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THE b-ADIC DIAPHONY OF DIGITAL SEQUENCES

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ABSTRACT. The b-adic diaphony is a quantitative measure for the irregularity of distribution of a sequence in the unit cube. In this article we give a formula for the b-adic diaphony of digital $(0, s)$-sequences over $\mathbb{Z}_b$, $s = 1, \dots, b$. This formula shows that for a fixed $s \in \{1, \dots, b\}$, the b-adic diaphony has the same values for any digital $(0, s)$-sequence over $\mathbb{Z}_b$. For $t > 0$ we show upper bounds on the b-adic diaphony of digital (t, s)-sequences over $\mathbb{Z}_b$. We also consider the asymptotic behavior of the b-adic diaphony of these digital sequences.

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1. Introduction

The b-adic diaphony is a quantitative measure for the irregularity of distribution of a sequence in the s-dimensional unit cube. This notion was introduced by Hellekalek and Leeb [11] for $b = 2$ and later generalized by Grozdanov and Stoilova [10] for general integers $b \geq 2$. The main difference to the classical diaphony is that the trigonometric functions are replaced by b-adic Walsh functions. Before we give the exact definition of the b-adic diaphony we recall the definition of Walsh functions.

Let $b \geq 2$ be an integer. For a nonnegative integer k with base b representation $k = \kappa_{a-1} b^{a-1} + \dots + \kappa_1 b + \kappa_0$, with $\kappa_i \in \{0, \dots, b-1\}$ and $\kappa_{a-1} \neq 0$, we define the Walsh function ${}_b\mathrm{wal}_k : [0, 1) \to \mathbb{C}$ by

$$ {}_b\mathrm{wal}_k(x) := \mathrm{e}^{2\pi i (x_1 \kappa_0 + \dots + x_a \kappa_{a-1})/b}, $$

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for $x \in [0,1)$ with base b representation $x = \frac{x_1}{b} + \frac{x_2}{b^2} + \cdots$ (unique in the sense that infinitely many of the x_i must be different from $b-1$).

For dimensions $s \geq 2$, $x_1, \ldots, x_s \in [0,1)$ and $k_1, \ldots, k_s \in \mathbb{N}_0$ we define ${}_b\mathrm{wal}_{k_1,\ldots,k_s} : [0,1)^s \to \mathbb{C}$ by

$$ {}_b\mathrm{wal}_{k_1,\ldots,k_s}(x_1, \ldots, x_s) := \prod_{j=1}^{s} {}_b\mathrm{wal}_{k_j}(x_j). $$

For vectors $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ and $\mathbf{x} = (x_1, \ldots, x_s) \in [0,1)^s$ we write

$$ {}_b\mathrm{wal}_{\mathbf{k}}(\mathbf{x}) := {}_b\mathrm{wal}_{k_1,\ldots,k_s}(x_1, \ldots, x_s). $$

Now we give the definition of the b-adic diaphony (see [10] or [11]).

Definition 1. Let $b \geq 2$ be an integer. The b-adic diaphony of the first N elements of a sequence $\omega = (\mathbf{x}_n)_{n \geq 0}$ in $[0,1)^s$ is defined by

$$ F_{b,N}(\omega) := \left(\frac{1}{(1+b)^s - 1} \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^s \\ \mathbf{k} \neq \mathbf{0}}} r_b(\mathbf{k}) \left| \frac{1}{N} \sum_{n=0}^{N-1} {}_b\mathrm{wal}_{\mathbf{k}}(\mathbf{x}_n) \right|^2 \right)^{1/2}, $$

where for $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$, $r_b(\mathbf{k}) := \prod_{j=1}^{s} r_b(k_j)$ and for $k \in \mathbb{Z}$,

$$ r_b(k) := \begin{cases} 1 & \text{if } k = 0, \\ b^{-2a} & \text{if } b^a \leq k < b^{a+1}, \quad \text{where } a \in \mathbb{N}_0. \end{cases} $$

Throughout this article we will write $a(k) = a$, if a is the unique determined integer such that $b^a \leq k < b^{a+1}$. If $b = 2$ we also speak of dyadic diaphony.

The b-adic diaphony is a quantitative measure for the irregularity of distribution of a sequence: a sequence ω in the s-dimensional unit cube is uniformly distributed modulo one if and only if $\lim_{N \to \infty} F_{b,N}(\omega) = 0$. This was shown in [11] for the case $b = 2$ and in [10] for the general case. Further it is shown in [3] that the b-adic diaphony is – up to a factor depending on b and s – the worst case error for quasi-Monte Carlo integration of functions from a certain Hilbert space $H_{\mathrm{wal},s,\gamma}$, which has been introduced in [5].

Throughout this paper let b be a prime and let $\mathbb{Z}_b$ be the finite field of prime order b. We consider the b-adic diaphony of a special class of sequences in $[0,1)^s$, namely of so-called digital (t,s)-sequences over $\mathbb{Z}_b$. Hereby s is the dimension of the sequence and $t \geq 0$ is a quality parameter; lower values of t imply stronger equidistribution properties. Digital (t,s)-sequences were introduced by Niederreiter [14, 15] in a more general setting and they provide at the moment the most efficient method to generate sequences with excellent distribution properties.

We remark that a digital $(0, s)$-sequence over $\mathbb{Z}_b$ only exists if $s \in \{1, \ldots, b\}$. For higher dimensions $s \geq b+1$ the concept of digital (t, s)-sequences over $\mathbb{Z}_b$ with $t > 0$ has to be stressed (see [14], [15] or [4]).

Before we give the definition of digital (t, s)-sequences over $\mathbb{Z}_b$ we introduce some notation: for a vector $\mathbf{c} = (c_1, c_2, \ldots) \in \mathbb{Z}_b^\infty$ and for $m \in \mathbb{N}$ we denote the vector in $\mathbb{Z}_b^m$ consisting of the first m components of $\mathbf{c}$ by $\mathbf{c}(m)$, i.e., $\mathbf{c}(m) = (c_1, \ldots, c_m)$. Further for an $\mathbb{N} \times \mathbb{N}$ matrix C over $\mathbb{Z}_b$ and for $m \in \mathbb{N}$ we denote by $C(m)$ the left upper $m \times m$ sub-matrix of C.

Definition 2. Let b be a prime. For $s \in \mathbb{N}$ and $t \in \mathbb{N}_0$, choose s $\mathbb{N} \times \mathbb{N}$ matrices $C_1, \ldots, C_s$ over $\mathbb{Z}_b$ with the following property: for every $m \in \mathbb{N}, m \geq t$ and every $d_1, \ldots, d_s \in \mathbb{N}_0$ with $d_1 + \cdots + d_s = m - t$ the vectors

$$\mathbf{c}_1^{(1)}(m), \ldots, \mathbf{c}_{d_1}^{(1)}(m), \ldots, \mathbf{c}_1^{(s)}(m), \ldots, \mathbf{c}_{d_s}^{(s)}(m)$$

are linearly independent in $\mathbb{Z}_b^m$. Here $\mathbf{c}_i^{(j)}$ is the ith row vector of the matrix C_j.

For $n \geq 0$ let $n = n_0 + n_1 b + n_2 b^2 + \cdots$ be the base b representation of n. For $j \in \{1, \ldots, s\}$ multiply the vector $\mathbf{n} = (n_0, n_1, \ldots)^\top$ by the matrix C_j,

$$C_j \cdot \mathbf{n} =: \left(x_n^j(1), x_n^j(2), \ldots\right)^\top \in \mathbb{Z}_b^\infty,$$

and set

$$x_n^{(j)} := \frac{x_n^j(1)}{b} + \frac{x_n^j(2)}{b^2} + \cdots$$

Finally, set

$$\mathbf{x}_n := \left(x_n^{(1)}, \ldots, x_n^{(s)}\right).$$

Every sequence $(\mathbf{x}_n)_{n \geq 0}$ constructed this way is called digital (t, s)-sequence over $\mathbb{Z}_b$. The matrices $C_1, \ldots, C_s$ are called the generator matrices of the sequence.

Remark 3. Let $(\mathbf{x}_n)_{n \geq 0}$ be a digital (t, s)-sequence over $\mathbb{Z}_b$. For $\emptyset \neq \mathfrak{u} \subseteq \{1, \ldots, s\}$ we define $\mathbf{x}_n^{(\mathfrak{u})}$ as the projection of $\mathbf{x}_n$ onto the coordinates in $\mathfrak{u}$. The sequence $(\mathbf{x}_n^{(\mathfrak{u})})_{n \geq 0}$ is now a digital $(t, |\mathfrak{u}|)$-sequence over $\mathbb{Z}_b$.

To guarantee that the points $\mathbf{x}_n$ belong to $[0, 1)^s$ (and not just to $[0, 1]^s$) and also for the analysis of the sequence we need the condition that for each $n \geq 0$ and $1 \leq j \leq s$, we have $x_n^j(i) = 0$ for infinitely many i. This condition is always satisfied if we assume that for each $1 \leq j \leq s$ and $r \geq 0$ we have $c_{i,r}^{(j)} = 0$ for all sufficiently large i, where $c_{i,r}^{(j)}$ are the entries of the matrix C_j. Throughout this article we assume that the generator matrices fulfil this condition (see [15, p. 72] where this condition is called (S6)). More information about (t, s)-sequences can be found in the books [15] and [4].

The classical diaphony F_N (see [17] or [7, Definition 1.29] or [13, Exercise 5.27, p. 162]) of the first N elements of a sequence $\omega = (\mathbf{x}_n)_{n\geq 0}$ in $[0,1)^s$ is given by

$$F_N(\omega) = \left(\sum_{\substack{\mathbf{k}\in\mathbb{Z}^s \\ \mathbf{k}\neq\mathbf{0}}} \frac{1}{\rho(\mathbf{k})^2} \left| \frac{1}{N} \sum_{n=0}^{N-1} \mathrm{e}^{2\pi i \langle \mathbf{k}, \mathbf{x}_n \rangle} \right|^2 \right)^{1/2},$$

where for $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{Z}^s$ it is $\rho(\mathbf{k}) = \prod_{i=1}^{s} \max(1, |k_i|)$ and $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^s$. It is well known that the diaphony is a quantitative measure for the irregularity of distribution of the first N points of a sequence: A sequence ω is uniformly distributed modulo one if and only if $\lim_{N\to\infty} F_N(\omega) = 0$.

For the classical diaphony it was proved by Faure [8, Theorem 4] that

$$\big(NF_N(\omega)\big)^2 = \frac{4\pi^2}{b^2} \sum_{j=1}^{\infty} \chi_b^{\delta_{j-1}} \left(\frac{N}{b^j} \right), \tag{1}$$

where ω is a digital $(0,1)$-sequence over $\mathbb{Z}_b$ whose generator matrix C is a non-singular upper triangular matrix. Faure (and we shall do so as well) called these sequences NUT-sequences. Here $\chi_b^{\delta_{j-1}}$ are certain functions depending on the generating matrix C. For an exact definition of $\chi_b^{\delta_{j-1}}$ see [8].

The aim of this paper is to prove a similar formula for the b-adic diaphony of digital $(0,s)$-sequences over $\mathbb{Z}_b$ for $s \in \{1, \ldots, b\}$. This formula shows that for fixed s the b-adic diaphony is invariant for digital $(0,s)$-sequences over $\mathbb{Z}_b$. For digital (t,s)-sequences over $\mathbb{Z}_b$, $t > 0$, we give upper bounds for the b-adic diaphony. We will also consider the asymptotic behavior. For $b = 2$ these results were already shown by Pillichshammer [16] and Kritzer and Pillichshammer in [12].

2. Results

We will show a formula for the b-adic diaphony of digital $(0,s)$-sequences over $\mathbb{Z}_b$ that has the same structure as (1). But instead of the functions $\chi_b^{\delta_{j-1}}$ we will get a function ψ_b, which does not depend on the generating matrices. In Section 3 we will show some similarities between ψ_b and $\chi_b^{\delta_{j-1}}$.

Definition 4. Let β be an integer in $\{1, \ldots, b-1\}$. For $x \in \left[\frac{j}{b}, \frac{j+1}{b}\right)$, $j \in \{0, \ldots, b-1\}$ we set

$$\psi_b^\beta(x) := \frac{b^2(b^2-1)}{12} \left| \frac{1}{b} \frac{z_\beta^j - 1}{z_\beta - 1} + z_\beta^j \left(x - \frac{j}{b} \right) \right|^2,$$

where $z_\beta = \mathrm{e}^{\frac{2\pi i}{b}\beta} = {}_b\mathrm{wal}_1(\beta)$; then the function is extended to the reals by periodicity. The function ψ_b is now defined as the mean of the functions ψ_b^β:

$$\psi_b(x) := \frac{1}{b-1} \sum_{\beta=1}^{b-1} \psi_b^\beta(x).$$

Remark 5. For $b = 2$ we have $\psi_2(x) = \| \cdot \|^2$, where $\| \cdot \|$ denotes the distance to the nearest integer function, i.e., $\|x\| := \min(x - \lfloor x \rfloor, 1 - (x - \lfloor x \rfloor))$.

Theorem 6. *Let b be a prime.*

1. *Let ω be a digital $(0, s)$-sequence over $\mathbb{Z}_b$. Then for any $N \geq 1$ we have*

$$\left(N F_{b,N}(\omega)\right)^2 = \sum_{u=1}^{\infty} \psi_b\left(\frac{N}{b^u}\right) P_s(u),$$

where $P_s(u)$ is a polynomial of degree $s-1$ in u defined by

$$P_s(u) := \frac{1}{(b+1)^s - 1} \frac{12}{b^2(b+1)} \left(\sum_{v=1}^{s} \binom{s}{v} (b-1)^v \sum_{w=0}^{\infty} \binom{w+u+v-1}{w+u} b^{-w} \right.$$
$$\left. + \sum_{v=1}^{s} \binom{s}{v} \sum_{l=1}^{v} b^{2l} \binom{u-l+v-1}{u-l} \sum_{j=0}^{v-l} \binom{v}{j} (-1)^j b^{v-l-j} \right).$$

2. *Let ω be a digital (t, s)-sequence over $\mathbb{Z}_b$. Then for any $N \geq 1$ we have*

$$\left(N F_{b,N}(\omega)\right)^2 \leq \frac{1}{(b+1)^s - 1} \sum_{v=1}^{s} \binom{s}{v} b^{2t+3v}$$
$$\times \left(\frac{b+1}{3b} + \frac{12}{b^2(b+1)} \sum_{u=t+v+1}^{\infty} \psi_b\left(\frac{N}{b^u}\right) (u-t-1)^{v-1} \right).$$

The proof of this Theorem will be given in Section 3.

Remark 7. This theorem shows that the b-adic diaphony is invariant for all digital $(0, s)$-sequences over $\mathbb{Z}_b$.

Remark 8. $P_s(u)$ is a polynomial in u with degree $s-1$ whose coefficients depend on b and s. The leading coefficient of $P_s(u)$ equals

$$\frac{12(b^2-1)^{s-1}}{(s-1)!b((b+1)^s-1)}.$$

The first polynomials are:

$$\begin{aligned} P_1(u) &= \frac{12}{b^2}, \\ P_2(u) &= -\frac{12\,(b-2)\,(b+1)}{b^2\,(b+2)} + \frac{12\,(b-1)\,(b+1)\,u}{b^2\,(b+2)}, \\ P_3(u) &= \frac{12(b+1)^2\left(b^2-3b+3\right)}{b^2\,(b^2+3b+3)} - \frac{6\,(b-1)\,(b+1)^2\,(3b-5)\,u}{b^2\,(b^2+3b+3)} \\ &\quad + \frac{6(b-1)^2(b+1)^2u^2}{b^2\,(b^2+3b+3)}. \end{aligned}$$

Hence for a digital $(0,1)$-sequence over $\mathbb{Z}_b$ and any $N \geq 1$ we have

$$\left(NF_{b,N}(\omega)\right)^2 = \frac{12}{b^2}\sum_{u=1}^{\infty}\psi_b\left(\frac{N}{b^u}\right). \tag{2}$$

We want to point out the similar structure of (1) and (2).

We now consider the asymptotic behavior of the b-adic diaphony of digital (t,s)-sequences.

Corollary 9. *The b-adic diaphony of a digital $(0,s)$-sequence ω over $\mathbb{Z}_b$ is of order*

$$F_{b,N}(\omega) = \mathcal{O}\left(\frac{(\log N)^{s/2}}{N}\right).$$

In particular we have

$$\limsup_{N\to\infty}\frac{NF_{b,N}(\omega)}{(\log N)^{s/2}} \leq \sqrt{\frac{2\cdot 3^{s-2}}{(\log 2)^s(s-1)!(3^s-1)}} \qquad \text{if } b=2, \tag{3}$$

and

$$\limsup_{N\to\infty}\frac{NF_{b,N}(\omega)}{(\log N)^{s/2}} \leq \sqrt{\frac{(b^2-1)^s}{4(\log b)^s(s-1)!((b+1)^s-1)}} \qquad \text{if } b \text{ is odd}. \tag{4}$$

For $s = 1$ we have equality in (3) *and* (4)*, i.e.,*

$$\limsup_{N\to\infty} \frac{NF_{b,N}(\omega)}{(\log N)^{1/2}} = \sqrt{\frac{1}{3\log 2}}, \qquad \text{if } b = 2,$$

and

$$\limsup_{N\to\infty} \frac{NF_{b,N}(\omega)}{(\log N)^{1/2}} = \sqrt{\frac{(b^2-1)}{4b\log b}} \qquad \text{if } b \text{ is odd}$$

if ω is a digital $(0,1)$-sequence over $\mathbb{Z}_b$.

The proof of this Corollary will be given in Section 3.

Corollary 10. *Let ω be a digital (t,s)-sequence over $\mathbb{Z}_b$ and let $b^{m-1} < N \le b^m$ with $m > t+s$. Then we have*

$$\big(NF_{b,N}(\omega)\big)^2 \le \frac{1}{(b+1)^s - 1} \sum_{v=1}^{s} \binom{s}{v} b^{2t+3v} \times \left(\frac{b+1}{3b} + \frac{b^2-1}{3b^2}(m-t-1)^v + (b-1)(m-t-1)^{v-1}c_v\right),$$

where

$$c_v = \sum_{u=1}^{\infty} \frac{(u+1)^{v-1}}{b^{2u}} < \infty.$$

In particular,

$$F_{b,N}(\omega) = \mathcal{O}\left(\frac{b^t(\log N)^{s/2}}{N}\right)$$

and we have

$$\limsup_{N\to\infty} \frac{NF_{b,N}(\omega)}{(\log N)^{s/2}} \le \frac{1}{\sqrt{(b+1)^s - 1}} b^t \left(\frac{b^3}{\log b}\right)^{s/2}.$$

The proof of this Corollary will be given in Section 3.

3. Proofs

We will show some useful facts about ψ_b: For $b = 2$ and $b = 3$ the functions $\chi_b^{\delta_{j-1}}$ and ψ_b are the same. Hence we have for any digital $(0,1)$-NUT-sequence ω

$$F_{2,N}(\omega) = \frac{\sqrt{3}}{\pi} F_N(\omega) \quad \text{and} \quad F_{3,N}(\omega) = \frac{\sqrt{3}}{\pi} F_N(\omega), \tag{5}$$

i.e., the classical diaphony and the b-adic diaphony are for $b=2,3$ up to the constant $\frac{\sqrt{3}}{\pi}$ the same. For $b=5$ the functions $\chi_b^{\delta_{j-1}}$ and ψ_b are no longer the same and there is no relation like (5), i.e.,

$$F_{5,1}(\omega)/F_1(\omega) \neq F_{5,2}(\omega)/F_2(\omega)$$

for all digital $(0,1)$-NUT-sequences ω. This can be easily checked by calculation of $\chi_b^{\delta_{j-1}}$ and ψ_b. Nonetheless, the functions still have a similar structure. The next lemma shows some properties of the function ψ_b. Compare this with properties of $\chi_b^{\delta_{j-1}}$ in [1, Propriété 3.3, Propriété 3.5 (ii)].

Lemma 11 (Properties of ψ_b). *We have:*

1. $\psi_b(x) = \frac{b^2(b^2-1)}{12}x^2$ *for* $x \in \left[0, \frac{1}{b}\right)$,
2. ψ_b *is continuous,*
3. *on intervals of the form* $\left[\frac{k}{b}, \frac{k+1}{b}\right)$ *the function* ψ_b *is a translation of the parabola* $\frac{b^2(b^2-1)}{12}x^2$.

Proof. These properties follow easily from the definition of ψ_b. □

Lemma 12. *The function ψ_b is bounded. In particular for all $x \in \mathbb{R}$ we have*

$$\psi_b(x) \leq \frac{(b+1)(b^2-1)}{36}.$$

Proof. Since ψ_b^β is on $\left[\frac{j}{b}, \frac{j+1}{b}\right]$, $j \in \{0,\dots,b-1\}$, a translation of the parabola $\frac{b^2(b^2-1)}{12}x^2$, it attains its maximum in $\frac{j}{b}$, $j \in \{0,\dots,b-1\}$. So there exists a $j \in \{0,\dots,b-1\}$ such that

$$\begin{aligned}
\psi_b^\beta(x) \leq \psi_b^\beta\left(\frac{j}{b}\right) &= \frac{b^2(b^2-1)}{12}\left|\frac{1}{b}\frac{\mathrm{e}^{\frac{2\pi i}{b}\beta j}-1}{\mathrm{e}^{\frac{2\pi i}{b}\beta}-1}\right|^2 \\
&\leq \frac{b^2-1}{12}\frac{2^2}{\left|\mathrm{e}^{\frac{\pi i}{b}\beta}\right|^2\left|2i\sin\left(\frac{\pi\beta}{b}\right)\right|^2} \\
&= \frac{b^2-1}{12}\frac{1}{\sin^2\left(\frac{\pi\beta}{b}\right)}.
\end{aligned}$$

So we get

$$\psi_b(x) = \frac{1}{b-1}\sum_{\beta=1}^{b-1}\psi_b^\beta(x) \leq \frac{b+1}{12}\sum_{\beta=1}^{b-1}\frac{1}{\sin^2\left(\frac{\pi\beta}{b}\right)} = \frac{(b+1)(b^2-1)}{36},$$

where we used the fact that

$$\sum_{\beta=1}^{b-1} \frac{1}{\sin^2\left(\frac{\pi\beta}{b}\right)} = \frac{b^2-1}{3},$$

see [5, Appendix C] or [4, Corollary A.23]. □

Lemma 13. 1. *$\psi_b\left(\frac{k}{b}\right) = \frac{(b+1)k(b-k)}{12}$ for $k \in \{0,\dots,b-1\}$,*

2. *ψ_b is symmetric, i.e., $\psi_b(x) = \psi_b(1-x)$ for all $x \in [0,1]$,*
3. *$\psi_b\left(\frac{1}{2}\right) = \frac{b(b^2-1)}{48}$ if b is odd.*

Proof. 1. Let $k \in \{0,\dots,b-1\}$. Then we have

$$\begin{aligned}
\psi_b\left(\frac{k}{b}\right) &= \frac{1}{b-1}\sum_{\beta=1}^{b-1}\frac{b^2(b^2-1)}{12}\left|\frac{1}{b}\frac{\mathrm{e}^{\frac{2\pi i}{b}\beta k}-1}{\mathrm{e}^{\frac{2\pi i}{b}\beta}-1}\right|^2 \\
&= \frac{b+1}{24}\sum_{\beta=1}^{b-1}\frac{1-\cos(2\pi\beta k/b)}{\sin^2(\pi\beta/b)} \\
&= \frac{b+1}{24}\sum_{\beta=1}^{b-1}\frac{1}{\sin^2(\pi\beta/b)} + \frac{b+1}{24}\operatorname{Re}\left(\sum_{\beta=1}^{b-1}\frac{\mathrm{e}^{\frac{2\pi i}{b}\beta k}}{\sin^2(\pi\beta/b)}\right) \\
&= \frac{(b+1)}{24}\frac{(b^2-1)}{3} - \frac{b+1}{24}\operatorname{Re}\left(\frac{b^2-1}{3}+2k(k-b)\right) \\
&= \frac{(b+1)k(b-k)}{12},
\end{aligned}$$

where we used the fact that

$$\sum_{\beta=1}^{b-1}\frac{1}{\sin^2\left(\frac{\pi\beta}{b}\right)} = \frac{b^2-1}{3} \quad\text{and}\quad \sum_{\beta=1}^{b-1}\frac{\mathrm{e}^{\frac{2\pi i}{b}\beta k}}{\sin^2\left(\frac{\beta\pi}{b}\right)} = \frac{b^2-1}{3} + 2|k|(|k|-b)$$

for $k \in \{-b,\dots,b\}$, see [5, Appendix C] or [4, Corollary A.23].

2. Since $\psi_b\left(\frac{k}{b}\right) = \psi_b\left(\frac{b-k}{b}\right)$ for $k \in \{0,\dots,b-1\}$ and ψ_b consists of translations of the same parabola (see Lemma 11), ψ_b must be symmetric.
3. Let $b = 2p+1$. Then we have $\frac{1}{2} \in \left[\frac{p}{b}, \frac{p+1}{b}\right)$. On this intervall ψ_b is of the form

$$\psi_b(x) = \frac{b^2(b^2-1)}{12}x^2 + ax + c.$$

We determine now the constants a and c. Since

$$\psi_b\left(\frac{p}{b}\right) = \psi_b\left(\frac{p+1}{b}\right)$$

we have

$$\psi_b'\left(\frac{1}{2}\right) = 0.$$

From this we obtain

$$a = -\frac{b^2(b^2-1)}{12}.$$

From

$$\psi_b\left(\frac{p}{b}\right) = \frac{(b+1)(b^2-1)}{48}$$

we obtain

$$c = \frac{b(b+1)(b^2-1)}{48}.$$

Now we can compute

$$\psi_b\left(\frac{1}{2}\right) = \frac{b^2(b^2-1)}{48} - \frac{b^2(b^2-1)}{24} + \frac{b(b+1)(b^2-1)}{48} = \frac{b(b^2-1)}{48}.$$

□

Lemma 14. *Let the nonnegative integer U have b-adic expansion*

$$U = U_0 + U_1 b + \cdots + U_{m-1} b^{m-1}.$$

For any nonnegative integer $n \le U-1$ let

$$n = n_0 + n_1 b + \cdots + n_{m-1} b^{m-1}$$

be the b-adic representation of n. For $0 \le p \le m-1$ let

$$U(p) := U_0 + \cdots + U_p b^p.$$

Let

$$\beta_0, \beta_1, \ldots, \beta_{m-1}$$

be arbitrary elements of $\mathbb{Z}_b$, not all zero. Then

$$\sum_{n=0}^{U-1} \mathrm{e}^{\frac{2\pi i}{b}(\beta_0 n_0 + \cdots + \beta_{m-1} n_{m-1})} = \mathrm{e}^{\frac{2\pi i}{b}(\beta_{w+1} U_{w+1} + \cdots + \beta_{m-1} U_{m-1})} b^{w+1} \theta,$$

where

$$\theta := \frac{1}{b}\frac{z^j - 1}{z - 1} + z^j\left(\frac{U(w)}{b^{w+1}} - \frac{j}{b}\right)$$

if

$$\frac{U(w)}{b^{w+1}} \in \left[\frac{j}{b}, \frac{j+1}{b}\right), \quad j = 0, \ldots, b-1, \quad z := \mathrm{e}^{\frac{2\pi i}{b}\beta_w}$$

and w is minimal such that $\beta_w \neq 0$.

Proof. From splitting the sum we obtain

$$\begin{aligned}
&\sum_{n=0}^{U-1} \mathrm{e}^{\frac{2\pi i}{b}(\beta_0 n_0+\cdots+\beta_{m-1}n_{m-1})}\\
&= \sum_{n=0}^{b^{w+1}(U_{w+1}+\cdots+U_{m-1}b^{m-w-2})-1} \mathrm{e}^{\frac{2\pi i}{b}(\beta_0 n_0+\cdots+\beta_{m-1}n_{m-1})}\\
&\quad+ \sum_{n=b^{w+1}(U_{w+1}+\cdots+U_{m-1}b^{m-w-2})}^{U-1} \mathrm{e}^{\frac{2\pi i}{b}(\beta_0 n_0+\cdots+\beta_{m-1}n_{m-1})}\\
&= 0 + \sum_{n=0}^{U(w)-1} \mathrm{e}^{\frac{2\pi i}{b}(\beta_w n_w+\beta_{w+1}U_{w+1}+\cdots+\beta_{m-1}U_{m-1})}\\
&= \mathrm{e}^{\frac{2\pi i}{b}(\beta_{w+1}U_{w+1}+\cdots+\beta_{m-1}U_{m-1})} \sum_{n=0}^{U(w)-1} \mathrm{e}^{\frac{2\pi i}{b}\beta_w n_w}.
\end{aligned}$$

We consider now the last sum. If $U_w = j$ for $j \in \{0, 1, \ldots, b-1\}$, then we have:

$$\begin{aligned}
\sum_{n=0}^{U(w)-1} \mathrm{e}^{\frac{2\pi i}{b}\beta_w n_w} &= \sum_{n=0}^{b^w-1} \mathrm{e}^{\frac{2\pi i}{b}\beta_w 0} + \sum_{n=b^w}^{2b^w-1} \mathrm{e}^{\frac{2\pi i}{b}\beta_w} + \cdots\\
&\quad \cdots + \sum_{n=(j-1)b^w}^{jb^w-1} \mathrm{e}^{\frac{2\pi i}{b}\beta_w (j-1)} + \sum_{n=jb^w}^{U(w)-1} \mathrm{e}^{\frac{2\pi i}{b}\beta_w j}\\
&= b^w + b^w \mathrm{e}^{\frac{2\pi i}{b}\beta_w} + \cdots + b^w \mathrm{e}^{\frac{2\pi i}{b}\beta_w (j-1)} + (U(w) - jb^w)\mathrm{e}^{\frac{2\pi i}{b}\beta_w j}\\
&= b^{w+1}\left(\frac{1}{b} + \frac{z}{b} + \cdots + \frac{z^{j-1}}{b} - j\frac{z^j}{b} + z^j \frac{U(w)}{b^{w+1}}\right)\\
&= b^{w+1}\left(\frac{1}{b}\frac{z^j-1}{z-1} + z^j\left(\frac{U(w)}{b^{w+1}} - \frac{j}{b}\right)\right).
\end{aligned}$$

Since $U_w = j$ means $U(w) \in [jb^w, (j+1)b^w)$ and $\frac{U(w)}{b^{w+1}} \in \left[\frac{j}{b}, \frac{j+1}{b}\right)$ the result follows. □

Proof of Theorem 6. Let ω be a digital (t,s)-sequence over $\mathbb{Z}_b$, $t \geq 0$. For a point $\mathbf{x}_n$ of ω and for $\emptyset \neq \mathfrak{u} \subseteq \{1, \ldots, s\}$, we define $\mathbf{x}_n^{(\mathfrak{u})}$ as the projection of $\mathbf{x}_n$ onto the coordinates in $\mathfrak{u}$. We have

$$\begin{aligned}
&\left(NF_{b,N}(\omega)\right)^2\\
&= \frac{1}{(b+1)^s-1}\sum_{\substack{\mathbf{k}\in\mathbb{N}_0^s\\ \mathbf{k}\neq\mathbf{0}}} r_b(\mathbf{k})\left|\sum_{n=0}^{N-1} {}_b\mathrm{wal}_{\mathbf{k}}(\mathbf{x}_n)\right|^2\\
&= \frac{1}{(b+1)^s-1}\sum_{\emptyset\neq\mathfrak{u}\subseteq\{1,\dots,s\}}\sum_{\mathbf{k}\in\mathbb{N}^{|\mathfrak{u}|}} r_b(\mathbf{k})\left|\sum_{n=0}^{N-1} {}_b\mathrm{wal}_{\mathbf{k}}\left(\mathbf{x}_n^{(\mathfrak{u})}\right)\right|^2\\
&= \frac{1}{(b+1)^s-1}\sum_{\substack{\emptyset\neq\mathfrak{u}\subseteq\{1,\dots,s\}\\ \mathfrak{u}=\{w_1,\dots,w_{|\mathfrak{u}|}\}}}\sum_{k_{w_1}=1}^{\infty}\cdots\\
&\qquad\cdots\sum_{k_{w_{|\mathfrak{u}|}}=1}^{\infty}\left(\prod_{j\in\mathfrak{u}}\frac{1}{b^{2a(k_j)}}\right)\left|\sum_{n=0}^{N-1} {}_b\mathrm{wal}_{(k_{w_1},\dots,k_{w_{|\mathfrak{u}|}})}\left(\mathbf{x}_n^{(\mathfrak{u})}\right)\right|^2.
\end{aligned}$$

Let now $\emptyset\neq\mathfrak{u}=\{w_1,\dots,w_{|\mathfrak{u}|}\}\subseteq\{1,\dots,s\}$ be fixed. We have to study

$$\Sigma(\mathfrak{u}) := \sum_{k_{w_1}=1}^{\infty}\cdots\sum_{k_{w_{|\mathfrak{u}|}}=1}^{\infty}\left(\prod_{j\in\mathfrak{u}}\frac{1}{b^{2a(k_j)}}\right)\left|\sum_{n=0}^{N-1} {}_b\mathrm{wal}_{(k_{w_1},\dots,k_{w_{|\mathfrak{u}|}})}\left(\mathbf{x}_n^{(\mathfrak{u})}\right)\right|^2.$$

For the sake of simplicity we assume in the following $\mathfrak{u}=\{1,\dots,\sigma\}$, $1\le\sigma\le s$. The other cases are dealt with a similar fashion. We have

$$\Sigma(\{1,\dots,\sigma\}) = \sum_{k_1=1}^{\infty}\cdots\sum_{k_\sigma=1}^{\infty}\left(\prod_{j=1}^{\sigma}\frac{1}{b^{2a(k_j)}}\right)\left|\sum_{n=0}^{N-1} {}_b\mathrm{wal}_{\mathbf{k}_\sigma}\left(\mathbf{x}_n^{(\{1,\dots,\sigma\})}\right)\right|^2,$$

where $\mathbf{k}_\sigma := (k_1,\dots,k_\sigma)$.
For $1\le j\le\sigma$, let $b^{a_j}\le k_j<b^{a_j+1}$, then $k_j=\kappa_0^{(j)}+\kappa_1^{(j)}b+\dots+\kappa_{a_j}^{(j)}b^{a_j}$ with $\kappa_v^{(j)}\in\{0,1,\dots,b-1\}, 0\le v<a_j$, and $\kappa_{a_j}^{(j)}\in\{1,\dots,b-1\}$.
Let $\mathbf{c}_i^{(j)}$ be the i-th row vector of the generator matrix $C_j, 1\le j\le\sigma$. Since the ith digit $x_n^{(j)}(i)$ of $x_n^{(j)}$ is given by $\left\langle\mathbf{c}_i^{(j)},\mathbf{n}\right\rangle$, we have

$$\begin{aligned}
\sum_{n=0}^{N-1} {}_b\mathrm{wal}_{\mathbf{k}_\sigma}\left(\mathbf{x}_n^{(\{1,\dots,\sigma\})}\right) &= \sum_{n=0}^{N-1}\mathrm{e}^{\frac{2\pi i}{b}\sum_{j=1}^{\sigma}\left(\kappa_0^{(j)}\left\langle\mathbf{c}_1^{(j)},\mathbf{n}\right\rangle+\dots+\kappa_{a_j}^{(j)}\left\langle\mathbf{c}_{a_j+1}^{(j)},\mathbf{n}\right\rangle\right)}\\
&= \sum_{n=0}^{N-1}\mathrm{e}^{\frac{2\pi i}{b}\left\langle\sum_{j=1}^{\sigma}\left(\kappa_0^{(j)}\mathbf{c}_1^{(j)}+\dots+\kappa_{a_j}^{(j)}\mathbf{c}_{a_j+1}^{(j)}\right),\mathbf{n}\right\rangle}.
\end{aligned}$$

Let

$$C_j = \left(c_{v,w}^{(j)}\right)_{v,w\geq 1}.$$

Define

$$u(\mathbf{k}_\sigma) := \min\left\{ l \geq 1 : \sum_{j=1}^{\sigma} \left(\kappa_0^{(j)} c_{1,l}^{(j)} + \cdots + \kappa_{a_j}^{(j)} c_{a_j+1,l}^{(j)}\right) \neq 0 \right\}$$

and

$$\boldsymbol{\beta}_{\mathbf{k}_\sigma} = (\beta_{\mathbf{k}_\sigma,0}, \beta_{\mathbf{k}_\sigma,1}, \ldots)^\top := \sum_{j=1}^{\sigma} \left(\kappa_0^{(j)} \mathbf{c}_1^{(j)} + \cdots + \kappa_{a_j}^{(j)} \mathbf{c}_{a_j+1}^{(j)}\right).$$

Since $C_1, \ldots, C_s$ generate a digital (t,s)-sequence over $\mathbb{Z}_b$, it is easy to verify $u(\mathbf{k}_\sigma) \leq \sum_{j=1}^{\sigma} a_j + \sigma + t =: R_\sigma + \sigma + t$. Indeed, let

$$\mathcal{C}(a_1, \ldots, a_\sigma) := \begin{pmatrix} c_{1,1}^{(1)} & \cdots & c_{a_1+1,1}^{(1)} & \cdots & c_{1,1}^{(\sigma)} & \cdots & c_{a_\sigma+1,1}^{(\sigma)} \\ c_{1,2}^{(1)} & \cdots & c_{a_1+1,2}^{(1)} & \cdots & c_{1,2}^{(\sigma)} & \cdots & c_{a_\sigma+1,2}^{(\sigma)} \\ \vdots & & \vdots & & \vdots & & \vdots \\ c_{1,R_\sigma+\sigma+t}^{(1)} & \cdots & c_{a_1+1,R_\sigma+\sigma+t}^{(1)} & \cdots & c_{1,R_\sigma+\sigma+t}^{(\sigma)} & \cdots & c_{a_\sigma+1,R_\sigma+\sigma+t}^{(\sigma)} \end{pmatrix}.$$

Note that $\mathcal{C} = \mathcal{C}(a_1, \ldots, a_\sigma)$ is an $(R_\sigma+\sigma+t)\times(R_\sigma+\sigma)$ matrix. Since $C_1, \ldots, C_s$ generate a digital (t,s)-sequence over $\mathbb{Z}_b$, it follows that $\mathcal{C}(a_1, \ldots, a_\sigma)$ has rank $R_\sigma + \sigma$. If however, $u(\mathbf{k}_\sigma) > R_\sigma + \sigma + t$, we would have

$$\begin{aligned} &\mathcal{C} \cdot \left(\kappa_0^{(1)}, \ldots, \kappa_{a_1-1}^{(1)}, \kappa_{a_1}^{(1)}, \ldots, \kappa_0^{(\sigma)}, \ldots, \kappa_{a_\sigma-1}^{(\sigma)}, \kappa_{a_\sigma}^{(\sigma)}\right)^\top \\ &\quad = (0, \ldots, 0)^\top, \end{aligned} \tag{6}$$

which would lead to a contradiction since the matrix $\mathcal{C}(a_1, \ldots, a_\sigma)$ has full rank and is multiplied by a nonzero vector in (6).

Let $N = N_0 + N_1 b + \cdots + N_{m-1} b^{m-1}$. If $u(\mathbf{k}_\sigma) \leq m$ we obtain from Lemma 14

$$\begin{aligned} \left| \sum_{n=0}^{N-1} {}_b\mathrm{wal}_{\mathbf{k}_\sigma}\left(\mathbf{x}_n^{(\{1,\ldots,\sigma\})}\right) \right|^2 &= \left| \sum_{n=0}^{N-1} \mathrm{e}^{\frac{2\pi i}{b}\langle \boldsymbol{\beta}_{\mathbf{k}_\sigma}, \mathbf{n}\rangle} \right|^2 \\ &= \left| \sum_{n=0}^{N-1} \mathrm{e}^{\frac{2\pi i}{b}(n_0\beta_{\mathbf{k}_\sigma,0} + \cdots + n_{m-1}\beta_{\mathbf{k}_\sigma,m-1})} \right|^2 \\ &= b^{2u(\mathbf{k}_\sigma)} \left| \frac{1}{b}\frac{z^j - 1}{z-1} + z^j \left(\frac{N(u(\mathbf{k}_\sigma)-1)}{b^{u(\mathbf{k}_\sigma)}} - \frac{j}{b} \right) \right|^2, \end{aligned}$$

where $z := \mathrm{e}^{\frac{2\pi i}{b}\beta_{\mathbf{k}_\sigma,u(\mathbf{k}_\sigma)-1}}$ and $j \in \{0,\dots,b-1\}$ such that $\frac{N(u(\mathbf{k}_\sigma)-1)}{b^{u(\mathbf{k}_\sigma)}} \in \left[\frac{j}{b}, \frac{j+1}{b}\right)$. So we have

$$\left|\sum_{n=0}^{N-1} {}_b\mathrm{wal}_{\mathbf{k}_\sigma}\left(\mathbf{x}_n^{(\{1,\dots,\sigma\})}\right)\right|^2 = b^{2u(\mathbf{k}_\sigma)} \frac{12}{b^2(b^2-1)} \psi_b^{\beta_{\mathbf{k}_\sigma,u(\mathbf{k}_\sigma)-1}} \left(\frac{N(u(\mathbf{k}_\sigma)-1)}{b^{u(\mathbf{k}_\sigma)}}\right).$$

Since $\psi_b^{\beta_{\mathbf{k}_\sigma,u(\mathbf{k}_\sigma)-1}}$ is 1-periodic and $\left\{\frac{N}{b^{u(\mathbf{k}_\sigma)}}\right\} = \frac{N(u(\mathbf{k}_\sigma)-1)}{b^{u(\mathbf{k}_\sigma)}}$ we get

$$\left|\sum_{n=0}^{N-1} {}_b\mathrm{wal}_{\mathbf{k}_\sigma}\left(\mathbf{x}_n^{(\{1,\dots,\sigma\})}\right)\right|^2 = b^{2u(\mathbf{k}_\sigma)} \frac{12}{b^2(b^2-1)} \psi_b^{\beta_{\mathbf{k}_\sigma,u(\mathbf{k}_\sigma)-1}} \left(\frac{N}{b^{u(\mathbf{k}_\sigma)}}\right).$$

If $u(k) > m$ we have

$$\begin{aligned}\left|\sum_{n=0}^{N-1} {}_b\mathrm{wal}_{\mathbf{k}_\sigma}\left(\mathbf{x}_n^{(\{1,\dots,\sigma\})}\right)\right|^2 &= \left|\sum_{n=0}^{N-1} \mathrm{e}^{\frac{2\pi i}{b}0}\right|^2 = N^2 \\ &= b^{2u(\mathbf{k}_\sigma)} \frac{12}{b^2(b^2-1)} \frac{b^2(b^2-1)}{12} \frac{N^2}{b^{2u(\mathbf{k}_\sigma)}} \\ &= b^{2u(\mathbf{k}_\sigma)} \frac{12}{b^2(b^2-1)} \psi_b^{\beta_{\mathbf{k}_\sigma,u(\mathbf{k}_\sigma)-1}} \left(\frac{N}{b^{u(\mathbf{k}_\sigma)}}\right),\end{aligned}$$

because $\frac{N}{b^{u(\mathbf{k}_\sigma)}} \in \left[0, \frac{1}{b}\right)$. Therefore, we have

$$\begin{aligned}&\Sigma(\{1,\dots,\sigma\}) \\ &= \sum_{k_1=1}^{\infty} \cdots \sum_{k_\sigma=1}^{\infty} \left(\prod_{j=1}^{\sigma} \frac{1}{b^{2a(k_j)}}\right) b^{2u(\mathbf{k}_\sigma)} \frac{12}{b^2(b^2-1)} \psi_b^{\beta_{\mathbf{k}_\sigma,u(\mathbf{k}_\sigma)-1}} \left(\frac{N}{b^{u(\mathbf{k}_\sigma)}}\right) \\ &= \frac{12}{b^2(b^2-1)} \sum_{a_1=0}^{\infty} \cdots \sum_{a_\sigma=0}^{\infty} \left(\prod_{j=1}^{\sigma} \frac{1}{b^{2a_j}}\right) \sum_{k_1=b^{a_1}}^{b^{a_1+1}-1} \cdots \\ &\qquad \cdots \sum_{k_\sigma=b^{a_\sigma}}^{b^{a_\sigma+1}-1} b^{2u(\mathbf{k}_\sigma)} \psi_b^{\beta_{\mathbf{k}_\sigma,u(\mathbf{k}_\sigma)-1}} \left(\frac{N}{b^{u(\mathbf{k}_\sigma)}}\right) \\ &= \frac{12}{b^2(b^2-1)} \sum_{a_1=0}^{\infty} \cdots \sum_{a_\sigma=0}^{\infty} \frac{1}{b^{2R_\sigma}} \sum_{u=1}^{R_\sigma+\sigma+t} \sum_{\beta=1}^{b-1} b^{2u} \psi_b^{\beta} \left(\frac{N}{b^u}\right) \underbrace{\sum_{k_1=b^{a_1}}^{b^{a_1+1}-1} \cdots \sum_{k_\sigma=b^{a_\sigma}}^{b^{a_\sigma+1}-1}}_{\substack{u(\mathbf{k}_\sigma)=u \\ \beta_{\mathbf{k}_\sigma,u(\mathbf{k}_\sigma)-1}=\beta}} 1.\end{aligned}$$

We need to evaluate the sum

$$\underbrace{\sum_{k_1=b^{a_1}}^{b^{a_1+1}-1} \cdots \sum_{k_\sigma=b^{a_\sigma}}^{b^{a_\sigma+1}-1} 1}_{\substack{u(\mathbf{k}_\sigma)=u \\ \beta_{\mathbf{k}_\sigma,u(\mathbf{k}_\sigma)-1}=\beta}}$$

for $1 \le u \le R_\sigma + \sigma + t$ and $\beta \in \{1,\dots,b-1\}$. This is the number of digits

$$\kappa_0^{(1)},\dots,\kappa_{a_1-1}^{(1)},\theta_1,\dots,\kappa_0^{(\sigma)},\dots,\kappa_{a_\sigma-1}^{(\sigma)},\quad \theta_\sigma \in \{0,\dots,b-1\},\quad \theta_1 \ne 0,\dots,\theta_\sigma \ne 0,$$

such that

$$\mathcal{C}(a_1,\dots,a_\sigma)\begin{pmatrix} \kappa_0^{(1)} \\ \vdots \\ \kappa_{a_1-1}^{(1)} \\ \theta_1 \\ \vdots \\ \kappa_0^{(\sigma)} \\ \vdots \\ \kappa_{a_\sigma-1}^{(\sigma)} \\ \theta_\sigma \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \beta \\ x_{u+1} \\ \vdots \\ x_{R_\sigma+\sigma+t} \end{pmatrix} \tag{7}$$

for arbitrary $x_{u+1},\dots,x_{R_\sigma+\sigma+t} \in \mathbb{Z}_b$.

Now we must distinguish between digital $(0,s)$-sequences and digital (t,s)-sequences, $t > 0$. In the case of digital $(0,s)$-sequences we can give the exact number of solutions of the required form of the above system. In the case of digital (t,s)-sequences, $t > 0$, we will use an upper bound.

First we deal with digital $(0,s)$-sequences over $\mathbb{Z}_b$. Note that since $C_1,\dots,C_s$ generate a digital $(0,s)$-sequence it follows that $\mathcal{C}(a_1,\dots,a_\sigma)$ is regular. So for any choice of the $x_{u+1},\dots,x_{R_\sigma+\sigma}$ there exists one digit vector such that (7) is fulfilled. We only count those that have nonzero entries for $\theta_1,\dots,\theta_\sigma$.

We consider two cases:

(i) Assume that $u = R_\sigma + l$, $l \in \{1,\dots,\sigma\}$. Then we have $b^{\sigma-l}$ choices for $x_{u+1},\dots,x_{R_\sigma+\sigma}$. Choose j, $j \le \sigma - l$, of the $\theta_1,\dots,\theta_\sigma$ to be zero. Now we construct a certain subsystem of (7) in the following way: Delete the column

$$\left(c_{a_i+1,1}^{(i)}, c_{a_i+1,2}^{(i)},\dots,c_{a_i+1,R_\sigma+\sigma}^{(i)}\right)^\top \quad \text{from} \quad \mathcal{C}(a_1,\dots,a_\sigma) \quad \text{if} \quad \theta_i = 0$$

by choice and delete the last j lines. From the digit vector we delete those θ_i we have chosen to be zero and from the right-hand-side vector we delete the last j entries. For arbitrary

$$x_{u+1},\dots,x_{R_\sigma+\sigma-j}$$

there exists one solution of the corresponding subsystem. The remaining $x_{R_\sigma+\sigma-j+1},\dots,x_{R_\sigma+\sigma}$ are chosen properly such that the solution of the subsystem becomes a solution of the whole system. Hence there are $b^{\sigma-l-j}$ solutions with j of the $\theta_1,\dots,\theta_\sigma$ chosen to be zero. If $j>\sigma-l$, then there is no solution. Hence the number of solutions such that $\theta_1\neq 0,\dots,\theta_\sigma\neq 0$ equals (using the include-exclude-principle)

$$\begin{aligned}&|\{\mathbf{h}:\mathbf{h}\text{ solution of (7)}\}|-|\{\mathbf{h}:\mathbf{h}\text{ solution of (7)},\theta_1=0\}|\\&\quad-\cdots-|\{\mathbf{h}:\mathbf{h}\text{ solution of (7)},\theta_\sigma=0\}|\\&\quad+|\{\mathbf{h}:\mathbf{h}\text{ solution of (7)},\theta_1=\theta_2=0\}|+-\cdots\\&=\sum_{j=0}^{\sigma-l}\binom{\sigma}{j}b^{\sigma-l-j}(-1)^j=:A(l).\end{aligned}$$

Hence

$$\underbrace{\sum_{k_1=b^{a_1}}^{b^{a_1+1}-1}\cdots\sum_{k_\sigma=b^{a_\sigma}}^{b^{a_\sigma+1}-1}}_{\substack{u(\mathbf{k}_\sigma)=u\\ \beta_{\mathbf{k}_\sigma,u(\mathbf{k}_\sigma)-1}=\beta}}1=A(l).$$

(ii) Assume that $u\le R_\sigma$. We rewrite system (7) in the form

$$\begin{pmatrix}c^{(1)}_{1,1} & \dots & c^{(1)}_{a_1,1} & \dots & c^{(\sigma)}_{1,1} & \dots & c^{(\sigma)}_{a_\sigma,1}\\ \multicolumn{7}{c}{\dots\dots\dots\dots\dots\dots}\\ c^{(1)}_{1,R_\sigma} & \dots & c^{(1)}_{a_1,R_\sigma} & \dots & c^{(\sigma)}_{1,R_\sigma} & \dots & c^{(\sigma)}_{a_\sigma,R_\sigma}\\ \multicolumn{7}{c}{\dots\dots\dots\dots\dots\dots}\\ c^{(1)}_{1,R_\sigma+\sigma} & \dots & c^{(1)}_{a_1,R_\sigma+\sigma} & \dots & c^{(\sigma)}_{1,R_\sigma+\sigma} & \dots & c^{(\sigma)}_{a_\sigma,R_\sigma+\sigma}\end{pmatrix}\begin{pmatrix}\kappa^{(1)}_0\\ \vdots\\ \kappa^{(1)}_{a_1-1}\\ \kappa^{(2)}_0\\ \vdots\\ \kappa^{(\sigma)}_0\\ \vdots\\ \kappa^{(\sigma)}_{a_\sigma-1}\end{pmatrix}=$$

$$(0,\dots,0,\beta,x_{u+1},\dots,x_{R_\sigma+\sigma})^\top-\sum_{j=1}^{\sigma}\theta_j\mathbf{c}^{(j)}_{a_j+1}(R_\sigma+\sigma).$$

Since the upper $R_\sigma \times R_\sigma$ sub-matrix of the above system is regular we find for arbitrary

$$x_{u+1}, \dots, x_{R_\sigma} \quad \text{and} \quad \theta_1 \neq 0, \dots, \theta_\sigma \neq 0$$

exactly one solution of the above subsystem. This solution can be made a solution of the whole system by an adequate choice of $x_{R_\sigma+1}, \dots, x_{R_\sigma+\sigma}$. Therefore, we have

$$\underbrace{\sum_{k_1=b^{a_1}}^{b^{a_1+1}-1} \cdots \sum_{k_\sigma=b^{a_\sigma}}^{b^{a_\sigma+1}-1}}_{\substack{u(\mathbf{k}_\sigma)=u \\ \beta_{\mathbf{k}_\sigma, u(\mathbf{k}_\sigma)-1}=\beta}} 1 = (b-1)^\sigma b^{R_\sigma - u}.$$

Now we have

$$\begin{aligned}
&\Sigma(\{1,\dots,\sigma\}) \\
&= \frac{12}{b^2(b^2-1)} \sum_{a_1,\dots,a_\sigma=0}^{\infty} \frac{1}{b^{2R_\sigma}} \sum_{u=1}^{R_\sigma+\sigma} \sum_{\beta=1}^{b-1} b^{2u} \psi_b^\beta\left(\frac{N}{b^u}\right) \underbrace{\sum_{k_1=b^{a_1}}^{b^{a_1+1}-1} \cdots \sum_{k_\sigma=b^{a_\sigma}}^{b^{a_\sigma+1}-1}}_{\substack{u(\mathbf{k}_\sigma)=u \\ \beta_{\mathbf{k}_\sigma, u(\mathbf{k}_\sigma)-1}=\beta}} 1 \\
&= \frac{12}{b^2(b+1)}(b-1)^\sigma \sum_{a_1,\dots,a_\sigma=0}^{\infty} \frac{1}{b^{R_\sigma}} \sum_{u=1}^{R_\sigma} b^u \psi_b\left(\frac{N}{b^u}\right) \\
&\quad + \frac{12}{b^2(b+1)} \sum_{a_1,\dots,a_\sigma=0}^{\infty} \sum_{l=1}^{\sigma} b^{2l} \psi_b\left(\frac{N}{b^{R_\sigma+l}}\right) A(l) \\
&= \frac{12}{b^2(b+1)}(b-1)^\sigma \sum_{u=1}^{\infty} b^u \psi_b\left(\frac{N}{b^u}\right) \sum_{\substack{a_1,\dots,a_\sigma=0 \\ R_\sigma \geq u}}^{\infty} \frac{1}{b^{R_\sigma}} \\
&\quad + \frac{12}{b^2(b+1)} \sum_{u=1}^{\infty} \psi_b\left(\frac{N}{b^u}\right) \sum_{l=1}^{\sigma} b^{2l} A(l) \sum_{\substack{a_1,\dots,a_\sigma=0 \\ R_\sigma = u-l}}^{\infty} 1 \\
&= \frac{12}{b^2(b+1)}(b-1)^\sigma \sum_{u=1}^{\infty} b^u \psi_b\left(\frac{N}{b^u}\right) \sum_{w=u}^{\infty} \binom{w+\sigma-1}{w} b^{-w} \\
&\quad + \frac{12}{b^2(b+1)} \sum_{u=1}^{\infty} \psi_b\left(\frac{N}{b^u}\right) \sum_{l=1}^{\sigma} b^{2l} A(l) \binom{u-l+\sigma-1}{u-l}.
\end{aligned}$$

So we get

$$\begin{aligned}
&\left(NF_{b,N}(\omega)\right)^2\\
&= \frac{1}{(b+1)^s-1} \sum_{\substack{\emptyset\neq\mathfrak{u}\subseteq\{1,\dots,s\}\\ \mathfrak{u}=\{w_1,\dots,w_{|\mathfrak{u}|}\}}} \Sigma(w_1,\dots,w_{|\mathfrak{u}|})\\
&= \frac{1}{(b+1)^s-1} \sum_{v=1}^{s} \binom{s}{v} \Bigg(\frac{12}{b^2(b+1)}(b-1)^v \sum_{u=1}^{\infty} b^u \psi_b\left(\frac{N}{b^u}\right) \sum_{w=u}^{\infty} \binom{w+v-1}{w} b^{-w}\\
&\quad + \frac{12}{b^2(b+1)} \sum_{u=1}^{\infty} \psi_b\left(\frac{N}{b^u}\right) \sum_{l=1}^{v} b^{2l} A(l) \binom{u-l+v-1}{u-l} \Bigg)\\
&= \frac{1}{(b+1)^s-1} \frac{12}{b^2(b+1)} \sum_{u=1}^{\infty} \psi_b\left(\frac{N}{b^u}\right) \Bigg(\sum_{v=1}^{s} \binom{s}{v} (b-1)^v \sum_{w=0}^{\infty} \binom{w+u+v-1}{w+u} b^{-w}\\
&\quad + \sum_{v=1}^{s} \binom{s}{v} \sum_{l=1}^{v} b^{2l} \binom{u-l+v-1}{u-l} \sum_{j=0}^{v-l} \binom{v}{j} (-1)^j b^{v-l-j} \Bigg)\\
&= \sum_{u=1}^{\infty} \psi_b\left(\frac{N}{b^u}\right) P_s(u).
\end{aligned}$$

This finishes the proof of 1. $\square$

Now let ω be a digital (t,s)-sequence over $\mathbb{Z}_b$, $t>0$. Let us rewrite system (7) as

$$\tilde{\mathcal{C}} \cdot \begin{pmatrix} \kappa_0^{(1)} \\ \vdots \\ \kappa_{a_1-1}^{(1)} \\ \vdots \\ \kappa_0^{(\sigma)} \\ \vdots \\ \kappa_{a_\sigma-1}^{(\sigma)} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \beta \\ x_{u+1} \\ \vdots \\ x_{R_\sigma+\sigma+t} \end{pmatrix} - \theta_1 \begin{pmatrix} c_{a_1+1,1}^{(1)} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ c_{a_1+1,R_\sigma+\sigma+t}^{(1)} \end{pmatrix} - \cdots - \theta_\sigma \begin{pmatrix} c_{a_\sigma+1,1}^{(\sigma)} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ c_{a_\sigma+1,R_\sigma+\sigma+t}^{(\sigma)} \end{pmatrix}, \tag{8}$$

where

$$\tilde{\mathcal{C}} := \begin{pmatrix} c_{1,1}^{(1)} & \ldots & c_{a_1,1}^{(1)} & \ldots & c_{1,1}^{(\sigma)} & \ldots & c_{a_\sigma,1}^{(\sigma)} \\ c_{1,2}^{(1)} & \ldots & c_{a_1,2}^{(1)} & \ldots & c_{1,2}^{(\sigma)} & \ldots & c_{a_\sigma,2}^{(\sigma)} \\ \vdots & & \vdots & & \vdots & & \vdots \\ c_{1,R_\sigma+\sigma+t}^{(1)} & \ldots & c_{a_1,R_\sigma+\sigma+t}^{(1)} & \ldots & c_{1,R_\sigma+\sigma+t}^{(\sigma)} & \ldots & c_{a_\sigma,R_\sigma+\sigma+t}^{(\sigma)} \end{pmatrix}.$$

Obviously, the matrix $\tilde{\mathcal{C}}$ has rank R_σ. Let now $1 \leq u \leq R_\sigma + \sigma + t$ and $\beta \in \{1, \ldots, b-1\}$ be fixed. For a fixed choice of $x_{u+1}, \ldots, x_{R_\sigma+\sigma+t}$ and $\theta_1, \ldots, \theta_\sigma$, it is clear that we have at most one solution of system (8). Therefore we have

$$\underbrace{\sum_{k_1=b^{a_1}}^{b^{a_1+1}-1} \cdots \sum_{k_\sigma=b^{a_\sigma}}^{b^{a_\sigma+1}-1}}_{\substack{u(\mathbf{k}_\sigma)=u \\ \beta_{\mathbf{k}_\sigma, u(\mathbf{k}_\sigma)-1}=\beta}} 1 \leq (b-1)^\sigma b^{R_\sigma+\sigma+t-u}.$$

Now we have

$$\begin{aligned}
&\Sigma(\{1,\ldots,\sigma\}) \\
&\leq \frac{12}{b^2(b^2-1)} \sum_{a_1=0}^{\infty} \cdots \sum_{a_\sigma=0}^{\infty} \frac{1}{b^{2R_\sigma}} \sum_{u=1}^{R_\sigma+\sigma+t} \sum_{\beta=1}^{b-1} b^{2u} \psi_b^\beta \left(\frac{N}{b^u}\right) (b-1)^\sigma b^{R_\sigma+\sigma+t-u} \\
&= \frac{12}{b^2(b+1)} (b-1)^\sigma b^{\sigma+t} \sum_{a_1=0}^{\infty} \cdots \sum_{a_\sigma=0}^{\infty} \frac{1}{b^{R_\sigma}} \sum_{u=1}^{R_\sigma+\sigma+t} b^u \psi_b \left(\frac{N}{b^u}\right) \\
&= \frac{12}{b^2(b+1)} (b-1)^\sigma b^{\sigma+t} \sum_{u=1}^{\infty} b^u \psi_b \left(\frac{N}{b^u}\right) \sum_{\substack{a_1,\ldots,a_\sigma=0 \\ R_\sigma \geq \max\{u-t-\sigma,0\}}}^{\infty} \frac{1}{b^{R_\sigma}} \\
&= \frac{12}{b^2(b+1)} (b-1)^\sigma b^{\sigma+t} \\
&\quad \times \left(\sum_{u=1}^{\sigma+t} b^u \psi_b \left(\frac{N}{b^u}\right) \sum_{a_1,\ldots,a_\sigma=0}^{\infty} \frac{1}{b^{R_\sigma}} + \sum_{u=\sigma+t+1}^{\infty} b^u \psi_b \left(\frac{N}{b^u}\right) \sum_{\substack{a_1,\ldots,a_\sigma=0 \\ R_\sigma \geq u-t-\sigma}} \frac{1}{b^{R_\sigma}} \right) \\
&= \frac{12}{b^2(b+1)} (b-1)^\sigma b^{\sigma+t} (\Sigma_1 + \Sigma_2).
\end{aligned}$$

For Σ_1 we have

$$\begin{aligned}\Sigma_1 &= \sum_{u=1}^{\sigma+t} b^u \psi_b\left(\frac{N}{b^u}\right) \sum_{a_1,\ldots,a_\sigma=0}^{\infty} \frac{1}{b^{a_1+\cdots+a_\sigma}} \\ &\le \frac{(b^2-1)(b+1)}{36}\left(\sum_{a=0}^{\infty}\frac{1}{b^a}\right)^{\sigma}\sum_{u=1}^{\sigma+t} b^u \\ &\le \frac{(b+1)^2}{36}\left(\frac{b}{b-1}\right)^{\sigma} b^{\sigma+t+1}.\end{aligned}$$

For Σ_2 we have

$$\begin{aligned}\Sigma_2 &= \sum_{u=\sigma+t+1}^{\infty} b^u \psi_b\left(\frac{N}{b^u}\right) \sum_{\substack{a_1,\ldots,a_\sigma=0\\ R_\sigma \ge u-t-\sigma}} \frac{1}{b^{R_\sigma}} \\ &= \sum_{u=\sigma+t+1}^{\infty} b^u \psi_b\left(\frac{N}{b^u}\right) \sum_{w=u-t-\sigma}^{\infty} \frac{1}{b^w}\binom{w+\sigma-1}{\sigma-1}.\end{aligned}$$

We now use [6, Lemma 6] to obtain

$$\begin{aligned}\Sigma_2 &\le \sum_{u=\sigma+t+1}^{\infty} b^u \psi_b\left(\frac{N}{b^u}\right) \frac{1}{b^{u-t-\sigma}}\binom{u-t-1}{\sigma-1}\left(\frac{b}{b-1}\right)^{\sigma} \\ &= b^{t+\sigma}\left(\frac{b}{b-1}\right)^{\sigma} \sum_{u=\sigma+t+1}^{\infty} \psi_b\left(\frac{N}{b^u}\right)\binom{u-t-1}{\sigma-1} \\ &\le b^{t+\sigma}\left(\frac{b}{b-1}\right)^{\sigma} \sum_{u=\sigma+t+1}^{\infty} \psi_b\left(\frac{N}{b^u}\right)(u-t-1)^{\sigma-1}.\end{aligned}$$

This yields

$$\begin{aligned}\Sigma(\{1,\ldots,\sigma\}) &\le \frac{12}{b^2(b+1)}(b-1)^{\sigma} b^{\sigma+t} \\ &\times\left(\frac{(b+1)^2}{36}\left(\frac{b}{b-1}\right)^{\sigma} b^{\sigma+t+1} + b^{t+\sigma}\left(\frac{b}{b-1}\right)^{\sigma}\sum_{u=\sigma+t+1}^{\infty}\psi_b\left(\frac{N}{b^u}\right)(u-t-1)^{\sigma-1}\right) \\ &= b^{3\sigma+2t}\left(\frac{b+1}{3b} + \frac{12}{b^2(b+1)}\sum_{u=\sigma+t+1}^{\infty}\psi_b\left(\frac{N}{b^u}\right)(u-t-1)^{\sigma-1}\right).\end{aligned}$$

Hence

$$\big(NF_{b,N}(\omega)\big)^2 = \frac{1}{(b+1)^s-1}\sum_{\substack{\emptyset\neq\mathfrak{u}\subseteq\{1,\dots,s\}\\ \mathfrak{u}=\{w_1,\dots,w_{|\mathfrak{u}|}\}}}\Sigma(\{w_1,\dots,w_{|\mathfrak{u}|}\})$$

$$\leq \frac{1}{(b+1)^s-1}\sum_{v=1}^{s}\binom{s}{v}b^{2t+3v}$$

$$\times\left(\frac{b+1}{3b}+\frac{12}{b^2(b+1)}\sum_{u=t+v+1}^{\infty}\psi_b\left(\frac{N}{b^u}\right)(u-t-1)^{v-1}\right).$$

This finishes the proof of 2. □

We will need the following lemma due to Chaix and Faure [1, Lemme 6.3.1.].

Lemma 15. *Let f be a real and bounded function; Set*

$$d_n := \sup_{x\in\mathbb{R}}\left|\sum_{j=1}^{n} f(x/b^j)\right| \qquad \textit{and} \qquad \alpha := \inf_{n\geq 1} d_n/n;$$

then we have

$$\alpha = \lim_{n\to\infty} d_n/n.$$

P r o o f o f C o r o l l a r y 9. For any $N \leq b^m$ we have

$$\big(NF_{b,N}(\omega)\big)^2 = \sum_{u=1}^{m}\psi_b\left(\frac{N}{b^u}\right)P_s(u) + \sum_{u=m+1}^{\infty}\frac{b^2(b^2-1)}{12}\frac{N^2}{b^{2u}}P_s(u)$$

$$\leq \sum_{u=1}^{m}\psi_b\left(\frac{N}{b^u}\right)P_s(u) + c\sum_{u=m+1}^{\infty}\frac{u^{s-1}}{b^{2u}}$$

$$= \sum_{u=1}^{m}\psi_b\left(\frac{N}{b^u}\right)P_s(u) + \mathcal{O}(1).$$

From this it follows immediately that

$$F_{b,N}(\omega) = \mathcal{O}\left(\frac{(\log N)^{s/2}}{N}\right).$$

Let

$$P_s(u) := \sum_{i=0}^{s-1}\alpha_i u^i.$$

Then we have

$$\frac{\left(NF_{b,N}(\omega)\right)^2}{(\log N)^s} = \alpha_{s-1}\sum_{u=1}^{m}\psi_b\left(\frac{N}{b^u}\right)\frac{u^{s-1}}{(\log N)^s} + \sum_{i=0}^{s-2}\alpha_i\sum_{u=1}^{m}\psi_b\left(\frac{N}{b^u}\right)\frac{u^i}{(\log N)^s} + \mathcal{O}\left(\frac{1}{(\log N)^s}\right).$$

As N tends to infinity the term $\mathcal{O}\left((\log N)^{-s}\right)$ goes to zero. For the second term we have

$$\sum_{i=0}^{s-2}\alpha_i\sum_{u=1}^{m}\psi_b\left(\frac{N}{b^u}\right)\frac{u^i}{(\log N)^s} \le \sum_{i=0}^{s-2}\alpha_i C\frac{(\log N)^{i+1}}{(\log N)^s} \overset{N\to\infty}{\longrightarrow} 0.$$

So it is sufficient to consider only the first term. We have

$$\limsup_{N\to\infty}\frac{NF_{b,N}(\omega)}{(\log N)^{s/2}} = \sqrt{\alpha_{s-1}\limsup_{N\to\infty}\sum_{u=1}^{m}\psi_b\left(\frac{N}{b^u}\right)\frac{u^{s-1}}{(\log N)^s}} \le \sqrt{\frac{\alpha_{s-1}}{(\log b)^{s-1}}\limsup_{N\to\infty}\sum_{u=1}^{m}\psi_b\left(\frac{N}{b^u}\right)\Big/\log N},$$

with equality if $s=1$. Let

$$d_n := \sup_{x\in\mathbb{R}}\left|\sum_{u=1}^{n}\psi_b\left(\frac{x}{b^u}\right)\right|.$$

ψ_b attains its maximum in a point $\frac{k}{b}$. The function $x\mapsto\sum_{u=1}^{m}\psi_b\left(\frac{x}{b^u}\right)$ is continuous and b^m-periodic, attains its maximum in a point N_m in $[b^m, 2b^m)$, $N_m\in\mathbb{N}$. So there exists a sequence of growing integers (N_m) such that

$$\frac{d_m}{(m+1)\log b} \le \sum_{u=1}^{m}\psi_b\left(\frac{N_m}{b^u}\right)\Big/\log N_m \le \frac{d_m}{m\log b}.$$

So we get with Lemma 15

$$\lim_{m\to\infty}\sum_{u=1}^{m}\psi_b\left(\frac{N_m}{b^u}\right)\Big/\log N_m = \lim_{m\to\infty}\frac{d_m}{m\log b} = \inf_{m\ge1}\frac{d_m}{m\log b} =: \frac{\gamma_b}{\log b}.$$

Hence

$$\limsup_{N\to\infty} \frac{NF_{b,N}(\omega)}{(\log N)^{s/2}} \leq \sqrt{\alpha_{s-1}\frac{\gamma_b}{(\log b)^s}}.$$

α_{s-1} is the leading coefficient from Remark 8. Since ψ_b is bounded by $\frac{(b+1)(b^2-1)}{36}$ we have

$$\gamma_b \leq \frac{(b+1)(b^2-1)}{36} < \infty.$$

Now we have to compute γ_b. We show that

$$\gamma_b = \begin{cases} \frac{1}{9} & \text{if } b = 2, \\ \frac{b(b^2-1)}{48} & \text{if } b \text{ is odd.} \end{cases}$$

The case $b = 2$ was already shown in [16, Corollary 2.3] since

$$\limsup_{N\to\infty} \frac{NF_{2,N}(\omega)}{(\log N)^{1/2}} = \sqrt{\frac{1}{3\log 2}}$$

implies $\gamma_2 = \frac{1}{9}$. It also follows from [1, Théorème 4.13] together with (5).
In the case $b = 2p + 1$ for an integer p, we can follow exactly the lines of the proof of [1, Théorème 4.13], since all properties of the function χ_b^I used in this proof also hold for our function ψ_b (see Lemma 11 and Lemma 13). One has only to use the values

$$\psi_b\left(\frac{k}{b}\right) \quad \text{and} \quad \psi_b\left(\frac{1}{2}\right) \quad \text{instead of} \quad \chi_b^I\left(\frac{k}{b}\right) \quad \text{and} \quad \chi_b^I\left(\frac{1}{2}\right).$$

□

Proof of Corollary 10. Suppose that $b^{m-1} < N \leq b^m$ with $m > t + s$, then

$$\begin{aligned} &\sum_{u=t+v+1}^{\infty} \psi_b\left(\frac{N}{b^u}\right)(u-t-1)^{v-1} \\ &= \sum_{u=t+v+1}^{m} \psi_b\left(\frac{N}{b^u}\right)(u-t-1)^{v-1} \\ &\quad + \sum_{u=m+1}^{\infty} \psi_b\left(\frac{N}{b^u}\right)(u-t-1)^{v-1} \\ &=: \Sigma_3 + \Sigma_4. \end{aligned}$$

Now,

$$\begin{aligned}\Sigma_3 &\leq \frac{(b^2-1)(b+1)}{36}\sum_{u=t+v+1}^{m}(u-t-1)^{v-1}\\ &\leq \frac{(b^2-1)(b+1)}{36}(m-t-1)^{v-1}\sum_{u=t+v+1}^{m}1\\ &\leq \frac{(b^2-1)(b+1)}{36}(m-t-1)^{v}\end{aligned}$$

and

$$\begin{aligned}\Sigma_4 &= \sum_{u=m+1}^{\infty}\psi_b\left(\frac{N}{b^u}\right)(u-t-1)^{v-1}\\ &= \sum_{u=m+1}^{\infty}\frac{b^2(b^2-1)}{12}\left(\frac{N}{b^u}\right)^2(u-t-1)^{v-1}\\ &= \sum_{u=1}^{\infty}\frac{b^2(b^2-1)}{12}\left(\frac{N}{b^{u+m}}\right)^2(u+m-t-1)^{v-1}\\ &= \frac{b^2(b^2-1)}{12}\left(\frac{N}{b^m}\right)^2\sum_{u=1}^{\infty}\frac{1}{b^{2u}}(u+m-t-1)^{v-1}\\ &\leq \frac{b^2(b^2-1)}{12}\sum_{u=1}^{\infty}\frac{1}{b^{2u}}\sum_{k=0}^{v-1}\binom{v-1}{k}(m-t-1)^k u^{v-1-k}\\ &\leq \frac{b^2(b^2-1)}{12}(m-t-1)^{v-1}\sum_{u=1}^{\infty}\frac{1}{b^{2u}}\sum_{k=0}^{v-1}\binom{v-1}{k}1^k u^{v-1-k}\\ &= \frac{b^2(b^2-1)}{12}(m-t-1)^{v-1}\sum_{u=1}^{\infty}\frac{(u+1)^{v-1}}{b^{2u}}\\ &= \frac{b^2(b^2-1)}{12}(m-t-1)^{v-1}c_v.\end{aligned}$$

Hence

$$\begin{aligned}&\big(NF_{b,N}(\omega)\big)^2\\ &\leq \frac{1}{(b+1)^s-1}\sum_{v=1}^{s}\binom{s}{v}b^{2t+3v}\\ &\quad\times\left(\frac{b+1}{3b}+\frac{b^2-1}{3b^2}(m-t-1)^v+(b-1)(m-t-1)^{v-1}c_v\right).\end{aligned}$$

The other results follow immediately. □

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