uniform
distribution
theory

APPROXIMATIONS TO TWO REAL NUMBERS

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ABSTRACT. For a real ξ put $\psi_\xi(t) = \min_{1 \leqslant x \leqslant t} ||x\xi||$. Let α, β be real numbers such that $\alpha \pm \beta \notin \mathbb{Z}$. We prove that the function $\psi_\alpha(t) - \psi_\beta(t)$ changes its sign infinitely many often as $t \to +\infty$. The proof uses continued fractions.

Communicated by Yann Bugeaud

Dedicated to the memory of Professor Edmund Hlawka

1. The result

For a real ξ denote the irrationality measure function

$$\psi_\xi(t) = \min_{1 \leqslant x \leqslant t} ||x\xi||.$$

The minimum is taken over all integer values of x in the range and $|| \cdot ||$ stands for the distance to the nearest integer.

The main result of this note is the following

THEOREM 1. *For any two different irrational numbers α, β such that $\alpha \pm \beta \notin \mathbb{Z}$ the difference function*

$$\psi_\alpha(t) - \psi_\beta(t)$$

changes its sign infinitely many times as $t \to +\infty$.

The phenomenon observed in Theorem 1 cannot be generalized to any dimension greater than one. In [2] the following two statements were proven.

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THEOREM 2 (A. Khintchine, 1926). *Let a function $\psi(t)$ decrease to zero as $t \to +\infty$. Then there exist two algebraically independent real numbers α^1, α^2 such that for all t large enough one has*

$$\psi_{(\alpha^1,\alpha^2)}(t) := \min_{1 \leqslant \max(|x_1|,|x_2|) \leqslant t} ||x_1\alpha^1 + x_2\alpha^2|| \leqslant \psi(t).$$

THEOREM 3 (A. Khintchine, 1926). *Let $\psi_1(t)$ decrease to zero as $t \to +\infty$ and the function $t \mapsto t\psi_1(t)$ increase to infinity as $t \to +\infty$. Then there exist two algebraically independent real numbers α_1, α_2 such that for all t large enough one has*

$$\psi_{\binom{\alpha_1}{\alpha_2}}(t) := \min_{1 \leqslant x \leqslant t} \max_{j=1,2} ||x\alpha_j|| \leqslant \psi_1(t).$$

Of course, in Theorems 2, 3 we suppose x_1, x_2, x to be integers.

From Theorem 2 and some basic facts on existence of badly approximable numbers (see [1, Ch. 3.]) it follows that the result of Theorem 1 cannot be generalized to the case of a linear form in two variables. Indeed, take

$$\psi(t) = o\big(t^{-2}\big), \quad t \to +\infty.$$

Let α^1, α^2 be the numbers from Theorem 2. So

$$\psi_{(\alpha^1,\alpha^2)}(t) = o\big(t^{-2}\big), \quad t \to +\infty.$$

Take β^1, β^2 to be numbers algebraically independent of α^1, α^2 such that they are badly approximable (in the sense of a linear form):

$$\inf_{(x_1,x_2)\in\mathbb{Z}^2\setminus\{(0,0)\}} \big(||x_1\beta^1 + x_2\beta^2|| \cdot \max(|x_1|,|x_2|)^2\big) > 0.$$

So for any $t \geqslant 1$ one has

$$\psi_{(\beta^1,\beta^2)}(t) \geqslant C_1 t^{-2}$$

with some positive C_1. We see that for all t large enough one has

$$\psi_{(\alpha^1,\alpha^2)}(t) < \psi_{(\beta^1,\beta^2)}(t).$$

The similar situation holds in the case of simultaneous approximations. From Theorem 3 it follows that the result of Theorem 1 cannot be generalized to the case of simultaneous approximations to two real numbers. Indeed, take

$$\psi_1(t) = o\left(t^{-1/2}\right), \quad t \to +\infty.$$

Let α_1, α_2 be the numbers from Theorem 3. Then for all t large enough one has

$$\psi_{\binom{\alpha_1}{\alpha_2}}(t) = o\left(t^{-1/2}\right), \quad t \to +\infty.$$

Take β_1, β_2 to be numbers algebraically independent of α^1, α^2 such that they are badly simultaneously approximable:

$$\inf_{x \in \mathbb{Z} \setminus \{0\}} \left(\max_{j=1,2} ||x\beta_j|| \cdot |x|^{1/2} \right) > 0.$$

Then for $t \geqslant 1$ we have

$$\psi_{\binom{\beta_1}{\beta_2}}(t) \geqslant C_2 t^{-1/2}$$

with some positive C_2. We see that

$$\psi_{\binom{\alpha_1}{\alpha_2}}(t) < \psi_{\binom{\beta_1}{\beta_2}}(t)$$

for all t large enough. (Of course, here $\psi_{(\beta^1, \beta^2)}, \psi_{\binom{\beta_1}{\beta_2}}$ are defined analogously to $\psi_{(\alpha^1, \alpha^2)}, \psi_{\binom{\alpha_1}{\alpha_2}}$.)

2. Lemmas about continued fractions

We can assume that $0 < \alpha, \beta < 1$. We consider continued fraction expansions

$$\alpha = [0; a_1 a_2, , \ldots, a_n, \ldots], \quad \beta = [0; b_1, b_2, \ldots, b_n, \ldots].$$

Define

$$\alpha_n = [a_n; a_{n+1}, a_{n+2}, \ldots], \quad \alpha_n^* = [0; a_n, a_{n-1}, \ldots, a_1],$$
$$\beta_n = [b_n; b_{n+1}, b_{n+2}, \ldots], \quad \beta_n^* = [0; b_n, b_{n-1}, \ldots, b_1],$$
$$\frac{r_n}{q_n} = [0; a_1, \ldots, a_n], \quad \frac{s_n}{p_n} = [0; b_1, \ldots, b_n].$$

LEMMA 1. *For $n \geqslant 2$ one has*

$$||q_{n-1}\alpha||q_{n+1} = \frac{\alpha_{n+1}(a_{n+1} + \alpha_n^*)}{\alpha_{n+1} + \alpha_n^*}.$$

P r o o f. It is a well known fact (see [1, Ch.1]) that

$$||q_{n-1}\alpha|| = \frac{1}{q_{n-1}\alpha_n + q_{n-2}} \tag{1}$$

or

$$q_{n-1}||q_{n-1}\alpha|| = q_{n-1}^2 \left| \alpha - \frac{r_{n-1}}{q_{n-1}} \right| = \frac{1}{\alpha_n + \alpha_{n-1}^*}, \tag{2}$$

where

$$\alpha_{n-1}^* = \frac{q_{n-2}}{q_{n-1}}.$$

So we see that

$$||q_{n-1}\alpha||q_{n+1} = ||q_{n-1}\alpha||q_{n-1}\frac{q_n}{q_{n-1}}\frac{q_{n+1}}{q_n} = \frac{1}{(\alpha_n + \alpha_{n-1}^*)\alpha_n^*\alpha_{n+1}^*}.$$

But as

$$\alpha_n = a_n + \frac{1}{\alpha_{n+1}}, \quad a_n + \alpha_{n-1}^* = \frac{1}{\alpha_n^*},$$

we see that

$$\alpha_n + \alpha_{n-1}^* = \frac{1}{\alpha_n^*} + \frac{1}{\alpha_{n+1}}.$$

So

$$||q_{n-1}\alpha||q_{n+1} = \frac{1}{\alpha_n^*\alpha_{n+1}^*\left(\dfrac{1}{\alpha_n^*} + \dfrac{1}{\alpha_{n+1}}\right)} = \frac{\alpha_{n+1}}{\alpha_{n+1}^*(\alpha_n^* + \alpha_{n+1})} = \frac{\alpha_{n+1}(a_{n+1} + \alpha_n^*)}{\alpha_{n+1} + \alpha_n^*}.$$

Lemma 1 is proved. □

As $a_{n+1} \geqslant 1$ and $\alpha_{n+1} > 1$ we obtain the following

COROLLARY. *For $n \geqslant 2$ one has*

$$||q_{n-1}\alpha||q_{n+1} > 1. \tag{3}$$

LEMMA 2. *Suppose that*

$$m, n \geqslant 2 \quad and \quad q_{n+1} \leqslant p_{m+1}. \tag{4}$$

Then

$$||q_{n-1}\alpha|| > ||p_m\beta||. \tag{5}$$

P r o o f. Suppose that (5) is not true. Then from (4) and (3) we see that

$$1 < ||q_{n-1}\alpha||q_{n+1} \leqslant ||p_m\beta||p_{m+1}.$$

As (see [1, Ch.1])

$$||p_m\beta||p_{m+1} = \frac{1}{1 + \dfrac{\beta_{m+1}^*}{\beta_{m+2}}} < 1$$

we have a contradiction. Lemma 5 is proved. □

3. Proof of Theorem 1

Now we are able to prove Theorem 1. We make use of the fact that q is a denominator of a convergent fraction to ξ if and only if $||q\xi|| = \min_{1\leqslant x\leqslant q} ||x\xi||$ (see [1], Ch.1, Theorem 5E). So given two consecutive denominators $q_\nu, q_{\nu+1}$ of convergent fraction to ξ in the interval

$$q_\nu \leqslant t < q_{\nu+1},$$

one has

$$\psi_\xi(t) = ||q_\nu \xi||.$$

Consider the sequences

$$q_0 \leqslant q_1 < \ldots < q_n < q_{n+1} < \ldots, \quad p_0 \leqslant p_1 < \ldots < p_m < p_{m+1} < \ldots$$

of convergents' denominators to α, β correspondingly. Suppose that the statement of Theorem 1 is false for certain irrationalities α, β. Without loss of generality assume that for all $t \geqslant p_{m_0} \geqslant q_{n_0-1}$ one has

$$\psi_\beta(t) \geqslant \psi_\alpha(t). \tag{6}$$

From Lemma 2 and the asumption (6) we see that between two consecutive denominators $p_m, p_{m+1}, m \geqslant m_0$ not more than one denominator of the form q_n may occur. Here we give a proof of this fact. Let q_{n-1} be the largest denominator of a convergent to α with $q_{n-1} \leqslant p_m$. Then

$$||p_m\beta|| = \psi_\beta(p_m) \geqslant \psi_\alpha(p_m) = \psi_\alpha(q_{n-1}) = ||q_{n-1}\alpha||$$

and (5) does not hold. By Lemma 2 we see that there is at most one denominator of a convergent to α, namely q_n, such that

$$p_m < q_n \leqslant p_{m+1}.$$

For $j \geqslant 0$ we define integer m_j from the condition that the denominator p_{m_j} is the largest denominator among the convergents to β such that

$$p_{m_j} < q_{n_0+j},$$

where n_0 is defined before formula (6). So

$$
\begin{aligned}
p_{m_0} < q_{n_0} &\leqslant p_{m_0+1} < \ldots < p_{m_1} &&< q_{n_0+1} &&\leqslant p_{m_1+1} &&< \ldots \\
&\ldots < p_{m_2} &&< q_{n_0+2} &&\leqslant p_{m_2+1} &&< \ldots \\
&\ldots < p_{m_{j-1}} &&< q_{n_0+j-1} &&\leqslant p_{m_{j-1}+1} &&< \ldots \\
&\ldots < p_{m_j} &&< q_{n_0+j} &&\leqslant p_{m_j+1} &&< \ldots \\
&\ldots < p_{m_{j+1}} &&< q_{n_0+j+1} &&\leqslant p_{m_{j+1}+1} &&< \ldots
\end{aligned}
$$

By (6) we see that for all $j \geqslant 0$ one has

$$||q_{n_0+j-1}\alpha|| = \psi_\alpha(q_{n_0+j-1}) = \psi_\alpha(p_{m_j}) \leqslant \psi_\beta(p_{m_j}) = ||p_{m_j}\beta||. \qquad (7)$$

From (6) we also have

$$||q_{n_0+j}\alpha|| = \psi_\alpha(q_{n_0+j}) = \psi_\alpha(p_{m_j+1}) \leqslant \psi_\beta(p_{m_j+1}) = ||p_{m_j+1}\beta||. \qquad (8)$$

We distinguish two cases. In the **first case** we suppose that for infinitely many j at least one of the inequalities in (7), (8) is strict, that is, there is the sign $<$ insead of $\leqslant$. In the **second case** for all j large enough we have equalities in both (7), (8).

Consider the **first case**. Without loss of generality we assume that

$$||q_{n_0+j-1}\alpha|| = \psi_\alpha(q_{n_0+j-1}) = \psi_\alpha(p_{m_j}) < \psi_\beta(p_{m_j}) = ||p_{m_j}\beta||. \qquad (9)$$

From (1) we have

$$||q_{n_0+j-1}\alpha|| = \frac{1}{q_{n_0+j-1}\alpha_{n_0+j} + q_{n_0+j-2}}, \quad ||p_{m_j}\beta|| = \frac{1}{p_{m_j}\beta_{m_j+1} + p_{m_j-1}}.$$

So

$$p_{m_j}\beta_{m_j+1} + p_{m_j-1} < q_{n_0+j-1}\alpha_{n_0+j} + q_{n_0+j-2}. \qquad (10)$$

As

$$\beta_{m_j+1} = b_{m_j+1} + \frac{1}{\beta_{m_j+2}}, \quad \alpha_{n_0+1} = a_{n_0+j} + \frac{1}{\alpha_{n_0+j+1}}$$

from (10) we deduce that

$$p_{m_j}\left(b_{m_j+1} + \frac{1}{\beta_{m_j+2}}\right) + p_{m_j-1} < q_{n_0+j-1}\left(a_{n_0+j} + \frac{1}{\alpha_{n_0+j+1}}\right) + q_{n_0+j-2}$$

or

$$p_{m_j+1} + \frac{p_{m_j}}{\beta_{m_j+2}} < q_{n_0+j} + \frac{q_{n_0+j-1}}{\alpha_{n_0+j+1}}.$$

But

$$p_{m_j+1} \geqslant q_{n_0+j}, \quad p_{m_j} \geqslant q_{n_0+j-1}.$$

So

$$\beta_{m_j+2} > \alpha_{n_0+j-1}. \qquad (11)$$

From the other hand, from (8) we deduce that

$$\frac{1}{q_{n_0+j}\alpha_{n_0+j+1} + q_{n_0+j-1}} = ||q_{n_0+j}\alpha|| = \psi_\alpha(q_{n_0+j}) = \psi_\alpha(p_{m_j+1})$$

$$\leqslant \psi_\beta(p_{m_j+1}) = ||p_{m_j+1}\beta|| = \frac{1}{p_{m_j+1}\beta_{m_j+2} + p_{m_j}}.$$

So

$$p_{m_j+1}\beta_{m_j+2} + p_{m_j} \leqslant q_{n_0+j}\alpha_{n_0+j+1} + q_{n_0+j-1}.$$

As
$$q_{n_0+j-1} \leqslant p_{m_j}, \quad q_{n_0+j} \leqslant p_{m_j+1},$$
we see that
$$\beta_{m_j+2} \leqslant \alpha_{n_0+j+1}.$$
This contradicts (11).

In the **second case** we see that for j large enough one has
$$\psi_\beta(p_{m_j+1}) = \psi_\alpha(q_{n_0+j}) = \psi_\beta(p_{m_{j+1}}). \tag{12}$$
Hence for j_0 large enough for all $j \geqslant j_0$ we have
$$m_{j+1} = m_j + 1 = m_{j_0} + j + 1 - j_0.$$
So for large j we have
$$p_{m_{j_0}+j+1-j_0}\beta - s_{m_{j_0}+j+1-j_0} = (-1)^{m_{j_0}+j+1-j_0}||p_{m_{j_0}+j+1-j_0}\beta|| =$$
$$(-1)^{m_{j_0}+j+1-j_0}\psi_\beta(p_{m_j+1}),$$
$$q_{n_0+j}\alpha - r_{n_0+j} = (-1)^{n_0+j}||q_{n_0+j}\alpha|| = (-1)^{n_0+j}\psi_\alpha(q_{n_0+j}).$$
So for all j large enough from (12) we deduce that
$$p_{m_{j_0}+j+1-j_0}\beta - s_{m_{j_0}+j+1-j_0} = (-1)^{m_{j_0}-j_0-n_0+1}(q_{n_0+j}\alpha - r_{n_0+j})$$
or
$$p_{m_{j_0}+j+1-j_0}\beta - (-1)^{m_{j_0}-j_0-n_0+1}q_{n_0+j}\alpha =$$
$$s_{m_{j_0}+j+1-j_0} - (-1)^{m_{j_0}-j_0-n_0+1}r_{n_0+j}.$$
Analogously,
$$p_{m_{j_0}+j+2-j_0}\beta - (-1)^{m_{j_0}-j_0-n_0+1}q_{n_0+j+1}\alpha =$$
$$s_{m_{j_0}+j+2-j_0} - (-1)^{m_{j_0}-j_0-n_0+1}r_{n_0+j+1}.$$
Recall that α, β are irrational numbers. So
$$p_{m_{j_0}+j+1-j_0}q_{n_0+j+1} - p_{m_{j_0}+j+2-j_0}q_{n_0+j} = 0$$
and
$$[0; b_{m_{j_0}+j+2-j_0}, b_{m_{j_0}+j+1-j_0}, \ldots, b_1] = \frac{p_{m_{j_0}+j+1-j_0}}{p_{m_{j_0}+j+2-j_0}} = \frac{q_{n_0+j}}{q_{n_0+j+1}} =$$
$$[0; a_{n_0+j+1}, a_{n_0+j}, \ldots, a_1],$$
for j large enough. As $\alpha, \beta \in (0,1)$ we see that $\alpha = \beta$ or $\alpha + \beta = 1$.

The proof of Theorem 1 is complete. $\qquad\qquad\square$

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