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uniform distribution theory

IRREGULARITY OF DISTRIBUTIONS AND MULTIPARAMETER A_p WEIGHTS

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ABSTRACT. Weighted multiparameter Littlewood-Paley theory is used to extend Roth's irregularity of distribution theorem to the case where Lebesgue measure on the unit cube is replaced by a product A_p weight.

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Dedicated to the memory of Professor Edmund Hlawka

1. Introduction

Let $\mathcal{A}_N$ denote a collection of N points in the unit cube $[0,1]^d \subset \mathbb{R}^d$ and, for $\vec{x}$ in the cube, $[\vec{0},\vec{x})$ denote the rectangle $\prod_{j=1}^{d}[0,x_j)$. The discrepancy function $\mathcal{D}_{\mathcal{N}} : \mathbb{R}^d \to \mathbb{R}$ is defined as

$$\mathcal{D}_N(\vec{x}) = \#\big(\mathcal{A}_N \cap [\vec{0},\vec{x})\big) - N\,\big|[\vec{0},\vec{x})\big| \tag{1}$$

(where $|E|$ denotes the Lebesgue measure of E), i.e., the discrepancy between the actual and the expected number of points of $\mathcal{A}_N$ found in $[\vec{0},\vec{x})$.

For $\mathcal{D}_N$ to have a small norm in some function space (L^p or Orlicz) means that the points of $\mathcal{A}_N$ are in some sense well-distributed, and the theory has, from its beginning, been concerned with examination of the fact that "no [point distribution] can, in a certain sense, be too evenly distributed" [16] (see also Beck and Chen [2]). The seminal result, by Roth in 1954, showed that in $\mathbb{R}^d$,

$$||\mathcal{D}_N||_{L^2} \gtrsim (\log N)^{(d-1)/2} \tag{2}$$

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(where $A \gtrsim B$ means $A \geq cB$, with the constant independent of N, though it may depend on the dimension d or the index p); this was later extended by Schmidt [17] to $1 < p < \infty$:

$$||\mathcal{D}_N||_{L^p} \gtrsim (\log N)^{(d-1)/2}. \tag{3}$$

More recently, using multiparameter Littlewood-Paley theory, Lacey [12] has obtained the delicate endpoint estimate

$$||\mathcal{D}_N||_{L(\log L)^{(d-2)/2}} \gtrsim (\log N)^{(d-1)/2}, \tag{4}$$

extending a result of Halász [11] from two to d dimensions. Also of interest is the endpoint case of L^∞, though it is not as easy to summarize the known results; see [2] and [3, 5] for the most recent information on this case.

All of the above results, it should be noted, are with respect to Lebesgue measure. In this paper, we will generalize Roth's theorem by extending the class of measures for which the Roth estimate continues to hold to include product $A_p(\mathbb{R}^d)$ weights. These important and well-studied objects (see, for example, the classic text of García-Cuerva and Rubio de Francia [10], or the upcoming book of Cruz-Uribe, Martell, and Pérez [7]) arise in harmonic analysis as the densities w with respect to which the fundamental constructs of harmonic analysis (maximal functions, singular integrals, etc.) are $L^p(w\,dx)$ bounded; it is also well-known and a subject of current research that they provide a powerful extrapolation tool (see, e.g., [8]). The extension of Roth's theorem to $A_p(\mathbb{R}^d)$ should, we hope, ultimately shed light on the delicate endpoint estimates, which are still unresolved.

2. Definitions

Let $\mathcal{D} = \{[2^j k, 2^j(k+1)) \mid j, k \in \mathbb{Z}\}$ denote the collection of dyadic intervals in $\mathbb{R}$. For each interval $I \in \mathcal{D}$, let $I_{\text{left}}, I_{\text{right}} \in \mathcal{D}$ denote its left and right halves, and let $h_I = -1_{I_{\text{left}}} + 1_{I_{\text{right}}}$ denote the (L^∞-normalized) Haar function supported on that interval. $\mathcal{D}^d$ will denote the collection of dyadic rectangles (parallelepipeds) in $\mathbb{R}^d$, and for each $R = I_1 \times \cdots \times I_d \in \mathcal{D}^d$, we let h_R denote the corresponding product of Haar functions: $h_R(\vec{x}) = \prod_{j=1}^d h_{I_j}(x_j)$.

For each $n \in \mathbb{N}$, we partition the rectangles $R \in \mathcal{D}^d$ of volume 2^{-n} as follows. Let $\mathbb{H}_n$ denote the collection of d-partitions of n (all d-tuples $\vec{r} \in \mathbb{N}^d$ such that $\sum_1^d r_i = n$); for each $\vec{r} \in \mathbb{H}_n$, let $\mathcal{R}_{\vec{r}}$ denote the set of all dyadic rectangles $R \in \mathcal{D}^d$ of dimensions $2^{-r_1} \times 2^{-r_2} \times \cdots \times 2^{-r_d}$ (and thus of volume $|R| = 2^{-n}$). Note that for fixed $\vec{r} \in \mathbb{H}_n$, each point $\vec{x} \in [0,1]^d$ lies in exactly one rectangle

$R \in \mathcal{R}_{\vec{r}}$; each point in $[0,1]^d$ lies in exactly $\#(\mathbb{H}_n) \approx (\log N)^{d-1}$ rectangles of dimension 2^{-n}.

Definition 2.1. Let $\vec{r} \in \mathbb{H}_n$. We call a function f an $\mathfrak{r}$-*function with parameter* $\vec{r}$ if

$$f = \sum_{R \in \mathcal{R}_{\vec{r}}} \epsilon_R h_R, \qquad \epsilon_R \in \{\pm 1\}, \tag{5}$$

i.e., f is a sum of signed Haar functions over (disjoint) rectangles of identical dimensions.

Definition 2.2. Let $w \in L^1_{\text{loc}}(\mathbb{R}^d)$ be a non-negative function. We call w a (dyadic) *product* $A_p(\mathbb{R}^d)$ *weight* if

$$A_p(w) := \sup_R \left(\frac{1}{|R|}\int_R w\right)\left(\frac{1}{|R|}\int_R w^{-\frac{1}{p-1}}\right)^{p-1} < \infty, \tag{6}$$

where the supremum (called the A_p *characteristic* of w) is taken over all dyadic rectangles R. $L^p(w)$ denotes the Banach space of functions for which

$$||f||_{L^p(w)} := \left(\int_{\mathbb{R}^d} |f|^p w\, dx\right)^{1/p} < \infty.$$

It is not difficult to see that the condition for product (multiparameter) A_p is equivalent to being in single-parameter A_p uniformly in each variable; e.g., any product $w(\vec{x}) := \prod_i w(x_i)$ of one-variable A_p weights w_i is a product A_p weight. The canonical examples of single-parameter weights are "power weights": $w(x) = |x|^\alpha$ is in A_p if and only if $-d < \alpha < d(p-1)$ (see [10, 18]).

Note also that it follows immediately from the definition that $w \in A_p$ if and only if the *dual weight* $w' := w^{-\frac{1}{p-1}} \in A_{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$; further, $(A_p(w))^{1/p} = (A_{p'}(w'))^{1/p'}$.

Definition 2.3. Given $f\colon [0,1]^d \to \mathbb{R}$, we define the *multiparameter square function* Sf by

$$S(f)(\vec{x}) = \left[\sum_{R \in \mathcal{R}_d} \frac{|\langle f, h_R\rangle|^2}{|R|^2} 1_R(\vec{x})\right]^{1/2}. \tag{7}$$

We will need the following theorem from multiparameter Littlewood-Paley theory.

Theorem 2.4. *Let $1 < p < \infty$. If w is a product $A_p(\mathbb{R}^d)$weight, and $f \in L^p(w)$, then there exist constants $c_w, C_w > 0$ such that*

$$c_w ||f||_{L^p(w)} \le ||Sf||_{L^p(w)} \le C_w ||f||_{L^p(w)}. \tag{8}$$

Note that both inequalities follow from the vector-valued versions (see [15] for the former, [1] for the latter) of the weighted one-dimensional results (see, for example, [20], [19], [13], [6]) by iteration.

3. Statement and Proof

Roth's theorem extends to the case of product A_p weights as follows:

Theorem 3.1. *If $w \in A_p(\mathbb{R}^d)$, then $||\mathcal{D}_N||_{L^p(w)} \gtrsim (\log N)^{(d-1)/2}$.*

We will give two proofs; a first approach, using duality, is as follows. The proof of Roth's theorem (as modified by Schmidt) hinges on the proposition that given any discrepancy function $\mathcal{D}_N$, for n sufficiently large ($\approx \log N$), we have guaranteed $\mathfrak{r}$-functions with parameters $\vec{r} \in \mathbb{H}_n$ that are strictly non-orthogonal to $\mathcal{D}_N$:

Proposition 3.2. *Let $\mathcal{A}_N$ be a collection of N points in the unit cube, and $\mathcal{D}_N$ be the corresponding discrepancy function. Let $n \in \mathbb{N}$ be determined by $2N \le 2^n < 4N$. Then for each $\vec{r} \in \mathbb{H}_n$, there is an $\mathfrak{r}$-function $f_{\vec{r}}$ of parameter $\vec{r}$ such that*

$$\langle \mathcal{D}_N, f_{\vec{r}} \rangle \ge c_d, \tag{9}$$

where c_d is a dimensional constant.

Given the above proposition (whose proof can be found in [17, 2]) the theorem follows:

P r o o f. Let $F = \sum_{\vec{r} \in \mathbb{H}_n} f_{\vec{r}}$, where $f_{\vec{r}}$ is the $\mathfrak{r}$-function constructed in the proposition above. Then by (9),

$$c_d (\log N)^{d-1} \lesssim \langle \mathcal{D}_N, F \rangle \tag{10}$$

$$= \int \left[\mathcal{D}_N(\vec{x}) (w(\vec{x}))^{1/p} \right] \left[F(\vec{x}) \frac{1}{(w(\vec{x}))^{1/p}} \right] d\vec{x} \tag{11}$$

$$\le ||\mathcal{D}_N||_{L^p(w)} ||F||_{L^{p'}(w')}, \tag{12}$$

the last by Hölder's inequality (where w' is the dual weight as before). By Theorem 2.4, we can pass to the multiparameter square function:

$$\le \frac{1}{c_w} ||\mathcal{D}_N||_{L^p(w)} ||S(F)||_{L^{p'}(w')} . \tag{13}$$

At this point, we observe that

$$S(F) = \left[\sum_{\vec{r} \in \mathbb{H}_n} |f_{\vec{r}}|^2 \right]^{1/2} = \left[\sum_{\vec{r} \in \mathbb{H}_n} 1 \right]^{1/2} \approx (\log N)^{(d-1)/2}$$

is constant, so

$$||S(F)||_{L^{p'}(w')} \approx (\log N)^{(d-1)/2} ||1||_{L^{p'}(w')([0,1]^d)} . \tag{14}$$

By the A_p condition,

$$||1||_{L^{p'}(w')([0,1]^d)} = \left(\int_{[0,1]^d} w' \right)^{1/p'} \leq \left(\frac{A_p(w)}{w([0,1]^d)} \right)^{1/p}, \tag{15}$$

so, all together, we get

$$||\mathcal{D}_N||_{L^p(w)} \gtrsim c_d \, c_w \left(\frac{w([0,1]^d)}{A_p(w)} \right)^{1/p} (\log N)^{(d-1)/2}. \tag{16}$$

□

A second proof circumvents duality and $\mathfrak{r}$-functions entirely, using Littlewood-Paley theory and three observations. The first is the basic observation of Roth; its proof can be found in [2]. The second is a probabilistic lemma.

Lemma 3.3. *Let $\vec{r} \in \mathbb{H}_n$, and let $R \in R_{\vec{r}}$ be a "good" rectangle (i.e., $R \cap \mathcal{A}_N = \emptyset$). Then*

$$|\langle \mathcal{D}_N, h_R \rangle| = 4^{-d} N |R|^2. \tag{17}$$

Lemma 3.4. *Let (Ω, μ) be a probability space, and $0 < p < \infty$. Given sets $E_1, \ldots, E_J \subset \Omega$, where $\mu(E_j) \geq \epsilon$ for $j = 1, \ldots, J$, there exists $\gamma_p > 0$ such that*

$$\left\| \sum_j 1_{E_j} \right\|_{L^p(\mu)} \geq \gamma_p J. \tag{18}$$

P r o o f. The statement is obvious for $p = 1$; monotonicity of L^p norms then proves it for $p > 1$. Consider the case of $0 < p < 1$. Suppose that for arbitrarily small γ_p, the inequality is false for some J and some selection of events E_j. Chebyshev's inequality then implies

$$\mu \left\{ \sum_j 1_{E_j} > \gamma_p^{1/2} J \right\} \leq \gamma_p^{p/2}. \tag{19}$$

However, we then have (with $\mathbb{E}$ denoting the expected value)

$$\epsilon J \leq \mathbb{E}\left\{\sum_j 1_{E_j}\right\} \tag{20}$$

$$\leq \mathbb{E}\left\{\left(1_{\left\{\sum_j 1_{E_j} > \gamma_p^{1/2} J\right\}} + 1_{\left\{\sum_j 1_{E_j} \leq \gamma_p^{1/2} J\right\}}\right) \sum_j 1_{E_j}\right\} \tag{21}$$

$$\leq \gamma_p^{p/2} J + \gamma_p^{1/2} J, \tag{22}$$

a contradiction for γ_p sufficiently small (e.g., $\gamma_p < (\frac{\epsilon}{2})^{2/p}$). □

The third fact we will need is one of the basic properties of A_∞ weights (where $A_\infty := \cup_{p>1} A_p$ denotes the union of all the A_p classes); it can be found, for example, in [10].

Lemma 3.5. *For any w in product A_∞, there exists an $\epsilon > 0$ such that for any dyadic rectangle R and subset $E \subset R$, if $\frac{|E|}{|R|} \geq \frac{1}{4}$, then $\frac{w(E)}{w(R)} \geq \epsilon$.*

Given the above three lemmas, the second proof of theorem 3.1 is as follows.

Proof. By Theorem 2.4,

$$||\mathcal{D}_N||_{L^p(w)} \geq \frac{1}{C_w} ||S(\mathcal{D}_N)||_{L^p(w)}. \tag{23}$$

Now, by Lemma 3.3,

$$S(\mathcal{D}_N)(\vec{x}) = \left[\sum_{R \in \mathcal{R}_d} \frac{|\langle \mathcal{D}_N, h_R \rangle|^2}{|R|^2} 1_R(\vec{x})\right]^{1/2} \tag{24}$$

$$\geq \left[\sum_{\vec{r} \in \mathbb{H}_n} \sum_{R \in \mathbb{R}_{\vec{r}}, R \text{ “good”}} \frac{|\langle \mathcal{D}_N, h_R \rangle|^2}{|R|^2} 1_R(\vec{x})\right]^{1/2} \tag{25}$$

$$\geq 4^{-d} N 2^{-n} \left[\sum_{\vec{r} \in \mathbb{H}_n} \sum_{R \in \mathbb{R}_{\vec{r}}, R \text{ “good”}} 1_R(\vec{x})\right]^{1/2} \tag{26}$$

$$\geq 4^{-d-1} \left[\sum_{\vec{r} \in \mathbb{H}_n} \sum_{R \in \mathbb{R}_{\vec{r}}, R \text{ “good”}} 1_R(\vec{x})\right]^{1/2}. \tag{27}$$

Taking norms of the above yields (letting $s = p/2$ and $E_{\vec{r}} := \cup_{R \in R_{\vec{r}}, R \text{ "good"}} R$)

$$||S(\mathcal{D}_N)||_{L^p(w)} \geq 4^{-d-1} \left\| \left[\sum_{\vec{r} \in \mathbb{H}_n} \sum_{R \in \mathbb{R}_{\vec{r}}, R \text{ "good"}} 1_R(\vec{x}) \right]^{1/2} \right\|_{L^p(w)} \tag{28}$$

$$\geq 4^{-d-1} \left\| \sum_{\vec{r} \in \mathbb{H}_n} 1_{E_{\vec{r}}}(\vec{x}) \right\|_{L^s(w)}^{1/2}. \tag{29}$$

Since there are at least N good rectangles, $\frac{|E_{\vec{r}}|}{|[0,1]^d|} \geq N 2^{-n} > \frac{1}{4}$; thus by Lemma 3.5, $\frac{w(E_{\vec{r}})}{w([0,1]^d)} > \epsilon$. In other words, the $E_{\vec{r}}$, with $\mu(E) := \frac{w(E)}{w([0,1]^d)}$, satisfy the conditions of Lemma 3.4; so we have

$$\geq 4^{-d-1} w\big([0,1]^d\big)^{1/p} \gamma_{p/2}^{1/2} n^{\frac{d-1}{2}}. \tag{30}$$

$n \approx \log N$; so we are done. □

4. Remarks

It would be of great interest to understand how the constants in the bounds above depend on the A_p characteristic of the weight; that is, to compute

$$\delta(p, W, N) \ := \sup_{A_p(w) = W} \inf_{\mathcal{P}_N} ||\mathcal{D}_N||_{L^p(w)}, \tag{31}$$

where the infimum is taken over all distributions of N points in $[0,1]^d$. Understanding that dependence (including, in particular, the determination of a "critical p" as suggested by extrapolation techniques (see [8])) could shed light on the delicate endpoint bounds for Roth's theorem as p tends towards infinity.

One obstacle to the above is the following. In the case of the scalar-valued single-parameter dyadic square function, it is known (see [19, 20]) that

$$||f||_{L^p(w)} \leq \alpha_{d,p} A_p(w)^{1/2} ||Sf||_{L^p(w)} \tag{32}$$

and ([13]),

$$||Sf||_{L^p(w)} \leq \beta_{d,p} A_p(w)^{\max\{1,p/2\}\frac{1}{p-1}} ||f||_{L^p(w)}, \tag{33}$$

the latter recently shown (in [6], using [14]) to be sharp. However, the dependence on the A_p characteristic in the crucial (for the multiparameter square function) *vector*-valued case is unknown. Note that if the bounds of the multiparameter square function in $\mathbb{R}^d$ are, as one might expect, the dth powers of the above bounds, then the first proof above would yield

$$||\mathcal{D}_N||_{L^p(w)} \gtrsim \frac{c_d\, w([0,1]^d)^{1/p}}{\alpha_{d,p}^d A_p(w)^{d/2(p-1)+1/p}} (\log N)^{(d-1)/2}, \tag{34}$$

and the second,

$$||\mathcal{D}_N||_{L^p(w)} \gtrsim \frac{\gamma_{p/2}^{1/2} w([0,1]^d)^{1/p}}{\beta_{d,p}^d 4^{d+1} A_p(w)^{\max\{1,p/2\}\frac{d}{p-1}}} (\log N)^{(d-1)/2}. \tag{35}$$

Another topic of interest is the construction of "low-discrepancy point distributions" for general product A_p weights, that is to answer the question: "Are there distributions for which the lower bounds above are attained?" This question is open in all dimensions $d \geq 2$; the calculations in [4] are relevant to the two-dimensional case.

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