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uniform distribution theory

DISCREPANCY ESTIMATE OF NORMAL VECTORS (THE CASE OF HYPERBOLIC MATRICES)

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ABSTRACT. Let A be a $t \times t$ invertible matrix with integer entries and with eigenvalues $|\lambda_i| \neq 1, \; i \in [1,t]$. In this paper we prove explicitly that there exists a vector α, such that discrepancy of the sequence $\{\alpha A^n\}_{n=1}^N$ is equal to $O(N^{-1}(\log N)^{t+5})$ for $N \longrightarrow \infty$. This estimate can be improved no more than on the logarithmic factor.

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1. Introduction

Let $(x_n)_{n\geq 0}$ be an infinite sequence of points in a $t-$dimensional unit cube $[0,1)^t$; $v = [0,\gamma_1) \times \cdots \times [0,\gamma_t)$ a box in $[0,1)^t$; and $J_v(N)$ a number of indexes $n \in [1,N]$ such that x_n lies in v. The sequence $(x_n)_{n\geq 0}$ is said to be *uniformly distributed* in $[0,1)^t$ if for every box v, $J_v(N)/N \to \gamma_1 \ldots \gamma_t$. The quantity

$$D\big((x_n)_{n=1}^N\big) = \sup_{v\in(0,1]^t} \left| \frac{1}{N} J_v(N) - \gamma_1 \ldots \gamma_t \right| \qquad (1)$$

is called the *discrepancy* of $(x_n)_{n=1}^N$.

In 1954, Roth (see [DrTi]) proved that for any sequence in $[0,1)^t$

$$\overline{\lim}_{N\to\infty} ND(N)/\log^{t/2} N > 0. \qquad (2)$$

Let A be a $t\times t$ invertible matrix with integer entries. A matrix A is said to be *ergodic* if for almost all $\alpha \in \mathbb{R}^t$, the sequence $\{\alpha A^n\}_{n\geq 1}$ is uniformly distributed.

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A vector $\alpha \in \mathbb{R}^t$ is said to be *normal (A normal)* if the sequence $\{\alpha A^n\}_{n\geq 1}$ is uniformly distributed.

Let $\lambda_i \quad (1 \leq i \leq t)$ denote the eigenvalues of a matrix A. For the case of $|\lambda_i| > 1, \quad i = 1, \dots, t$, normal vectors were constructed by Postnikov ($t = 2$) and by Polosuev ($t \geq 2$) (see [Po]). Normal vectors were constructed for the general case of an ergodic matrix in [Le1]. The author [Le1] obtained also the following discrepancy estimate

$$D\Big(\{\alpha A^n\}_{n=1}^N\Big) = O\left(N^{-\frac{1}{2}}(\log N)^{t+3}\right).$$

In [Ko1], Korobov posed the problem of finding a function $\psi(N)$ with maximum decay, such that there exists α with

$$D\Big(\{\alpha A^n\}_{n=1}^N\Big) = O\big(\psi(N)\big), \quad \text{for } N \longrightarrow \infty.$$

The author [Le2] proved that $\psi(N) = N^{-1}(\log N)^{2t+3}$ for the case of a diagonal ergodic matrix. In [LeVo], we extended this result to the general case of an integer matrix with $|\lambda_i| > 1, \quad i = 1, \dots, t$. In this paper, we obtain a similar result for the case of hyperbolic matrix A (i.e., $|\lambda_i| \neq 1$ for $i \in [1, t]$). By (2) this result not be improved more than on the logarithmic factor.

2. Construction

Let $f(x)$ be the characteristic polynomial of the matrix A, $\varphi(x) = x^s - a_{s-1}x^{s-1} - \ldots - a_0$ an irreducible factor of $f(x)$, $s = \deg \varphi$. In this paper we will consider only the case of $f(x) = (\varphi(x))^d$. The construction and the proof for the general case

$$f(x) = \varphi_1^{d_1}(x) \ldots \varphi_r^{d_r}(x)$$

are completely similar, but technically more difficult. Let's translate A to a general Jordan form: $A = W^{-1}\tilde{A}W$,

$$\tilde{A} = \begin{pmatrix} \hat{A}E & & \\ & \ddots & \\ & & E \\ & & \hat{A} \end{pmatrix}, \qquad \hat{A} = \begin{pmatrix} 01 & & 0 \\ & \ddots & \\ & & 1 \\ a_0 & \ldots & a_{s-1} \end{pmatrix}, \tag{3}$$

where $\hat{A}$ is the companion matrix of the polynomial $\varphi(x)$, $W = (w_{i,j})_{1\leq i,j\leq sd}$ is the matrix with integer coefficients, and E the identity matrix. Let $\lambda_1, \dots, \lambda_s$

be roots of $\varphi(x)$, and let $|\lambda_i| > 1$, for $i = 1, \dots, r$, $|\lambda_i| < 1$, for $i = r+1, \dots, s$,

$$\lambda_0 = \min_{1 \le i \le r} |\lambda_i|, \quad \lambda_{-1} = \min_{r+1 \le i \le s} |\lambda_i^{-1}|. \tag{4}$$

Let $r < s$. (We considered the case of $r = s$ in [LeVo]. Using the method of this paper we can get the improved discrepancy estimate (12) also for the case $r = s$.) Let

$$\delta(x) = \begin{cases} 1, & \text{if } \ x = 0, \\ 0, & \text{otherwise }, \end{cases} \tag{5}$$

$$C = (c_{ij})_{1 \le i,j \le ds} \quad \text{with} \quad c_{(i_1-1)s+i_0,(\mu_0-1)d+\mu_1} = \lambda_{\mu_0}^{i_0-1} \delta(i_1 - \mu_1), \tag{6}$$
$$1 \le i_1, \mu_1 \le d, \ \ 1 \le i_0, \mu_0 \le s,$$

$$\widetilde{\Lambda} = \begin{pmatrix} \Lambda_1 & & \\ & \ddots & \\ & & \Lambda_s \end{pmatrix}, \quad \Lambda_i = \begin{pmatrix} \lambda_i 1 & & & \\ & \ddots & & \\ & & \lambda_i 1 & \\ & & & \lambda_i \end{pmatrix}, \tag{7}$$

and

$$\overline{\Lambda} = \begin{pmatrix} \Lambda_1^{-1} & & & & & \\ & \ddots & & & & \\ & & \Lambda_r^{-1} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}, \quad \Lambda^{'} = \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & \Lambda_{r+1} & & \\ & & & & \ddots & \\ & & & & & \Lambda_s \end{pmatrix},$$

$$A_1 = C\overline{\Lambda}C^{-1}, \quad A_{-1} = C\Lambda^{'}C^{-1},$$
$$A_i = A_1 \quad \text{for} \quad i > 0, \quad \text{and} \quad A_i = A_{-1} \quad \text{for} \quad i \le 0. \tag{8}$$

Let

$$\kappa_0 = 2 + s + ([\log_2 \lambda_0] + 1)(d+1), \quad \kappa_2 = \left[\frac{\kappa_0 + d + 1}{\log_2 \lambda_0}\right] + 3,$$
$$\kappa_3 = \left[\frac{\kappa_0 + d + 1}{\log_2 \lambda_{-1}}\right] + 1, \qquad \kappa_1 = \kappa_2 + \kappa_3 + 1, \tag{9}$$

$F_m \in [2^{\kappa_0 m}, 2^{\kappa_0 m+1})$ be prime $(m = 1, 2, \dots)$,

$$n_1 = 0, \quad n_m = n_{m-1} + \kappa_1 m 2^m, \quad m = 2, 3, \dots, \tag{10}$$

$$\tilde{\alpha} = \sum_{m=1}^{\infty} \sum_{n=0}^{2^m-1} \sum_{\nu=0}^{\kappa_1-1} F_m \left\{ \frac{n\mathbf{b}_{\mathbf{m},\nu}}{F_m} \right\} A_1^{n_m+m(\kappa_1 n+\nu)}, \quad \text{and} \quad \alpha = \tilde{\alpha} W, \tag{11}$$

where $\mathbf{b}_{\nu,m} \in [0, F_m)^{sd}$.

Theorem. *There exist integer vectors*

$$\mathbf{b}_{\nu,\mathbf{m}} \in [0, F_m)^{sd} \quad (m = 1, 2, \dots, \ \nu \in [0, \kappa_1))$$

such that

$$D\left(\{\alpha A^n\}_{n=1}^N\right) = O\left(\frac{\log^{sd+5} N}{N}\right), \quad \text{for} \quad N \to \infty. \tag{12}$$

We prove this result in Section 4.

Remark. The theorem can be extended also to negative values of n. Let

$$\tilde{\alpha}' = \sum_{m=1}^{\infty} \sum_{n=0}^{2^m-1} \sum_{\nu=0}^{\kappa_1-1} F_m \left\{ \frac{n\mathbf{b}'_{\mathbf{m},\nu}}{F_m} \right\} A_{-1}^{n_m+m(\kappa_1 n+\nu)}$$

and let

$$\alpha = (\tilde{\alpha} + \tilde{\alpha}')W.$$

Then there exist integer vectors $\mathbf{b}_{\nu,\mathbf{m}}, \mathbf{b}'_{\nu,\mathbf{m}} \in [0, F_m)^{sd}$, such that

$$D\left(\{\alpha A^n\}_{n=-N_1}^{N_2}\right) = O\left(\frac{\log^{sd+5} N}{N}\right),$$

where $N = \max(N_1, N_2)$.

3. Auxiliary results

We will need the following inequalities: The Erdős-Turan-Koksma inequality (see [DrTi, p. 18]):

$$ND\Big((x_n)_{n=0}^{N-1}\Big) \le \left(\frac{3}{2}\right)^t \left(\frac{N}{M} + \sum_{0<\max|m_i|\le M} \frac{\left|\sum_{n=0}^{N-1} e\big(\langle \mathbf{m}, \mathbf{x}_n \rangle\big)\right|}{\overline{m_1} \dots \overline{m_t}} \right), \tag{13}$$

where

$e(y) = \exp(2\pi i y)$, $\mathbf{x}_n = (x_{n,1}, \dots, x_{n,t})$, $\mathbf{m} = (m_1, \dots m_t)$, $\overline{m_i} = \max(1, |m_i|)$,

and

$$\langle (a_1, \dots, a_t), (b_1, \dots, b_t) \rangle = a_1 b_1 + \dots + a_t b_t, \ t = sd.$$

Lemma A (see [Ko2, p. 1]). *Let β be a real number, M and N natural, then*

$$\left|\sum_{n=M}^{M+N-1} e(n\beta)\right| \leq \min\left(N, \frac{1}{2 \parallel \beta \parallel}\right),$$

where $\|\beta\| = \min(\{\beta\}, 1-\{\beta\})$.

Lemma B (see [Ko2, p. 72]). *Let $P \geq 2, (a, P) = 1$, then for any real φ,*

$$\sum_{n=1}^{P} \min\left(P, \frac{1}{\parallel an/P + \varphi \parallel}\right) \leq 8P(1 + \ln P).$$

Lemma C (see [Ko2, p. 2]). *Let*

$$\delta_q(a) = \begin{cases} 1 & if \quad a \equiv 0 \pmod q, \\ 0, & otherwise, \end{cases}$$

where $q \geq 1, a \in \mathbb{Z}$. Then

$$\delta_q(a) = \frac{1}{q} \sum_{x=1}^{q} e\left(\frac{ax}{q}\right).$$

For the sake of the clarity we will prove three lemmas from [Le1].

We consider roots $\lambda_1, \ldots, \lambda_s$ of the polynomial $\varphi(x)$. Let $Q(\lambda_\nu)$ (respectively, $Q(\lambda_1, \ldots, \lambda_s)$) be the field extension of Q, by adding the element λ_ν (respectively, $\lambda_1, \ldots, \lambda_s$); σ_ν (respectively, $\sigma_{\nu_1\nu_2}$) isomorphism of the field $Q(\lambda_1)$ (respectively, $Q(\lambda_{\nu_1})$) into the field $Q(\lambda_1, \ldots, \lambda_s)$ (respectively, $Q(\lambda_{\nu_2})$), with the following rule: $\lambda_1 \to \lambda_\nu$ (respectively, $\lambda_{\nu_1} \to \lambda_{\nu_2}$) and $a \to a$ for all $a \in Q$ $(\nu, \nu_1, \nu_2 = 1, \ldots, s)$. Let

$$G = \begin{pmatrix} 1 & \ldots & 1 \\ \lambda_1 & \ldots & \lambda_s \\ & \ldots & \\ \lambda_1^{s-1} & \ldots & \lambda_s^{s-1} \end{pmatrix}, \qquad G^{-1} = \begin{pmatrix} d_{11} & \ldots & d_{1s} \\ & & \\ & \ldots & \\ d_{s1} & \ldots & d_{ss} \end{pmatrix}. \tag{14}$$

The numbers $\lambda_1, \ldots, \lambda_s$ are the roots of an irreducible polynomial under Q $\varphi(x)$. Hence $\lambda_i \neq \lambda_j$ for $i \neq j$, and the Vandermonde determinant of the matrix G is not zero. Thus the numbers $d_{\nu,j}$ $(1 \leq \nu, j \leq s)$ are defined correctly.

Lemma 1 ([Le1, Lemma 1]). *With notation as above, we have*

$$d_{\nu j} \in Q(\lambda_\nu), \quad \sigma_\nu(d_{1j}) = d_{\nu j}, \quad \sigma_{\nu_1,\nu_2}(d_{\nu_1 j}) = d_{\nu_2 j}, \quad where \;\; \nu, \nu_1, \nu_2, j = 1, \ldots, s.$$

P r o o f. Let us take the following system of linear equations :

$$\sum_{k=1}^{s} y_{jk} \sum_{\nu=1}^{s} \lambda_\nu^{i+k-2} = \delta(i-j), \qquad i, j = 1, \ldots, s. \tag{15}$$

For fixing $j \quad (j = 1, \ldots, s)$, we have the system of s linear equations with s variables $y_{j1}, \ldots, y_{js}$ and the matrix

$$\left(\sum_{\nu=1}^{s} \lambda_\nu^{i+k-2} \right)_{1 \le i,k \le s} . \tag{16}$$

So, from the system of s^2 equations (15) we get s systems of linear equations with the similar matrices (16). By [BS, p. 404], we have that the matrix (16) is nonsingular, their elements are rational numbers (see [BS, p. 404]),

$$\sum_{\nu=1}^{s} \lambda_\nu^{i+k-2} = Tr_{Q(\lambda_1)/Q} \lambda_1^{i+k-2} \in Q,$$

and

$$\det \left(\sum_{\nu=1}^{s} \lambda_\nu^{i+k-2} \right)_{1 \le i,k \le s} = \prod_{i<j} (\lambda_i - \lambda_j)^2 \neq 0.$$

Hence, there exist rational $\tilde{f}_{jk} \; (j, k = 1, \ldots, s)$ that are solutions of the system (15). Let

$$\tilde{d}_{\nu j} = \sum_{k=1}^{s} \tilde{f}_{jk} \lambda_\nu^{k-1}, \qquad \nu, j = 1, \ldots, s. \tag{17}$$

From (15), we get

$$\begin{aligned} \sum_{\nu=1}^{s} \lambda_\nu^{i-1} \tilde{d}_{\nu j} &= \sum_{\nu=1}^{s} \lambda_\nu^{i-1} \sum_{k=1}^{s} \tilde{f}_{jk} \lambda_\nu^{k-1} \\ &= \sum_{k=1}^{s} \tilde{f}_{jk} \sum_{\nu=1}^{s} \lambda_\nu^{i+k-2} \\ &= \delta(i-j) \qquad \text{for } \; i, j = 1, \ldots, s. \end{aligned} \tag{18}$$

Bearing in mind that there exists only one matrix inverse to G, we obtain that the numbers $d_{\nu j}$ $(\nu, j = 1, \ldots, s)$ are uniquely defined from the following system of linear equations

$$\sum_{\nu=1}^{s} \lambda_\nu^{i-1} x_{\nu j} = \delta(i-j), \qquad i, j = 1, \ldots, s.$$

By (17) and (18), we have

$$d_{\nu j} = \tilde{d}_{\nu j} = \sum_{k=1}^{s} \tilde{f}_{jk} \lambda_\nu^{k-1}, \qquad i, j = 1, \ldots, s, \tag{19}$$

with rational $\tilde{f}_{jk}$. Using (19), we obtain the assertion of the lemma. □

We will use the function $\delta(x)$ (5) to obtain the coefficients $a_{i,j}$ of the matrix $\tilde{A}$ (3), and coefficients $l_{i,j}$ of the matrix $\tilde{\Lambda}_i$ (7):

$$\begin{aligned} a_{(i_1-1)s+i_0,(j_1-1)s+j_0} = {} & \delta(i_1-j_1+1)\delta(i_0-j_0)+\delta(i_1-j_1) \\ & \times \Big\{(1-\delta(s-i_0))\delta(i_0-j_0+1) \\ & + \delta(s-i_0)\sum_{n=0}^{s-1} a_n\delta(n-j_0+1)\Big\}, \end{aligned} \tag{20}$$

$$\begin{aligned} l_{(\mu_0-1)d+\mu_1,(\nu_0-1)d+\nu_1} = \delta(\mu_0-\nu_0)\left(\delta(\mu_1-\nu_1)\lambda_{\nu_0}+\delta(\mu_1+1-\nu_1)\right), \\ 1 \le \mu_1,\nu_1 \le d, \quad 1 \le \mu_0,\nu_0 \le s. \end{aligned} \tag{21}$$

Let us take $ds \times ds$ matrix

$$\begin{aligned} F = (f_{\nu j}) \ \text{ with } \ f_{(\nu_0-1)d+\nu_1,(j_1-1)s+j_0} = d_{\nu_0 j_0}\delta(\nu_1-j_1), \\ 1 \le j_1,\nu_1 \le d, \quad 1 \le j_0,\nu_0 \le s. \end{aligned} \tag{22}$$

Lemma 2 ([Le1, Lemma 2]). *The following equations are true:*

$$F = C^{-1}, \qquad \tilde{A} = C\tilde{\Lambda}C^{-1}. \tag{23}$$

P r o o f. Let $g_{\nu\mu}$ be elements of the matrix FC. By (14), (6) and (22), we get

$$\begin{aligned} & g_{(\nu_0-1)d+\nu_1,(\mu_0-1)d+\mu_1} \\ & = \sum_{j_1=1}^{d}\sum_{j_0=1}^{s} f_{(\nu_0-1)d+\nu_1,(j_1-1)s+j_0}\, c_{(j_1-1)s+j_0,(\mu_0-1)d+\mu_1} \\ & = \sum_{j_1=1}^{d}\sum_{j_0=1}^{s} d_{\nu_0 j_0}\delta(\nu_1-j_1)\lambda_{\mu_0}^{j_0-1}\delta(j_1-\mu_1) \\ & = \delta(\nu_1-\mu_1)\sum_{j_0=1}^{s} d_{\nu_0 j_0}\lambda_{\mu_0}^{j_0-1} = \delta(\nu_1-\mu_1)\delta(\nu_0-\mu_0), \\ & \qquad\qquad 1 \le \nu_0,\mu_0 \le s, \quad 1 \le \nu_1,\mu_1 \le d. \end{aligned}$$

Therefore, FC is the identity matrix, and C is the nonsingular matrix, so

$$F = C^{-1}.$$

Let us denote by $\tilde{a}_{ij}$ elements of the matrix

$$C\tilde{\Lambda}C^{-1} = C\tilde{\Lambda}F.$$

Bearing in mind that

$$\lambda_\nu^s = a_{s-1}\lambda_\nu^{s-1} + \cdots + a_0, \quad 1 \le \nu \le s,$$

we get from (6), (14) and (20)–(22) that

$$\begin{aligned}
&\tilde{a}_{(i_1-1)s+i_0,(j_1-1)s+j_0} \\
&= \sum_{\mu_0,\nu_0=1}^{s} \sum_{\mu_1,\nu_1=1}^{d} c_{(i_1-1)s+i_0,(\mu_0-1)d+\mu_1} \\
&\quad \times l_{(\mu_0-1)d+\mu_1,(\nu_0-1)d+\nu_1} f_{(\nu_0-1)d+\nu_1,(j_1-1)s+j_0} \\
&= \sum_{\mu_0,\nu_0=1}^{s} \sum_{\mu_1,\nu_1=1}^{d} \lambda_{\mu_0}^{i_0-1} \delta(i_1-\mu_1)\delta(\mu_0-\nu_0) \\
&\quad \times \Big((\delta(\mu_1-\nu_1)\lambda_{\nu_0} + \delta(\mu_1-\nu_1+1) \Big) \delta(\nu_1-j_1) d_{\nu_0 j_0} \\
&= \sum_{\nu_0=1}^{s} \lambda_{\nu_0}^{i_0-1} \left(\delta(i_1-j_1)\lambda_{\nu_0} + \delta(i_1-j_1+1) \right) d_{\nu_0 j_0} \\
&= \delta(i_1-j_1+1) \sum_{\nu_0=1}^{s} \lambda_{\nu_0}^{i_0-1} d_{\nu_0 j_0} + \delta(i_1-j_1) \sum_{\nu_0=1}^{s} d_{\nu_0 j_0} \lambda_{\nu_0}^{i_0} \\
&= \delta(i_1-j_1+1)\delta(i_0-j_0) + \delta(i_1-j_1) \\
&\quad \times \sum_{\nu_0=1}^{s} d_{\nu_0 j_0} \Big((1-\delta(s-i_0))\lambda_{\nu_0}^{i_0} + \delta(s-i_0) \sum_{n=0}^{s-1} a_n \lambda_{\nu_0}^{n} \Big) \\
&= \delta(i_1-j_1+1)\delta(i_0-j_0) + \delta(i_1-j_1) \\
&\quad \times \Big((1-\delta(s-i_0))\delta(i_0-j_0+1) + \delta(s-i_0) \sum_{n=0}^{s-1} a_n \delta(n-j_0+1) \Big).
\end{aligned}$$

Now, by (20),

$$\tilde{a}_{ij} = a_{ij} \quad (1 \le i, j \le ds), \quad \text{and} \qquad \tilde{A} = C\tilde{\Lambda}C^{-1}.$$

Lemma 2 is proved. □

According to [Ga, p. 100] we have that

$$\Lambda_i^x = \begin{pmatrix} \lambda_i^x & \binom{x}{1}\lambda_i^{x-1} & \dots & \binom{x}{d-1}\lambda_i^{x-d+1} \\ & \ddots & \ddots & \\ & & \ddots & \binom{x}{1}\lambda_i^{x-1} \\ & & & \lambda_i^x \end{pmatrix}.$$

Let $l^{(x)}_{\mu,\nu}$ be coefficients of the matrix $\widetilde{\Lambda}^x$. By (5) and (7), we obtain

$$l^{(x)}_{(\mu_0-1)d+\mu_1,(\nu_0-1)d+\nu_1} = \delta(\mu_0-\nu_0)\binom{x}{\nu_1-\mu_1}\lambda_{\mu_0}^{x-\nu_1+\mu_1}. \tag{24}$$

Let

$$\lambda = \max_{1\le i\le s}|\lambda_i|, \qquad \widetilde{\lambda} = \max_{1\le i\le s}(1,|\lambda_i|^{-1}), \qquad g = |\det G|. \tag{25}$$

Let $\mathbf{m} = (m_1,\dots,m_{ds})$ be a $1\times sd$ matrix with integer elements, $\mathbf{m}^{(t)}$ the transposed matrix.

Lemma 3 ([Le1, Lemma 3]). *Let* $c_0 = (gsd)^{-2}(\lambda\widetilde{\lambda})^{-rd}\left(\max_{1\le\nu,j\le s}|d_{\nu,j}|\right)^{-2}$, $0 < \max_{1\le i\le ds}|m_i| \le c_0 2^j$; $j \ge [-\log_2 c_0]+1$, $s\ge 2$, x *be an integer,* $2^j > x \ge 0$. *Then*

$$2^{jd} \ge \lambda^{-x}\max_{1\le i_1\le d,1\le i_0\le s}|a_{(i_1-1)s+i_0}(x,\mathbf{m})| \ge 2^{-j(s-1)}, \tag{26}$$

where $a_i(x,\mathbf{m})$ *are elements of the matrix* $\widetilde{A}^x\mathbf{m}^{(t)}$.

P r o o f. Let $a^{(x)}_{i,j}$ be the elements of the matrix $\widetilde{A}^x = C\widetilde{\Lambda}^x C^{-1}$. Using (6), (24), (22) and Lemma 2, we obtain

$$\begin{aligned} &a^{(x)}_{(i_1-1)s+i_0,(j_1-1)s+j_0}\\ &= \sum_{\nu_0,\mu_0=1}^{s}\sum_{\nu_1,\mu_1=1}^{d}\lambda_{\mu_0}^{i_0-1}\delta(i_1-\mu_1)\delta(\mu_0-\nu_0)\lambda_{\nu_0}^{x-\nu_1+\mu_1}\\ &\quad\times\binom{x}{\nu_1-\mu_1}d_{\nu_0 j_0}\delta(\nu_1-j_1)\\ &= \binom{x}{j_1-i_1}\sum_{\nu_0=1}^{s}\lambda_{\nu_0}^{x+i_1-j_1+i_0-1}d_{\nu_0 j_0}, \end{aligned}$$

where $\binom{x}{i}$ are equal to zero for $i>x$ and $i<0$, $1\le i_1,j_1\le d$, $1\le i_0,j_0\le s$. We see that

$$\begin{aligned} &a_{(i_1-1)s+i_0}(x,\mathbf{m})\\ &= \sum_{\nu=1}^{s}\sum_{j_1=i_1}^{d}\binom{x}{j_1-i_1}\lambda_{\nu}^{x+i_1-j_1+i_0-1}\sum_{j_0=1}^{s}d_{\nu j_0}m_{(j_1-1)s+j_0}. \end{aligned} \tag{27}$$

From (25) and conditions of the lemma, we get

$$\max_{1\le i\le ds} |a_i(x,\mathbf{m})| \le c_0 s^2 d\lambda^x \binom{x}{d-1} 2^j (\lambda\widetilde{\lambda})^{s+d} \max_{1\le \nu,j\le s} |d_{\nu,j}|$$
$$\le \lambda^x 2^{jd)} c_0 s^2 d(\lambda\widetilde{\lambda})^{s+d} \max_{1\le \nu,j\le s} |d_{\nu,j}| \le \lambda^x 2^{jd}. \qquad (28)$$

So the left side of (26) is proved. Let us prove the right side of (26). The vector $\mathbf{m}$ is non zero, so there exists an integer $k_0 \in [1,d]$ such that

$$\sum_{j=1}^{s} |m_{(j_1-1)s+j}| = 0, \quad \sum_{j=1}^{s} |m_{(k_0-1)s+j}| > 0 \qquad \text{for} \;\; k_0 < j_1. \qquad (29)$$

By (27),

$$a_{(k_0-1)s+i}(x,\mathbf{m}) = \sum_{\nu=1}^{s} \lambda_\nu^{x+i-1} \sum_{j=1}^{s} d_{\nu,j} m_{(k_0-1)s+j}. \qquad (30)$$

The determinant of the matrix $(d_{\nu j})_{1\le j,\nu\le s}$ is not zero. Hence, there exists $\nu_0 \in [1,s]$ such that

$$\sum_{j=1}^{s} d_{\nu_0 j} m_{(k_0-1)s+j} \ne 0.$$

Using Lemma 1, we have

$$\sigma_{\nu_0,\nu}\left(\sum_{j=1}^{s} d_{\nu_0,j} m_{(k_0-1)s+j}\right) = \sum_{j=1}^{s} d_{\nu j} m_{(k_0-1)s+j} \ne 0, \qquad \nu = 1,\dots,s. \quad (31)$$

Let $M_{i,j}$ be the minor of the matrix G (14). It is known that

$$d_{i,j} = (-1)^{i+j} M_{i,j} (\det G)^{-1}.$$

Bearing in mind that $\lambda_1,\dots,\lambda_s$ are algebraic integers, we obtain that $d_{i,j}\det G$ are also algebraic integers. From (25) and (31), we get

$$\left| N_{Q(\lambda_\nu)/Q}\left(g\sum_{j=1}^{s} d_{\nu j} m_{(k_0-1)s+j}\right)\right| \ge 1.$$

Taking into account that $\lambda_i \ne \lambda_j$ for $i \ne j$, we have that the isomorphisms $\sigma_1,\dots,\sigma_s$ are different isomorphisms of the field $Q(\lambda_1)$ into the field $Q(\lambda_1,\dots,\lambda_s)$. Using Lemma 1, we obtain

$$N_{Q(\lambda_\nu)/Q}\left(g\sum_{j_0=1}^{s} d_{\nu j_0} m_{(k_0-1)s+j_0}\right) = \prod_{\nu=1}^{s}\left(g\sum_{j_0=1}^{s} d_{\nu j_0} m_{(k_0-1)s+j_0}\right).$$

Hence

$$\left|\sum_{j_0=1}^{s} d_{\nu j_0} m_{(k_0-1)s+j_0}\right| \geq \frac{1}{g^s \left(\max_{1\leq\nu\leq s}\left|\sum_{j_0=1}^{s} m_{(k_0-1)s+j_0} d_{\nu,j_0}\right|\right)^{s-1}}$$

$$\geq g^{-s}\left(s \max_{1\leq\nu,j_0\leq s} |d_{\nu,j_0}|\right)^{-s+1} c_0^{-s+1} 2^{-j_0(s-1)}. \tag{32}$$

By (30), we get

$$a_{(k_0-1)s+i_0}(x,\mathbf{m}) = \sum_{\nu=1}^{s} z_\nu \lambda_\nu^{i_0-1}, \qquad z_\nu = \lambda_\nu^x \sum_{j_0=1}^{s} d_{\nu j_0} m_{(k_0-1)s+j_0}, \tag{33}$$

with $1 \leq i_0 \leq s$. Let us consider (33) as a system of linear equations with variables

$$z_\nu \ (1 \leq \nu \leq s) \quad \text{and the matrix} \quad (\lambda_\nu^{i_0-1})_{1\leq\nu,i_0\leq s}.$$

Applying (14), we obtain

$$G(z_1,\ldots,z_s)^{(t)} = \big(a_{(k_0-1)s+1}(x,\mathbf{m}),\ldots,a_{k_0 s}(x,\mathbf{m})\big)^{(t)},$$
$$(z_1,\ldots,z_s)^{(t)} = G^{-1}\big(a_{(k_0-1)s+1}(x,\mathbf{m}),\ldots,a_{k_0 s}(x,\mathbf{m})\big)^{(t)}$$

and

$$z_\nu = \sum_{i_0=1}^{s} d_{\nu,i_0} a_{(k_0-1)s+i_0}(x,\mathbf{m}). \tag{34}$$

Therefore,

$$\max_{1\leq\nu\leq s} |z_\nu| \leq s \max_{1\leq i_0\leq s} |a_{(k_0-1)s+i_0}(x,\mathbf{m})| \max_{1\leq\nu,j_0\leq s} |d_{\nu,j_0}|.$$

By (32), (33) and conditions of the lemma, we get

$$\lambda^x \max_{1\leq i_1\leq d, 1\leq i_0\leq s} |a_{(i_1-1)s+i_0}(x,\mathbf{m})|$$
$$\geq g^{-s}\left(s \max_{1\leq\nu,j_0\leq s} |d_{\nu,j_0}|\right)^{-s} c_0^{-s+1} 2^{-j_0(s-1)} \geq 2^{-j_0(s-1)}.$$

Lemma 3 is proved. □

Let: $a_i(0,x,\mathbf{m})$ be elements of the matrix $A_1^x \mathbf{m}^{(t)}$,
$a_i(1,x,\mathbf{m})$ be elements of the matrix $(A_{-1})^x \mathbf{m}^{(t)}$,
G_0 be the matrix $(\lambda_\nu^{j_0-1})_{1\leq\nu,\, j_0\leq r}$,
G_{-1} be the matrix $(\lambda_\nu^{j_0-1})_{r\leq\nu,\, j_0\leq s}$, and
$d_{\nu,\, j_0,\, k}$ be the elements of the matrix G_k^{-1}, $\quad (k = 0,-1)$.

Lemma 4. *Let*

$$c_1 = \Big(\max_{1\le\nu,\, j_0\le s} |d_{\nu,\, j_0}|\Big)^{-1} \Big(\max_{1\le\nu,\, j_0\le r,\ k=0,-1} (1, |d_{\nu,\, j_0,k})|\Big)^{-1} .(gsd)^{-2}\lambda_0^{-s-d},$$

$$0 < \max_{1\le i\le ds} |m_i| \le c_1 2^j, \quad j \ge -[\log_2 c_1] + 1, \quad s \ge 2, \quad x \text{ be an integer,}$$

$$2^j \ge x \ge 0.$$

Then

$$2^{jd} \ge \lambda_k^x \max_{1\le i\le ds} |a_i(k, x, \mathbf{m})| \ge 2^{-j(s-1)}, \qquad k = 0, -1. \tag{35}$$

Proof. We will prove (35) for the case of $k = 0$. The proof for the case of $k = -1$ is similar.

Let $a_{i,\, j}^{(x)}$ be elements of the matrix

$$A_1^x = C\bar{\Lambda}^x C^{-1}.$$

By (7), (6), (22) and (24), we have

$$\begin{aligned} &a^{(x)}_{(i_1-1)s+i_0,(j_1-1)s+j_0} \\ &= \sum_{\nu_0,\mu_0=1}^{r} \sum_{\nu_1,\mu_1=1}^{d} \lambda_{\mu_0}^{i_0-1} \delta(i_1-\mu_1)\delta(\mu_0-\nu_0)\lambda_{\nu_0}^{-x+\nu_1-\mu_1} \\ &\quad\times \binom{x}{\nu_1-\mu_1} d_{\nu_0,\, j_0}\delta(\nu_1-j_1) \\ &= \binom{x}{j_1-i_1} \sum_{\nu_0=1}^{r} \lambda_{\nu_0}^{-x-i_1+j_1+i_0-1} d_{\nu_0,\, j_0}. \end{aligned}$$

Hence

$$\begin{aligned} &a_{(i_1-1)s+i_0}(0, x, \mathbf{m}) \\ &= \sum_{\nu=1}^{r} \sum_{j_1=i_1}^{d} \binom{x}{j_1-i_1} \lambda_\nu^{-x-i_1+j_1+i_0-1} \sum_{j_0=1}^{s} d_{\nu,\, j_0} m_{(j_1-1)s+j_0}. \end{aligned}$$

According to Lemma 1, $a_i(0, x, \mathbf{m})$ is the sum of algebraic conjugate numbers. By the definition of r (4), we get that $a_i(0, x, \mathbf{m})$ also is the sum of complex conjugate numbers. Therefore, $a_i(0, x, \mathbf{m})$ are real numbers. From (4), we have, similarly to (28), that

$$\begin{aligned} \max_{1\le i\le ds} |a_i(0, x, \mathbf{m})| &\le c_1 s^2 d\lambda_0^{-x+s+d} \binom{2^j}{d-1} \max_{1\le\nu,\, j_0\le s} |d_{\nu,j_0}| 2^j \\ &\le \lambda_0^{-x} 2^{jd} c_1 s^2 d\lambda_0^{s+d} \max_{1\le\nu,j_0\le s} d_{\nu,\, j_0} \le \lambda_0^{-x} 2^{jd}. \end{aligned}$$

Let us prove the right side of (35). By (4) and (29)–(31), we have similarly to (32) that

$$\left|\sum_{j_0=1}^{s} d_{\nu,\,j_0} m_{(k_0-1)s+j_0}\right| \geq g^{-s}\left(s \max_{1\leq\nu,\,j_0\leq s}|d_{\nu,j_0}|\right)^{-s+1} c_1^{-s+1}2^{-j(s-1)}, \qquad (36)$$

and

$$a_{(k_0-1)s+i_0}(0,x,\mathbf{m}) = \sum_{\nu=1}^{r}\lambda_\nu^{-x+i_0-1}\sum_{j_0=1}^{s} d_{\nu,\,j_0} m_{(k_0-1)s+j_0}.$$

Therefore,

$$a_{(k_0-1)s+i_0}(0,x,\mathbf{m}) = \sum_{\nu=1}^{r} z_\nu \lambda_\nu^{i_0-1},$$

where

$$z_\nu = \lambda_\nu^{-x}\sum_{j_0=1}^{s} d_{\nu,j_0} m_{(k_0-1)s+j_0}. \qquad (37)$$

Let consider (37) as a system of linear equations with variables $z_\nu,\ (1\leq\nu\leq r)$ and the matrix $G_0 = (\lambda_\nu^{i_0-1})_{1\leq\nu,i_0\leq r}$. Similarly to (34), we get

$$z_\nu = \sum_{i_0=1}^{r} d_{\nu,i_0,0}\, a_{(k_0-1)s+i_0}(0,x,\mathbf{m}) \quad \text{with} \quad (d_{\nu,i_0,0})_{1\leq\nu,i_0\leq r} = G_0^{-1}.$$

We derive from (4) and (37)

$$\begin{aligned}\lambda_0^{-x}\max_{1\leq\nu\leq r}\left|\sum_{j_0=1}^{s} d_{\nu,\,j_0} m_{(k_0-1)s+j_0}\right| &\leq \max_{1\leq\nu\leq r}|z_\nu| \\ &\leq r\max_{1\leq i_0\leq s}|a_{(k_0-1)s+i_0}(0,x,\mathbf{m})| \max_{1\leq\nu,i_0\leq r}|d_{\nu,i_0,0}|.\end{aligned}$$

By (36) and conditions of the lemma, we obtain

$$\begin{aligned}&\lambda_0^{x}\max_{1\leq i\leq ds}|a_i(0,x,\mathbf{m})| \\ &\geq (gs)^{-s}\left(\max_{1\leq\nu,\,j_0\leq s}|d_{\nu,j_0}|\right)^{-s+1} c_1^{-s+1}2^{-j(s-1)}\left(\max_{1\leq\nu,\,j_0\leq r}|d_{\nu,\,j_0,0}|\right)^{-1} \\ &\geq 2^{-j(s-1)}.\end{aligned}$$

Lemma 4 is proved. □

4. Proof of the Theorem

4.1. Several lemmas and the Main Lemma

Lemma 5. *Let* $\mathbf{d}_\nu = (d_{\nu,1}, \dots, d_{\nu,sd})$, $\mathbf{z}_\nu = (z_{\nu,1}, \dots, z_{\nu,sd})$ *be integer vectors,* $(\mathbf{z}_{-\kappa_3}, \dots, \mathbf{z}_{\kappa_2}) \neq (\mathbf{0}, \dots, \mathbf{0})$ *and* $z_{\nu,\mu} \in [0, F_j)$, $\nu \in [-\kappa_3, \kappa_2]$, $\mu \in [1, sd]$. *Then*

$$\sigma := \frac{1}{F_j^{sd\kappa_1}} \sum_{\mathbf{d}_{-\kappa_3},\dots,\mathbf{d}_{\kappa_2} \in [0,F_j)^{sd}} \min\left(2^j, \frac{1}{2\left\|\sum_{\nu=-\kappa_3}^{\kappa_2} \left\langle \mathbf{z}_\nu, \frac{\mathbf{d}_\nu}{F_j} \right\rangle + \frac{m_{sd+1}}{F_j}\right\|}\right) = O(j)\,.$$

P r o o f. Let $z_{\nu_0,\mu_0} \neq 0$, for some $\nu_0 \in [-\kappa_3, \kappa_2]$ and $\mu_0 \in [1, sd]$. Then

$$(z_{\nu_0,\mu_0}, F_j) = 1$$

and

$$\sigma \leq \max_{d_{\nu,\mu} \in [0,F_j),\ (\nu,\mu) \neq (\nu_0,\mu_0)} \left(\frac{1}{F_j} \sum_{d_{\nu_0,\mu_0}=0}^{F_j-1} \min\left(F_j, \frac{1}{2\left\|\frac{z_{\nu_0,\mu_0} d_{\nu_0,\mu_0}}{F_j} + \psi\right\|}\right)\right),$$

where ψ do not depend on d_{ν_0,μ_0}. Using Lemma B, and (9)), we obtain

$$\frac{1}{F_j} \sum_{d_{\nu_0,\mu_0}=0}^{F_j-1} \min\left(F_j, \frac{1}{2\left\|\frac{z_{\nu_0,\mu_0} d_{\nu_0,\mu_0}}{F_j} + \psi\right\|}\right)$$

$$\leq 8\,(1 + \ln F_j) \leq 8(1 + \ln 2^{\kappa_0 j+1}) \leq 8(1 + \kappa_0 j + 1) = O(j)\,.$$

Lemma 5 is proved. □

Lemma 6. *Let* $0 < \max_\mu |(\mathbf{m})_\mu| \leq c_2 2^j$, *with* $c_2 = c_1/(s \max |w_{i,j}|)$, *and* $0 \leq l < j$. *Then*

$$\prod_{\nu=-\kappa_3}^{\kappa_2} \prod_{\mu=1}^{sd} \min\left(1, \frac{1}{2F_j \left\|\left(A_\nu^{|j\nu-l|} W \mathbf{m}^{(t)}\right)_\mu\right\|}\right) = O\left(\frac{1}{2^j}\right), \qquad (38)$$

where the O *-constant does not depend on* $\mathbf{m}$ *and* l.

P r o o f. Let us take: $\widetilde{\mathbf{m}} = W\mathbf{m}^{(t)}$, $\nu_0 = \left[\frac{d+1}{\log_2 \lambda_0}\right] + 2 < \kappa_2$,

$$\left|\left(A_1^{\nu_0 j-l} \widetilde{\mathbf{m}}\right)_{\mu_0}\right| = \left|\left(A_{\nu_0}^{|\nu_0 j-l|} \widetilde{\mathbf{m}}\right)_{\mu_0}\right| = \max_{1\leq\mu\leq sd} \left|\left(A_{\nu_0}^{|\nu_0 j-l|} \widetilde{\mathbf{m}}\right)_\mu\right|$$

for some $\mu_0 \in [1, sd]$ (see (8) and (9)). We have $0 < \max_\mu |(\widetilde{\mathbf{m}})_\mu| \leq c_1 2^j$. Using Lemma 4, we obtain

$$\left| \left(A_1^{\nu_0 j - l} \widetilde{\mathbf{m}} \right)_{\mu_0} \right| \leq 2^{jd} \lambda_0^{-\nu_0 j + l} \leq 2^{jd} \lambda_0^{-\nu_0 j + j} = 2^{-jv}$$

with $v = (\nu_0 - 1)\log_2 \lambda_0 - d \geq \big((d + 1/\log_2 \lambda_0)\big)\log_2 \lambda_0 - d = 1$. Hence

$$\left\| \left(A_1^{\nu_0 j - l} \widetilde{\mathbf{m}} \right)_{\mu_0} \right\| = \left| \left(A_1^{\nu_0 j - l} \widetilde{\mathbf{m}} \right)_{\mu_0} \right|.$$

Now we use Lemma 4 to obtain the lower bound

$$F_j \left\| \left(A_1^{\nu_0 j - l} \widetilde{\mathbf{m}} \right)_{\mu_0} \right\| \geq 2^{j\kappa_0} 2^{-j(s-1)} \lambda_0^{-(\nu_0 j - l)} > 2^{j(\kappa_0 - s + 1)} \lambda_0^{-\nu_0 j} = 2^{j v_1},$$

$$\text{with} \quad v_1 = \kappa_0 - s + 1 - \nu_0 \log_2 \lambda_0.$$

By (9), we have

$$\kappa_0 = 2 + s + ([\log_2 \lambda_0] + 1)(d + 1)$$

and

$$\begin{aligned} v_1 &= 2 + s + ([\log_2 \lambda_0] + 1)(d + 1) - s + 1 - \left(\left[\frac{d+1}{\log_2 \lambda_0} \right] + 2 \right) \log_2 \lambda_0 \\ &\geq 3 + ([\log_2 \lambda_0] + 1)(d + 1) - ((d + 1)/\log_2 \lambda_0 + 2) \log_2 \lambda_0 \\ &\geq 3 + [\log_2 \lambda_0](d + 1) + (d + 1) - (d + 1) - 2[\log_2 \lambda_0] - 2\{\log_2 \lambda_0\} \\ &\geq 3 - 2\{\log_2 \lambda_0\} \geq 1. \end{aligned}$$

Thus

$$\left(F_j \left\| \left(A_{\nu_0}^{\nu_0 j - l} W \mathbf{m}^{(t)} \right)_{\mu_0} \right\| \right)^{-1} \leq \; 2^{-j}.$$

Now we take in (38) the trivial estimate for $\mu \neq \mu_0$, and for $\mu = \mu_0$, $\nu \neq \nu_0$. Lemma 6 is proved. □

Let

$$S(P) = \sum_{n^*=2}^{P-3} e\Big(\Big\langle W \mathbf{m}^{(t)}, \alpha_{n^*} \Big\rangle \Big), \tag{39}$$

where

$$\alpha_{n^*} = \sum_{\nu=-\kappa_3}^{\kappa_2} \mathrm{sign}(\nu) F_j \left\{ \frac{(n^* + u_{\nu+\nu^*}) \mathbf{d}_{\nu+\nu^*}}{F_j} \right\} A_\nu^{|j\nu - l|} \tag{40}$$

$$\operatorname{sign}(\nu)=\begin{cases}1, & \text{if} \quad \nu>0,\\ -1, & \text{otherwise,}\end{cases} \qquad u_\nu=\begin{cases}1, & \text{if} \quad \nu\geq\kappa_1,\\ 0, & \text{if} \quad \nu\in[0,\kappa_1-1],\\ -1, & \text{otherwise}\end{cases} \tag{41}$$

and

$$\mathbf{d}_{\nu+\kappa_1 i}=\mathbf{d}_\nu \quad \text{for} \quad i\in\mathbb{Z}$$

and

$$\nu\in[-\kappa_3,\kappa_2].$$

Lemma 7. *Let $5\leq P\leq 2^j$. With notations as above we have:*

$$|S(P)|\leq\sum_{m_{sd+1}=-2^{j-1}}^{2^{j-1}-1}\frac{|S_1(\mathbf{m})|}{\bar{m}_{sd+1}},$$

where

$$S_1(\mathbf{m})=\sum_{n^*=0}^{2^j-1}e\left(\left\langle W\mathbf{m}^{(t)},\alpha_{n^*}\right\rangle+\frac{m_{sd+1}n^*}{2^j}\right).$$

Proof. Similarly to [Ko2, p. 13], applying Lemma C and Lemma A we have that

$$|S(P)|=\left|\sum_{n^*=2}^{P-3}\sum_{k=0}^{2^j-1}e\left(\left\langle W\mathbf{m}^{(t)},\alpha_k\right\rangle\right)\delta_{2^j}(k-n^*)\right|$$

$$=\left|\sum_{n^*=2}^{P-3}\sum_{k=0}^{2^j-1}\frac{1}{2^j}\sum_{m_{sd+1}=-2^{j-1}}^{2^{j-1}-1}e\left(\left\langle W\mathbf{m}^{(t)},\alpha_k\right\rangle\right)e\left(\frac{(k-n^*)m_{sd+1}}{2^j}\right)\right|$$

$$=\left|\frac{1}{2^j}\sum_{m_{sd+1}=-2^{j-1}}^{2^{j-1}-1}\sum_{n^*=2}^{P-3}e\left(-\frac{m_{sd+1}n^*}{2^j}\right)\sum_{k=0}^{2^j-1}e\left(\left\langle W\mathbf{m}^{(t)},\alpha_k\right\rangle+\frac{m_{sd+1}k}{2^j}\right)\right|$$

$$\leq\sum_{m_{sd+1}=-2^{j-1}}^{2^{j-1}-1}\frac{1}{\bar{m}_{sd+1}}\left|\sum_{k=0}^{2^j-1}e\left(\left\langle W\mathbf{m}^{(t)},\alpha_k\right\rangle+\frac{m_{sd+1}k}{2^j}\right)\right|.$$

Lemma 7 is proved. □

Lemma 8. *With notations as above we have:*

$$|S_1(\mathbf{m})| \leq \sum_{\mathbf{z_0},\dots,\mathbf{z}_{\kappa_1-1}\in[0,F_j)^{sd}} \min\left(2^j, \frac{1}{2\left\|\sum_{\nu=-\kappa_3}^{\kappa_2}\left\langle \mathbf{z}_\nu, \frac{\mathbf{d}_{\nu+\nu^*}}{F_j}\right\rangle + \frac{m_{sd+1}}{2^j}\right\|}\right)$$
$$\times \prod_{\nu=-\kappa_3}^{\kappa_2}\prod_{\mu=1}^{sd}\min\left(1, \frac{1}{2F_j\left\|\left(-\frac{\mathbf{z}_\nu}{F_j}+\operatorname{sign}(\nu)A_\nu^{|j\nu-l|}W\mathbf{m}^{(t)}\right)_\mu\right\|}\right).$$

P r o o f. Let

$$(n^*+u_{\nu+\nu^*})\mathbf{d}_{\nu+\nu^*} \equiv \mathbf{k}_\nu \mod F_j,$$

where

$$\mathbf{k}_\nu \in [0,F_j)^{sd},\ -\kappa_3 \leq \nu \leq \kappa_2, \quad \text{and} \quad \mathbf{k}_\nu = (k_{\nu,1},\dots,k_{\nu,sd}).$$

From (40), we have

$$S_1(\mathbf{m}) = \sum_{n^*=0}^{2^j-1}\left(\prod_{\nu=-\kappa_3}^{\kappa_2}\sum_{\mathbf{k}_\nu\in[0,F_j)^{sd}}\prod_{i=1}^{sd}\delta_{F_j}\left((n^*+u_{\nu+\nu^*})\mathbf{d}_{\nu+\nu^*}-k_{\nu,i}\right)\right)$$
$$\times e\left(\left\langle W\mathbf{m}^{(t)}, \sum_{\nu=-\kappa_3}^{\kappa_2}\operatorname{sign}(\nu)\mathbf{k}_\nu A_\nu^{|j\nu-l|}\right\rangle + \frac{m_{sd+1}n^*}{2^j}\right).$$

By Lemma C, we get

$$S_1(\mathbf{m}) = \sum_{n^*=0}^{2^j-1}\sum_{\mathbf{k}_{-\kappa_3},\dots,\mathbf{k}_{\kappa_2}\in[0,F_j)^{sd}}\sum_{\mathbf{z}_{-\kappa_3},\dots,\mathbf{z}_{\kappa_2}\in[0,F_j)^{sd}}\frac{1}{F_j^{\kappa_1 sd}}$$
$$\times e\left(\sum_{\nu=-\kappa_3}^{\kappa_2}\sum_{i=1}^{sd} z_{\nu,i}\times\frac{\left((n^*+u_{\nu+\nu^*})d_{\nu+\nu^*,i}-k_{\nu,i}\right)}{F_j}\right)$$
$$\times e\left(\left\langle W\mathbf{m}^{(t)}, \sum_{\nu=-\kappa_3}^{\kappa_2}\operatorname{sign}(\nu)\mathbf{k}_\nu A_\nu^{|j\nu-l|}\right\rangle + \frac{m_{sd+1}n^*}{2^j}\right).$$

Changing the order of summation, we obtain

$$|S_1(\mathbf{m})| = \Bigg| \frac{1}{F_j^{\kappa_1 sd}} \sum_{\mathbf{k}_{-\kappa_3},\dots,\mathbf{k}_{\kappa_2} \in [0,F_j)^{sd}} \ \sum_{\mathbf{z}_{-\kappa_3},\dots,\mathbf{z}_{\kappa_2} \in [0,F_j)^{sd}}$$

$$e\Bigg(\sum_{\nu=-\kappa_3}^{\kappa_2} \left\langle W\mathbf{m}^{(t)}, \operatorname{sign}(\nu)\mathbf{k}_\nu A_\nu^{|j\nu-l|} \right\rangle - \frac{\langle \mathbf{z}_\nu, \mathbf{k}_\nu \rangle}{F_j} \Bigg) \sum_{n^*=0}^{2^j-1}$$

$$e\Bigg(n^* \Bigg(\frac{m_{sd+1}}{2^j} + \sum_{\nu=-\kappa_3}^{\kappa_2} \frac{1}{F_j} \langle \mathbf{z}_\nu, \mathbf{d}_{\nu+\nu^*} \rangle \Bigg) + \sum_{\nu=-\kappa_3}^{\kappa_2} \frac{u_{\nu+\nu^*}}{F_j} \langle \mathbf{z}_\nu, \mathbf{d}_{\nu+\nu^*} \rangle \Bigg) \Bigg|$$

$$\leq \sum_{\mathbf{z}_{-\kappa_3},\dots,\mathbf{z}_{\kappa_2} \in [0,F_j)^{sd}} \frac{1}{F_j^{\kappa_1 sd}} \Bigg| \sum_{\mathbf{k}_{-\kappa_3},\dots,\mathbf{k}_{\kappa_2} \in [0,F_j)^{sd}}$$

$$e\Bigg(\sum_{\nu=-\kappa_3}^{\kappa_2} \left\langle \mathbf{k}_\nu, \operatorname{sign}(\nu) A_\nu^{|j\nu-l|} W\mathbf{m}^{(t)} - \mathbf{z}_\nu / F_j \right\rangle \Bigg) \Bigg|$$

$$\times \Bigg| \sum_{n^*=0}^{2^j-1} e\Bigg(n^* \Bigg(\frac{m_{sd+1}}{2^j} + \sum_{\nu=-\kappa_3}^{\kappa_2} \frac{1}{F_j} \langle \mathbf{z}_\nu, \mathbf{d}_{\nu+\nu^*} \rangle \Bigg)\Bigg) \Bigg|. \tag{42}$$

Using Lemma A, we get

$$\frac{1}{F_j^{\kappa_1 sd}} \Bigg| \sum_{\mathbf{k}_{-\kappa_3},\dots,\mathbf{k}_{\kappa_2} \in [0,F_j)^{sd}} e\Bigg(\sum_{\nu=-\kappa_3}^{\kappa_2} \left\langle \mathbf{k}_\nu, \operatorname{sign}(\nu) A_\nu^{|j\nu-l|} W\mathbf{m}^{(t)} - \frac{\mathbf{z}_\nu}{F_j} \right\rangle \Bigg) \Bigg|$$

$$\leq \prod_{\nu=-\kappa_3}^{\kappa_2} \prod_{\mu=1}^{sd} \min\Bigg(1, \frac{1}{2F_j \left\| \left(-\frac{\mathbf{z}_\nu}{F_j} + \operatorname{sign}(\nu) A_\nu^{|j\nu-l|} W\mathbf{m}^{(t)} \right)_\mu \right\|} \Bigg), \tag{43}$$

and

$$\Bigg| \sum_{n^*=0}^{2^j-1} e\Bigg(n^* \Bigg(\frac{m_{sd+1}}{2^j} + \sum_{\nu=-\kappa_3}^{\kappa_2} \left\langle \mathbf{z}_\nu, \frac{\mathbf{d}_{\nu+\nu^*}}{F_j} \right\rangle \Bigg)\Bigg) \Bigg|$$

$$\leq \min\Bigg(2^j, \frac{1}{2 \left\| \sum_{\nu=-\kappa_3}^{\kappa_2} \left\langle \mathbf{z}_\nu, \frac{\mathbf{d}_{\nu+\nu^*}}{F_j} \right\rangle + \frac{m_{sd+1}}{2^j} \right\|} \Bigg). \tag{44}$$

Now from (42), (43) and (44), we obtain the assertion of the lemma. □

Let us denote

$$T(\mathbf{d}_{-\kappa_3},\dots,\mathbf{d}_{\kappa_2})$$
$$=\sum_{l=0}^{j-1}\sum_{\nu^*=0}^{\kappa_1-1}\sum_{0<\max_{1\le i\le sd}|m_i|\le c_2 2^j}\frac{1}{\bar{m}_1\dots\bar{m}_{sd}}\sum_{m_{sd+1}=-2^{j-1}}^{2^{j-1}}\frac{1}{\bar{m}_{sd+1}}$$
$$\times\sum_{\mathbf{z_0},\dots,\mathbf{z}_{\kappa_1-1}\in[0,F_j)^{sd}}\min\left(2^j,\frac{1}{2\left\|\sum_{\nu=-\kappa_3}^{\kappa_2}\left\langle\mathbf{z}_\nu,\frac{\mathbf{d}_{\nu+\nu^*}}{F_j}\right\rangle+\frac{m_{sd+1}}{2^j}\right\|}\right)$$
$$\times\prod_{\nu=-\kappa_3}^{\kappa_2}\prod_{\mu=1}^{sd}\min\left(1,\frac{1}{2F_j\left\|\left(-\frac{\mathbf{z}_\nu}{F_j}+\operatorname{sign}(\nu)A_\nu^{|j\nu-l|}W\mathbf{m}^{(t)}\right)_\mu\right\|}\right).$$

Main Lemma. *Let us take*

$$\mathbf{d}_{j,-\kappa_3},\ \dots\ ,\mathbf{d}_{j,\kappa_2}\in[0,F_j)^{sd}\quad \textit{so that}\quad T(\mathbf{d}_{j,-\kappa_3},\ \dots\ ,\mathbf{d}_{j,\kappa_2})$$

will be minimal. Then

$$T(\mathbf{d}_{j,-\kappa_3},\dots,\mathbf{d}_{j,\kappa_2})=O\left(j^{sd+4}\right).$$

P r o o f. Consider the mean values

$$\widetilde{T}=\frac{1}{F_j^{sd\kappa_1}}\sum_{\mathbf{d}_{-\kappa_3},\dots,\mathbf{d}_{\kappa_2}\in[0,F_j)^{sd}}T(\mathbf{d}_{-\kappa_3},\dots,\mathbf{d}_{\kappa_2}).\tag{45}$$

We see

$$\widetilde{T}\le\sum_{l=0}^{j-1}\sum_{\nu^*=0}^{\kappa_1-1}\sum_{0<\max_{1\le i\le sd}|m_i|\le 2^j}\sum_{m_{sd+1}=-2^j}^{2^j-1}\frac{1}{\bar{m}_1\dots\bar{m}_{sd}\bar{m}_{sd+1}}\sigma(\mathbf{m}),$$

where

$$\sigma(\mathbf{m})=\frac{1}{F_j^{sd\kappa_1}}\sum_{\mathbf{z}_{-\kappa_3},\dots,\mathbf{z}_{\kappa_2}\in[0,F_j)^{sd}}\ \sum_{\mathbf{d}_{-\kappa_3},\dots,\mathbf{d}_{\kappa_2}\in[0,F_j)^{sd}}$$
$$\min\left(2^j,\frac{1}{2\left\|\sum_{\nu=-\kappa_3}^{\kappa_2}\left\langle\mathbf{z}_\nu,\frac{\mathbf{d}_{\nu+\nu^*}}{F_j}\right\rangle+\frac{m_{sd+1}}{2^j}\right\|}\right)\prod_{\nu=-\kappa_3}^{\kappa_2}\prod_{\mu=1}^{sd}$$
$$\min\left(1,\frac{1}{2F_j\left\|\left(-\frac{\mathbf{z}_\nu}{F_j}+\operatorname{sign}(\nu)A_\nu^{|j\nu-l|}W\mathbf{m}^{(t)}\right)_\mu\right\|}\right).$$

Let

$$\sigma_1 := \frac{1}{F_j^{sd\kappa_1}} \sum_{\mathbf{d}_{-\kappa_3},\dots,\mathbf{d}_{\kappa_2}\in[0,F_j)^{sd}} \min\left(2^j, \frac{1}{2\left\|\sum_{\nu=-\kappa_3}^{\kappa_2}\left\langle \mathbf{z}_\nu, \frac{\mathbf{d}_{\nu+\nu^*}}{F_j}\right\rangle + \frac{m_{sd+1}}{2^j}\right\|}\right).$$

In the case of $(\mathbf{z}_{-\kappa_3},\dots,\mathbf{z}_{\kappa_2}) \neq (\mathbf{0},\dots,\mathbf{0})$, we will use Lemma $5\sigma_1 = O\left(j\right).$ If $(\mathbf{z}_{-\kappa_3},\dots,\mathbf{z}_{\kappa_2}) = (\mathbf{0},\dots,\mathbf{0})$, we will use the trivial estimate $\sigma_1 = O\left(2^j\right).$ Therefore, $\sigma(\mathbf{m}) = O\left(\sigma_2+\sigma_3\right)$, where

$$\sigma_2 = j \sum_{\mathbf{z}_{-\kappa_3},\dots,\mathbf{z}_{\kappa_2}\in[0,F_j)^{sd}} \prod_{\nu=-\kappa_3}^{\kappa_2}\prod_{\mu=1}^{sd} \min\left(1, \frac{1}{2F_j\left\|\left(-\frac{\mathbf{z}_\nu}{F_j} + \operatorname{sign}(\nu)A_\nu^{|j\nu-l|}W\mathbf{m}^{(t)}\right)_\mu\right\|}\right), \tag{46}$$

and

$$\sigma_3 = 2^j \prod_{\nu=-\kappa_3}^{\kappa_2}\prod_{\mu=1}^{sd} \min\left(1, \frac{1}{2F_j\left\|\left(A_\nu^{|j\nu-l|}W\mathbf{m}^{(t)}\right)_\mu\right\|}\right).$$

By Lemma 6, we have $\sigma_3 = O(1)$. Using Lemma B, we get

$$\sum_{z_{\nu\mu}\in[0,F_j)} \min\left(1, \frac{1}{2F_j\left\|\left(-\frac{\mathbf{z}_{nu}}{F_j} + \operatorname{sign}(\nu)A_\nu^{|j\nu-l|}W\mathbf{m}^{(t)}\right)_\mu\right\|}\right)$$

$$\leq 8\ (1+\ln F_j) \leq 8(1+\ln 2^{\kappa_0 j+1}) \leq 8(1+\kappa_0 j+1) = O(j).$$

From (46), we obtain $\sigma_2 = O\left(j^2\right)$. Hence $\sigma(\mathbf{m}) = O\left(j^2\right)$ and

$$\widetilde{T} = O\left(j^2 \sum_{l=0}^{j-1}\sum_{\nu^*=0}^{\kappa_1-1} \sum_{0<\max_{1\leq i\leq sd}|m_i|\leq 2^j} \sum_{m_{sd+1}=-2^j}^{2^j-1} \frac{1}{\bar{m}_1\dots\bar{m}_{sd}\bar{m}_{sd+1}}\right).$$

Thus,

$$\widetilde{T} = O\left(j^{sd+4}\right).$$

By (45), there exist vectors

$$\mathbf{d}_{j,-\kappa_3},\dots,\mathbf{d}_{j,\kappa_2} \in [0,F_j)^{sd} \quad \text{with} \quad T(\mathbf{d}_{j,0},\dots,\mathbf{d}_{j,\kappa_1-1}) = O\left(j^{sd+4}\right).$$

The Main Lemma is proved. □

We will use vectors $\mathbf{b}_{j,\nu} = \mathbf{d}_{j,\nu}$ in (11) and (40), where

$$\mathbf{d}_{j,\nu_1+\kappa_1 i} = \mathbf{d}_{j,\nu_1} \text{ for } i \in \mathbb{Z} \quad \text{and} \quad \nu_1 \in [-\kappa_3, \kappa_2],\ j = 1, 2, \dots$$

Applying Lemma 7, Lemma 8 and the Main Lemma, we get

Corollary. *Let* $5 \le P \le 2^j$. *Then*

$$\sum_{l=0}^{j-1} \sum_{\nu^*=0}^{\kappa_1-1} \sum_{0<\max_{1\le i\le sd}|m_i|\le c_2 2^j} \frac{1}{\bar{m}_1, \dots, \bar{m}_{sd}} \left| \sum_{n^*=2}^{P-3} e\left(\left\langle W\mathbf{m}^{(t)}, \alpha_{n^*}\right\rangle\right)\right| = O\left(j^{sd+4}\right).$$

4.2. End of the proof of the Theorem

Let us decompose our interval $[1, N]$ into subintervals: $[1, n_2), [n_2, n_3), \dots$ $\dots, [n_{r-1}, n_r), [n_r, N]$, where $n_{r+1} > N \ge n_r$. By (10) we have that

$$N = n_r + r\kappa_1(\widetilde{n} + \widetilde{\nu}) + \widetilde{l},$$

where

$$0 \le \widetilde{l} \le r-1, \quad 0 \le \widetilde{n} \le 2^r - 1, \quad 0 \le \widetilde{\nu} \le \kappa_1 - 1.$$

From (1), we have the trivial estimate

$$LD\left((x_n)_{n=J}^{J+L-1}\right) \le L, \quad L = 1, 2, \dots \tag{47}$$

Using (1) and (10), we get

$$\begin{aligned} ND\left(\left\{\alpha A^k\right\}_{k=1}^{N}\right) \le &(n_2-1)D\left(\left\{\alpha A^k\right\}_{k=1}^{n_2}\right) \\ &+ \sum_{j=2}^{r-1} \sum_{\nu^*=0}^{\kappa_1-1} \sum_{l=0}^{j-1} 2^j D\left(\left\{\alpha A^{n_j+j(\kappa_1 n^*+\nu^*)+l}\right\}_{n^*=0}^{2^j-1}\right) \\ &+ \sum_{\nu^*=0}^{\kappa_1-1} \sum_{l=0}^{r-1} \widetilde{n} D\left(\left\{\alpha A^{n_r+r(\kappa_1 n^*+\nu^*)+l}\right\}_{n^*=0}^{\widetilde{n}-1}\right) + O(r^2). \end{aligned}$$

Hence to obtain (12), it is sufficient to prove that

$$v := \sum_{l=0}^{j-1} \sum_{\nu^*=0}^{\kappa_1-1} PD\left(\left\{\alpha A^{n_j+j(\kappa_1 n^*+\nu^*)+l}\right\}_{n^*=0}^{P-1}\right) = O(j^{sd+4}) \text{ for } P \in [1, 2^j].$$

By (1) and (47), we have

$$v \le 4\kappa_1 j + \sum_{l=0}^{j-1} \sum_{\nu^*=0}^{\kappa_1-1} (P-4)D\left(\left\{\alpha A^{n_j+j(\kappa_1 n^*+\nu^*)+l}\right\}_{n^*=2}^{P-3}\right).$$

Thus, to obtain (12), it is sufficient to prove that

$$\sum_{l=0}^{j-1}\sum_{\nu^*=0}^{\kappa_1-1}(P-4)D\left(\left\{\alpha A^{n_j+j(\kappa_1 n^*+\nu^*)+l}\right\}_{n^*=2}^{P-3}\right)=O(j^{sd+4})\quad\text{for}\quad P\in\left[5,2^j\right]. \tag{48}$$

Applying the Erdös-Turan-Koksma inequality (13) with $M = c_2 2^j$, we get that (48) is obtained from the following estimate

$$\sum_{l=0}^{j-1}\sum_{\nu^*=0}^{\kappa_1-1}\sum_{0<\max|m_i|\le c_2 2^j}\frac{\left|\sum_{n^*=2}^{P-3} e\left(\langle \mathbf{m},\alpha A^{n_j+j(\kappa_1 n^*+\nu^*)+l}\rangle\right)\right|}{\overline{m_1}\dots\overline{m_{sd}}}=O\left(j^{sd+4}\right). \tag{49}$$

Let

$$\widetilde{\alpha}_{n^*}=\alpha_{n^*}A^{-k},\quad \widehat{\alpha}_{n^*}=\widetilde{\alpha}_{n^*}W,\quad\text{with}\quad k=n_j+j(\kappa_1 n^*+\nu^*)+l. \tag{50}$$

From (3), we have

$$\langle\mathbf{m},\alpha A^k\rangle=\langle W\mathbf{m}^{(t)},\widetilde{\alpha}\widetilde{A}^k\rangle,\quad \langle\mathbf{m},\widehat{\alpha}_{n^*}A^k\rangle=\langle W\mathbf{m}^{(t)},\widetilde{\alpha}_{n^*}\widetilde{A}^k\rangle=\langle W\mathbf{m}^{(t)},\alpha_{n^*}\rangle.$$

Using the inequality

$$|e(x)-1|=|2\sin(\pi x)|\le 2\pi\,\|x\|,$$

we get

$$\left|e\left(\langle W\mathbf{m}^{(t)},\widetilde{\alpha}\widetilde{A}^k\rangle\right)-e\left(\langle W\mathbf{m}^{(t)},\widetilde{\alpha}_{n^*}\widetilde{A}^k\rangle\right)\right|\le 2\pi\left\|\langle W\mathbf{m}^{(t)},(\widetilde{\alpha}-\widetilde{\alpha}_{n^*})\widetilde{A}^k\rangle\right\|.$$

Hence

$$\begin{aligned}\left|\sum_{n^*=2}^{P-3}e\left(\langle\mathbf{m},\alpha A^k\rangle\right)\right| &= \left|\sum_{n^*=2}^{P-3}e\left(\langle W\mathbf{m}^{(t)},\widetilde{\alpha}\widetilde{A}^k\rangle\right)\right|\\ &=\left|\sum_{n^*=2}^{P-3}e\left(\langle W\mathbf{m}^{(t)},\widetilde{\alpha}_{n^*}\widetilde{A}^k+(\widetilde{\alpha}-\widetilde{\alpha}_{n^*})\widetilde{A}^k\rangle\right)\right|\\ &\le\left|\sum_{n^*=2}^{P-3}e\left(\langle W\mathbf{m}^{(t)},\alpha_{n^*}\rangle\right)\right|+2\pi\sum_{n^*=2}^{P-3}\left\|\langle W\mathbf{m}^{(t)},(\widetilde{\alpha}-\widetilde{\alpha}_{n^*})\widetilde{A}^k\rangle\right\|.\end{aligned}$$

Therefore, (49)is obtained from the Corollary and the following estimate

$$\left\|\langle W\mathbf{m}^{(t)},(\widetilde{\alpha}-\widetilde{\alpha}_{n^*})\widetilde{A}^k\rangle\right\|=O(2^{-j})\quad\text{with}\quad n^*\in[2,2^j-3] \tag{51}$$

for $0<\max|m_i|\le c_2 2^j,\ k=n_j+j(\kappa_1 n^*+\nu^*)+l$.

Let

$$\begin{aligned}\Omega \quad &= \{(m,n,\nu) \mid m \in \mathbb{N}, n \in [0,2^m-1], \nu \in [0,\kappa_1-1]\}\\ &= \Omega_{-3} \cup \Omega_{-2} \cup \Omega_{-1} \cup \Omega_0 \cup \Omega_1 \cup \Omega_2 \cup \Omega_3,\end{aligned}$$

where

$$\begin{aligned}\Omega_{-3} &= \{(m,n,\nu) \mid m \in [0,j-1], n \in [0,2^m-1], \nu \in [0,\kappa_1-1]\},\\ \Omega_3 &= \{(m,n,\nu) \mid m = j+1, j+2, \dots, n \in [0,2^m-1], \nu \in [0,\kappa_1-1]\},\\ \Omega_{-2} &= \{(j,n,\nu) \mid n \in [0,n^*-3], \nu \in [0,\kappa_1-1]\},\\ \Omega_2 &= \{(j,n,\nu) \mid n \in [n^*+2, 2^j-1], \nu \in [0,\kappa_1-1]\},\\ \Omega_{-1} &= \{(j,n^*-2,\nu) \mid \nu \in [0,\nu^*-\kappa_3+2\kappa_1-1]\},\\ \Omega_1 &= \{(j,n^*,\nu) \mid \nu \in [\nu^*+\kappa_2+1, 2\kappa_1-1]\},\\ \Omega_0 &= \{(j,n^*,\nu) \mid \nu \in [\nu^*-\kappa_3, \nu^*+\kappa_2]\}.\end{aligned}$$

Let

$$\beta_i = \sum_{(m,n,\nu)\in\Omega_i} F_m \left\{\frac{n_1 \mathbf{b}_{\mathbf{m},\nu_1}}{F_m}\right\} A_1^{n_m+m(\kappa_1 n+\nu)}, \tag{52}$$

with $\kappa_1 n_1 + \nu_1 = \kappa_1 n + \nu$ for $n_1 \in [0,2^m-1]$, $\nu_1 \in [0,\kappa_1-1]$ and $i \in [-3,3]$. By (11), we see that

$$\widetilde{\alpha} = \sum_{i=-3}^{3} \beta_i. \tag{53}$$

From (3), (7), (8) and (23), we get

$$A_1^n \widetilde{A}^k = A_1^{n-k} \text{ for } n \geq k, \text{ and } A_1^n \widetilde{A}^k = A^{k-n} - A_{-1}^{k-n} \quad \text{for} \quad n < k. \tag{54}$$

Applying (41), we have

$$\begin{aligned}\beta_0 \widetilde{A}^k &= \sum_{\nu\in[\nu^*-\kappa_3,\nu^*+\kappa_2]} F_j \left\{\frac{(n^*+u_\nu)\mathbf{b}_{j,\nu}}{F_j}\right\} A_1^{n_j+j(\kappa_1 n^*+\nu)} \widetilde{A}^{n_j+j(\kappa_1 n^*+\nu^*)+l}\\ &\equiv \sum_{\nu\in[\nu^*+1,\nu^*+\kappa_2]} F_j \left\{\frac{(n^*+u_\nu)\mathbf{b}_{j,\nu}}{F_j}\right\} A_1^{j(\nu-\nu^*)-l}\\ &\quad - \sum_{\nu\in[\nu^*-\kappa_3,\nu^*]} F_j \left\{\frac{(n^*+u_\nu)\mathbf{b}_{j,\nu}}{F_j}\right\} A_{-1}^{j(\nu^*-\nu)+l} \mod 1.\end{aligned}$$

Hence

$$\beta_0 \widetilde{A}^k \equiv \sum_{\nu \in [1,\kappa_2]} F_j \left\{ \frac{(n^* + u_{\nu+\nu^*})\mathbf{b}_{\mathbf{j},\nu+\nu^*}}{F_j} \right\} A_1^{j\nu - l}$$
$$- \sum_{\nu \in [-\kappa_3, 0]} F_j \left\{ \frac{(n^* + u_{\nu+\nu^*})\mathbf{b}_{\mathbf{j},\nu+\nu^*}}{F_j} \right\} A_{-1}^{-j\nu + l} \bmod 1.$$

By (40), we obtain

$$\beta_0 \widetilde{A}^k \equiv \alpha_{n^*} \bmod 1.$$

Now from (50) and (53), we get

$$(\widetilde{\alpha} - \widetilde{\alpha}_{n^*})\widetilde{A}^k = \sum_{i \in [-3,3]} \beta_i \widetilde{A}^k - \alpha_{n^*} \equiv \sum_{i \in [-3,3] \setminus 0} \beta_i \widetilde{A}^k \bmod 1.$$

We derive from (51) that (49) is obtained from the following estimate:

$$\left\| \langle W\mathbf{m}^{(t)}, \beta_i \widetilde{A}^k \rangle \right\| = O(2^{-j}), \quad i \in [-3,3] \setminus 0 \tag{55}$$

for $0 < \max |m_i| \le c_2 2^j,\ k = n_j + j(\kappa_1 n^* + \nu^*) + l$.
Consider the case $i > 0$. Bearing in mind that

$$|\mathbf{m}| \le c_2 2^j \quad \text{with} \quad c_2 = c_1/(s \max |w_{i,j}|),$$

we get that $|(W\mathbf{m}^{(t)})_i| \le c_1 2^j$, $1 \le i \le sd$. Hence we can apply Lemma 4. According to (52) and (54), we have

$$\left| \langle W\mathbf{m}^{(t)}, \beta_i \widetilde{A}^k \rangle \right| \le \sum_{(m,n,\nu) \in \Omega_i} sdF_m \left| A_1^{n_m + m(\kappa_1 n + \nu) - k} W\mathbf{m}^{(t)} \right|$$
$$\le \sum_{(m,n,\nu) \in \Omega_i} sdF_m 2^{md} \lambda_0^{k - n_m - m(\kappa_1 n + \nu)}. \tag{56}$$

Let $i = 1$. Using (9) and (56), we get

$$\left| \langle W\mathbf{m}^{(t)}, \beta_1 \widetilde{A}^k \rangle \right| \le sdF_j 2^{jd} \sum_{\nu=\nu^*+\kappa_2+1}^{2\kappa_1 - 1} \lambda_0^{k - n_j - j(\kappa_1 n^* + \nu)}$$
$$\le 2sd 2^{j(\kappa_0 + d)} \sum_{\nu=\nu^*+\kappa_2+1}^{2\kappa_1 - 1} \lambda_0^{j(\nu^* - \nu) + l}$$
$$= 2sd \sum_{\nu=\nu^*+\kappa_2+1}^{2\kappa_1 - 1} 2^{j a_\nu},$$

where

$$\begin{aligned} a_\nu &= \kappa_0 + d - (\log_2 \lambda_0)(\nu - \nu^* - l/j) \\ &\le \kappa_0 + d - (\log_2 \lambda_0)\kappa_2 \\ &\le \kappa_0 + d - (\log_2 \lambda_0)(\kappa_0 + d + 1)/\log_2 \lambda_0 \\ &= -1. \end{aligned} \tag{57}$$

Thus (55) is true for $i = 1$.

Let $i = 2$. Then

$$\begin{aligned} \left|\left\langle W\mathbf{m}^{(t)}, \beta_2 \widetilde{A}^k \right\rangle\right| &\le sdF_j 2^{jd} \sum_{n=n^*+2}^{2^j-1} \sum_{\nu=0}^{\kappa_1-1} \lambda_0^{k-n_j-j(\kappa_1 n+\nu)} \\ &\le 2sd \sum_{n=n^*+2}^{2^j-1} \sum_{\nu=0}^{\kappa_1-1} 2^{ja_{n,\nu}}, \end{aligned}$$

where

$$\begin{aligned} a_{n,\nu} &= \kappa_0 + d - (\log_2 \lambda_0)(\kappa_1(n-n^*) + \nu - \nu^* - l/j) \\ &\le \kappa_0 + d - (\log_2 \lambda_0)\kappa_1(n - n^* - 1). \end{aligned}$$

Hence

$$\left|\left\langle W\mathbf{m}^{(t)}, \beta_2 \widetilde{A}^k \right\rangle\right| = O(2^{ja_{n^*+2,0}}).$$

Using (9) and (4.2), we have

$$\begin{aligned} \left|\left\langle W\mathbf{m}^{(t)}, \beta_2 \widetilde{A}^k \right\rangle\right| &= O\left(2^{j(\kappa_0+d-(\log_2 \lambda_0)\kappa_1)}\right) \\ &= O(2^{-j}). \end{aligned}$$

Thus (55) is true for $i = 2$.

Let $i = 3$. Then

$$\left|\left\langle W\mathbf{m}^{(t)}, \beta_3 \widetilde{A}^k \right\rangle\right| \le sd \sum_{m\ge j+1} \sum_{n=0}^{2^j-1} \sum_{\nu=0}^{\kappa_1-1} F_m 2^{md} \lambda_0^{k-n_m-m(\kappa_1 n+\nu)}.$$

It is easy to see that

$$\left|\left\langle W\mathbf{m}^{(t)}, \beta_3 \widetilde{A}^k \right\rangle\right| = O\left(\sum_{m\ge j+1} 2^{m(\kappa_0+d)} \lambda_0^{k-n_m}\right).$$

Let

$$a_m = \log_2(2^{m(\kappa_0+d)} \lambda_0^{k-n_m}) = m(\kappa_0 + d) + (\log_2 \lambda_0)(k - n_m).$$

By (10), we have

$$\left|\left\langle W\mathbf{m}^{(t)}, \beta_3 \widetilde{A}^k \right\rangle\right| = O(2^{a_{j+1}}). \tag{58}$$

From (50) and (51), we get

$$k = n_j + j(\kappa_1 n^* + \nu^*) + l \quad \text{with} \quad n^* \in [2, 2^j - 3].$$

Thus

$$\begin{aligned} a_{j+1} &= (j+1)(\kappa_0 + d) + (\log_2 \lambda_0)(n_j + j(\kappa_1 n^* + \nu^*) + l - n_{j+1}) \\ &\le (j+1)(\kappa_0 + d) - 2j\kappa_1 \log_2 \lambda_0 \\ &\le (j+1)(\kappa_0 + d) - 2j([(\kappa_0 + d + 1)/\log_2 \lambda_0] + 3)/\log_2 \lambda_0 \\ &\le (j+1)(\kappa_0 + d) - 2j(\kappa_0 + d + 1) \\ &\le \; -j(\kappa_0 + d + 1). \end{aligned}$$

By (58), we obtain that (55) is true for $i = 3$.

Consider the case $i < 0$. Applying (54) and Lemma 4, we get

$$\begin{aligned} \left\| \langle W\mathbf{m}^{(t)}, \beta_i \widetilde{A}^k \rangle \right\| &\le \sum_{(m,n,\nu)\in\Omega_i} sdF_m \left| A_{-1}^{k-n_m-m(\kappa_1 n+\nu)} W\mathbf{m}^{(t)} \right| \\ &\le \sum_{(m,n,\nu)\in\Omega_i} sdF_m 2^{md} \lambda_{-1}^{n_m+m(\kappa_1 n+\nu)-k}. \end{aligned}$$

Now, repeating the proofs for the case $i > 0$, we get that (55) is also true for $i < 0$. The Theorem is proved. □

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