)\Big) = \Delta\Big(H_m, \mathbf{0}, \big(\alpha^{(1)}, \beta^{(2)}\big)\Big).$$

A similar observation yields

$$\Delta\Big(H_m, \mathbf{0}, \big(\alpha^{(1)}, \beta^{(1)}\big)\Big) = \Delta\Big(H_m, \mathbf{0}, \big(\alpha^{(2)}, \beta^{(2)}\big)\Big).$$

From this we obtain

$$\Delta\Big(H_m, \big(\alpha^{(1)}, \beta^{(1)}\big), \big(\alpha^{(2)}, \beta^{(2)}\big)\Big) = \frac{10m}{21} + \frac{4}{49} - \frac{2^{2-m}}{49}.$$

□

Similarly to Proposition 1, we have the following two propositions, the proofs of which are carried out analogously to the proof of Proposition 1.

Proposition 2. *Let $k \geq 1$ and let $m = 3k+1$. Furthermore, let $\gamma^{(1)}, \gamma^{(2)}, \delta^{(1)}, \delta^{(2)}$ have the following base 2 representation:*

$$\begin{aligned}
\gamma^{(1)} &= 0.\underbrace{001001001\ldots001001}_{m-1 \text{ bits}}1, \\
\gamma^{(2)} &= 0.\underbrace{110110110\ldots110111}_{m-1 \text{ bits}}0, \\
\delta^{(1)} &= 0.1\underbrace{001001001\ldots001001}_{m-1 \text{ bits}}, \\
\delta^{(2)} &= 0.1\underbrace{110110110\ldots110111}_{m-1 \text{ bits}}.
\end{aligned}$$

Then we have

$$\Delta\left(H_m, \left(\gamma^{(1)}, \delta^{(1)}\right), \left(\gamma^{(2)}, \delta^{(2)}\right)\right) = \frac{10m}{21} - \frac{11}{294} - \frac{3 \cdot 2^{1-m}}{49}.$$

Proposition 3. *Let $k \geq 1$ and let $m = 3k+2$. Furthermore, let $\kappa^{(1)}, \kappa^{(2)}, \eta^{(1)}, \eta^{(2)}$ have the following base 2 representation:*

$$\begin{aligned}
\kappa^{(1)} &= 0.\underbrace{001001001\ldots001001}_{m-2 \text{ bits}}01, \\
\kappa^{(2)} &= 0.\underbrace{110110110\ldots110}_{m-2 \text{ bits}}11, \\
\eta^{(1)} &= 0.01\underbrace{001001001\ldots001001}_{m-2 \text{ bits}}, \\
\eta^{(2)} &= 0.10\underbrace{110110110\ldots110111}_{m-2 \text{ bits}}.
\end{aligned}$$

Then we have

$$\Delta\left(H_m, \left(\kappa^{(1)}, \eta^{(1)}\right), \left(\kappa^{(2)}, \eta^{(2)}\right)\right) = \frac{10m}{21} - \frac{2}{147} + \frac{3 \cdot 2^{2-m}}{49}.$$

We sum up in the following theorem.

Theorem 1. *For $m \geq 3$, the extreme discrepancy of the Hammersley net H_m with 2^m points is bounded below by*

$$2^m D_{2^m}(H_m) \geq \frac{10m}{21} + c, \tag{5}$$

where c is a constant not depending on m.

Proof. By Propositions 1, 2, and 3, it follows immediately that Equation (5) holds if we only consider intervals that are bounded by m-bit numbers. On the other hand, all the points of H_m are m-bit, and the one-dimensional projections do not have any double points. Therefore, as pointed out in, e.g., [12], it is easily seen that the local discrepancy function evaluated at arbitrary numbers in $[0, 1)$, differs from the discrepancy function evaluated at the nearest m-bit numbers by at most a constant. □

Remark 1. Theorem 1 implies that an equivalent of Equation (3) does not hold for the Hammersley net. Indeed we see from Theorem 1 that the leading term in the extreme discrepancy of H_m is at least $10m/21$, and it cannot be $m/3$, as one might suspect from the results for the van der Corput sequence.

3. Remarks on upper bounds

In Section 2, we have dealt with lower bounds on the extreme discrepancy of the Hammersley net. A natural question is to consider upper bounds. As it has turned out in our research so far, the problem of finding strong upper bounds is considerably harder than finding lower bounds. Nevertheless, let us give some remarks on upper bounds of the extreme discrepancy of H_m here. There is a general relation between the extreme and the star discrepancy of any point set P of N points in the s-dimensional unit cube (see, e.g., [5, 14]), which is

$$D_N(P) \le 2^s D_N^*(P),$$

such that we would deduce

$$D_{2^m}(H_m) \le 4 D_{2^m}^*(H_m).$$

However, since it is well known that the local discrepancy function of H_m is always non-negative (see, e.g., [2]), we immediately obtain

$$D_{2^m}(H_m) \le 2 D_{2^m}^*(H_m)$$

from Equation (4). Thus, using Equation (1), we obtain

$$2^m D_{2^m}(H_m) \le \frac{2m}{3} + \frac{26}{9} - (-1)^m \frac{8}{9 \cdot 2^m}.$$

Furthermore, we have the following conjecture.

Conjecture 1. *The maximum*

$$\max_{\substack{\sigma^{(1)},\sigma^{(2)}\\ \tau^{(1)},\tau^{(2)}\\ m-\text{bit}}} \left| \Delta\Big(H_m, \big(\sigma^{(1)}, \tau^{(1)}\big), \big(\sigma^{(2)}, \tau^{(2)}\big)\Big) \right|$$

is attained for

- $\alpha^{(1)}, \alpha^{(2)}, \beta^{(1)}, \beta^{(2)}$ *as given in Proposition 1 if* $m \equiv 0 \pmod 3$,
- $\gamma^{(1)}, \gamma^{(2)}, \delta^{(1)}, \delta^{(2)}$ *as given in Proposition 2 if* $m \equiv 1 \pmod 3$,
- $\kappa^{(1)}, \kappa^{(2)}, \eta^{(1)}, \eta^{(2)}$ *as given in Proposition 3 if* $m \equiv 2 \pmod 3$.

Note that it would follow from Conjecture 1 that $D_{2^m}(H_m) \le 10m/21 + \widetilde{c}$ for some constant $\widetilde{c}$ that does not depend on m. However, the proof of this conjecture seems to be technically very difficult. Indeed, in [12], the star discrepancy of H_m was analyzed, and the authors of [12] studied the problem of maximizing the expression

$$\Delta\Big(H_m, \mathbf{0}, \big(\alpha, \beta\big)\Big) = \sum_{u=0}^{m-1} \|2^u \beta\| \left(\alpha_{m-u} \oplus \alpha_{m+1-j(u,\alpha,\beta,m)}\right)$$

for m-bit α and β.

If, however, we would like to consider the extreme discrepancy instead of the star discrepancy, this would mean we would have to maximize a term involving four sums of the above form, with each m-bit number occurring in two sums at the same time. As the derivation of the results for the star discrepancy in [12] was technically very involved, we actually think that a similar method of proof for the extreme discrepancy seems hardly possible, so one most likely will have to resort to different techniques for estimating the local discrepancy function, which is a problem left open for future research.

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