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CONTINUED FRACTIONS WITH MINIMAL REMAINDERS

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ABSTRACT. Consider the representation of a rational number as a continued fraction, associated with "centered" Euclidean algorithm. We prove a new formula for the limit distribution function for sequences of rationals with bounded sum of partial quotients.

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Dedicated to the memory of Professor Edmund Hlawka

1. Introduction and main results

The classical Euclidean algorithm for $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ uses the division of the form

$$a = bq + r, \quad q \in \mathbb{Z}, \quad b > 0, \quad 0 \leqslant r < b.$$

It leads to a continued fraction expansion of a real number

$$x = [a_0; a_1, a_2, \ldots, a_m, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_m + \cdots}}}, \tag{1}$$

where $a_0 \in \mathbb{Z}$, $a_j \in \mathbb{N}$ for $j \geqslant 1$. Numbers a_i are called partial quotients of fraction (1).

For $x \in \mathbb{Q}$ the representation (1) is finite. We assume for the uniqueness that the last partial quotient a_l is greater than or equal 2. Let

$$S^{[0]}(a/b) := a_0 + a_1 + \ldots + a_l.$$

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Define the set

$$\mathcal{F}_n := \left\{x \in \mathbb{Q}, x \in [0,1] : S^{[0]}(x) \leqslant n+1\right\}. \tag{2}$$

The limit distribution function

$$F^{[0]}(x) := \lim_{n\to\infty} \frac{\sharp\left\{\xi \in \mathcal{F}_n : \xi \leqslant x\right\}}{\sharp\mathcal{F}_n}, \qquad x \in [0,1]$$

is the famous Minkowski's question mark function $?(x)$. Properties of $?(x)$ were investigated, for example, in [1], [14] and [2].

There are different kinds of Euclidean algorithms. For example, "by-excess" Euclidean algorithm uses the division "by-excess"

$$a = bq + r, \quad -b < r \leqslant 0,$$

This algorithm leads to regular reduced continued fraction ([3], [5]) expansion of a real number x, that is

$$x = [[a_0; a_1, a_2, \ldots, a_m, \ldots]] = a_0 - \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cdots - \cfrac{1}{a_m - \cdots}}}, \tag{3}$$

where $a_0 \in \mathbb{Z}$, $a_j \geqslant 2$ for $j \geqslant 1$. Numbers a_i are called partial quotients of fraction (3).

For a rational x representation (3) is finite. For rational x we denote the sum of partial quotients in the representation of x in the form (3) by $S^{[1]}(x)$. We put

$$\Xi_n := \left\{x \in \mathbb{Q}, x \in [0,1] : S^{[1]}(x) \leqslant n+2\right\}$$

Consider the limit distribution function

$$F^{[1]}(x) := \lim_{n\to\infty} \frac{\sharp\left\{\xi \in \Xi_n : \xi \leqslant x\right\}}{\sharp\Xi_n}, \qquad x \in [0,1],$$

In 1995 R. F. Tichy and J. Uitz [8] considered a one parameter family $g_\lambda(x)$, $\lambda \in (0,1)$, $x \in [0,1]$, of singular functions. Functions $F^{[0]}$, $F^{[1]}$ belong to this family with $\lambda = \frac{1}{2}$ and $\lambda = \frac{3-\sqrt{5}}{2}$, correspondingly. Similar functions $\kappa(x,\alpha)$, $x \in [0,\infty)$, $\lambda \in (0,1)$, were introduced by A. Denjoy [1] much more earlier, in 1938. For $x \in [0,1]$ functions $\kappa(x,\alpha)$ and $g_\lambda(x)$ are related in the following way:

$$\kappa(x,\alpha) = 1 - (1-\alpha)g_{1-\alpha}(x).$$

In the same paper A. Denjoy proved that

$$\kappa(x,\alpha) = \alpha^{a_0} - \alpha^{a_0}(1-\alpha)^{a_1} + \alpha^{a_0+a_2}(1-\alpha)^{a_1+a_3} - \cdots,$$

where $a_0, a_1, \ldots, a_m, \ldots$ are partial quotients of representation (1) of number x. Similar formula for $g_\lambda(x)$, $x \in (0,1)$ is given in the paper [5]:

$$g_\lambda(x) = \lambda^{a_1-1} - \lambda^{a_1-1}(1-\lambda)^{a_2} + \lambda^{a_1-1}(1-\lambda)^{a_2}\lambda^{a_3} - \cdots$$
$$\cdots + (-1)^{m+1}\lambda^{\sum\limits_{\substack{1\leqslant i\leqslant m\\ i\equiv 1 \bmod 2}} a_i - 1}(1-\lambda)^{\sum\limits_{\substack{1\leqslant i\leqslant m\\ i\equiv 0 \bmod 2}} a_i} + \cdots \quad (4)$$

For $\lambda = \frac{1}{2}$ formula (4) gives a well-known result by R. Salem [7], namely

$$g_{1/2}(x) = ?(x) = \frac{1}{2^{a_1-1}} - \frac{1}{2^{a_1+a_2-1}} + \frac{1}{2^{a_1+a_2+a_3-1}} - \cdots$$

Let us consider a "centered" version of the Euclidean algorithm. This algorithm uses "centered" division

$$a = bq + r, \quad -\frac{b}{2} < r \leqslant \frac{b}{2} \quad (5)$$

and leads to the following representation of a real number x:

$$x = \left[a_0; \frac{\varepsilon_1}{a_1}, \cdots, \frac{\varepsilon_l}{a_l}, \cdots\right] = a_0 + \cfrac{\varepsilon_1}{a_1 + \cfrac{\varepsilon_2}{a_2 + \cdots + \cfrac{\varepsilon_l}{a_l + \cdots}}}. \quad (6)$$

This representation is known as a continued fraction with minimal remainders. Numbers a_i are called partial quotients of fraction (6). Here $a_0 \in \mathbb{Z}$, $\varepsilon_i = \pm 1$ and $a_j \geqslant 2$, $a_j + \varepsilon_{j+1} > 2$ for $j \geqslant 1$. For rational x, if $a_s = 2$ is the last partial quotient, then $\varepsilon_s = 1$ for uniqueness of the representation. Such fractions can be found in the book [6] by O. Perron.

Statistical properties of various Euclidean algorithms were investigated by B. Vallee and V. Baladi in papers [11], [12], [13]. The most precise asymptotic formulae for the mean length for the classical Euclidean algorithm and the centered Euclidean algorithm are proved in papers [9], [10] by A. V. Ustinov. A similar formula for "by-excess" Euclidean algorithm was obtained in author's paper [5].

For rational x let us denote by $S(x)$ the sum of partial quotients of representation (6), and put

$$\mathcal{Z}_n := \{x \in \mathbb{Q} \cap [0,1] : S(x) \leqslant n+1\}. \quad (7)$$

In present paper we investigate the limit distribution function

$$F(x) := \lim_{n\to\infty} \frac{\sharp\{\xi \in \mathcal{Z}_n : \xi \leqslant x\}}{\sharp \mathcal{Z}_n}, \qquad x \in [0,1].$$

The main result is the following theorem.

Theorem 1. *Let $x \in [0,1]$ be represented in the form* (6)*, then*

$$F(x) = a_0 - c\lambda\left(\frac{E_1}{\lambda^{A_1}} + \frac{E_2}{\lambda^{A_2}} + \cdots + \frac{E_j}{\lambda^{A_j}} + \cdots\right), \tag{8}$$

where

$$E_j = \prod_{1 \leqslant i \leqslant j} (-\varepsilon_i), \qquad A_j = \sum_{0 \leqslant i \leqslant j} a_i, \qquad c = 1/(\lambda - 1),$$

and λ is the unique real root of the equation

$$\lambda^3 - \lambda^2 - \lambda - 1 = 0.$$

For rational x the sum in formula (8) *is finite.*

In this paper we also prove

Theorem 2. *Let for $x \in [0,1]$ the derivative $F'(x)$ (finite or infinite) exists. Then either $F'(x) = 0$ or $F'(x) = \infty$.*

As function $F(x)$ is monotonic, then by Lebesgue's theorem, the derivative $F'(x)$ exists and is finite almost everywhere (in the sense of Lebesgue measure). Therefore $F'(x) = 0$ almost everywhere. In other words, $F(x)$ is a singular function.

In the proof of Theorem 1 we need the following result.

Proposition 1. *For $x \in [0,1/2]$ the function $F(x)$ satisfies the following functional equation*

$$F(1-x) = 1 - \frac{F(x)}{\lambda}. \tag{9}$$

The proof of Proposition 1 is given in section 3. Theorem 1 uses Proposition 1 and it's proof is given in section 4. The proof of Theorem 2 is in section 5.

2. Properties of a continued fraction with minimal remainders

It follows immediately from the definition of continued fraction with minimal remainders (6), that

- $a_i \geqslant 2$ for $i \geqslant 1$,
- If $a_i = 2$, then $\varepsilon_{i+1} = 1$ for $i \geqslant 1$,
- If the last partial quotient $a_l = 2$, then $\varepsilon_l = 1$.

Let $x = [b_0; b_1, \ldots, b_s, \ldots]$ be represented in the form of ordinary continued fraction (1). We describe the algorithm for converting this fraction into a fraction of the form (6) (see [6]).

Fraction (1) is constructed by the classical Euclidean algorithm

$$r_0 = \frac{b}{a}, \quad r_{i+1} = \frac{1}{r_i} - b_i, \quad b_i = \left[\frac{1}{r_i}\right], \quad 0 \leqslant r_i < 1.$$

The remainder r_{i+1} is less then $\frac{1}{2}$ if and only if $b_{i+1} > 1$. So while $b_{i+1} > 1$ partial quotients of the classical Euclidean algorithm coincide with partial quotients of "centered" Euclidean algorithm.

For the first i such that $b_{i+1} = 1$, we use the identity

$$b_i + \cfrac{1}{1 + \cfrac{1}{b_{i+2} + \alpha}} = b_i + 1 - \frac{1}{b_{i+2} + 1 + \alpha}, \quad \alpha \geqslant 0.$$

And since $\frac{1}{b_{i+1}+1} < \frac{1}{2}$, we have

$$[b_0; b_1, \ldots, b_i, 1, b_{i+2}] = \left[b_0; \frac{1}{b_1}, \cdots, \frac{1}{b_i + 1}, \frac{-1}{b_{i+2} + 1}\right]$$

Then we apply the same procedure to the "tail"

$$b_{i+2} + 1 + \cfrac{1}{b_{i+3} + \cdots + \cfrac{1}{b_s + \cdots}}$$

of the fraction (1).

We define the convergents of the continued fraction with minimal remainders of the number $x = \left[a_0; \frac{\varepsilon_1}{a_1}, \cdots, \frac{\varepsilon_l}{a_l}, \cdots\right]$ as

$$\frac{P_n}{Q_n} = \left[a_0; \frac{\varepsilon_1}{a_1}, \cdots, \frac{\varepsilon_n}{a_n}\right], \quad (P_n, Q_n) = 1, \quad n \geqslant 1.$$

To get a recurrence formulas for P_n/Q_n, $n \geqslant 0$ we put formally

$$\frac{P_{-1}}{Q_{-1}} = \frac{1}{0}, \quad \frac{P_0}{Q_0} = \frac{a_0}{1}.$$

Then for $\varepsilon_{n+1} = 1$ we have

$$\frac{P_{n+1}}{Q_{n+1}} = \frac{a_n P_n + P_{n-1}}{a_n Q_n + Q_{n-1}},$$

otherwise,

$$\frac{P_{n+1}}{Q_{n+1}} = \frac{a_n P_n - P_{n-1}}{a_n Q_n - Q_{n-1}}.$$

3. Definition and properties of sets $\mathcal{Z}_n$

We define a sequence of sets $\mathcal{X}_k$ by

$$\mathcal{X}_k = \{x \in \mathbb{Q} \cap [0,1] : S(x) = k+1\}, \quad n \geqslant 1.$$

It is clear that

$$\mathcal{Z}_n = \bigcup_{1 \leqslant k \leqslant n} \mathcal{X}_k,$$

where $\mathcal{Z}_n$ is defined by (7). Suppose that the elements of $\mathcal{Z}_k$ are arranged in the increasing order. The number of elements of $\mathcal{Z}_n$, $\mathcal{X}_n$ we denote by Z_n, X_n, respectively.

In particular,

$$\mathcal{X}_1 = \left\{\frac{1}{2}\right\}, \quad \mathcal{X}_2 = \left\{\frac{1}{3}\right\}, \quad \mathcal{X}_3 = \left\{\frac{1}{4}, \frac{2}{5}, \frac{2}{3}\right\}, \quad \mathcal{X}_4 = \left\{\frac{1}{5}, \frac{2}{7}, \frac{3}{7}, \frac{3}{5}, \frac{3}{4}\right\}.$$

So

$$X_1 = X_2 = 1, \quad X_3 = 3, \quad X_4 = 5.$$

Lemma 1. *For $n \geqslant 1$ we have*

$$X_{n+3} = X_{n+2} + X_{n+1} + X_n.$$

P r o o f. We construct one-to-one correspondence Φ between elements of sets $\mathcal{X}_{n+2} \cup \mathcal{X}_{n+1} \cup \mathcal{X}_n$ and $\mathcal{X}_{n+3}$.

Let

$$x \in \mathcal{X}_{n+2} \cup \mathcal{X}_{n+1} \cup \mathcal{X}_n, x = \left[a_0; \frac{\varepsilon_1}{a_1}, \cdots, \frac{\varepsilon_l}{a_l}\right],$$

we define $\Phi(x) : \mathcal{X}_{n+2} \cup \mathcal{X}_{n+1} \cup \mathcal{X}_n \to \mathcal{X}_{n+3}$ in the following way:

- If $x \in \mathcal{X}_{n+2}$, then
$$\Phi(x) = \left[a_0; \frac{\varepsilon_1}{a_1}, \cdots, \frac{\varepsilon_{l-1}}{a_{l-1}}, \frac{\varepsilon_l}{a_l+1}\right] \in \mathcal{X}_{n+3}.$$
- If $x \in \mathcal{X}_{n+1}$, then
$$\Phi(x) = \left[a_0; \frac{\varepsilon_1}{a_1}, \cdots, \frac{\varepsilon_l}{a_l}, \frac{1}{2}\right] \in \mathcal{X}_{n+3}.$$
- If $x \in \mathcal{X}_n$ and $a_l > 2$, then
$$\Phi(x) = \left[a_0; \frac{\varepsilon_1}{a_1}, \cdots, \frac{\varepsilon_l}{a_l}, \frac{-1}{3}\right] \in \mathcal{X}_{n+3}.$$
- If $x \in \mathcal{X}_n$ and $a_l = 2$, then
$$\Phi(x) = \left[a_0; \frac{\varepsilon_1}{a_1}, \cdots, \frac{\varepsilon_{l-2}}{a_{l-2}}, \frac{\varepsilon_{l-1}}{a_{l-1}+1}, \frac{-1}{2}, \frac{1}{2}\right] \in \mathcal{X}_{n+3}.$$

The correspondence $\Phi(x)$ is injective by the construction. Let us show that it is surjective. For any $y \in \mathcal{X}_{n+3}$, $y = \left[a_0; \frac{\varepsilon_1}{a_1}, \cdots, \frac{\varepsilon_l}{a_l}\right]$ we find the preimage x of y.

- If $a_l > 3$ or $a_l = 3$ and $\varepsilon_l = 1$, then
$$x = \left[a_0; \frac{\varepsilon_1}{a_1}, \cdots, \frac{\varepsilon_{l-1}}{a_{l-1}}, \frac{\varepsilon_l}{a_l - 1}\right] \in \mathcal{X}_{n+2}.$$
- If $a_l = 2$ and either $a_{l-1} > 2$ or $a_{l-1} = 2$ and $\varepsilon_{l-1} = 1$, then
$$x = \left[a_0; \frac{\varepsilon_1}{a_1}, \cdots, \frac{\varepsilon_{l-1}}{a_{l-1}}\right] \in \mathcal{X}_{n+1}.$$
- If $a_l = 3$, $\varepsilon_l = -1$, then $a_{l-1} > 2$, therefore,
$$x = \left[a_0; \frac{\varepsilon_1}{a_1}, \cdots, \frac{\varepsilon_{l-1}}{a_{l-1}}\right] \in \mathcal{X}_n.$$
- If $a_l = a_{l-1} = 2$, $\varepsilon_{l-1} = -1$, then $a_{l-2} > 2$, therefore,
$$x = \left[a_0; \frac{\varepsilon_1}{a_1}, \cdots, \frac{\varepsilon_{l-3}}{a_{l-3}}, \frac{\varepsilon_{l-2}}{a_{l-2} - 1}, \frac{1}{2}\right] \in \mathcal{X}_n.$$

Lemma is proved. □

Corollary 1. *For $n \geqslant 1$ we have*
$$Z_{n+3} = Z_{n+2} + Z_{n+1} + Z_n + 2. \tag{10}$$

Proof. By the definition of $\mathcal{Z}_n$ and Lemma 1, we get
$$\begin{aligned} Z_{n+2} + Z_{n+1} + Z_n &= (X_1 + \cdots + X_{n+2}) + (X_1 + \cdots + X_{n+1}) + (X_1 + \cdots + X_n) \\ &= X_1 + X_2 + X_3 + X_4 + \cdots + X_{n+3} + (X_1 - X_3) \\ &= Z_{n+3} - 2. \end{aligned}$$
□

We remind the definition of the Stern-Brocot sequences $\mathcal{F}_n$, $n = 0, 1, 2, \ldots$ Consider two-point set $\mathcal{F}_0 = \left\{\frac{0}{1}, \frac{1}{1}\right\}$. Let
$$n \geqslant 0 \quad \text{and} \quad \mathcal{F}_n = \left\{0 = x_{0,n} < x_{1,n} < \cdots < x_{N(n),n} = 1\right\},$$
where
$$x_{j,n} = p_{j,n}/q_{j,n}, \quad (p_{j,n}, q_{j,n}) = 1, \quad j = 0, \ldots, N(n) \quad \text{and} \quad N(n) = 2^n + 1.$$
Then
$$\mathcal{F}_{n+1} = \mathcal{F}_n \cup Q_{n+1} \quad \text{with} \quad Q_{n+1} = \{x_{j-1,n} \oplus x_{j,n}, \quad j = 1, \ldots, N(n)\}.$$
Here
$$\frac{a}{b} \oplus \frac{c}{d} = \frac{a+b}{c+d}$$
is the mediant of fractions $\frac{a}{b}$ and $\frac{c}{d}$.

So the first sequences are

$$Q_1 = \left\{ \frac{1}{2} \right\}, \quad Q_2 = \left\{ \frac{1}{3}, \frac{2}{3} \right\}, \quad Q_3 = \left\{ \frac{1}{4}, \frac{2}{5}, \frac{3}{5}, \frac{3}{4} \right\}.$$

It is clear that for any rational number q there exists such number n that $q \in Q_n$. Note that sum $S^{[0]}(x)$ of partial quotients of ordinary continued fraction of a number $x \in Q_n$ equals to $n+1$. Formula (2) gives an equivalent definition of $\mathcal{F}_n$.

It is convenient to represent sequences $\mathcal{F}_n$ by means of the binary tree $\mathcal{D}^{[0]}$ (Figure 1). This tree is called Stern-Brocot's tree. Nodes of the tree are labeled

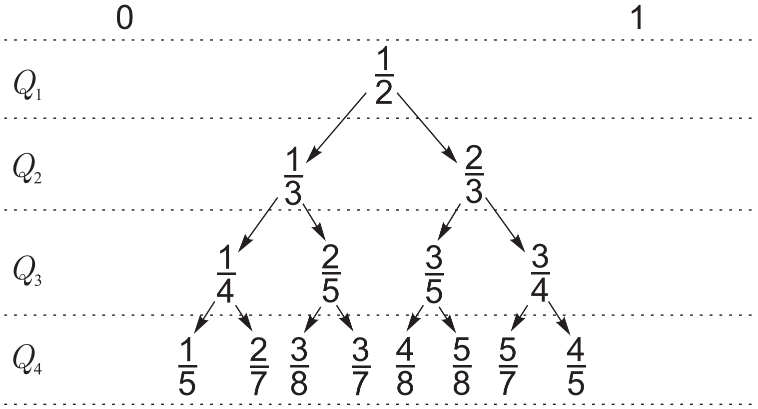

FIGURE 1.

by rationals from $(0, 1)$ and partitioned into levels by the following rule: the nth level consists of nodes labeled by numbers x, such that $S^{[0]}(x) = n + 1$ (i.e., the nth level consists of nodes, labeled by numbers from Q_n).

It is possible to distribute nodes of the tree into levels by another way. For example, we can use a rule: the nth level consists of nodes labeled by numbers x, such that sum $S^{[1]}(x)$ of partial quotients of regular reduced continued fraction of number x equals $n + 1$. Then we get tree $\mathcal{D}^{[1]}$ (Figure 2) from paper [4].

Now let us distribute nodes of the tree into levels by the following rule: the nth level consists of nodes labeled by numbers x, such that $S(x) = n + 1$ (i.e., $x \in \mathcal{X}_n$). We denote this tree by $\mathcal{D}$ (Figure 3).

Any node ξ of the tree $\mathcal{D}$ is a root of a subtree, which we denote by $\mathcal{D}^{(\xi)}$. Nodes of $\mathcal{D}^{(\xi)}$ are also partitioned into levels: ξ itself belongs to the level 1, and a node of the tree $\mathcal{D}^{(\xi)}$, labeled by number x belongs to the level $S(x) - S(\xi) + 1$ in the tree $\mathcal{D}^{(\xi)}$. The number of nodes of $\mathcal{D}^{(\xi)}$ from the level 1 to the level n we denote by $D_n^{(\xi)}$.

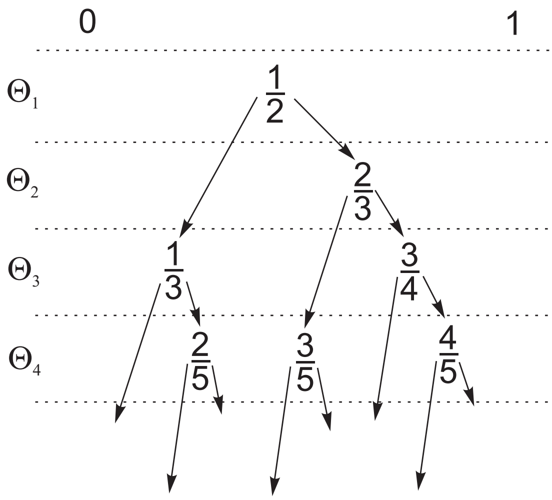

FIGURE 2.

Let us consider more detailed structure of the tree $\mathcal{D}$. From every node ξ of $\mathcal{D}$ we issue two arrows: the left one and the right one. The left one goes to the node labeled by x^l and the right one goes to node labeled by x^r. Note that if $\xi = x \oplus y$, where x, y are consecutive elements of $\mathcal{F}_n$, then $\xi^l = x \oplus \xi$, $\xi^r = \xi \oplus y$.

REMARK 1. Let

$$\xi \in \mathcal{X}_n \quad \text{and} \quad \xi = \left[a_0;\ \frac{\varepsilon_1}{a_1}, \cdots, \frac{\varepsilon_l}{a_l}\right] = x \oplus y,$$

where $x, y -$ neighboring elements of $\mathcal{F}_n$, $S(x) < S(y)$. If $a_l > 2$, then

$$x \oplus \xi = \left[a_0; \frac{\varepsilon_1}{a_1}, \cdots, \frac{\varepsilon_{l-1}}{a_{l-1}}, \frac{\varepsilon_l}{a_l + 1}\right] \in X_{n+1},$$

$$y \oplus \xi = \left[a_0; \frac{\varepsilon_1}{a_1}, \cdots, \frac{\varepsilon_{l-1}}{a_{l-1}}, \frac{\varepsilon_l}{a_l - 1}, \frac{1}{2}\right] \in X_{n+1},$$

If $a_l = 2$, then $\varepsilon_l = 1$ and

$$x \oplus \xi = \left[a_0; \frac{\varepsilon_1}{a_1}, \cdots, \frac{\varepsilon_{l-1}}{a_{l-1}}, \frac{1}{3}\right] \in X_{n+1},$$

$$y \oplus \xi = \left[a_0; \frac{\varepsilon_1}{a_1}, \cdots, \frac{\varepsilon_{l-1}}{a_{l-1}}, \frac{\varepsilon_{l-1}}{a_{l-1} + 1}, \frac{-1}{3}\right] \in X_{n+2}.$$

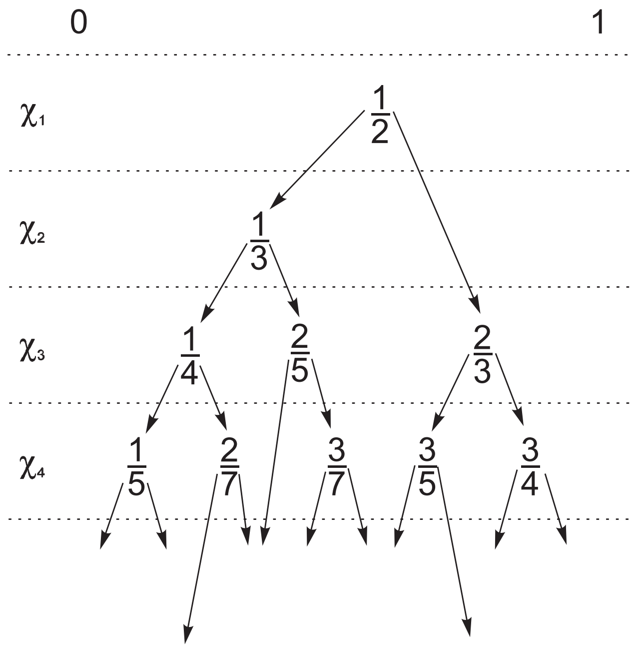

FIGURE 3.

From Remark 1 we deduce the following statement.

LEMMA 2. *Let* $\xi = \left[a_0; \frac{\varepsilon_1}{a_1}, \cdots, \frac{\varepsilon_l}{a_l}\right]$, *then*

$$D_n^{(\xi)} = \begin{cases} D_n^{(1/2)} & \textit{if} \quad a_l = 2 \\ D_n^{(1/3)} & \textit{if} \quad a_l > 2. \end{cases} \tag{11}$$

Note that $D_n^{(1/2)} = Z_n$. For brevity we put $Y_n = D_n^{(1/3)}$. So

$$Y_1 = 1, \quad Y_2 = 2, \quad Y_3 = 3.$$

For Z_n we have recurrence formula (10). It is easy to prove a similar formula for Y_n:

$$Y_{n+3} = Y_{n+2} + Y_{n+1} + Y_n + 2. \tag{12}$$

LEMMA 3. *Let* λ *be the unique real root of the equation*

$$\lambda^3 - \lambda^2 - \lambda - 1 = 0, \quad \textit{and} \quad c = 1/(\lambda - 1). \tag{13}$$

Then

$$\lim_{n\to\infty} \frac{Y_n}{Y_{n+1}} = \lim_{n\to\infty} \frac{Z_n}{Z_{n+1}} = \frac{1}{\lambda}, \quad \lim_{n\to\infty} \frac{Y_n}{Z_n} = c.$$

P r o o f. Equation (12) can be reduced to a homogeneous one by the substitution $Y'_n = Y_n + 1$:

$$Y'_{n+3} = Y'_{n+2} + Y'_{n+1} + Y'_n.$$

The characteristic equation is

$$\lambda^3 - \lambda^2 - \lambda = 1.$$

This equation has the unique real root $\lambda \approx 1,839292$ and two complex roots λ_1, λ_2, such that

$$|\lambda_1| = |\lambda_2| < 1.$$

So

$$Y_n + 1 = Y'_n = C_1\lambda^n + C_2\lambda_2^n + C_3\lambda_3^n,$$
$$Z_n + 1 = Z'_n = D_1\lambda^n + D_2\lambda_2^n + D_3\lambda_3^n$$

with certain constants C_1, C_2, C_3, D_1, D_2, D_3. Put $c = C_1/D_1$. From construction of the tree $\mathcal{D}$ it is clear that

$$Z_n = Y_{n-1} + Y_{n-2} + 1.$$

Dividing both parts of this equality by Z_n and taking the limit we get

$$1 = \lim_{n\to\infty} \frac{C_1\left(\lambda_1^{n-1} + \lambda_1^{n-2}\right) + C_2\left(\lambda_2^{n-1} + \lambda_2^{n-2}\right) + C_3\left(\lambda_3^{n-1} + \lambda_3^{n-2}\right) + 1}{D_1\lambda_1^n + D_2\lambda_2^n + D_3\lambda_3^n + 1} = \frac{C_1 + C_1\lambda}{D_1\lambda^2}.$$

Since λ is the root of equation (13), we get the following relation between c and λ:

$$c = \frac{\lambda^2}{1+\lambda} = \frac{1}{\lambda - 1} \approx 1,1915. \tag{14}$$

Lemma is proved. □

4. Properties of the limit distribution function $F(x)$ of sequence $\mathcal{Z}_n$

In this section we prove some auxiliary results about function $F(x)$.

Lemma 4. *Let x, y be consecutive elements of the sequence $\mathcal{Z}_n$. Then*

$$F(y) - F(x) = \lim_{n\to\infty} \frac{D_{n+2-S(x\oplus y)}^{(x\oplus y)}}{D_n^{(1/2)}}.$$

P r o o f. Note that nodes of tree $\mathcal{D}^{(x\oplus y)}$ are labeled exactly by the numbers from the set $\{\xi \in \mathbb{Q} : x < \xi < y\}$. So

$$F(y) - F(x) = \lim_{n\to\infty} \frac{\sharp\{\xi \in \mathcal{Z}_n : x < \xi \leqslant y\}}{Z_n} = \lim_{n\to\infty} \frac{D^{(x\oplus y)}_{n+2-S(x\oplus y)}}{D^{(1/2)}_n}.$$

□

Lemma 5. *Let x, y be consecutive elements of Z_n, $S(x) < S(y)$, and let a_l be the last partial quotient in continued fraction with minimal remainders representation of the number $x \oplus y$.*

If $a_l = 2$, then

$$F(x \oplus y) - F(x) = \frac{c}{\lambda}\left(F(y) - F(x)\right), \tag{15}$$

$$F(x \oplus y) - F(y) = \frac{c}{\lambda^2}\left(F(x) - F(y)\right). \tag{16}$$

If $a_l > 2$, then

$$F(x \oplus y) - F(x) = \frac{1}{\lambda}\left(F(y) - F(x)\right), \tag{17}$$

$$F(x \oplus y) - F(y) = \frac{1}{c\lambda}\left(F(x) - F(y)\right). \tag{18}$$

P r o o f. We suppose that $x < y$ (in case $x > y$ the proof is similar). According to Lemma 4 one has

$$F(x \oplus y) - F(x) = \lim_{n\to\infty} \frac{D^{\left((x\oplus y)^l\right)}_{n+2-S((x\oplus y)^l)}}{D_n},$$

$$F(y) - F(x \oplus y) = \lim_{n\to\infty} \frac{D^{\left((x\oplus y)^r\right)}_{n+2-S((x\oplus y)^r)}}{D_n}.$$

By Remark 1, if $a_l = 2$, then the last partial quotients of continued fractions with minimal remainders of numbers $(x \oplus y)^l$, $(x \oplus y)^r$ are greater than 2 and

$$S((x \oplus y)^l) = S(x \oplus y) + 1, \qquad S((x \oplus y)^r) = S(x \oplus y) + 2.$$

So

$$\frac{F(x \oplus y) - F(x)}{F(y) - F(x \oplus y)} = \lim_{n\to\infty} \frac{D^{\left((x\oplus y)^l\right)}_{n+2-S((x\oplus y)^l)}}{D^{\left((x\oplus y)^r\right)}_{n+2-S((x\oplus y)^r)}} = \lim_{n\to\infty} \frac{Y_{n+1-S(x\oplus y)}}{Y_{n-S(x\oplus y)}} = \lambda, \tag{19}$$

i.e., $F(x \oplus y)$ divides the segment $[F(x), F(y)]$ in the relation $\lambda : 1$. Taking into account that

$$\frac{c}{\lambda^2} + \frac{c}{\lambda} = 1$$

we get formulas (15), (16).

If $a_l > 2$, then by Remark 1 the last partial quotient of the continued fraction with minimal remainders for the number $(x \oplus y)^r$ is 2. For the number $(x \oplus y)^l$ the last partial quotient is greater than 2 and

$$S((x \oplus y)^l) = S\left((x \oplus y)^r\right) = S(x \oplus y) + 1.$$

Therefore

$$\frac{F(x \oplus y) - F(x)}{F(y) - F(x \oplus y)} = \lim_{n\to\infty} \frac{D_{n+2-S(x\oplus y)^l)}^{\left((x\oplus y)^l\right)}}{D_{n+2-S(x\oplus y)^r)}^{((x\oplus y)^r)}} = \lim_{n\to\infty} \frac{Y_{n+1-S(x\oplus y)}}{Z_{n+1-S(x\oplus y)}} = c, \tag{20}$$

i.e., the number $F(x \oplus y)$ divides the segment $[F(x), F(y)]$ in the relation $c:1$. Taking into account that

$$\frac{1}{\lambda} + \frac{1}{c\lambda} = 1$$

we get formulas (17), (18). Lemma is proved. □

Now we are able to prove Proposition 1.

P r o o f o f P r o p o s i t i o n 1. Suppose $x \in [0, 1/2]$. By the definition

$$F(x) = \lim_{n\to\infty} \frac{\sharp\left\{\xi \in \mathcal{Z}_n : \xi \leqslant x\right\}}{Z_n}.$$

So

$$\lim_{n\to\infty} \frac{\sharp\left\{\xi \in \mathcal{Z}_{n-1} : \xi \leqslant x\right\}}{Z_{n-1}} \frac{Z_{n-1}}{Z_n} = \frac{F(x)}{\lambda}.$$

Taking into account that $S(1-\xi) = 1 + S(\xi)$ for $\xi \in \mathbb{Q} \cap [0, 1/2)$, for $x \in [0, 1/2]$ we have

$$\left\{\xi \in \mathcal{Z}_{n-1} : \xi < x\right\} = \left\{1 - \xi = \eta \in \mathcal{Z}_n : 1 - \eta < x\right\} = \left\{\eta \in \mathcal{Z}_n : \eta > 1 - x\right\}.$$

So we get

$$\begin{aligned} \frac{F(x)}{\lambda} + F(1-x) &= \lim_{n\to\infty} \frac{\sharp\left\{\xi \in \mathcal{Z}_{n-1} : \xi \leqslant x\right\}}{Z_n} + \lim_{n\to\infty} \frac{\sharp\left\{\xi \in \mathcal{Z}_n : \xi \leqslant 1-x\right\}}{Z_n} \\ &= \lim_{n\to\infty} \frac{\sharp\left\{\eta \in \mathcal{Z}_n : \eta > 1-x\right\} \cup \left\{\xi \in \mathcal{Z}_n : \xi \leqslant 1-x\right\}}{Z_n} = 1. \end{aligned}$$

This equality proves formula (9). □

5. Proof of Theorem 1

Let us prove the theorem for rational $x \in [0, 1/2]$ by induction on $S(x)$. The equality

$$F(1/a_1) = \frac{c}{\lambda^{a_1-1}},$$

follows from formula (15), since $1/a_1 = \underbrace{0 \oplus \cdots \oplus 0 \oplus}_{(a_1-1)\text{ times}} 1$.

Suppose that the formula (8) is proved for

$$x = \left[0; \frac{1}{a_1}, \frac{\varepsilon_2}{a_2}, \cdots, \frac{\varepsilon_m}{a_m}\right].$$

Then it is enough to prove it for

$$y = \left[0; \frac{1}{a_1}, \frac{\varepsilon_2}{a_2}, \cdots, \frac{\varepsilon_{m-1}}{a_{m-1}}, \frac{\varepsilon_m}{a_m+1}\right]$$

and for

$$z = \left[0; \frac{1}{a_1}, \frac{\varepsilon_2}{a_2}, \cdots, \frac{\varepsilon_m}{a_m-1}, \frac{1}{2}\right], \qquad \text{if} \quad a_m > 2,$$

$$w = \left[0; \frac{1}{a_1}, \frac{\varepsilon_2}{a_2}, \cdots, \frac{\varepsilon_{m-2}}{a_{m-2}}, \frac{\varepsilon_{m-1}}{a_{m-1}+1}, \frac{-1}{3}\right], \qquad \text{if} \quad a_m = 2.$$

We see that

$$y = \left[0; \frac{1}{a_1}, \frac{\varepsilon_2}{a_2}, \cdots, \frac{\varepsilon_{m-1}}{a_{m-1}}\right] \oplus \left[0; \frac{1}{a_1}, \frac{\varepsilon_2}{a_2}, \cdots, \frac{\varepsilon_{m-1}}{a_{m-1}}, \frac{\varepsilon_m}{a_m}\right]$$

and the last partial quotient $a_m + 1$ of continued fraction with minimal remainders expression of number y is greater than 2. From (17) and the inductive assumption we get

$$\begin{aligned}
F(y) =& F\left(\left[0; \frac{1}{a_1}, \frac{\varepsilon_2}{a_2}, \cdots, \frac{\varepsilon_{m-1}}{a_{m-1}}\right]\right) \\
&+ \frac{1}{\lambda}\left(F\left(\left[0; \frac{1}{a_1}, \frac{\varepsilon_2}{a_2}, \cdots, \frac{\varepsilon_m}{a_m}\right]\right) - F\left(\left[0; \frac{1}{a_1}, \frac{\varepsilon_2}{a_2}, \cdots, \frac{\varepsilon_{m-1}}{a_{m-1}}\right]\right)\right) \\
=& F\left(\left[0; \frac{1}{a_1}, \frac{\varepsilon_2}{a_2}, \cdots, \frac{\varepsilon_{m-1}}{a_{m-1}}\right]\right) - \frac{1}{\lambda} c\lambda \frac{E_m}{\lambda^{A_m}} \\
=& F\left(\left[0; \frac{1}{a_1}, \frac{\varepsilon_2}{a_2}, \cdots, \frac{\varepsilon_{m-1}}{a_{m-1}}\right]\right) - c\lambda \frac{E_m}{\lambda^{A_{m-1}+(a_m+1)}}.
\end{aligned}$$

If $a_m > 2$, we must prove the formula (8) for z. We see that

$$z = \left[0; \frac{1}{a_1}, \frac{\varepsilon_2}{a_2}, \cdots, \frac{\varepsilon_m}{a_m}\right] \oplus \left[0; \frac{1}{a_1}, \frac{\varepsilon_2}{a_2}, \cdots, \frac{\varepsilon_{m-1}}{a_{m-1}}, \frac{\varepsilon_m}{a_m-1}\right]$$

and the last partial quotient of number z is 2. So by (15) we have

$$\begin{aligned}F(z) =&F\left(\left[0;\frac{1}{a_1},\frac{\varepsilon_2}{a_2},\cdots,\frac{\varepsilon_{m-1}}{a_{m-1}},\frac{\varepsilon_m}{a_m-1}\right]\right)\\&-\frac{c}{\lambda}\left(F\left(\left[0;\frac{1}{a_1},\frac{\varepsilon_2}{a_2},\cdots,\frac{\varepsilon_{m-1}}{a_{m-1}},\frac{\varepsilon_m}{a_m-1}\right]\right)-F\left(\left[0;\frac{1}{a_1},\frac{\varepsilon_2}{a_2},\cdots,\frac{\varepsilon_m}{a_m}\right]\right)\right)\\=&F\left(\left[0;\frac{1}{a_1},\frac{\varepsilon_2}{a_2},\cdots,\frac{\varepsilon_{m-1}}{a_{m-1}},\frac{1}{a_m}\right]\right)+\frac{c}{\lambda}c\lambda\left(\frac{E_m}{\lambda^{A_{m-1}+(a_m-1)}}-\frac{E_m}{\lambda^{A_m}}\right)\\=&F\left(\left[0;\frac{1}{a_1},\frac{\varepsilon_2}{a_2},\cdots,\frac{\varepsilon_{m-1}}{a_{m-1}},\frac{1}{a_m}\right]\right)-c\lambda\frac{E_m(-1)}{\lambda^{A_{m-1}+(a_m-1)+2}}.\end{aligned}$$

If $a_m = 2$, we must prove formula (8) for w. We see that

$$w=\left[0;\frac{1}{a_1},\frac{\varepsilon_2}{a_2},\cdots,\frac{\varepsilon_{m-1}}{a_{m-1}},\frac{\varepsilon_m}{a_m+1}\right]\oplus\left[0;\frac{1}{a_1},\frac{\varepsilon_2}{a_2},\cdots,\frac{\varepsilon_m}{a_m},\frac{1}{2}\right]$$

and the last partial quotient of number w is greater than 2. So by (17) we have

$$\begin{aligned}F(w) =&F\left(\left[0;\frac{1}{a_1},\frac{\varepsilon_2}{a_2},\cdots,\frac{\varepsilon_{m-2}}{a_{m-2}},\frac{\varepsilon_{m-1}}{a_{m-1}+1}\right]\right)\\&+\frac{1}{\lambda}\left(F\left(\left[0;\frac{1}{a_1},\frac{\varepsilon_2}{a_2},\cdots,\frac{\varepsilon_{m-1}}{a_{m-1}},\frac{1}{2}\right]\right)\right.\\&\left.-F\left(\left[0;\frac{1}{a_1},\frac{\varepsilon_2}{a_2},\cdots,\frac{\varepsilon_{m-2}}{a_{m-2}},\frac{\varepsilon_{m-1}}{a_{m-1}+1}\right]\right)\right),\end{aligned}\tag{21}$$

As

$$\begin{aligned}\left[0;\frac{1}{a_1},\frac{\varepsilon_2}{a_2},\cdots,\frac{\varepsilon_{m-1}}{a_{m-1}},\frac{1}{2}\right]=&\left[0;\frac{1}{a_1},\frac{\varepsilon_2}{a_2},\cdots,\frac{\varepsilon_{m-1}}{a_{m-1}+1}\right]\\&\oplus\left[0;\frac{1}{a_1},\frac{\varepsilon_2}{a_2},\cdots,\frac{\varepsilon_{m-2}}{a_{m-2}},\frac{\varepsilon_{m-1}}{a_{m-1}}\right],\end{aligned}$$

by (16) we have

$$\begin{aligned}&F\left(\left[0;\frac{1}{a_1},\frac{\varepsilon_2}{a_2},\cdots,\frac{\varepsilon_{m-1}}{a_{m-1}},\frac{1}{2}\right]\right)-F\left(\left[0;\frac{1}{a_1},\frac{\varepsilon_2}{a_2},\cdots,\frac{\varepsilon_{m-2}}{a_{m-2}},\frac{\varepsilon_{m-1}}{a_{m-1}+1}\right]\right)\\&=\frac{c}{\lambda^2}\left(F\left(\left[0;\frac{1}{a_1},\frac{\varepsilon_2}{a_2},\cdots,\frac{\varepsilon_{m-1}}{a_{m-1}}\right]\right)-F\left(\left[0;\frac{1}{a_1},\frac{\varepsilon_2}{a_2},\cdots,\frac{\varepsilon_{m-2}}{a_{m-2}},\frac{\varepsilon_{m-1}}{a_{m-1}+1}\right]\right)\right)\\&=-\frac{c}{\lambda^2}c\lambda E_{m-1}\left(\frac{1}{\lambda^{\sum\limits_{0\leqslant i\leqslant m-2}a_i+a_{m-1}}}-\frac{1}{\lambda^{\sum\limits_{0\leqslant i\leqslant m-2}a_i+(a_{m-1}+1)}}\right)\\&=-c\lambda E_{m-1}\frac{1}{\lambda^{\sum\limits_{0\leqslant i\leqslant m-1}a_i+3}}.\end{aligned}$$

Substituting this result in (21), we finally get

$$F(w) = F\left(\left[0; \frac{1}{a_1}, \frac{\varepsilon_2}{a_2}, \cdots, \frac{\varepsilon_{m-2}}{a_{m-2}}, \frac{\varepsilon_{m-1}}{a_{m-1}+1}\right]\right) - c\lambda E_{m-1}\frac{1}{\lambda^{A_{m-1}+4}}.$$

So we have proven Theorem 1 for rational $x \in [0, 1/2]$. For rational $x \in (1/2, 1]$ it follows from formula (9). For irrational $x \in [0, 1]$ we should take into account the continuity of $F(x)$. □

6. Singularity of the function $F(x)$

In this section we prove Theorem 2. First of all let us consider the case $x \in \mathbb{Q}$.

Lemma 6. *For rational $x \in [0, 1]$ we have $F'(x) = 0$.*

Let us prove that the right derivative $F'_+(x)$ exists and $F'_+(x) = 0$ (for $F'_-(x)$ the prove is similar).

P r o o f. Let $x = a/b$, $a, b \in \mathbb{N}$, then there exists such n that $a/b \in \mathcal{X}_n$. Denote by a'/b' the right neighboring to a/b element in $\mathcal{Z}_n$. Sequence of mediants $y_k = \left\{\frac{ka+a'}{kb+b'}\right\}$, converges to a/b from the right as $k \to \infty$. So for $\xi > x$ sufficiently close to x there exists such m that $x < y_{m+1} \leqslant \xi \leqslant y_m$ and so

$$\frac{|F(\xi) - F(a/b)|}{\xi - a/b} \leqslant \frac{F(y_m) - F(a/b)}{y_{m+1} - a/b}.$$

Remind that $a/b \in \mathcal{Z}_n$, but $a/b \notin \mathcal{Z}_{n-1}$. So $S(a/b) > S(a'/b')$. By Lemma 5 we see that

$$\begin{aligned} F(a/b \oplus a'/b') - F(a/b) &\leqslant \max\left(\frac{c}{\lambda^2}, \frac{1}{c\lambda}\right)\big(F(a'/b') - F(a/b)\big) \\ &= \frac{1}{c\lambda}\big(F(a'/b') - F(a/b)\big). \end{aligned}$$

Similarly, $S\left(\frac{ka+a'}{kb+b'}\right) > S(a/b)$, $k = 1, \ldots, m-1$, and

$$\begin{aligned} F\left(\frac{(k+1)a+a'}{(k+1)b+b'}\right) - F(a/b) &\leqslant \max\left(\frac{c}{\lambda}, \frac{1}{\lambda}\right)\left(F\left(\frac{ka+a'}{kb+b'}\right) - F(a/b)\right) \\ &\leqslant \frac{c^k}{\lambda^k}\big(F(a/b \oplus a'/b') - F(a/b)\big). \end{aligned}$$

So

$$0 \leqslant F'_+(x) = \lim_{\xi \to x_+} \frac{F(\xi) - F(x)}{\xi - x} \leqslant \lim_{m\to\infty} \frac{F(y_m) - F(x)}{y_{m+1} - x} \tag{22}$$

$$= \lim_{m\to\infty} \frac{\frac{c^{m-1}}{\lambda^{m+1}}\big(F(a'/b') - F(a/b)\big)}{\frac{1}{((m+1)b+b')b}} = 0. \qquad \square$$

Let $x \notin \mathbb{Q}$. Suppose that $F'(x) = a$, where a is finite and $a \neq 0$. We should prove that it is not possible. We shall use the Stern-Brocot sequences $\mathcal{F}_n$.

Given n we can find two consecutive elements $p_n/q_n < p'_n/q'_n$ from the set $\mathcal{F}_n$ such that $p_n/q_n < x < p'_n/q'_n$. In such a way we obtain an infinite sequence of pairs of elements $\{p_n/q_n, p'_n/q'_n\}$, converging to x from the left and from the right, respectively. So

$$\lim_{n\to\infty} \frac{F(p'_n/q'_n) - F(p_n/q_n)}{p'_n/q'_n - p_n/q_n} = a,$$

Put

$$G_n(x) = \frac{F(p'_{n+1}/q'_{n+1}) - F(p_{n+1}/q_{n+1})}{F(p'_n/q'_n) - F(p_n/q_n)} \frac{q_{n+1}q'_{n+1}}{q_n q'_n}.$$

Then

$$G_n(x) = \frac{F(p'_{n+1}/q'_{n+1}) - F(p_{n+1}/q_{n+1})}{p'_{n+1}/q'_{n+1} - p_{n+1}/q_{n+1}} \frac{p'_n/q'_n - p_n/q_n}{F(p'_n/q'_n) - F(p_n/q_n)} \xrightarrow[n\to\infty]{} 1. \tag{23}$$

It is clear that if $x \in (p_n/q_n, p_n/q_n \oplus p'_n/q'_n)$, then the pair

$$\{p_{n+1}/q_{n+1}, p'_{n+1}/q'_{n+1}\} \quad \text{coincides with} \quad \{p_n/q_n, p_n/q_n \oplus p'_n/q'_n\}.$$

Also if $x \in (p_n/q_n \oplus p'_n/q'_n, p'_n/q'_n)$, then the pair

$$\{p_{n+1}/q_{n+1}, p'_{n+1}/q'_{n+1}\} \quad \text{coincides with} \quad \{p_n/q_n \oplus p'_n/q'_n, p'_n/q'_n\}.$$

Note that p_n/q_n, p'_n/q'_n are always among the intermediate and convergent fractions to x in the sense of ordinary continued fraction.

Let us show that for an irrational x one can find an infinite subsequence $\{p_{n_k}/q_{n_k}, p'_{n_k}/q'_{n_k}\}$ of the sequence $\{p_n/q_n, p'_n/q'_n\}$ with the following property: the last partial quotient in the continued fraction with minimal remainders expression of $p_{n_k}/q_{n_k} \oplus p'_{n_k}/q'_{n_k}$ is equal to 2.

Let

$$x = \left[0; \frac{1}{a_1}, \frac{\varepsilon_2}{a_2}, \cdots, \frac{\varepsilon_m}{a_m}, \cdots\right].$$

Then either

$$p_n/q_n \oplus p'_n/q'_n = \left[0; \frac{1}{a_1}, \frac{\varepsilon_2}{a_2}, \cdots, \frac{\varepsilon_{m-1}}{a_{m-1}}, \frac{1}{2}\right]$$

or

$$p_n/q_n \oplus p'_n/q'_n = \left[0; \frac{1}{a_1}, \frac{\varepsilon_2}{a_2}, \cdots, \frac{\varepsilon_{m-1}}{a_{m-1}}, \frac{\varepsilon_m}{b_m}\right]$$

for some natural m, $b_m \leqslant a_m + 1$. In the first case the pair $\{p_n/q_n, p'_n/q'_n\}$ satisfies the necessary property.

In the other case we consider the pair

$$\{p_{n+1}/q_{n+1},\ p'_{n+1}/q'_{n+1}\}$$

and for $p_{n+1}/q_{n+1} \oplus p'_{n+1}/q'_{n+1} \in \{p_{n+2}/q_{n+2},\ p'_{n+2}/q'_{n+2}\}$ by Remark 1 we have two possibilities: either

$$\left[0; \frac{1}{a_1}, \frac{\varepsilon_2}{a_2}, \cdots, \frac{\varepsilon_{m-1}}{a_{m-1}}, \frac{\varepsilon_m}{b_m - 1}, \frac{1}{2}\right]$$

or

$$\left[0; \frac{1}{a_1}, \frac{\varepsilon_2}{a_2}, \cdots, \frac{\varepsilon_{m-1}}{a_{m-1}}, \frac{\varepsilon_m}{b_m + 1}\right],$$

where $b_m + 1 \leqslant a_m + 1$. But the last inequality can occur only finitely many times. So for any m we can find a pair $\{p_{n_k}/q_{n_k}, p'_{n_k}/q'_{n_k}\}$ satisfying the necessary property.

In the sequel we consider such a subsequence $\{p_{n_k}/q_{n_k}, p'_{n_k}/q'_{n_k}\}$. From (23), we see that

$$\lim_{k\to\infty} G_{n_k}(x) = 1. \tag{24}$$

For a fixed k we consider following cases:

Case 1: $x \in (p_{n_k}/q_{n_k}, p_{n_k}/q_{n_k} \oplus p'_{n_k}/q'_{n_k})$,

Case 2: $x \in (p_{n_k}/q_{n_k} \oplus p'_{n_k}/q'_{n_k}, p'_{n_k}/q'_{n_k})$.

Consider **Case 1**. As the last partial quotient of $p_n/q_n \oplus p'_n/q'_n$ is equal to 2, Lemma 5 leads to

$$\frac{F(p'_{n_k+1}/q'_{n_k+1}) - F(p_{n_k+1}/q_{n_k+1})}{F(p'_{n_k}/q'_{n_k}) - F(p_{n_k}/q_{n_k})} = \frac{F(p_{n_k}/q_{n_k} \oplus p'_{n_k}/q'_{n_k}) - F(p_{n_k}/q_{n_k})}{F(p'_{n_k}/q'_{n_k}) - F(p_{n_k}/q_{n_k})}$$

$$= \begin{cases} \frac{c}{\lambda} & \text{if } S(p_{n_k}/q_{n_k}) < S(p'_{n_k}/q'_{n_k}). \\ \frac{c}{\lambda^2} & \text{if } S(p_{n_k}/q_{n_k}) > S(p'_{n_k}/q'_{n_k}). \end{cases} \tag{25}$$

Then there are two possibilities:

a) $x \in \big(p_{n_k}/q_{n_k}, (p_{n_k}/q_{n_k} \oplus p'_{n_k}/q'_{n_k})^l\big)$,

b) $x \in \big((p_{n_k}/q_{n_k} \oplus p'_{n_k}/q'_{n_k})^l, p_{n_k}/q_{n_k} \oplus p'_{n_k}/q'_{n_k}\big)$, where

$$(p_{n_k}/q_{n_k} \oplus p'_{n_k}/q'_{n_k})^l = p_{n_k}/q_{n_k} \oplus (p_{n_k}/q_{n_k} \oplus p'_{n_k}/q'_{n_k}),$$
$$(p_{n_k}/q_{n_k} \oplus p'_{n_k}/q'_{n_k})^r = (p_{n_k}/q_{n_k} \oplus p'_{n_k}/q'_{n_k}) \oplus p'_{n_k}/q'_{n_k}.$$

As the last partial quotient of $p_{n_k}/q_{n_k} \oplus p'_{n_k}/q'_{n_k}$ is equal to 2, then by Remark 1, the last partial quotients of $(p_{n_k}/q_{n_k} \oplus p'_{n_k}/q'_{n_k})^r$ and $(p_{n_k}/q_{n_k} \oplus p'_{n_k}/q'_{n_k})^l$ are not equal to 2. As $S(p_{n_k}/q_{n_k} \oplus p'_{n_k}/q'_{n_k}) > S(p_{n_k}/q_{n_k})$, we deduce from Lemma 5 that

$$\frac{F(p'_{n_k+2}/q'_{n_k+2}) - F(p_{n_k+2}/q_{n_k+2})}{F(p'_{n_k+1}/q'_{n_k+1}) - F(p_{n_k+1}/q_{n_k+1})} = \begin{cases} \frac{1}{\lambda}, & \text{in case a)}, \\ \frac{1}{c\lambda}, & \text{in case b)}. \end{cases} \tag{26}$$

Consider **Case 2**. Analogously to the case 1 we get

$$\frac{F(p'_{n_k+1}/q'_{n_k+1}) - F(p_{n_k+1}/q_{n_k+1})}{F(p'_{n_k}/q'_{n_k}) - F(p_{n_k}/q_{n_k})} = \frac{F(p'_{n_k}/q'_{n_k}) - F(p_{n_k}/q_{n_k} \oplus p'_{n_k}/q'_{n_k})}{F(p'_{n_k}/q'_{n_k}) - F(p_{n_k}/q_{n_k})}$$
$$= \begin{cases} \frac{c}{\lambda^2} & \text{if} \quad S(p_{n_k}/q_{n_k}) < S(p'_{n_k}/q'_{n_k}). \\ \frac{c}{\lambda} & \text{if} \quad S(p_{n_k}/q_{n_k}) > S(p'_{n_k}/q'_{n_k}). \end{cases} \tag{27}$$

We shall consider the following subcases:

a) $x \in \big(p_n/q_n \oplus p'_n/q'_n, (p_n/q_n \oplus p'_n/q'_n)^r\big)$,

b) $x \in \big((p_n/q_n \oplus p'_n/q'_n)^r, p'_n/q'_n\big)$.

As $S(p_{n_k}/q_{n_k} \oplus p'_{n_k}/q'_{n_k}) > S(p'_{n_k}/q'_{n_k})$, from Lemma 5 we see that

$$\frac{F(p'_{n_k+2}/q'_{n_k+2}) - F(p_{n_k+2}/q_{n_k+2})}{F(p'_{n_k+1}/q'_{n_k+1}) - F(p_{n_k+1}/q_{n_k+1})} = \begin{cases} \frac{1}{c\lambda}, & \text{in case a)}, \\ \frac{1}{\lambda}, & \text{in case b)}. \end{cases} \tag{28}$$

As the sequence $\{p_{n_k}/q_{n_k}, p'_{n_k}/q'_{n_k}\}$ is infinite, then at least one case (from the cases 1a), 1b), 2a), 2b)) will occur infinitely often. So there exists a subsequence $\{p_{n_{k_m}}/q_{n_{k_m}}, p'_{n_{k_m}}/q'_{n_{k_m}}\}$ such that

$$\frac{F(p'_{n_{k_m}+1}/q'_{n_{k_m}+1}) - F(p_{n_{k_m}+1}/q_{n_{k_m}+1})}{F(p'_{n_{k_m}}/q'_{n_{k_m}}) - F(p_{n_{k_m}}/q_{n_{k_m}})} = \alpha,$$

$$\frac{F(p'_{n_{k_m}+2}/q'_{n_{k_m}+2}) - F(p_{n_{k_m}+2}/q_{n_{k_m}+2})}{F(p'_{n_{k_m}+1}/q'_{n_{k_m}+1}) - F(p_{n_{k_m}+1}/q_{n_{k_m}+1})} = \beta,$$

where α − is one of the numbers $\frac{c}{\lambda}$, $\frac{c}{\lambda^2}$, and β − is one of the numbers $\frac{1}{\lambda}$, $\frac{1}{c\lambda}$. Now

$$G_{n_{k_m}}(x) = \alpha \frac{q_{n_{k_m}+1} q'_{n_{k_m}+1}}{q_{n_{k_m}} q'_{n_{k_m}}}, \qquad G_{n_{k_m}+1}(x) = \beta \frac{q_{n_{k_m}+2} q'_{n_{k_m}+2}}{q_{n_{k_m}+1} q'_{n_{k_m}+1}}.$$

From (24) we see that

$$\lim_{m\to\infty} \frac{q_{n_{k_m}} q'_{n_{k_m}}}{q_{n_{k_m}+1} q'_{n_{k_m}+1}} = \alpha, \qquad \lim_{m\to\infty} \frac{q_{n_{k_m}+1} q'_{n_{k_m}+1}}{q_{n_{k_m}+2} q'_{n_{k_m}+2}} = \beta. \tag{29}$$

Now we must show that (29) is not possible. To do this we distinguish the cases again.

1,a) In this case

$$\left\{p_{n_{k_m}+1}/q_{n_{k_m}+1}, p'_{n_{k_m}+1}/q'_{n_{k_m}+1}\right\} = \left\{p_{n_{k_m}}/q_{n_{k_m}}, \left(p_{n_{k_m}} + p'_{n_{k_m}} right\right) / \left(q_{n_{k_m}} + q'_{n_{k_m}}\right)\right\},$$

$$\left\{p_{n_{k_m}+2}/q_{n_{k_m}+2}, p'_{n_{k_m}+2}/q'_{n_{k_m}+2}\right\} = \left\{p_{n_{k_m}}/q_{n_{k_m}}, \left(2p_{n_{k_m}} + p'_{n_{k_m}}\right) / \left(2q_{n_{k_m}} + q'_{n_{k_m}}\right)\right\}.$$

Now (29) leads to

$$\lim_{m\to\infty} \frac{q'_{n_{k_m}}}{q_{n_{k_m}} + q'_{n_{k_m}}} = \alpha, \qquad \lim_{m\to\infty} \frac{q_{n_{k_m}} + q'_{n_{k_m}}}{2q_{n_{k_m}} + q'_{n_{k_m}}} = \beta,$$

where by (25) and (26) one has

$$\beta = \frac{1}{\lambda}, \quad \alpha = \frac{c}{\lambda} \quad \text{for} \quad S\left(p_{n_{k_m}}/q_{n_{k_m}}\right) < S\left(p'_{n_{k_m}}/q'_{n_{k_m}}\right),$$

and

$$\alpha = \frac{c}{\lambda^2} \quad \text{for} \quad S\left(p_{n_{k_m}}/q_{n_{k_m}}\right) > S\left(p'_{n_{k_m}}/q'_{n_{k_m}}\right).$$

Note that

$$\frac{2q_{n_{k_m}} + q'_{n_{k_m}}}{q_{n_{k_m}} + q'_{n_{k_m}}} = 2 - \frac{q'_{n_{k_m}}}{q_{n_{k_m}} + q'_{n_{k_m}}}.$$

So we have

$$\frac{1}{\beta} = 2 - \alpha.$$

For $\beta = \frac{1}{\lambda}$, $\alpha = \frac{c}{\lambda}$ we get

$$\lambda^2 - 2 = 0.$$

For $\beta = \frac{1}{\lambda}$, $\alpha = \frac{c}{\lambda^2}$ we get

$$\lambda^2 - \lambda - 1 = 0.$$

In both cases we have a contradiction with the fact that λ is a root of equation (13).

1,b) In this case

$$\left\{p_{n_{k_m}+1}/q_{n_{k_m}+1}, p'_{n_{k_m}+1}/q'_{n_{k_m}+1}\right\} = \left\{p_{n_{k_m}}/q_{n_{k_m}}, (p_{n_{k_m}} + p'_{n_{k_m}})/(q_{n_{k_m}} + q'_{n_{k_m}})\right\},$$

$$\left\{p_{n_{k_m}+2}/q_{n_{k_m}+2}, p'_{n_{k_m}+2}/q'_{n_{k_m}+2}\right\} = \left\{\left(2p_{n_{k_m}} + p'_{n_{k_m}}\right) / \left(2q_{n_{k_m}} + q'_{n_{k_m}}\right), \left(p_{n_{k_m}} + p'_{n_{k_m}}\right) / \left(q_{n_{k_m}} + q'_{n_{k_m}}\right)\right\},$$

Now (29) leads to

$$\lim_{m\to\infty} \frac{q'_{n_{k_m}}}{q_{n_{k_m}} + q'_{n_{k_m}}} = \alpha, \quad \lim_{m\to\infty} \frac{q_{n_{k_m}}}{2q_{n_{k_m}} + q'_{n_{k_m}}} = \beta,$$

where by (25) and (26) one has

$$\beta = \frac{1}{c\lambda}, \quad \alpha = \frac{c}{\lambda} \quad \text{for } S\left(p_{n_{k_m}}/q_{n_{k_m}}\right) < S\left(p'_{n_{k_m}}/q'_{n_{k_m}}\right),$$

and

$$\alpha = \frac{c}{\lambda^2} \quad \text{for} \quad S\left(p_{n_{k_m}}/q_{n_{k_m}}\right) > S\left(p'_{n_{k_m}}/q'_{n_{k_m}}\right).$$

Note that

$$\frac{2q_{n_{k_m}} + q'_{n_{k_m}}}{q_{n_{k_m}}} = 1 + \frac{q_{n_{k_m}} + q'_{n_{k_m}}}{q_{n_{k_m}}} = 1 + \frac{1}{1 - \frac{q'_{n_{k_m}}}{q_{n_{k_m}} + q'_{n_{k_m}}}}.$$

So we have

$$\frac{1}{\beta} = 1 + \frac{1}{1-\alpha}.$$

For $\beta = \frac{1}{c\lambda}$, $\alpha = \frac{c}{\lambda}$ we get

$$\lambda^2 - 2 = 0.$$

For $\beta = \frac{1}{c\lambda}$, $\alpha = \frac{c}{\lambda^2}$

$$\lambda^2 - \lambda - 1 = 0.$$

Again in both cases we have the contradiction with the fact that λ is a root of equation (13).

2,a) In this case

$$\left\{p_{n_{k_m}+1}/q_{n_{k_m}+1}, p'_{n_{k_m}+1}/q'_{n_{k_m}+1}\right\} = \left\{\left(p_{n_{k_m}} + p'_{n_{k_m}}\right) / \left(q_{n_{k_m}} + q'_{n_{k_m}}\right), p'_{n_{k_m}}/q'_{n_{k_m}}\right\},$$

$$\left\{p_{n_{k_m}+2}/q_{n_{k_m}+2}, p'_{n_{k_m}+2}/q'_{n_{k_m}+2}\right\} = \\ \left\{\left(p_{n_{k_m}} + p'_{n_{k_m}}\right) / \left(q_{n_{k_m}} + q'_{n_{k_m}}\right), \left(p_{n_{k_m}} + 2p'_{n_{k_m}}\right) / \left(q_{n_{k_m}} + 2q'_{n_{k_m}}\right)\right\}.$$

So (29) leads to

$$\lim_{m\to\infty} \frac{q_{n_{k_m}}}{q_{n_{k_m}} + q'_{n_{k_m}}} = \alpha, \qquad \lim_{m\to\infty} \frac{q'_{n_{k_m}}}{q_{n_{k_m}} + 2q'_{n_{k_m}}} = \beta,$$

where by (27) and (28) one has

$$\beta = \frac{1}{c\lambda}, \quad \alpha = \frac{c}{\lambda^2} \quad \text{for} \quad S\left(p_{n_{k_m}}/q_{n_{k_m}}\right) < S(p'_{n_{k_m}}/q'_{n_{k_m}}),$$

and

$$\alpha = \frac{c}{\lambda} \quad \text{for} \quad S\left(p_{n_{k_m}}/q_{n_{k_m}}\right) > S\left(p'_{n_{k_m}}/q'_{n_{k_m}}\right).$$

Note that

$$\frac{q_{n_{k_m}} + 2q'_{n_{k_m}}}{q'_{n_{k_m}}} = 1 + \frac{q_{n_{k_m}} + q'_{n_{k_m}}}{q'_{n_{k_m}}} = 1 + \frac{1}{1 - \frac{q_{n_{k_m}}}{q_{n_{k_m}} + q'_{n_{k_m}}}}.$$

So we have

$$\frac{1}{\beta} = 1 + \frac{1}{1-\alpha},$$

and this case is reduced to the case 1,b).

2,b) In this case

$$\left\{p_{n_{k_m}+1}/q_{n_{k_m}+1}, p'_{n_{k_m}+1}/q'_{n_{k_m}+1}\right\} = \\ \left\{\left(p_{n_{k_m}} + p'_{n_{k_m}}\right) / \left(q_{n_{k_m}} + q'_{n_{k_m}}\right), p'_{n_{k_m}}/q'_{n_{k_m}}\right\},$$

$$\left\{p_{n_{k_m}+2}/q_{n_{k_m}+2}, p'_{n_{k_m}+2}/q'_{n_{k_m}+2}\right\} = \\ \left\{\left(p_{n_{k_m}} + 2p'_{n_{k_m}}\right) / \left(q_{n_{k_m}} + 2q'_{n_{k_m}}\right), p'_{n_{k_m}}/q'_{n_{k_m}}\right\}.$$

Now (29) leads to

$$\lim_{m\to\infty} \frac{q_{n_{k_m}}}{q_{n_{k_m}} + q'_{n_{k_m}}} = \alpha, \quad \lim_{m\to\infty} \frac{q_{n_{k_m}} + q'_{n_{k_m}}}{q_{n_{k_m}} + 2q'_{n_{k_m}}} = \beta,$$

where by (27) and (28) one has

$$\beta = \frac{1}{\lambda}, \quad \alpha = \frac{c}{\lambda^2} \quad \text{for} \quad S(p_{n_{k_m}}/q_{n_{k_m}}) < S(p'_{n_{k_m}}/q'_{n_{k_m}}),$$

and

$$\alpha = \frac{c}{\lambda} \quad \text{for} \quad S\left(p_{n_{k_m}}/q_{n_{k_m}}\right) > S\left(p'_{n_{k_m}}/q'_{n_{k_m}}\right).$$

Note that

$$\frac{q_{n_{k_m}} + 2q'_{n_{k_m}}}{q_{n_{k_m}} + q'_{n_{k_m}}} = 2 - \frac{q_{n_{k_m}}}{q_{n_{k_m}} + q'_{n_{k_m}}}.$$

So we have

$$\frac{1}{\beta} = 2 - \alpha,$$

and we have the same situation as in the case 1,a).

Theorem 2 is proved. □

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