SUPREMUM OF RANDOM DIRICHLET POLYNOMIALS WITH SUB-MULTIPLICATIVE COEFFICIENTS

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ABSTRACT. We study the supremum of random Dirichlet polynomials $D_N(s) = \sum_{n=1}^{N} \varepsilon_n d(n) n^{-s}$, where (ε_n) is a sequence of independent Rademacher random variables, and d is a sub-multiplicative function. The approach is gaussian and entirely based on comparison properties of Gaussian processes, with no use of the metric entropy method. As in preceding related works, the proof uses a sieve argument due to Queffélec.

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1. Introduction

Let $\varepsilon = \{\varepsilon_n, n \geq 1\}$ denote a sequence of independent Rademacher random variables ($\mathbf{P}\{\varepsilon_i = \pm 1\} = 1/2$) defined on a basic probability space $(\Omega, \mathcal{A}, \mathbf{P})$. Consider the random Dirichlet polynomials $\mathcal{D}(s) = \sum_{n=1}^{N} \varepsilon_n d(n) n^{-s}$, where $s = \sigma + it$. In a recent work [8], (see references therein for related results, notably Queffélec's works) we obtained sharp estimates of the supremum of $\mathcal{D}(s)$, under moderate growth condition on coefficients. Put

$$D_1(M) = \sum_{m=1}^{M} d(m), \qquad \widetilde{D}_1(M) = \max_{1 \leq m \leq M} \frac{D_1(m)}{m},$$

$$D_2(M) = \sum_{m=1}^{M} d(m)^2, \qquad \widetilde{D}_2^2(M) = \max_{1 \leq m \leq M} \frac{D_2(m)}{m}. \qquad (1)$$

We showed

Theorem 1. *Let $0 \leq \sigma \leq 1/2$ and assume that*

$$d(kp^j) \leq Cd(k)j^H \tag{2}$$

for some positive C, H, any positive integers k, j and any prime p. Then there exists a constant $C_{\sigma,d}$ depending on d and σ such that for any integer $N \geq 2$

$$\mathbf{E} \sup_{t \in \mathbf{R}} \left| \mathcal{D}(\sigma + it) \right| \leq C_{\sigma,d} \frac{N^{1-\sigma} \widetilde{D}_2(N)}{\log N} . \tag{3}$$

Moreover, if for some $b < 1/(\sqrt{5} + 1) \approx 0.31$

$$\widetilde{D}_2(M) \leq CM^b, \tag{4}$$

then

$$\mathbf{E} \sup_{t \in \mathbf{R}} \left| \mathcal{D}(\sigma + it) \right| \leq C_{\sigma,d} \frac{N^{1-\sigma}}{\log N} . \tag{5}$$

Suppose $d(n)$ is a multiplicative function: $d(nm) = d(n)d(m)$ if n, m are coprime. Then condition (2) is satisfied if and only if

$$d(p^{r+j}) \leq Cd(p^r)j^H, \tag{6}$$

for some $C > 0$, $H > 0$ and any $j \geq 1$, $r \geq 0$. This last condition is fulfilled when, for instance,

$$\frac{d\left(p^{k+1}\right)}{d\left(p^k\right)} \leq \left(1 + \frac{1}{k}\right)^H, \qquad k = 0, 1, \dots \tag{7}$$

a property which is satisfied for a relatively wide class of multiplicative functions, among them, the divisor function and $d_1(n) = \lambda^{\omega(n)}$, where $\lambda > 1$ and $\omega(n) = \#\{p : p \mid n\}$ is the well-known additive prime divisor function.

However, condition (2) requires that $d(p^j) = \mathcal{O}(j^H)$. Thus Theorem 1 does not apply to some classical multiplicative functions such as

$$d_2(n) = \lambda^{\Omega(n)},$$

where $\Omega(n) = \sum_{p^\nu \| n} \nu$ is the other prime divisor function.

The main concern of this work is to show that the approach used in [8] can be still adapted and further, simplified, to obtain extensions for a much larger class of multiplicative functions including these examples, and also for sub-multiplicative functions, namely functions satisfying the weaker condition:

$$d(nm) \leq d(n)d(m) \qquad \text{provided} \quad (n, m) = 1. \tag{8}$$

For instance, $d(n) = e^{(\log n)^\alpha}$, $0 < \alpha < 1$ is sub-multiplicative, as well as function $d_K(n) = \chi\{(n, K) = 1\}$ in Example 2. The related random Dirichlet polynomials are studied in this paper.

We obtain a general upper bound, which also contains and improves the main results in [7], [8] (Theorem 1.1 and Theorem 1, respectively). Introduce some notation. Let $2 = p_1 < p_2 < \ldots$ be the sequence of all primes, and let $\pi(N)$ denote the number of prime numbers less or equal to N. The following decomposition is basic

$$\{2, \ldots, N\} = \sum_{j=1}^{\pi(N)} E_j, \qquad \text{where} \qquad E_j = \{2 \leq n \leq N : P^+(n) = p_j\},$$

$P^+(n)$ being the largest prime divisor of n. It is natural to disregard cells E_j such that $d(n) \equiv 0$, $n \in E_j$. We thus set

$$\mathcal{H}_d = \{1 \leq j \leq \pi(N) : d_{|E_j} \not\equiv 0\}, \qquad \tau_d = \max(\mathcal{H}_d).$$

Consider now the following condition:

$$p \mid n \implies d(n) \leq C \, d\left(\frac{n}{p}\right), \qquad \text{and} \qquad d(p^j) \leq C_1 \lambda^j, \tag{9}$$

for some positive C, C_1, λ with $\lambda < \sqrt{2}$, any prime number p, any integers n, j. Clearly, if $C < \sqrt{2}$, the second property is implied by the first. But this is not always so. Consider the following example. Fix some prime number P_1 as well as some real numbers $1 < \lambda_1 < \sqrt{2}$, $C_1 \geq 1$, and put

$$d(n) = \begin{cases} C_1 \lambda^j & \text{if} \quad P_1^j \parallel n, \\ 1 & \text{if} \quad (n, P_1) = 1. \end{cases} \tag{10}$$

Then d is sub-multiplicative, and satisfies condition (9) with a constant C which has to be larger than $C_1 \lambda$.

That d be sub-multiplicative is easy: let n, m be coprime integers. If $(n, P_1) = 1$ and $(m, P_1) = 1$, then $d(n) = d(m) = d(nm) = 1$. If $P_1^j \parallel n$ and $(m, P_1) = 1$, then $d(n) = C_1 \lambda^j$, $d(m) = 1$; so $d(nm) = d(n) = d(n)d(m)$. Finally if $P_1^j \parallel n$ and $P_1^k \parallel m$, then $d(nm) = C_1 \lambda^{\max(j,k)} \leq C_1^2 \lambda^{j+k} = d(n)d(m)$.

Now let p be such that $p|n$. If $p \neq P_1$, either $P_1 \nmid n$ and then $d(n) = d(n/p) = 1$, or $P_1^j \parallel n$ and $d(n) = d(n/p) = C_1 \lambda^j$. If $p = P_1$, assume first $P_1 \parallel n$, then $d(n/p) = 1$, and in order that $d(n) = C_1 \lambda \leq C d(n/p)$, one must take $C \geq C_1 \lambda$. Finally, if $P_1^j \parallel n$ with $j \geq 2$, then $d(n) = C_1 \lambda^j = \lambda d(n/p) \leq C d(n/p)$. It remains to observe that $d(p^j) = 1 \leq C_1 \lambda^j$, if $p \neq P_1$; and by definition $d(P_1^j) = C_1 \lambda^j$. This proves our claim.

More generally, let $P_1 < \cdots < P_J$ be J prime numbers, together with reals $C_1 < \ldots < C_J$ and $\lambda_1 < \cdots < \lambda_J$ such that $1 < \lambda_j < \sqrt{2}$ and $C_j \geq 1$ for all j, and form the corresponding functions $d_1, \cdots d_J$. The product of sub-multiplicative functions being again a sub-multiplicative function, we deduce

that the product $d = d_1 \ldots d_J$ is another example of sub-multiplicative function satisfying condition (9), with a constant C which has to be greater than $C_1\lambda_1 \ldots C_J\lambda_J$.

2. Main result

The following theorem is the main result of the paper.

THEOREM 2. *Let d be a non-negative sub-multiplicative function. Assume that condition (9) is realized. Let $0 \leq \sigma < 1/2$. Then there exists a constant $C_{\sigma,d}$ depending on σ and d only, such that for any integer $N \geq 2$,*

$$\mathbf{E} \sup_{t \in \mathbf{R}} |\mathcal{D}(\sigma + it)| \leq C_{\sigma,d} \, \widetilde{D}_2(N) \, B,$$

where

$$B = \begin{cases} \frac{N^{1/2-\sigma}\tau_d^{1/2}}{(\log N)^{1/2}} & \text{if } \left(\frac{N \log \log N}{\log N}\right)^{1/2} \leq \tau_d \leq \pi(N), \\ \frac{N^{3/4-\sigma}(\log \log N)^{1/4}}{(\log N)^{3/4}} & \text{if } \left(\frac{N}{(\log N)\log \log N}\right)^{1/2} \leq \tau_d \leq \left(\frac{N \log \log N}{\log N}\right)^{1/2}, \\ N^{1/2-\sigma}\left(\frac{\tau_d \log \log \tau_d}{\log \tau_d}\right)^{1/2} & \text{if } 1 \leq \tau_d \leq \left(\frac{N}{(\log N)\log \log N}\right)^{1/2}. \end{cases}$$

Observe that condition (2) implies condition (9). Indeed, write $n = kp$ and take $j = 1$. We get $d(n) = d(kp) \leq Cd(k) = Cd(n/p)$. Fix some real λ, $1 < \lambda < \sqrt{2}$. Then $d(p^j) \leq Cd(1)j^H \leq C_1\lambda^j$, for some suitable constant C_1.

Further, function d_1 obviously satisfies condition (9), whereas we know that it does not satisfy condition (2).

The bounds given in Theorem 2 being all less than $C_{\sigma,\lambda}\widetilde{D}_2(N)\frac{N^{1-\sigma}}{\log N}$, we therefore deduce that Theorem 1 is strictly included in Theorem 2. We give two classes of examples of application.

EXAMPLE 1. Consider multiplicative functions satisfying the following condition:

$$\frac{d(p^a)}{d(p^{a-1})} \leq \lambda, \qquad a = 1, 2, \ldots \tag{11}$$

Clearly (11) is strictly weaker than (7). Further it implies (9). First $d(p^j) \leq d(1)\lambda^j$. Next, let $p \mid n$ and a denote the p-valuation of n: $p^a || n$. By multiplicativity of $d(.)$ and condition (11)

$$d(n) = d\left(\frac{n}{p^a}\right) d(p^a) = d\left(\frac{n}{p^a}\right) d(p^{a-1}) \frac{d(p^a)}{d(p^{a-1})} = d\left(\frac{n}{p}\right) \frac{d(p^a)}{d(p^{a-1})} \leq \lambda \, d\left(\frac{n}{p}\right).$$

Thus (9) is fulfilled. Notice that (11) implies

$$M_d := \sup_p d(p) < \infty \quad \text{with} \quad M_d \leq \lambda d(1). \tag{12}$$

Under condition (11), estimates for $\widetilde{D}_2(N)$ are known. By theorem 2 of [4] (see also [3]), any non-negative multiplicative function d satisfying a Wirsing type condition

$$d(p^m) \leq \lambda_1 \lambda_2^m, \tag{13}$$

for some constants $\lambda_1 > 0$ and $0 < \lambda_2 < 2$ and all prime powers $p^m \leq x$, also satisfies

$$\frac{1}{x} \sum_{n \leq x} d(n) \ \leq \ C(\lambda_1, \lambda_2) \exp\left\{ \sum_{p \leq x} \frac{d(p) - 1}{p} \right\}, \tag{14}$$

where $C(\lambda_1, \lambda_2)$ depends on λ_1, λ_2 only.

As d satisfies (11), if $\lambda < \sqrt{2}$, condition (13) is verified with $\lambda_1 = M_d$, $\lambda_2 = \lambda$. Since d^2 is multiplicative and satisfies (11) with $\lambda^2 < 2$, we also have that d^2 verifies condition (13) as well. Consequently, from (14) it follows that

$$\widetilde{D}_1(N) \ \leq \ C(\lambda) \exp\left\{ \sum_{p \leq N} \frac{d(p) - 1}{p} \right\},$$

$$\widetilde{D}_2(N) \ \leq \ C(\lambda) \exp\left\{ \sum_{p \leq N} \frac{d^2(p) - 1}{p} \right\} \tag{15}$$

for some constant $C(\lambda)$ depending on λ only. Recall that there exists an absolute constant c_1 such that for $x \geq 2$

$$\left| \sum_{p \leq x} \frac{1}{p} - \log \log x - c_1 \right| < \frac{5}{\log x}. \tag{16}$$

Thus

$$\sum_{p \leq x} \frac{d(p)}{p} \ \leq \ M_d \sum_{p \leq x} \frac{1}{p} \ \leq \ M_d \log(c_2 \log x)$$

and similarly

$$\sum_{p \leq x} \frac{d^2(p)}{p} \ \leq \ M_d^2 \log(c_2 \log x).$$

Thereby under condition (11), we have the following estimates

$$\widetilde{D}_1(N) \leq C(\lambda)(\log N)^{M_d}, \qquad \widetilde{D}_2(N) \leq C(\lambda)(\log N)^{M_d^2}. \tag{17}$$

For functions d_1, d_2, there is also a standard way to proceed. Letting $\tau = \pi(N)$, we have for d_2, for instance,

$$\frac{1}{N} \sum_{n \leq N} \lambda^{\Omega(n)} \leq \sum_{n \leq N} \frac{\lambda^{\Omega(n)}}{n} \leq \sum_{\alpha_1=0}^{\infty} \cdots \sum_{\alpha_\tau=0}^{\infty} \frac{\lambda^{\alpha_1 + \cdots + \alpha_\tau}}{p_1^{\alpha_1} \cdots p_\tau^{\alpha_\tau}} = \prod_{j=1}^{\tau} \left(1 - \frac{\lambda}{p_j} \right)^{-1}$$

which can be evaluated by means of (16).

The restriction $\lambda < 2$ can be relaxed into $\lambda < q$, when considering, instead of $\mathcal{D}(s)$, random Dirichlet polynomials based on sets of integers having all their prime divisors greater or equal to q, e.g., on some arithmetic progressions. To go beyond a condition of type (11), notably to work under the weaker condition (13), one has probably to perform another approach than the one based on a decomposition into random processes as appearing in (30) below.

EXAMPLE 2. Take some positive integer K, and put

$$d_K(n) = \begin{cases} 1 & \text{if } (n, K) = 1, \\ 0 & \text{if } (n, K) > 1. \end{cases}$$

Then d_K is sub-multiplicative. Let $p \,|\, n$. By definition, $d_K(n/p) = 0$ if and only if $(n/p, K) > 1$, in which case $(n, K) > 1$ and so $d_K(n) = 0$. Thus $d_K(n) \leq d_K(n/p)$. Now if $d_K(n/p) = 1$, that $d_K(n) \leq d_K(n/p)$ is trivial. Besides $d_K(p^j) = d_K(p) \leq 1$. Therefore condition (9) is satisfied with $C = 1 = \lambda$. This defines the remarkable class of random Dirichlet polynomials,

$$\mathcal{D}(s) = \sum_{\substack{(n, K)=1 \\ 1 \leq n \leq N}} \frac{\varepsilon_n}{n^s}, \tag{18}$$

which naturally extends the one of $\mathcal{E}_\tau$-based Dirichlet polynomials considered in [10] and [7]. Indeed, recall that $\mathcal{E}_\tau = \left\{ 2 \leq n \leq N : P^+(n) \leq p_\tau \right\}$. Define

$$K_\tau = \begin{cases} \prod_{\tau < \ell \leq \pi(N)} p_\ell & \text{if } \tau < \pi(N), \\ 1 & \text{if } \tau = \pi(N). \end{cases} \tag{19}$$

Then $n \in \mathcal{E}_\tau$, $n \leq N$, if and only if $(n, K_\tau) = 1$, namely $d_{K_\tau}(n) = 1$. So that

$$\sum_{n \in \mathcal{E}_\tau} \frac{\varepsilon_n}{n^s} = \sum_{n=1}^{N} d_{K_\tau}(n) \frac{\varepsilon_n}{n^s}. \tag{20}$$

Consequently, the $\mathcal{E}_\tau$-based Dirichlet polynomials are one example of Dirichlet polynomials with sub-multiplicative weights. Here $\mathcal{H}_{d_{K_\tau}} = \sum_{j \leq \tau} E_j$. We therefore neglect cells E_j, $j > \tau$. Further, we have $\widetilde{D}_1(N) = \widetilde{D}_2(N) \leq 1$.

If we now specify Theorem 2 to this case, we get

COROLLARY 3. *Let $0 < \sigma < 1/2$. We have*

$$\mathbf{E} \sup_{t \in \mathbf{R}} \left| \sum_{n \in \mathcal{E}_\tau} \frac{\varepsilon_n}{n^{\sigma+it}} \right| \leq C_\sigma B, \qquad (21)$$

where

$$B = \begin{cases} \frac{N^{1/2-\sigma}\tau^{1/2}}{(\log N)^{1/2}} & \text{if } \left(\frac{N \log \log N}{\log N}\right)^{1/2} \leq \tau \leq \pi(N), \\ \frac{N^{3/4-\sigma}(\log \log N)^{1/4}}{(\log N)^{3/4}} & \text{if } \left(\frac{N}{(\log N)\log \log N}\right)^{1/2} \leq \tau \leq \left(\frac{N \log \log N}{\log N}\right)^{1/2}, \\ N^{1/2-\sigma}\left(\frac{\tau \log \log \tau}{\log \tau}\right)^{1/2} & \text{if } 1 \leq \tau \leq \left(\frac{N}{(\log N)\log \log N}\right)^{1/2}. \end{cases}$$

REMARK 4. By comparing this with the upper bound part of Theorem 1.1 in [7], we observe that the bounds obtained are either identical (if $N^{1/2} \leq \tau \leq \pi(N)$), or strictly better. For instance, when

$$\left(\frac{N}{(\log N)\log \log N}\right)^{1/2} \leq \tau \leq \left(\frac{N \log \log N}{\log N}\right)^{1/2},$$

we have

$$\frac{N^{3/4-\sigma}(\log \log N)^{1/4}}{(\log N)^{3/4}} \ll \frac{N^{3/4-\sigma}}{(\log N)^{1/2}},$$

thereby yielding a better bound. In the discussion following Theorem 1.1 in [7], we compared both the upper bounds and lower bounds obtained, and showed that the gap between them was of order

$$(\log N)^{1/2} \quad \text{if } N^{1/2}/\log N \leq \tau \leq N^{1/2},$$

and of order

$$(\log \tau)^{1/2} \quad \text{if } 1 \leq \tau \leq N^{1/2}/\log N.$$

In the case $N^{1/2} \leq \tau \leq N$, the bounds are optimal.

Corollary 3 also yields, when compared with these lower bounds, a much smaller gap of order $\log \log N$ instead of $\log N$. For instance, the gap is of order

$$(\log \log N)^{1/4} \quad \text{if } \left(\frac{N}{(\log N)\log \log N}\right)^{1/2} \leq \tau \leq \left(\frac{N \log \log N}{\log N}\right)^{1/2},$$

whereas it is of order

$$(\log \log \tau)^{1/2} \quad \text{if } 1 \leq \tau \leq \left(\frac{N}{(\log N)\log \log N}\right)^{1/2}.$$

EXAMPLE 3. Fix some integer $N \geq 1$, and consider the truncated divisor function

$$d_N(n) = \#\{k \leq N : k|n\}.$$

This function, which occurs in many important arithmetical questions, is sub-multiplicative. Take n and m coprimes. If $k \leq N$ is such that $k|mn$, then k is uniquely written

$$k = k_1 k_2, \ (k_1, k_2) = 1, \ k_1|m, \ k_2|n; \quad \text{and naturally,} \quad k_1 \leq N, \ k_2 \leq N.$$

We infer that $d_N(mn) \leq d_N(m)d_N(n)$.

Further, it satisfies our condition (9). Let $p|n$, if $p > N$, then $d_N(n) = d_N(\frac{n}{p})$. Otherwise, if $p \leq N$, let $\mathcal{K} = \{k \leq N : (k, p) = 1\}$. For $k \in \mathcal{K}$ such that $k|n$, the p-height $p(k)$ of k denotes the largest integer a so that $p^a k|n$ and $p^a k \leq N$. The divisors of n are of type $p^a k$, $k \in \mathcal{K}$. Further if $p^a k_1 = p^b k_2$, $k_1, k_2 \in \mathcal{K}$, necessarily $k_1 = k_2$. Indeed, it is obvious if $a = b$; and if $a > b$ we get $p|k_2$, which excluded. Consequently,

$$d_N(n) = \sum_{\substack{k \in \mathcal{K} \\ k|n}} \big(1 + p(k)\big), \qquad d_N\left(\frac{n}{p}\right) = \sum_{\substack{k \in \mathcal{K} \\ k|n}} [1 + (p(k) - 1)^+].$$

As for any integer $a \geq 0$, $1 + a \leq 2[1 + (a-1)^+]$, we deduce

$$d_N(n) \leq 2d_N\left(\frac{n}{p}\right).$$

And choosing any $\lambda > 1$, we obviously have

$$d_N(p^j) = \#\{\ell \leq j : p^\ell \leq N\} \leq j \leq C\lambda^j.$$

Although the proof is much in the spirit of proofs of the main results in [7], [8], there are substantial changes and simplifications. First, we work from the beginning with Gaussian processes. Further, the delicate step of estimating some related metric and computing associated entropy numbers is notably simplified. Cauchy-Schwarz's inequality and the comparison properties of Gaussian processes indeed allow to avoid any computation (see before (52)), and also give rise to strictly better estimates.

This further allowed us to consider random Dirichlet polynomials with more complicated arithmetical structure, like the one of "Hall type" built from the sub-multiplicative functions d_K, where entropy numbers seem hard to estimate efficiently.

Let $\tau = \tau_d$ and let $a_j(n)$ denote the p_j-valuation of integer n. Put

$$\underline{a}(n) = \big\{a_j(n), 1 \leq j \leq \tau\big\} \qquad (n \leq N).$$

Let also $\mathbf{T} = [0,1[= \mathbf{R}/\mathbf{Z}$ be the torus. A first classical reduction allows to replace the Dirichlet polynomial by some relevant trigonometric polynomial. To any Dirichlet polynomial

$$P(s) = \sum_{n=1}^{N} a_n n^{-s},$$

associate the trigonometric polynomial $Q(\underline{z})$ defined by

$$Q(\underline{z}) = \sum_{n=1}^{N} a_n n^{-\sigma} e^{2i\pi \langle \underline{a}(n), \underline{z} \rangle}, \qquad \underline{z} = (z_1, \ldots, z_\tau) \in \mathbf{T}^\tau.$$

By Kronecker's Theorem ([5], Theorem 442)

$$\sup_{t \in \mathbf{R}} \left| P(\sigma + it) \right| = \sup_{\underline{z} \in \mathbf{T}^\tau} \left| Q(\underline{z}) \right|, \tag{22}$$

as observed in [1].

REMARK 5. Naturally, no similar reduction occurs when considering the supremum over a given bounded interval I. However, when the length of I is of exponential size with respect to the degree of P, precisely when

$$|I| \geq e^{(1+\varepsilon)\omega N (\log N\omega) \log N},$$

the related supremum becomes comparable, for ω large, to the one taken on the real line, with an error term of order $\mathcal{O}(\omega^{-1})$. This is in turn a rather general phenomenon due to existence of "localized" versions of Kronecker's theorem, and in the present case to Turán's estimate (see [14] for a slightly improved form of it, and references therein). When the length is of sub-exponential order, the study still seems to belong to the field of application of the general theory of regularity of stochastic processes.

In the technical lemma below, we collected some useful estimates, which have already appeared in [8], and are easily deduced from the fact that if a_n are complex numbers and $b \in \mathcal{C}^1([1, x])$, then (see, for instance, [11] p. 3)

$$\sum_{1 \leq n \leq x} a_n b(n) = A(x)b(x) - \int_1^x A(t)b'(t)dt, \tag{23}$$

where we let $A(t) = \sum_{n \leq t} a_n$.

LEMMA 6. *Let $M \leq N$ and $0 < \sigma < 1/2$. Then*

$$\sum_{m \leq M} \frac{d(m)^2}{m^{2\sigma}} \leq C\widetilde{D}_2^2(M)M^{1-2\sigma}. \tag{24}$$

$$\sum_{m \leq M} \left(\frac{N}{m}\right)^{1/2} \left(\log\left(\frac{N}{m}\right)\right)^{-1/2} d(m) \leq C\widetilde{D}_1(M)(NM)^{1/2}\left(\log\left(\frac{N}{M}\right)\right)^{-1/2}. \tag{25}$$

$$\sum_{k \leq M} \frac{d(k)^2}{k^{2\sigma}} \leq C\widetilde{D}_2(M)^2(M)^{1-2\sigma}. \tag{26}$$

Now we can pass to the

Proof of Theorem 2. Fix some integer ν in $[1, \tau]$. We denote

$$F_\nu = \sum_{1 \leq j \leq \nu} E_j, \qquad F^\nu = \sum_{\nu < j \leq \tau} E_j.$$

Consider as in [7], [8] the decomposition $Q = Q_1^\varepsilon + Q_2^\varepsilon$, where

$$Q_1^\varepsilon(\underline{z}) = \sum_{n \in F_\nu} \varepsilon_n d(n) n^{-\sigma} e^{2i\pi\langle \underline{a}(n), \underline{z}\rangle},$$

$$Q_2^\varepsilon(\underline{z}) = \sum_{n \in F^\nu} \varepsilon_n d(n) n^{-\sigma} e^{2i\pi\langle \underline{a}(n), \underline{z}\rangle}.$$

By the contraction principle ([6] p. 16–17)

$$\mathbf{E} \sup_{\underline{z} \in \mathbf{T}^\tau} \left|Q_i^\varepsilon(\underline{z})\right| \leq 4 \sqrt{\frac{\pi}{2}} \, \mathbf{E} \sup_{\underline{z} \in \mathbf{T}^\tau} \left|Q_i(\underline{z})\right| \qquad (i = 1, 2), \tag{27}$$

where Q_i is the same process as Q_i^ε except that the Rademacher random variables ε_n are replaced by independent $\mathcal{N}(0,1)$ random variables μ_n. Consequently, both the supremums of Q_1 and of Q_2 can be estimated, via their associated L^2-metric.

We will establish the two following estimates:

$$\mathbf{E} \sup_{\underline{z} \in \mathbf{T}^\tau} \left|Q_1(\underline{z})\right| \leq CN^{1/2-\sigma}\widetilde{D}_2(N)\left(\frac{\nu \log\log\nu}{\log\nu}\right)^{1/2}, \tag{28}$$

and

$$\mathbf{E} \sup_{\underline{z} \in \mathbf{T}^\tau} \left|Q_2(\underline{z})\right| \leq C\left(N^{1/2-\sigma}\widetilde{D}_2(N/p_\nu)\frac{\tau^{1/2}}{(\log\tau)^{1/2}} + \frac{N^{1-\sigma}\widetilde{D}_1(N/p_\nu)}{\nu^{1/2}\log\nu}\right). \tag{29}$$

First, evaluate the supremum of Q_2. Writing

$$Q_2(\underline{z}) = \sum_{\nu < j \leq \tau} e^{2i\pi z_j} \sum_{n \in E_j} \mu_n d(n) n^{-\sigma} e^{2i\pi \{ \sum_{k \neq j} a_k(n) z_k + [a_j(n) - 1] z_j \}}$$

$$= \sum_{\nu < j \leq \tau} e^{2i\pi z_j} \sum_{n \in E_j} \mu_n d(n) n^{-\sigma} e^{2i\pi \left\{ \sum_k a_k(\frac{n}{p_j}) z_k \right\}}$$

next expanding yields that Q_2 maybe be written as a sum of four terms, each of them being, up to a factor $1, i, -1$, one of the possible realizations of the random process X defined by

$$X(\gamma) = \sum_{\nu < j \leq \tau} \alpha_j \sum_{n \in E_j} \mu_n \frac{d(n)}{n^\sigma} \beta_{\frac{n}{p_j}}, \qquad \gamma \in \Gamma, \tag{30}$$

where

$$\gamma = \left((\alpha_j)_{\nu < j \leq \tau}, (\beta_m)_{1 \leq m \leq N/2} \right)$$

and

$$\Gamma = \left\{ \gamma : |\alpha_j| \vee |\beta_m| \leq 1, \ \nu < j \leq \tau, \ 1 \leq m \leq N/2 \right\}.$$

Here, indeed,

$$\alpha_j = \alpha_j(\underline{z}) = \begin{cases} \cos(2\pi z_j), \\ \text{or} \\ \sin(2\pi z_j), \end{cases} \qquad \nu < j \leq \tau;$$

and

$$\beta_m = \beta_m(\underline{z}) = \begin{cases} \cos\left(2\pi \sum_k a_k(m) z_k\right), \\ \text{or} \\ \sin\left(2\pi \sum_k a_k(m) z_k\right), \end{cases} \qquad 1 \leq m \leq \frac{N}{2}.$$

Consequently,

$$\sup_{\underline{z} \in \mathbf{T}^\tau} |Q_2(\underline{z})| \leq 4 \sup_{\gamma \in \Gamma} |X(\gamma)|. \tag{31}$$

This sieve principle, letting the two parameters α and β appear, is due to Quéffelec (see [10]). This is naturally an important step in the elaboration of the proof.

The problem now reduces to estimating the supremum over Γ of the real valued Gaussian process X. We observe that

$$\|X_\gamma - X_{\gamma'}\|_2^2 = \sum_{\nu < j \leq \tau} \sum_{n \in E_j} d(n)^2 n^{-2\sigma} \left[\alpha_j \beta_{\frac{n}{p_j}} - \alpha'_j \beta'_{\frac{n}{p_j}} \right]^2$$

$$\leq 2 \sum_{\nu < j \leq \tau} \sum_{n \in E_j} d(n)^2 n^{-2\sigma} \left[(\alpha_j - \alpha'_j)^2 + (\beta_{\frac{n}{p_j}} - \beta'_{\frac{n}{p_j}})^2 \right].$$

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As $p_j \mid n$, by condition (9), $d(n) \le \lambda \, d(\frac{n}{p_j})$; and so

$$\sum_{\nu < j \le \tau} \sum_{n \in E_j} \frac{d(n)^2}{n^{2\sigma}} (\alpha_j - \alpha_j')^2 \le \lambda^2 \sum_{\nu < j \le \tau} (\alpha_j - \alpha_j')^2 p_j^{-2\sigma} \sum_{m \le N/p_j} \frac{d(m)^2}{m^{2\sigma}}$$

$$\le \lambda^2 \sum_{\nu < j \le \tau} (\alpha_j - \alpha_j')^2 \frac{N^{1-2\sigma} \widetilde{D}_2^2(N/p_j)}{p_j}, \quad (32)$$

where we used estimate (24) of Lemma 6.

Besides, by condition (9) again, we obtain

$$\sum_{\nu < j \le \tau} \sum_{n \in E_j} \frac{d(n)^2 \left(\beta_{\frac{n}{p_j}} - \beta'_{\frac{n}{p_j}} \right)^2}{n^{2\sigma}} \le C\lambda^2 \sum_{m \le N/p_\nu} (\beta_m - \beta_m')^2 \left(\sum_{\substack{\nu < j \le \tau \\ mp_j \le N}} \frac{d(m)^2}{(mp_j)^{2\sigma}} \right)$$

$$:= C\lambda^2 \sum_{m \le N/p_\nu} K_m^2 (\beta_m - \beta_m')^2. \quad (33)$$

Let $k \in (\nu, \tau]$ be such that $N/p_k < m \le N/p_{k-1}$. Since $p_j \sim j \log j$, we have

$$K_m^2 = \sum_{\nu < j \le k-1} d(m)^2 (mp_j)^{-2\sigma} \le d(m)^2 m^{-2\sigma} \sum_{j \le k-1} p_j^{-2\sigma}$$

$$\le C \, d(m)^2 m^{-2\sigma} \sum_{j \le k} (j \log j)^{-2\sigma} \le C \, d(m)^2 m^{-2\sigma} \frac{k^{1-2\sigma}}{(\log k)^{2\sigma}}$$

$$\le C \, d(m)^2 m^{-2\sigma} \frac{k}{p_k^{2\sigma}} \le C \, m^{-2\sigma} d(m)^2 \frac{k}{(N/m)^{2\sigma}}$$

$$= C \, d(m)^2 \frac{k}{N^{2\sigma}} \, .$$

We have $k \log k \le Cp_k \le C \, \frac{N}{m}$, and so $k \le C \, \frac{N}{m} \left(\log(\frac{N}{m}) \right)^{-1}$. Thus

$$K_m \le C \, d(m) N^{-\sigma} \left(\frac{N}{m} \right)^{1/2} \left(\log\left(\frac{N}{m} \right) \right)^{-1/2}. \quad (34)$$

By using estimate (25) of Lemma 6

$$\sum_{m \le N/p_\nu} K_m \le CN^{-\sigma} \sum_{m \le N/p_\nu} \left(\frac{N}{m} \right)^{1/2} \left(\log\left(\frac{N}{m} \right) \right)^{-1/2} d(m)$$

$$\le \frac{CN^{1-\sigma} \widetilde{D}_1(N/p_\nu)}{\nu^{1/2} \log \nu} \, . \quad (35)$$

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Now define a second Gaussian process by putting for all $\gamma \in \Gamma$

$$Y(\gamma) = \sum_{\nu < j \leq \tau} \left(\frac{\widetilde{D}_2^2(N/p_j) N^{1-2\sigma}}{p_j} \right)^{1/2} \alpha_j \xi_j' + \sum_{m \leq N/p_\nu} K_m \, \beta_m \xi_m'' := \, Y_\gamma' + Y_\gamma'',$$

where ξ_i', ξ_j'' are independent $\mathcal{N}(0,1)$ random variables. It follows from (32) and (33) that for some suitable constant C, one has the comparison relations: for all $\gamma, \gamma' \in \Gamma$,

$$\|X_\gamma - X_{\gamma'}\|_2 \leq C \|Y_\gamma - Y_{\gamma'}\|_2.$$

By the Slepian comparison lemma ([15], Lemma 10.2.3), since $X_0 = Y_0 = 0$, we have

$$\mathbf{E} \sup_{\gamma \in \Gamma} |X_\gamma| \leq \ 2\mathbf{E} \sup_{\gamma \in \Gamma} X_\gamma \leq \ 2C \mathbf{E} \sup_{\gamma \in \Gamma} Y_\gamma \leq \ 2C \mathbf{E} \sup_{\gamma \in \Gamma} |Y_\gamma|. \tag{36}$$

And with (31)

$$\mathbf{E} \sup_{\underline{z} \in \mathbf{T}^\tau} \big|Q_2(\underline{z})\big| \leq \ C \mathbf{E} \sup_{\gamma \in \Gamma} \big|Y(\gamma)\big|. \tag{37}$$

It remains to evaluate the supremum of Y. First of all,

$$\mathbf{E} \sup_{\gamma \in \Gamma} |Y'(\gamma)| \ \leq \ N^{\frac{1}{2}-\sigma} \sum_{\nu < j \leq \tau} p_j^{-1/2} \widetilde{D}_2(N/p_j).$$

As $p_j \sim j \log j$, we have

$$\sum_{\nu < j \leq \tau} p_j^{-1/2} \leq \sum_{1 < j \leq \tau} p_j^{-1/2} \leq \frac{C \tau^{1/2}}{(\log \tau)^{1/2}},$$

thus

$$\mathbf{E} \sup_{\gamma \in \Gamma} |Y'(\gamma)| \leq C \ N^{\frac{1}{2}-\sigma} \widetilde{D}_2(N/p_\nu) \frac{\tau^{1/2}}{(\log \tau)^{1/2}}. \tag{38}$$

To control the supremum of Y'', we use our estimates for the sums of K_m and write that

$$\mathbf{E} \sup_{\gamma \in \Gamma} |Y''(\gamma)| \ \leq \ \sum_{m \leq N/p_\nu} K_m \leq \frac{C N^{1-\sigma} \widetilde{D}_1(N/p_\nu)}{\nu^{1/2} \log \nu} \ . \tag{39}$$

Therefore by reporting (38), (39) into (37), we get (29).

Now, we turn to the supremum of Q_1. Introduce the auxiliary Gaussian process

$$\Upsilon(\underline{z}) = \sum_{n \in F_\nu} d(n) n^{-\sigma} \big\{ \vartheta_n \cos 2\pi \langle \underline{a}(n), \underline{z} \rangle + \vartheta_n' \sin 2\pi \langle \underline{a}(n), \underline{z} \rangle \big\}, \qquad \underline{z} \in \mathbf{T}^\nu,$$

where ϑ_i, ϑ_j' are independent $\mathcal{N}(0,1)$ random variables. By symmetrization (see, e.g., Lemma 2.3 p. 269 in [9]),

$$\mathbf{E} \sup_{\underline{z} \in \mathbf{T}^\nu} \big|Q_1(\underline{z})\big| \ \leq \ \sqrt{8\pi} \mathbf{E} \sup_{\underline{z} \in \mathbf{T}^\nu} \big|\Upsilon(\underline{z})\big|. \tag{40}$$

Further

$$\|\Upsilon(\underline{z}) - \Upsilon(\underline{z})\|_2^2 = 4 \sum_{n \in F_\nu} \frac{d(n)^2}{n^{2\sigma}} \sin^2(\pi \langle \underline{a}(n), \underline{z} - \underline{z}' \rangle)$$

$$\leq 4\pi^2 \sum_{n \in F_\nu} \frac{d(n)^2}{n^{2\sigma}} |\langle \underline{a}(n), \underline{z} - \underline{z}' \rangle|^2$$

$$\leq 4\pi^2 \sum_{n \in F_\nu} \frac{d(n)^2}{n^{2\sigma}} \left[\sum_{j=1}^\nu a_j(n) \left| z_j - z_j' \right| \right]^2.$$

Now

$$\sum_{n \in F_\nu} \frac{d(n)^2}{n^{2\sigma}} \left[\sum_{j=1}^\nu a_j(n) |z_j - z_j'| \right]^2 = \sum_{j=1}^\nu |z_j - z_j'|^2 \sum_{n \in F_\nu} \frac{a_j(n)^2 d(n)^2}{n^{2\sigma}} +$$

$$\sum_{\substack{1 \leq j_1, j_2 \leq \nu \\ j_1 \neq j_2}} |z_{j_1} - z_{j_1}'| \, |z_{j_2} - z_{j_2}'| \sum_{n \in F_\nu} \frac{a_{j_1}(n) a_{j_2}(n) d(n)^2}{n^{2\sigma}} := S + R.$$

Examine first the contribution of the rectangle terms. Only those integers n such that $a_{j_1}(n) \geq 1$ and $a_{j_2}(n) \geq 1$ are to be considered. Using sub-multiplicativity, we have

$$R \leq \sum_{\substack{1 \leq j_1, j_2 \leq \nu \\ j_1 \neq j_2}} |z_{j_1} - z_{j_1}'| \, |z_{j_2} - z_{j_2}'| \sum_{b_1, b_2 = 1}^\infty b_1 b_2 \sum_{\substack{n \leq N, a_{j_1}(n) = b_1, \\ a_{j_2}(n) = b_2}} \frac{d(n)^2}{n^{2\sigma}}$$

$$\leq C \sum_{\substack{1 \leq j_1, j_2 \leq \nu \\ j_1 \neq j_2}} |z_{j_1} - z_{j_1}'| \, |z_{j_2} - z_{j_2}'| \sum_{b_1, b_2 = 1}^\infty \frac{b_1 d(p_{j_1}^{b_1})^2}{p_{j_1}^{2b_1 \sigma}} \frac{b_2 d(p_{j_2}^{b_2})^2}{p_{j_2}^{2b_2 \sigma}}$$

$$\times \left[\sum_{k \leq \frac{N}{p_{j_1}^{b_1} p_{j_2}^{b_2}}} \frac{d(k)^2}{k^{2\sigma}} \right]. \tag{41}$$

Examine now the contribution of the square terms. We have

$$S \leq \sum_{j=1}^\nu |z_j - z_j'|^2 \sum_{b=1}^\infty \sum_{\substack{n \in F_\nu \\ a_j(n) = b}} \frac{b^2 d(n)^2}{n^{2\sigma}}$$

$$\leq \sum_{j=1}^\nu |z_j - z_j'|^2 \sum_{b=1}^\infty \frac{b^2 d(p_j^b)^2}{p_j^{2b\sigma}} \sum_{m \leq \frac{N}{p_j^b}} \frac{d(m)^2}{m^{2\sigma}}. \tag{42}$$

By estimate (26) of Lemma 6, we have

$$\sum_{k \le \frac{N}{p_{j_1}^{b_1} p_{j_2}^{b_2}}} \frac{d(k)^2}{k^{2\sigma}} \le C\widetilde{D}_2(N)^2 \left[\frac{N}{p_{j_1}^{b_1} p_{j_2}^{b_2}} \right]^{1-2\sigma}. \tag{43}$$

Hence

$$R \le C\widetilde{D}_2(N)^2 \sum_{\substack{1 \le j_1, j_2 \le \nu \\ j_1 \ne j_2}} |z_{j_1} - z'_{j_1}| |z_{j_2} - z'_{j_2}| \sum_{b_1, b_2 = 1}^{\infty} \frac{b_1 d(p_{j_1}^{b_1})^2}{p_{j_1}^{2b_1\sigma}} \frac{b_2 d(p_{j_2}^{b_2})^2}{p_{j_2}^{2b_2\sigma}} \left[\frac{N}{p_{j_1}^{b_1} p_{j_2}^{b_2}} \right]^{1-2\sigma}$$

$$= C\widetilde{D}_2(N)^2 N^{1-2\sigma} \sum_{\substack{1 \le j_1, j_2 \le \nu \\ j_1 \ne j_2}} |z_{j_1} - z'_{j_1}| |z_{j_2} - z'_{j_2}| \sum_{b_1, b_2 = 1}^{\infty} \frac{b_1 d(p_{j_1}^{b_1})^2}{p_{j_1}^{b_1}} \frac{b_2 d(p_{j_2}^{b_2})^2}{p_{j_2}^{b_2}}.$$

But, by condition (9)

$$\sum_{b \ge 1} b \frac{d(p_j^b)^2}{p_j^b} \le C \sum_{b \ge 1} b \left(\frac{\lambda^2}{2} \right)^b \left(\frac{2}{p_j} \right)^b \le C \left(\frac{2}{p_j} \right) \sum_{b \ge 1} b \left(\frac{\lambda^2}{2} \right)^b \le \frac{C_\lambda}{p_j}. \tag{44}$$

From this follows that

$$R \le C_\lambda \widetilde{D}_2(N)^2 N^{1-2\sigma} \left[\sum_{j=1}^{\nu} \frac{|z_j - z'_j|}{p_j} \right]^2. \tag{45}$$

Further

$$\begin{aligned}
S &\le C\widetilde{D}_2(N)^2 \sum_{j=1}^{\nu} |z_j - z'_j|^2 \sum_{b=1}^{\infty} \frac{b^2 d(p_j^b)^2}{p_j^{2b\sigma}} \left[\frac{N}{p_j^b} \right]^{1-2\sigma} \\
&\le C\widetilde{D}_2(N)^2 N^{1-2\sigma} \sum_{j=1}^{\nu} |z_j - z'_j|^2 \sum_{b=1}^{\infty} \frac{b^2 d(p_j^b)^2}{p_j^b} \\
&\le C\widetilde{D}_2(N)^2 N^{1-2\sigma} \left[\sum_{j=1}^{\nu} \frac{|z_j - z'_j|^2}{p_j} \right],
\end{aligned} \tag{46}$$

by arguing as in (44) for getting the last inequality.

Consequently,

$$\left\| \Upsilon(\underline{z}) - \Upsilon(\underline{z}) \right\|_2$$
$$\le C_\lambda N^{1/2-\sigma} \widetilde{D}_2(N) \max \left(\sum_{j=1}^{\nu} \frac{|z_j - z'_j|}{p_j}, \left[\sum_{j=1}^{\nu} \frac{|z_j - z'_j|^2}{p_j} \right]^{1/2} \right). \tag{47}$$

We shall control the Gaussian process Υ in a more simple and more efficient way than in [7], [8]. By the Cauchy-Schwarz inequality

$$
\sum_{j=1}^{\nu} \frac{|z_j - z_j'|}{p_j} \leq \left(\sum_{j=1}^{\nu} \frac{|z_j - z_j'|^2}{p_j} \right)^{1/2} \left(\sum_{j=1}^{\nu} \frac{1}{p_j} \right)^{1/2}
$$

$$
\leq \left(\sum_{j=1}^{\nu} \frac{|z_j - z_j'|^2}{p_j} \right)^{1/2} \left(\sum_{j=1}^{\nu} \frac{1}{j \log j} \right)^{1/2}
$$

$$
\leq (\log \log \nu)^{1/2} \left(\sum_{j=1}^{\nu} \frac{|z_j - z_j'|^2}{p_j} \right)^{1/2}. \tag{48}
$$

Therefore,

$$
\left\| \Upsilon(\underline{z}) - \Upsilon(\underline{z}) \right\|_2 \leq C_\lambda N^{1/2-\sigma} \widetilde{D}_2(N) (\log \log \nu)^{1/2} \left(\sum_{j=1}^{\nu} \frac{|z_j - z_j'|^2}{p_j} \right)^{1/2}. \tag{49}
$$

A Gaussian metric appears: let indeed $g_1, \ldots, g_\nu$ be independent $\mathcal{N}(0,1)$ distributed random variables. Then $U(z) := \sum_{j=1}^{\nu} g_j p_j^{-1/2} z_j$ satisfies

$$
\|U(z) - U(z')\|_2 = \left(\sum_{j=1}^{\nu} \frac{|z_j - z_j'|^2}{p_j} \right)^{1/2}.
$$

And so

$$
\left\| \Upsilon(\underline{z}) - \Upsilon(\underline{z}) \right\|_2 \leq C_\lambda N^{1/2-\sigma} \widetilde{D}_2(N) (\log \log \nu)^{1/2} \|U(z) - U(z')\|_2. \tag{50}
$$

Now we take again advantage of the comparison properties of Gaussian processes, and deduce from Slepian's Lemma

$$
\mathbf{E} \sup_{\underline{z}, \underline{z}' \in T^\nu} |\Upsilon(\underline{z}') - \Upsilon(\underline{z})| \leq C_\lambda N^{1/2-\sigma} \widetilde{D}_2(N) (\log \log \nu)^{1/2} \mathbf{E} \sup_{\underline{z}, \underline{z}' \in T^\nu} |U(\underline{z}') - U(\underline{z})|.
$$

But obviously

$$
\sup_{\underline{z} \in T^\nu} |U(z)| = \sum_{j=1}^{\nu} |g_j| p_j^{-1/2}.
$$

Thereby

$$
\mathbf{E} \sup_{\underline{z}' \in T^\nu} |U(\underline{z}') - U(\underline{z})| \leq C \sum_{j=1}^{\nu} p_j^{-1/2} \leq C \sum_{j=1}^{\nu} \frac{1}{(j \log j)^{1/2}} \leq C \left(\frac{\nu}{\log \nu} \right)^{1/2}.
$$

And by substituting

$$
\mathbf{E} \sup_{\underline{z}' \in T^\nu} |\Upsilon(\underline{z}') - \Upsilon(\underline{z})| \leq C_\lambda N^{1/2-\sigma} \widetilde{D}_2(N) \left(\frac{\nu \log \log \nu}{\log \nu} \right)^{1/2}.
$$

Observe also that

$$\|\Upsilon(\underline{z})\|_2 \leq C N^{1/2-\sigma} \widetilde{D}_2(N), \quad \underline{z} \in \mathbf{T}^\nu. \tag{51}$$

Thus

$$\mathbf{E} \sup_{\underline{z}' \in T^\nu} |\Upsilon(\underline{z}')| \leq C N^{1/2-\sigma} \widetilde{D}_2(N) \left(\frac{\nu \log \log \nu}{\log \nu}\right)^{1/2}. \tag{52}$$

This is slightly better than in [8], inequality (22), where one has the bound $C N^{1/2-\sigma} \widetilde{D}_2(N) \nu^{1/2}$. By substituting in (40) we get

$$\mathbf{E} \sup_{\underline{z} \in \mathbf{T}^\nu} |Q_1(\underline{z})| \leq C_{\sigma,\lambda} N^{1/2-\sigma} \widetilde{D}_2(N) \left(\frac{\nu \log \log \nu}{\log \nu}\right)^{1/2}, \tag{53}$$

which is (28).

Since

$$\widetilde{D}_1(N/p_\nu) \leq \widetilde{D}_2(N/p_\nu) \leq \widetilde{D}_2(N),$$

we consequently get from (28), (29) and (22),

$$\mathbf{E} \sup_{t \in \mathbf{R}} |\mathcal{D}(\sigma + it)| \leq$$
$$C_{\sigma,\lambda} N^{1/2-\sigma} \widetilde{D}_2(N) \left[\left(\frac{\nu \log \log \nu}{\log \nu}\right)^{1/2} + \frac{\tau^{1/2}}{(\log \tau)^{1/2}} + \frac{N^{1/2}}{\nu^{1/2} \log \nu}\right]. \tag{54}$$

We now observe that

$$f(x) := (x \log \log x)^{1/2} (\log x)^{-1/2} + N^{1/2} x^{-1/2} (\log x)^{-1}$$

satisfies

$$f'(x) \sim \frac{1}{2} x^{-1/2} (\log x)^{-1/2} \left[(\log \log x)^{1/2} - N^{1/2} x^{-1} (\log x)^{-1/2}\right].$$

Thus we choose

$$\nu \sim \frac{N^{1/2}}{(\log \log N)^{1/2} (\log N)^{1/2}}.$$

We get

$$\frac{N^{1/2}}{\nu^{1/2} \log \nu} \approx \frac{N^{1/4} (\log \log N)^{1/4}}{(\log N)^{3/4}} \approx \left(\frac{\nu \log \log \nu}{\log \nu}\right)^{1/2}.$$

We find

$$\mathbf{E} \sup_{t \in \mathbf{R}} |\mathcal{D}(\sigma + it)| \leq C_{\sigma,\lambda} N^{1/2-\sigma} \widetilde{D}_2(N) \left[\frac{N^{1/4} (\log \log N)^{1/4}}{(\log N)^{3/4}} + \frac{\tau^{1/2}}{(\log \tau)^{1/2}}\right]. \tag{55}$$

We also observe that

$$\frac{N^{1/4} (\log \log N)^{1/4}}{(\log N)^{3/4}} \leq \frac{\tau^{1/2}}{(\log \tau)^{1/2}}, \quad \text{if and only if} \quad \tau \geq \left(\frac{N \log \log N}{\log N}\right)^{1/2}.$$

Further when $\tau \leq \left(\frac{N \log\log N}{\log N}\right)^{1/2}$, we may also just set $\nu = \tau$ in the initial decomposition, and thus ignore Q_2^ε. It means that we use the bound (53) in place of (54). This makes sense when τ is sufficiently small, namely when

$$\left(\frac{\tau \log\log \tau}{\log \tau}\right)^{1/2} \leq \frac{N^{1/4}(\log\log N)^{1/4}}{(\log N)^{3/4}},$$

which is so when

$$\tau \leq \left(\frac{N}{(\log N)\log\log N}\right)^{1/2}.$$

We consequently have to distinguish three cases:

Case 1. $\left(\frac{N \log\log N}{\log N}\right)^{1/2} \leq \tau \leq \pi(N)$. We get from (55)

$$\mathbf{E}\sup_{t\in\mathbf{R}}|\mathcal{D}(\sigma + it)| \ \leq \ C_{\sigma,\lambda}\frac{N^{1/2-\sigma}\widetilde{D}_2(N)\tau^{1/2}}{(\log N)^{1/2}}. \tag{56}$$

Case 2. $\left(\frac{N}{(\log N)\log\log N}\right)^{1/2} \leq \tau \leq \left(\frac{N \log\log N}{\log N}\right)^{1/2}$. In this case we obtain from (55)

$$\mathbf{E}\sup_{t\in\mathbf{R}}|\mathcal{D}(\sigma + it)| \ \leq \ C_{\sigma,\lambda}\frac{N^{3/4-\sigma}\widetilde{D}_2(N)(\log\log N)^{1/4}}{(\log N)^{3/4}}. \tag{57}$$

Case 3. $1 \leq \tau \leq \left(\frac{N}{(\log N)\log\log N}\right)^{1/2}$. By the comment made above, τ is small enough, and we forget Q_2^ε. We obtain from (53) directly

$$\mathbf{E}\sup_{t\in\mathbf{R}}|\mathcal{D}(\sigma + it)| \ \leq \ C_{\sigma,\lambda}N^{1/2-\sigma}\widetilde{D}_2(N)\left(\frac{\tau \log\log \tau}{\log \tau}\right)^{1/2}. \tag{58}$$

Summarizing

$$\mathbf{E}\sup_{t\in\mathbf{R}}|\mathcal{D}(\sigma + it)| \ \leq \ C_{\sigma,\lambda}\,\widetilde{D}_2(N)\,B,$$

where

$$B = \begin{cases} \frac{N^{1/2-\sigma}\tau^{1/2}}{(\log N)^{1/2}} & \text{if } \left(\frac{N \log\log N}{\log N}\right)^{1/2} \leq \tau \leq \pi(N), \\[2ex] \frac{N^{3/4-\sigma}(\log\log N)^{1/4}}{(\log N)^{3/4}} & \text{if } \left(\frac{N}{(\log N)\log\log N}\right)^{1/2} \leq \tau \leq \left(\frac{N \log\log N}{\log N}\right)^{1/2}, \\[2ex] N^{1/2-\sigma}\left(\frac{\tau \log\log \tau}{\log \tau}\right)^{1/2} & \text{if } 1 \leq \tau \leq \left(\frac{N}{(\log N)\log\log N}\right)^{1/2}. \end{cases}$$

This achieves the proof. $\qquad\qquad\square$

3. Number theoretic tools

In this section, we collect some tools from number theory used in the proofs. We do not think that these one are new. The following result of Hall will be useful. Let f be defined on positive integers and satisfying $f(1)=1, 0 \leq f(n) \leq 1$, and being sub-multiplicative.

Put
$$\mathbf{\Pi}(f, x) = \prod_{p \leq x} \left(1 - \frac{1}{p}\right) \left(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \dots\right).$$

Then ([2], theorem 2)
$$\sum_{n \leq x} f(n) \leq C\, x\mathbf{\Pi}(f, x), \tag{59}$$

C being an absolute constant. This estimate allows in turn a similar control for bounded non-negative sub-multiplicative functions.

Apply it to $f = d_K$. As
$$1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \dots = \frac{p}{p-1}, \quad \text{if } (p, K) = 1,$$

we have
$$\mathbf{\Pi}(f, x) = \prod_{\substack{p \leq x \\ (p,K)>1}} \left(1 - \frac{1}{p}\right) = \prod_{\substack{p \leq x \\ p|K}} \left(1 - \frac{1}{p}\right). \tag{60}$$

Hence the classical estimate, (see [2] for references)
$$\varphi_K(x) := \#\{k \leq x : (k, K) = 1\} \leq C\, x \prod_{\substack{p|K \\ p \leq x}} \left(1 - \frac{1}{p}\right). \tag{61}$$

We will need the following technical Lemma.

LEMMA 7.

a) *Let a real $\beta > 0$ and integer $L > 0$. Then*
$$\sum_{\substack{(n,L)=1 \\ n \leq x}} n^{-\beta} \leq C_\beta x^{1-\beta} \prod_{\substack{p|L \\ p \leq x}} \left(1 - \frac{1}{p}\right). \tag{62}$$

b) *Let $0 \leq \beta < 1$. Then*
$$\sum_{\substack{n \leq x \\ P^+(n) \leq y}} \frac{1}{n^\beta} \leq C_\beta x^{1-\beta} e^{-\frac{1}{2}\frac{\log x}{\log y}}, \tag{63}$$

for some constant C_β, $y \geq y_\beta$, $y/x \leq c_\beta$.

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c) *If $\beta = 1$, then*

$$\sum_{\substack{y \leq n \leq x \\ P^+(n) \leq y}} \frac{1}{n} \leq C \log y. \tag{64}$$

REMARK 8. It is natural to compare, in our setting, estimates a) and b), via the relation

$$\sum_{\substack{P^+(n) \leq p_\tau \\ n \leq N}} \frac{1}{n^\beta} = \sum_{\substack{(n, K_\tau) = 1 \\ n \leq N}} n^{-\beta}$$

where K_τ is defined in (19). By a) and Mertens Theorem we get

$$\sum_{\substack{P^+(n) \leq p_\tau \\ n \leq N}} \frac{1}{n^\beta} \leq C_\beta N^{1-\beta} \prod_{\tau < \ell \leq \pi(N)} \left(1 - \frac{1}{p}\right) \leq C_\beta N^{1-\beta} \frac{\log p_\tau}{\log N}.$$

However, by using b) we get the much better bound $C_\beta N^{1-\beta} e^{-\frac{1}{2} \frac{\log N}{\log p_\nu}}$.

P r o o f.

a) By applying formula (23) with $a_m = \chi\{(m, L) = 1\}$, $b_m = m^{-\sigma}$, $1 \leq m \leq x$,

$$\sum_{\substack{(m, L) = 1 \\ m \leq x}} m^{-\beta} \leq \frac{A(x)}{x^\beta} + \beta \int_1^x A(t) \frac{dt}{t^{\beta+1}},$$

where $A(t) = \sum_{n < t} d_L(n)$.

But by Hall's estimate (61), $A(t) \leq Ct \prod_{\substack{p \mid L \\ p \leq t}} \left(1 - \frac{1}{p}\right)$. Thus

$$\sum_{\substack{(m, L) = 1 \\ m \leq x}} m^{-\beta} \leq Cx \prod_{\substack{p \mid L \\ p \leq x}} \left(1 - \frac{1}{p}\right) \frac{1}{x^\beta} + C\beta \int_1^x \prod_{\substack{p \mid L \\ p \leq t}} \left(1 - \frac{1}{p}\right) \frac{dt}{t^\beta}$$

$$\leq C_\beta \int_1^x \prod_{\substack{p \mid L \\ p \leq t}} \left(1 - \frac{1}{p}\right) \frac{dt}{t^\beta}.$$

Applying now twice Mertens's theorem, gives

$$\prod_{\substack{p \mid L \\ p \leq t}} \left(1 - \frac{1}{p}\right) = \frac{\prod_{p \leq t} \left(1 - \frac{1}{p}\right)}{\prod_{\substack{p \nmid L \\ p \leq t}} \left(1 - \frac{1}{p}\right)} \leq \frac{C}{\log t \prod_{\substack{p \nmid L \\ p \leq x}} \left(1 - \frac{1}{p}\right)}$$

$$\leq \frac{C \prod_{\substack{p \mid L \\ p \leq x}} \left(1 - \frac{1}{p}\right)}{\log t \prod_{p \leq x} \left(1 - \frac{1}{p}\right)} \leq C \frac{\log x}{\log t} \prod_{\substack{p \mid L \\ p \leq x}} \left(1 - \frac{1}{p}\right). \tag{65}$$

Hence

$$\sum_{\substack{(m,L)=1 \\ m \leq x}} m^{-\beta} \leq C_\beta \log x \prod_{\substack{p|L \\ p \leq x}} \left(1 - \frac{1}{p}\right) \int_1^x \frac{dt}{t^\beta \log t} \leq C_\beta x^{1-\beta} \prod_{\substack{p|L \\ p \leq x}} \left(1 - \frac{1}{p}\right).$$

b) Let $\Psi(x,y) := \#\{n \leq x : P^+(n) \leq y\}$. By using this time (23) with $a_n = \chi\{P^+(n) \leq y\}$ $1 \leq n \leq N$, we obtain

$$\sum_{\substack{1 \leq n \leq x \\ P^+(n) \leq y}} \frac{1}{n^\beta} = \frac{\#\{1 \leq n \leq x : P^+(n) \leq y\}}{x^\beta}$$

$$+ \beta \int_1^x \frac{\#\{1 \leq n \leq t : P^+(n) \leq y\}}{t^{\beta+1}} dt$$

$$= \frac{\Psi(x,y)}{x^\beta} + \beta \int_1^y \frac{dt}{t^\beta} + \beta \int_y^x \frac{\Psi(t,y)}{t^{\beta+1}} dt. \qquad (66)$$

Recall that $\Psi(x,y) \leq x e^{-\frac{1}{2} \frac{\log x}{\log y}}$, $x \geq y \geq 2$, ([11], Chapter III. 5). Thus, for y sufficiently large to have $1 - \beta > \frac{1}{\log y}$,

$$\int_y^x \frac{\Psi(t,y)}{t^{\beta+1}} dt \leq \int_y^x e^{-\frac{1}{2} \frac{\log t}{\log y}} \frac{dt}{t^\beta} = \int_y^x t^{-\frac{1}{2 \log y} - \beta} dt$$

$$= \frac{1}{1 - \frac{1}{2 \log y} - \beta} \left(t^{1 - \frac{1}{2 \log y} - \beta} \Big|_{t=y}^{t=x}\right) \leq \frac{2}{1 - \beta} x^{1 - \frac{1}{2 \log y} - \beta}$$

$$= \frac{2}{1 - \beta} x^{1-\beta} e^{-\frac{1}{2} \frac{\log x}{\log y}}. \qquad (67)$$

Therefore,

$$\sum_{\substack{y \leq n \leq x \\ P^+(n) \leq y}} \frac{1}{n^\beta} \leq C_\beta \left[x^{1-\beta} e^{-\frac{1}{2} \frac{\log x}{\log y}} + y^{1-\beta}\right]. \qquad (68)$$

Now, we have

$$x^{1-\beta} e^{-\frac{1}{2} \frac{\log x}{\log y}} \geq y^{1-\beta} \quad \text{if and only if} \quad \log \frac{x}{y} \geq \frac{1}{2(1-\beta)} \frac{\log x}{\log y}.$$

Write $x = \theta y$, $\theta \geq 1$. This means

$$\log \theta \geq \frac{1}{2(1-\beta)} \frac{\log \theta y}{\log y} = \frac{1}{2(1-\beta)} \left\{\frac{\log \theta}{\log y} + 1\right\},$$

or

$$\log \theta \left\{1 - \frac{1}{2(1-\beta) \log y}\right\} \geq \frac{1}{2(1-\beta)}.$$

If y is large enough, $y \geq y_\beta$, y/x small enough, $y \leq c_\beta x$, then the above condition is satisfied. Consequently

$$\sum_{\substack{n \leq x \\ P^+(n) \leq y}} \frac{1}{n^\beta} \leq C_\beta x^{1-\beta} e^{-\frac{1}{2}\frac{\log x}{\log y}} . \tag{69}$$

c) The case $\beta = 1$ can be treated as before:

$$\sum_{\substack{1 \leq n \leq x \\ P^+(n) \leq y}} \frac{1}{n} = \frac{\Psi(x,y)}{x} + \int_1^y \frac{dt}{t} + \int_y^x \frac{\Psi(t,y)}{t^2} \, dt . \tag{70}$$

And

$$\int_y^x \frac{\Psi(t,y)}{t^2} \, dt \leq \int_y^x e^{-\frac{1}{2}\frac{\log t}{\log y}} \frac{dt}{t}$$

$$= \int_y^x t^{-\frac{1}{2\log y}-1} dt = \frac{1}{-\frac{1}{2\log y}} \left[t^{-\frac{1}{2\log y}} \Big|_{t=y}^{t=x} \right]$$

$$\leq \frac{1}{\frac{1}{2\log y}} y^{-\frac{1}{2\log y}} \leq C \log y. \tag{71}$$

Therefore,

$$\sum_{\substack{y \leq n \leq x \\ P^+(n) \leq y}} \frac{1}{n} \leq C \left[e^{-\frac{1}{2}\frac{\log x}{\log y}} + \log y \right] \leq C \log y. \tag{72}$$

One can, however, get this directly. Let

$$j = j_y = \max\{\ell : p_\ell \leq y\}.$$

Then, for any $\beta > 0$,

$$\sum_{\substack{1 \leq n \leq x \\ P^+(n) \leq y}} \frac{1}{n^\beta} \leq \sum_{\alpha_1=0}^{\infty} \cdots \sum_{\alpha_j=0}^{\infty} \frac{1}{p_1^{\alpha_1 \beta} \cdots p_j^{\alpha_j \beta}} = \prod_{\ell=1}^{j} \left(\frac{1}{1 - \frac{1}{p_\ell^\beta}} \right) . \tag{73}$$

And when $\beta = 1$, by Mertens Theorem, the latter is less than $\leq C \log y$. $\quad \square$

This last argument can be used to get a two-sided estimate when y is not too large. In this case, the estimates depend on y only. Put for $\beta > 0$, $j \geq 1$,

$$\Pi_\beta(j) = \prod_{\ell=1}^{j} \left[\frac{1}{1 - p_\ell^{-\beta}} \right] .$$

Lemma 9. *If $y = o(\log x)$, letting $j_y = \max\{j : p_j \leq y\}$, we have for any $\beta > 0$,*

$$c_\beta \Pi_\beta(j_y) \;\leq\; \sum_{\substack{1 \leq n \leq x \\ P^+(n) \leq y}} \frac{1}{n^\beta} \;\leq\; C_\beta \Pi_\beta(j_y). \tag{74}$$

And the involved constants c_β, C_β depend on β only. In particular

$$C_1 \log y \;\leq\; \sum_{\substack{1 \leq n \leq x \\ P^+(n) \leq y}} \frac{1}{n} \;\leq\; C_2 \log y. \tag{75}$$

P r o o f. Indeed, notice first, as $p_\ell \sim \ell \log \ell$, that we have $j_y \leq Cy/\log y$. In what follows, we write more simply $j = j_y$. Now consider integers $n = p_1^{\alpha_1} \ldots p_j^{\alpha_j}$, such that $\max\{\alpha_\ell, \ell \leq j\} \leq H := (\log x)/Cy$. Thus

$$n \leq y^{j \max\{\alpha_\ell, \ell \leq j\}} \;=\; e^{j(\log y) \max\{\alpha_\ell, \ell \leq j\}} \;\leq\; e^{\frac{Cy}{\log y}(\log y)\{\frac{\log x}{Cy}\}} \;\leq\; x.$$

We may also assume that $(H+1)\beta \geq 2$. Therefore,

$$\sum_{\substack{1 \leq n \leq x \\ P^+(n) \leq y}} \frac{1}{n^\beta} \geq \sum_{\alpha_1=0}^{H} \cdots \sum_{\alpha_j=0}^{H} \frac{1}{p_1^{\alpha_1\beta} \cdots p_j^{\alpha_j\beta}}$$

$$= \prod_{\ell=1}^{j} \left[\frac{1}{1 - \frac{1}{p_\ell^\beta}} - \sum_{\alpha_j=H+1}^{\infty} \frac{1}{p_\ell^{\alpha_\ell\beta}} \right] = \prod_{\ell=1}^{j} \left[\frac{1}{1 - \frac{1}{p_\ell^\beta}} \right] \prod_{\ell=1}^{j} \left[1 - \frac{1}{p_\ell^{(H+1)\beta}} \right]$$

$$\geq c_\beta \prod_{\ell=1}^{j} \left[\frac{1}{1 - \frac{1}{p_\ell^\beta}} \right].$$

But

$$\prod_{\ell=1}^{j} \left[1 - \frac{1}{p_\ell^{(H+1)\beta}} \right] \geq \prod_{\ell=1}^{j} \left[1 - \frac{1}{p_\ell^2} \right] \geq e^{-C' \sum_{\ell=1}^{\infty} p_\ell^{-2}} > 0.$$

since the series $\sum_{\ell=1}^{\infty} p_\ell^{-2}$ is obviously convergent. And so, in view of (73)

$$c_\beta \prod_{\ell=1}^{j} \left[\frac{1}{1 - \frac{1}{p_\ell^\beta}} \right] \leq \sum_{\substack{1 \leq n \leq x \\ P^+(n) \leq y}} \frac{1}{n^\beta} \leq C_\beta \prod_{\ell=1}^{j} \left[\frac{1}{1 - \frac{1}{p_\ell^\beta}} \right]. \tag{76}$$

When $\beta = 1$, by using Mertens Theorem

$$C_1 \log y \;\leq\; \sum_{\substack{1 \leq n \leq x \\ P^+(n) \leq y}} \frac{1}{n} \;\leq\; C_2 \log y.$$

$\square$

We continue with some useful observation.

REMARK 10. Let $u := \frac{\log x}{\log y}$ and $\rho(.)$ denote Dickman's function. According to ([11], p. 435),

$$\sum_{\substack{n \leq x \\ P^+(n) \leq y}} \frac{1}{n} = \log y \int_0^u \rho(v)dv + \mathcal{O}(u) = \log y \left(e^\gamma + \mathcal{O}\left(\frac{u}{\log y} + e^{-u/2}\right)\right) + \mathcal{O}(u)$$

$$= e^\gamma \log y + \mathcal{O}(u), \tag{77}$$

for $x \geq y \geq 2$, γ being Euler's constant.

4. Special cases of Theorem 2

We have obtained in Corollary 3 the following estimate

$$\mathbf{E} \sup_{t \in \mathbf{R}} \left| \sum_{n \in \mathcal{E}_\tau} \frac{\varepsilon_n}{n^{\sigma+it}} \right| \leq C_\sigma N^{1/2-\sigma} \left(\frac{\tau \log \log \tau}{\log \tau}\right)^{1/2},$$

valid for the range of values $1 \leq \tau \leq (\frac{N}{(\log N) \log \log N})^{1/2}$. When the order of τ is small, we shall prove the following strengthening in which N disappears from the estimates (there is no power of N).

THEOREM 11. *Assume that $\tau = o(\log N)$. Let $0 < \sigma < 1/2$. Then, there are c_σ, C_σ depending on σ only, such that*

$$c_\sigma \frac{\Pi_{2\sigma}(\tau)^{1/2} \tau^{1-\sigma}}{(\log \tau)^\sigma} \leq \mathbf{E} \sup_{t \in \mathbf{R}} \left| \sum_{\substack{n \leq N \\ P^+(n) \leq p_\tau}} \frac{\varepsilon_n}{n^{\sigma+it}} \right| \leq C_\sigma \frac{\Pi_{2\sigma}(\tau)^{1/2} \tau^{\frac{3}{2}-2\sigma}}{(\log \tau)^{2\sigma}}. \tag{78}$$

And if $\sigma = 1/2$, there are absolute constants C_1, C_2 such that

$$C_1 \tau^{1/2} \leq \mathbf{E} \sup_{t \in \mathbf{R}} \left| \sum_{j=1}^\tau \sum_{n \in E_j} \frac{\varepsilon_n}{\sqrt{n}} n^{-it} \right| \leq C_2 \tau^{1/2} (\log \log \tau)^{1/2}.$$

REMARK 12. It is easy to show that $\Pi_{2\sigma}(\tau) \leq C_\sigma \tau^{1-2\sigma} (\log \tau)^{-2\sigma}$. We deduce from the first part of Theorem 11,

$$\mathbf{E} \sup_{t \in \mathbf{R}} \left| \sum_{\substack{n \leq N \\ P^+(n) \leq p_\tau}} \frac{\varepsilon_n}{n^{\sigma+it}} \right| \leq C_\sigma \frac{\tau^{2-3\sigma}}{(\log \tau)^{3\sigma}} \ll_\sigma \frac{(\log N)^{2-3\sigma}}{(\log \log N)^{3\sigma}},$$

which is thus far much better than the one obtained in Corollary 3.

As the latter is a direct application of Theorem 2, Theorem 11 cannot therefore be deduced from this one. A more general statement than Theorem 2 including Theorem 11 should certainly have a much heavier formulation. If $\sigma = 1/2$, there is a gap of order $(\log \log \tau)^{1/2}$. Finally remark that the case considered there is not covered by [7, Theorem 1.2].

Proof of Theorem 11. First, we shall give the proof of the upper bound. It will be necessary to adapt to this case the step where Slepian's Lemma is used. Next we give the proof of the lower bound and will recall some elements of the approach initiated in [7], which is used in this part of the proof.

Proof of the upper bound. We have

$$\Upsilon(\underline{z}) = \sum_{n \in F_\tau} \frac{1}{n^\sigma} \left\{ \vartheta_n \cos 2\pi \langle \underline{a}(n), \underline{z} \rangle + \vartheta'_n \sin 2\pi \langle \underline{a}(n), \underline{z} \rangle \right\}.$$

And

$$\| \Upsilon(\underline{z}) - \Upsilon(\underline{z}') \|_2^2 \leq 4\pi^2 \sum_{n \in F_\tau} \frac{1}{n^{2\sigma}} \left[\sum_{j=1}^\tau a_j(n) |z_j - z'_j| \right]^2.$$

Now

$$\sum_{n \in F_\tau} \frac{1}{n^{2\sigma}} \left[\sum_{j=1}^\tau a_j(n) |z_j - z'_j| \right]^2 = \sum_{n \in F_\tau} \frac{1}{n^{2\sigma}} \sum_{j=1}^\tau a_j(n)^2 |z_j - z'_j|^2 +$$

$$\sum_{n \in F_\tau} \frac{1}{n^{2\sigma}} \sum_{1 \leq j_1 \neq j_2 \leq \tau} a_{j_1}(n) a_{j_2}(n) |z_{j_1} - z'_{j_1}| \, |z_{j_2} - z'_{j_2}| \; := \; S + R.$$

Further, by using Lemma 9

$$R \; \leq \; \sum_{1 \leq j_1 \neq j_2 \leq \tau} |z_{j_1} - z'_{j_1}| \, |z_{j_2} - z'_{j_2}| \sum_{b_1, b_2 = 1}^\infty b_1 b_2 \sum_{\substack{n \in F_\tau \\ a_{j_1}(n) = b_1 \\ a_{j_2}(n) = b_2}} \frac{1}{n^{2\sigma}}$$

$$\leq \; C \sum_{1 \leq j_1 \neq j_2 \leq \tau} |z_{j_1} - z'_{j_1}| \, |z_{j_2} - z'_{j_2}| \sum_{b_1, b_2 = 1}^\infty \frac{b_1 b_2}{p_{j_1}^{2b_1 \sigma} p_{j_2}^{2b_2 \sigma}} \left[\sum_{\substack{m \leq N/(p_{j_1}^{b_1} p_{j_2}^{b_2}) \\ P^+(m) \leq p_\tau}} \frac{1}{m^{2\sigma}} \right]$$

$$\leq \; C_\sigma \Pi_{2\sigma}(\tau) \sum_{1 \leq j_1 \neq j_2 \leq \tau} \frac{|z_{j_1} - z'_{j_1}|}{p_{j_1}^{2\sigma}} \frac{|z_{j_2} - z'_{j_2}|}{p_{j_2}^{2\sigma}}.$$

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Thus

$$
\begin{aligned}
R &\leq C_\sigma \Pi_{2\sigma}(\tau) \left(\sum_{j=1}^{\tau} \frac{|z_j - z'_j|}{p_j^{2\sigma}} \right)^2 \\
&\leq C_\sigma \Pi_{2\sigma}(\tau) \left(\sum_{j=1}^{\tau} \frac{1}{p_j^{2\sigma}} \right) \left(\sum_{j=1}^{\tau} \frac{|z_j - z'_j|^2}{p_j^{2\sigma}} \right) \\
&\leq C_\sigma \Pi_{2\sigma}(\tau) \left(\frac{\tau^{1-2\sigma}}{(\log \tau)^{2\sigma}} \right) \left(\sum_{j=1}^{\tau} \frac{|z_j - z'_j|^2}{p_j^{2\sigma}} \right).
\end{aligned}
\tag{79}
$$

And

$$
\begin{aligned}
S &\leq \sum_{n \in F_\tau} \frac{1}{n^{2\sigma}} \sum_{j=1}^{\tau} a_j(n)^2 |z_j - z'_j|^2 \leq \sum_{j=1}^{\tau} |z_j - z'_j|^2 \sum_{b=1}^{\infty} b^2 \sum_{\substack{n \in F_\tau \\ a_j(n)=b}} \frac{1}{n^{2\sigma}} \\
&\leq \sum_{j=1}^{\tau} |z_j - z'_j|^2 \sum_{b=1}^{\infty} \frac{b^2}{p_j^{2b\sigma}} \sum_{\substack{m \leq N/p_j^b \\ P^+(m) \leq p_\tau}} \frac{1}{m^{2\sigma}} \\
&\leq C_\sigma \Pi_{2\sigma}(\tau) \sum_{j=1}^{\tau} |z_j - z'_j|^2 \sum_{b=1}^{\infty} \frac{b^2}{p_j^{2b\sigma}} \leq C_\sigma \Pi_{2\sigma}(\tau) \sum_{j=1}^{\tau} \frac{|z_j - z'_j|^2}{p_j^{2\sigma}}.
\end{aligned}
\tag{80}
$$

Consequently,

$$
\left\| \Upsilon(\underline{z}) - \Upsilon(\underline{z}) \right\|_2^2 \leq C_\sigma \Pi_{2\sigma}(\tau) \left(\frac{\tau^{1-2\sigma}}{(\log \tau)^{2\sigma}} \right) \left[\sum_{j=1}^{\tau} \frac{|z_j - z'_j|^2}{p_j^{2\sigma}} \right].
\tag{81}
$$

We deduce from Slepian's Lemma, noting that $\log p_\tau \sim \log \tau$

$$
\begin{aligned}
\mathbf{E} \sup_{\underline{z}, \underline{z}' \in T^\tau} |\Upsilon(\underline{z}') - \Upsilon(\underline{z})| &\leq C_\sigma \Pi_{2\sigma}(\tau)^{1/2} \left(\frac{\tau^{\frac{1}{2}-\sigma}}{(\log \tau)^{\sigma}} \right) \left[\sum_{j=1}^{\tau} \frac{1}{p_j^{\sigma}} \right] \\
&\leq C_\sigma \Pi_{2\sigma}(\tau)^{1/2} \left(\frac{\tau^{\frac{1}{2}-\sigma}}{(\log \tau)^{\sigma}} \right) \left(\frac{\tau^{1-\sigma}}{(\log \tau)^{\sigma}} \right) \\
&= C_\sigma \Pi_{2\sigma}(\tau)^{1/2} \left(\frac{\tau^{\frac{3}{2}-2\sigma}}{(\log \tau)^{2\sigma}} \right).
\end{aligned}
$$

But

$$
\| \Upsilon(\underline{z}) \|_2 \leq \left[\sum_{n \in F_\tau} \frac{1}{n^{2\sigma}} \right]^{1/2} \leq C_\sigma \Pi_{2\sigma}(\tau)^{1/2}, \qquad \underline{z} \in \mathbf{T}^\tau.
\tag{82}
$$

Thus

$$\mathbf{E} \sup_{\underline{z} \in T^\tau} |\Upsilon(\underline{z})| \leq C_\sigma \Pi_{2\sigma}(\tau)^{1/2} \left(\frac{\tau^{\frac{3}{2} - 2\sigma}}{(\log \tau)^{2\sigma}} \right). \tag{83}$$

Recall that we have denoted $\Pi_{2\sigma}(\tau) = \prod_{\ell=1}^\tau (1 - p_\ell^{-2\sigma})^{-1}$. By combining (83) with (93), we get

$$c_\sigma \frac{\Pi_{2\sigma}(\tau)^{1/2} \tau^{1-\sigma}}{(\log \tau)^\sigma} \leq \mathbf{E} \sup_{t \in \mathbf{R}} \left| \sum_{\substack{n \leq N \\ P^+(n) \leq p_\tau}} \frac{\varepsilon_n}{n^{\sigma + it}} \right| \leq C_\sigma \left(\frac{\Pi_{2\sigma}(\tau)^{1/2} \tau^{\frac{3}{2} - 2\sigma}}{(\log \tau)^{2\sigma}} \right). \tag{84}$$

If $\sigma = 1/2$, the modifications for R and S are, by using Mertens Theorem

$$\begin{aligned}
R &\leq C\Pi_1(\tau) \left(\sum_{j=1}^\tau \frac{|z_j - z_j'|}{p_j} \right)^2 \leq C(\log \tau) \left(\sum_{j=1}^\tau \frac{1}{p_j} \right) \left(\sum_{j=1}^\tau \frac{|z_j - z_j'|^2}{p_j} \right) \\
&\leq C(\log \tau)(\log \log \tau) \left(\sum_{j=1}^\tau \frac{|z_j - z_j'|^2}{p_j} \right),
\end{aligned} \tag{85}$$

and

$$\begin{aligned}
S &\leq \sum_{j=1}^\tau |z_j - z_j'|^2 \sum_{b=1}^\infty \frac{b^2}{p_j^b} \sum_{\substack{m \leq N/p_j^b \\ P^+(m) \leq p_\tau}} \frac{1}{m} \leq C\Pi_1(\tau) \sum_{j=1}^\tau |z_j - z_j'|^2 \sum_{b=1}^\infty \frac{b^2}{p_j} \\
&\leq C(\log \tau) \sum_{j=1}^\tau \frac{|z_j - z_j'|^2}{p_j}.
\end{aligned} \tag{86}$$

Hence

$$\left\| \Upsilon(\underline{z}) - \Upsilon(\underline{z}) \right\|_2^2 \leq C(\log \tau)(\log \log \tau) \left(\sum_{j=1}^\tau \frac{|z_j - z_j'|^2}{p_j} \right). \tag{87}$$

And by Slepian's Lemma

$$\begin{aligned}
\mathbf{E} \sup_{\underline{z}, \underline{z}' \in T^\tau} |\Upsilon(\underline{z}') - \Upsilon(\underline{z})| &\leq C(\log \tau)^{1/2} (\log \log \tau)^{1/2} \left(\sum_{j=1}^\tau \frac{1}{p_j^{1/2}} \right) \\
&\leq C \left(\frac{\tau \log \log \tau}{\log \tau} \right)^{1/2}.
\end{aligned}$$

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As

$$\|\Upsilon(\underline{z})\|_2 \ \leq \ \left[\sum_{n\in F_\tau} \frac{1}{n}\right]^{1/2} \ \leq \ C\Pi_1(\tau)^{1/2} \ \leq \ C(\log \tau)^{1/2}, \qquad \underline{z} \in \mathbf{T}^\tau,$$

we conclude to

$$\mathbf{E} \sup_{\underline{z}\in T^\tau} |\Upsilon(\underline{z})| \ \leq \ C\tau^{1/2}(\log\log \tau)^{1/2}. \tag{88}$$

Combining this estimate with (94) finally gives

$$C_1\tau^{1/2} \leq \mathbf{E} \sup_{\underline{z}\in T^\tau} |\Upsilon(\underline{z})| \ \leq \ C_2\tau^{1/2}(\log\log \tau)^{1/2}. \tag{89}$$

$\square$

Now we pass to the proof of the lower bound. As already mentioned, we use a new approach to lower bounds introduced in [7]. Recall briefly its principle. We begin with a useful Lemma ([7], Lemma 3.1).

LEMMA 13. *Let $X = \{X_z, z \in Z\}$ and $Y = \{Y_z, z \in Z\}$ be two finite sets of random variables defined on a common probability space. We assume that X and Y are independent and that the random variables Y_z are all centered. Then*

$$\mathbf{E} \sup_{z\in Z} |X_z + Y_z| \geq \mathbf{E} \sup_{z\in Z} |X_z|.$$

Let $\underline{d} = \{d_n, n \geq 1\}$ be a sequence of real numbers. By the reduction step (22)

$$\sup_{t\in\mathbf{R}} \left|\sum_{j=1}^\tau \sum_{n\in E_j} d_n\varepsilon_n n^{-\sigma-it}\right| = \sup_{\underline{z}\in\mathbf{T}^\tau} \left|Q(\underline{z})\right|,$$

where

$$Q(\underline{z}) = \sum_{j=1}^\tau \sum_{n\in E_j} d_n\varepsilon_n n^{-\sigma} e^{2i\pi\langle\underline{a}(n),\underline{z}\rangle}.$$

Introduce the following subset of $\mathbf{T}^\tau$,

$$\mathcal{Z} = \left\{\underline{z} = \{z_j, 1 \leq j \leq \tau\} : z_j = 0 \text{ if } j \leq \tau/2, \text{ and } z_j \in \{0,1/2\} \text{ if } j \in]\tau/2, \tau]\right\}.$$

Observe that for any $\underline{z} \in \mathcal{Z}$, any n, $e^{2i\pi\langle\underline{a}(n),\underline{z}\rangle} = \cos(2\pi\langle\underline{a}(n),\underline{z}\rangle) = (-1)^{2\langle\underline{a}(n),\underline{z}\rangle}$. It follows that

$$\Im Q(\underline{z}) = 0,$$

and so

$$Q(\underline{z}) = \sum_{\tau/2<j\leq\tau} \sum_{n\in E_j} d_n\varepsilon_n n^{-\sigma}(-1)^{2\langle\underline{a}(n),\underline{z}\rangle}, \qquad \underline{z} \in \mathcal{Z}.$$

Thereby the restriction of Q to $\mathcal{Z}$ is just a finite rank Rademacher process. Now define

$$\mathcal{L}_j = \left\{ n = p_j\, \tilde{n} \ : \ \tilde{n} \le \frac{N}{p_j} \quad \text{and} \ \ P^+(\tilde{n}) \le p_{\tau/2} \right\}, \qquad j \in (\tau/2, \tau].$$

Since $E_j \supset \mathcal{L}_j$, $j = 1, \ldots \tau$, the sets $\mathcal{L}_j$ are pairwise disjoint. Put for $z \in \mathcal{Z}$,

$$Q'(\underline{z}) = \sum_{\tau/2 < j \le \tau} \sum_{n \in \mathcal{L}_j} \varepsilon_n n^{-\sigma}(-1)^{2\langle \underline{a}(n), \underline{z}\rangle}.$$

Since

$$\{Q(\underline{z}) - Q'(\underline{z}), \underline{z} \in \mathcal{Z}\} \quad \text{and} \quad \{Q'(\underline{z}), \underline{z} \in \mathcal{Z}\}$$

are independent, we deduce from the above Lemma that

$$\mathbf{E} \sup_{\underline{z} \in \mathcal{Z}} |Q(\underline{z})| \ \ge \ \mathbf{E} \sup_{\underline{z} \in \mathcal{Z}} |Q'(\underline{z})|.$$

It is possible to proceed to a direct evaluation of $Q'(\underline{z})$ and we recall that

$$\sup_{\underline{z} \in \mathcal{Z}} |Q'(\underline{z})| = \sum_{\tau/2 < j \le \tau} \left| \sum_{n \in \mathcal{L}_j} d_n \varepsilon_n n^{-\sigma} \right|$$

which, in view of the Khintchine inequalities for Rademacher sums, allows to get ([7], Proposition 3.2)

PROPOSITION 14. *There exists a universal constant c such that for any system of coefficients (d_n)*

$$c \sum_{\tau/2 < j \le \tau} \left| \sum_{n \in \mathcal{L}_j} d_n^2 n^{-2\sigma} \right|^{1/2} \ \le \ \mathbf{E} \sup_{\underline{z} \in \mathcal{Z}} |Q'(\underline{z})| \ \le \ \sum_{\tau/2 < j \le \tau} \left| \sum_{n \in \mathcal{L}_j} d_n^2 n^{-2\sigma} \right|^{1/2}.$$

Consequently,

$$\mathbf{E} \sup_{t \in \mathbf{R}} \left| \sum_{j=1}^{\tau} \sum_{n \in E_j} d_n \varepsilon_n n^{-\sigma - it} \right| \ \ge \ c \sum_{\tau/2 < j \le \tau} \left| \sum_{n \in \mathcal{L}_j} d_n^2 n^{-2\sigma} \right|^{1/2}. \tag{90}$$

P r o o f   o f   t h e   l o w e r   b o u n d. Take $d_n \equiv 1$ in estimate (90). We get

$$\mathbf{E} \sup_{t \in \mathbf{R}} \left| \sum_{\substack{n \le N \\ P^+(n) \le p_\tau}} \frac{\varepsilon_n}{n^{\sigma + it}} \right| \ = \ \mathbf{E} \sup_{t \in \mathbf{R}} \left| \sum_{j=1}^{\tau} \sum_{n \in E_j} \frac{\varepsilon_n}{n^{\sigma + it}} \right|$$

$$\ge \ c \sum_{\tau/2 < j \le \tau} \left| \sum_{n \in \mathcal{L}_j} \frac{1}{n^{2\sigma}} \right|^{1/2}. \tag{91}$$

By assumption

$$\log \frac{N}{p_j} \geq \log \frac{N}{p_{\tau/2}} = \log N - \log p_{\tau/2} \gg p_{\tau/2}.$$

Owing to the very definition of the sets $\mathcal{L}_j$, and using Lemma 9, we get

$$\sum_{\tau/2<j\leq\tau} \left| \sum_{n\in\mathcal{L}_j} \frac{1}{n^{2\sigma}} \right| = \sum_{\tau/2<j\leq\tau} \frac{1}{p_j^\sigma} \left[\sum_{\substack{\tilde{n}\leq\frac{N}{p_j} \\ P^+(\tilde{n})\leq p_{\tau/2}}} \frac{1}{\tilde{n}^{2\sigma}} \right]^{1/2}$$

$$\geq C_\sigma \Pi_{2\sigma}(\tau)^{1/2} \sum_{\tau/2<j\leq\tau} \frac{1}{p_j^\sigma}$$

$$\geq C_\sigma \Pi_{2\sigma}(\tau)^{1/2} \frac{\tau^{1-\sigma}}{(\log\tau)^\sigma}. \tag{92}$$

Consequently,

$$\mathbf{E} \sup_{t\in\mathbf{R}} \left| \sum_{\substack{n\leq N \\ P^+(n)\leq p_\tau}} \frac{\varepsilon_n}{n^{\sigma+it}} \right| \geq C_\sigma \Pi_{2\sigma}(\tau)^{1/2} \frac{\tau^{1-\sigma}}{(\log\tau)^\sigma}. \tag{93}$$

And if $\sigma = 1/2$, by Mertens Theorem,

$$\mathbf{E} \sup_{t\in\mathbf{R}} \left| \sum_{\substack{n\leq N \\ P^+(n)\leq p_\tau}} \frac{\varepsilon_n}{n^{\frac{1}{2}+it}} \right| \geq C\tau^{1/2}. \tag{94}$$
$$\square$$

We conclude with another case. Return to Example 2 and let now K be unspecified. There is no loss to assume K is square-free. First examine the case when K has few prime divisors. Suppose

$$\sum_{\substack{p|K \\ p\leq N}} \frac{1}{p^\sigma} = o\left(\frac{N^{1-\sigma}}{\log N}\right). \tag{95}$$

Using Bohr's lower bound

$$\mathbf{E} \sup_{t\in\mathbf{R}} \left| \sum_{\substack{(n,K)=1 \\ 1\leq n\leq N}} \frac{\varepsilon_n}{n^s} \right| \geq C \sum_{\substack{(p,K)=1 \\ p\leq N}} \frac{1}{p^\sigma}. \tag{96}$$

We get with 2 a two-sided estimate

$$C\frac{N^{1-\sigma}}{\log N} \leq \mathbf{E} \sup_{t\in\mathbf{R}} \left| \sum_{\substack{(n,K)=1 \\ 1\leq n\leq N}} \frac{\varepsilon_n}{n^s} \right| \leq C\frac{N^{1-\sigma}}{\log N}. \tag{97}$$

The case of a number K with many prime divisors is more complicated. By the comment previously made, this concerns the case

$$\sum_{\substack{p \mid K \\ p \leq N}} \frac{1}{p^\sigma} \asymp \frac{N^{1-\sigma}}{\log N}. \tag{98}$$

We restrict ourselves to integers K of type

$$K = \prod_{\substack{p \mid K \\ p \leq p_\nu}} p \cdot \prod_{p_\nu < p \leq N} p,$$

where $1 \leq \nu < \pi(N)$. This amounts to consider the random Dirichlet polynomials

$$\sum_{\substack{1 \leq n \leq N \\ (n,K)=1}} \frac{\varepsilon_n}{n^s} = \sum_{\substack{n \in F_\nu \\ (n,K)=1}} \frac{\varepsilon_n}{n^s}.$$

We will assume ν to be not too large. More precisely, we assume, in accordance with Corollary 3

$$\nu \leq \left(\frac{N}{(\log N) \log \log N} \right)^{1/2}.$$

THEOREM 15. *Let $0 < \sigma < 1/2$. There exists a constant C_σ depending on σ only such that*

$$\mathbf{E} \sup_{t \in \mathbf{R}} \left| \sum_{\substack{(n,K)=1 \\ 1 \leq n \leq N}} \frac{\varepsilon_n}{n^s} \right| \leq C_\sigma N^{1/2-\sigma} \max \left(1, \sum_{\substack{k \leq \nu \\ p_k \nmid K}} \frac{1}{p_k} \right)^{1/2} \left[\sum_{\substack{k \leq \nu \\ p_k \nmid K}} \frac{1}{\sqrt{p_k}} \right].$$

P r o o f o f T h e o r e m 15. We examine more specifically the increments of the Gaussian process Υ. There is no loss to assume

$$p \mid K \implies p \leq p_\nu.$$

We have here

$$\Upsilon(\underline{z}) = \sum_{\substack{n \in F_\nu \\ (n,K)=1}} n^{-\sigma} \left\{ \vartheta_n \cos 2\pi \langle \underline{a}(n), \underline{z} \rangle + \vartheta_n' \sin 2\pi \langle \underline{a}(n), \underline{z} \rangle \right\}. \tag{99}$$

And as

$$(n,K) = 1 \quad \text{if and only if} \quad a_\ell(n) > 0 \implies (p_\ell, K) = 1,$$

$$\|\Upsilon(\underline{z}) - \Upsilon(\underline{z})\|_2^2 = 4 \sum_{\substack{n \in F_\nu \\ (n,K)=1}} n^{-2\sigma} \sin^2(\pi \langle \underline{a}(n), \underline{z} - \underline{z}' \rangle)$$

$$\leq 4\pi^2 \sum_{\substack{n \in F_\nu \\ (n,K)=1}} n^{-2\sigma} \left[\sum_{\substack{1 \leq j \leq \nu \\ (p_j, K)=1}} a_j(n) |z_j - z_j'| \right]^2.$$

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Now

$$
\sum_{\substack{n \in F_\nu \\ (n,K)=1}} n^{-2\sigma} \left[\sum_{\substack{1 \le j \le \nu \\ (p_j,K)=1}} a_j(n)|z_j - z_j'| \right]^2 = \sum_{\substack{n \in F_\nu \\ (n,K)=1}} n^{-2\sigma} \sum_{\substack{1 \le j \le \nu \\ (p_j,K)=1}} a_j(n)^2 |z_j - z_j'|^2 +
$$

$$
\sum_{\substack{n \in F_\nu \\ (n,K)=1}} n^{-2\sigma} \sum_{\substack{1 \le j_1 \ne j_2 \le \nu \\ (p_{j_1} p_{j_2}, K)=1}} a_{j_1}(n) a_{j_2}(n) |z_{j_1} - z_{j_1}'| \, |z_{j_2} - z_{j_2}'| \;:=\; S + R.
$$

Further

$$
\begin{aligned}
R \;\le\; & \sum_{\substack{1 \le j_1 \ne j_2 \le \nu \\ (p_{j_1} p_{j_2}, K)=1}} |z_{j_1} - z_{j_1}'| \, |z_{j_2} - z_{j_2}'| \sum_{b_1,b_2=1}^{\infty} b_1 b_2 \sum_{\substack{n \in F_\nu, \, (n,K)=1 \\ a_{j_1}(n)=b_1, a_{j_2}(n)=b_2}} \frac{1}{n^{2\sigma}} \\[2mm]
\le\; & C \sum_{\substack{1 \le j_1 \ne j_2 \le \nu \\ (p_{j_1} p_{j_2}, K)=1}} |z_{j_1} - z_{j_1}'| \, |z_{j_2} - z_{j_2}'| \sum_{b_1,b_2=1}^{\infty} \frac{b_1 b_2}{(p_{j_1}^{b_1} p_{j_2}^{b_2})^{2\sigma}} \\[2mm]
& \times \left[\sum_{m \le N/(p_{j_1}^{b_1} p_{j_2}^{b_2})} m^{-2\sigma} \right] \\[2mm]
\le\; & CN^{1-2\sigma} \sum_{\substack{1 \le j_1 \ne j_2 \le \nu \\ (p_{j_1} p_{j_2}, K)=1}} |z_{j_1} - z_{j_1}'| \, |z_{j_2} - z_{j_2}'| \sum_{b_1,b_2=1}^{\infty} \frac{b_1 b_2}{p_{j_1}^{b_1} p_{j_2}^{b_2}}. \quad (100)
\end{aligned}
$$

But

$$
\sum_{b=1}^{\infty} \frac{b}{p_k^b} = \sum_{b=1}^{\infty} \frac{b}{2^b} \left[\frac{2}{p_k} \right]^b \;\le\; \frac{2}{p_k} \sum_{b=1}^{\infty} \frac{b}{2^b} \;\le\; C p_k^{-1}.
$$

Thus

$$
\begin{aligned}
R \;\le\; & CN^{1-2\sigma} \left(\sum_{\substack{1 \le j \le \nu \\ (p_j,K)=1}} \frac{|z_j - z_j'|}{p_j} \right)^2 \\[2mm]
\le\; & CN^{1-2\sigma} \left(\sum_{\substack{1 \le j \le \nu \\ (p_j,K)=1}} \frac{1}{p_j} \right) \left(\sum_{\substack{1 \le j \le \nu \\ (p_j,K)=1}} \frac{|z_j - z_j'|^2}{p_j} \right). \quad (101)
\end{aligned}
$$

And

$$
\begin{aligned}
S \ &\leq\ \sum_{\substack{n \in F_\nu \\ (n,K)=1}} n^{-2\sigma} \sum_{\substack{1 \leq j \leq \nu \\ (p_j,K)=1}} a_j(n)^2 |z_j - z_j'|^2 \\[2mm]
&\leq\ \sum_{\substack{1 \leq j \leq \nu \\ (p_j,K)=1}} |z_j - z_j'|^2 \sum_{b=1}^{\infty} b^2 \sum_{\substack{n \in F_\nu \\ (n,K)=1 \\ a_j(n)=b}} \frac{1}{n^{2\sigma}} \\[2mm]
&\leq\ \sum_{\substack{1 \leq j \leq \nu \\ (p_j,K)=1}} |z_j - z_j'|^2 \sum_{b=1}^{\infty} \frac{b^2}{p_j^{2b\sigma}} \sum_{m \leq N/p_j^b} \frac{1}{m^{2\sigma}} \\[2mm]
&\leq\ \sum_{\substack{1 \leq j \leq \nu \\ (p_j,K)=1}} |z_j - z_j'|^2 \sum_{b=1}^{\infty} \frac{b^2}{p_j^{b}} \ \leq\ \sum_{\substack{1 \leq j \leq \nu \\ (p_j,K)=1}} \frac{|z_j - z_j'|^2}{p_j}. \qquad (102)
\end{aligned}
$$

Therefore,

$$
\left\| \Upsilon(\underline{z}) - \Upsilon(\underline{z}) \right\|_2^2 \ \leq\ C_\sigma N^{1-2\sigma} \left[\sum_{\substack{k \leq \nu \\ p_k \nmid K}} \frac{|z_k - z_k'|^2}{p_k} \right] \max\left(1, \sum_{\substack{1 \leq j \leq \nu \\ (p_j,K)=1}} \frac{1}{p_j} \right). \quad (103)
$$

Let

$$
\Delta \ :=\ N^{1/2-\sigma} \max\left(1, \sum_{\substack{1 \leq j \leq \nu \\ (p_j,K)=1}} \frac{1}{p_j} \right)^{1/2}.
$$

We obtain

$$
\left\| \Upsilon(\underline{z}) - \Upsilon(\underline{z}) \right\|_2 \ \leq\ C_\sigma \Delta \left[\sum_{\substack{k \leq \nu \\ p_k \nmid K}} \frac{|z_k - z_k'|^2}{p_k} \right]^{1/2}. \quad (104)
$$

Let $g_1, \ldots, g_\nu$ be independent $\mathcal{N}(0,1)$ distributed random variables and define

$$
U(z) \ :=\ \sum_{\substack{k \leq \nu \\ p_k \nmid K}} g_k p_k^{-1/2} z_k.
$$

Then

$$
\left\| \Upsilon(\underline{z}) - \Upsilon(\underline{z}) \right\|_2 \ \leq\ C_\sigma \Delta \| U(z) - U(z') \|_2. \quad (105)
$$

We deduce from Slepian's Lemma

$$
\mathbf{E} \sup_{\underline{z}' \in T^\nu} |\Upsilon(\underline{z}') - \Upsilon(\underline{z})| \ \leq\ C_\sigma \Delta \mathbf{E} \sup_{\underline{z}' \in T^\nu} |U(\underline{z}') - U(\underline{z})|.
$$

Obviously,

$$\sup_{\underline{z} \in T^\nu} |U(z)| \;=\; \sum_{\substack{k \le \nu \\ p_k \nmid K}} \frac{|g_k|}{p_k^{1/2}}.$$

Thereby

$$\mathbf{E} \sup_{\underline{z}' \in T^\nu} |U(\underline{z}') - U(\underline{z})| \le C \sum_{\substack{k \le \nu \\ p_k \nmid K}} p_k^{-1/2}.$$

And by reporting

$$\mathbf{E} \sup_{\underline{z}' \in T^\nu} |\Upsilon(\underline{z}') - \Upsilon(\underline{z})| \;\le\; C_\sigma \Delta \left[\sum_{\substack{k \le \nu \\ p_k \nmid K}} \frac{1}{\sqrt{p_k}} \right].$$

But

$$\|\Upsilon(\underline{z})\|_2 \;\le\; \left[\sum_{\substack{n \in F_\nu \\ (n,K)=1}} \frac{1}{n^{2\sigma}} \right]^{1/2} \;\le\; C_\sigma N^{1/2-\sigma} \left[\sum_{\substack{k \le \nu \\ p_k \nmid K}} \frac{1}{p_j} \right]^{1/2}, \qquad \underline{z} \in \mathbf{T}^\nu. \quad (106)$$

Thus

$$\mathbf{E} \sup_{\underline{z}' \in T^\nu} |\Upsilon(\underline{z}')| \le C_\sigma \Delta \left[\sum_{\substack{k \le \nu \\ p_k \nmid K}} \frac{1}{\sqrt{p_k}} \right], \qquad (107)$$

or

$$\mathbf{E} \sup_{\underline{z}' \in T^\nu} |\Upsilon(\underline{z}')| \le N^{1/2-\sigma} \max\left(1, \sum_{\substack{k \le \nu \\ p_k \nmid K}} \frac{1}{p_k} \right)^{1/2} \left[\sum_{\substack{k \le \nu \\ p_k \nmid K}} \frac{1}{\sqrt{p_k}} \right]. \qquad (108)$$

$\square$

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