A CHARACTERIZATION OF HIGHER ORDER NETS USING WEYL SUMS AND ITS APPLICATIONS

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ABSTRACT. Point sets referred to as $(t, \alpha, \beta, n, m, s)$-nets were recently introduced and shown to generalize both digital $(t, \alpha, \beta, n \times m, s)$-nets and classical (t, m, s)-nets. Their definition captures the geometrical properties of their digital analogue, which has recently been shown to yield quadrature points for quasi-Monte Carlo rules which can achieve arbitrary high convergence rates of the integration error for sufficiently smooth functions. In this paper, we characterize $(t, \alpha, \beta, n, m, s)$-nets using Weyl sums generalizing the analogous result for (t, m, s)-nets.

As an application of this characterization we study numerical integration using such higher order nets. It is shown that for functions having square integrable mixed partial derivatives of order α in each variable, integration errors converge at a rate of $N^{-(\alpha-1)+\delta}$ for any $\delta > 0$, establishing that $(t, \alpha, \beta, n, m, s)$-nets can exploit the smoothness of the function under consideration.

The characterization is consequently employed to study the randomization of $(t, \alpha, \beta, n, m, s)$-nets and the application of randomized $(t, \alpha, \beta, n, m, s)$-nets to numerical integration. It is found that the root mean-square error converges at a rate of $N^{-(\alpha-\frac{1}{2})+\delta}$ for any $\delta > 0$, improving on the result on integration errors associated with $(t, \alpha, \beta, n, m, s)$-nets.

As a further application, it can be used for the construction of new $(t, \alpha, \beta, n, m, s)$-nets itself: We introduce an analogue of the $(u, u + v)$-construction for digital $(t, \alpha, \beta, n \times m, s)$-nets and (t, m, s)-nets.

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1. Introduction

Quasi-Monte Carlo rules are equal weight integration formulas used to approximate integrals over the unit cube $[0,1]^s$, where the dimension s is typically large. Quasi-Monte Carlo point sets can roughly be divided into integration lattices, see [21, 28], and nets, see [14, 21]. In this paper, we will focus on nets, which were formally introduced by Niederreiter [20] with a view towards providing deterministic sample points for quasi-Monte Carlo rules, see [14, 21, 22]. Furthermore, for important results in this direction which examine the discrepancy of nets, see [4, 26, 27].

Digital nets, see [14, 20, 21], are an important special case of nets and were recently generalized by Dick, see [6, 7]. Dick introduced higher order digital nets and sequences in [6, 7], see also [14], and showed that point sets $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}$, obtained from a higher order digital net or sequence, can be used in a quasi-Monte Carlo rule $b^{-m} \sum_{h=0}^{b^m-1} f(\boldsymbol{x}_h)$ to approximate the integral $\int_{[0,1]^s} f(\boldsymbol{x}) \, d\boldsymbol{x}$, and that the integration error can achieve an arbitrary high rate of convergence for sufficiently smooth functions.

In [10], the geometrical properties of those higher order digital nets and sequences, called digital $(t, \alpha, \beta, n \times m, s)$-nets and digital $(t, \alpha, \beta, \sigma, s)$-sequences, were analyzed, see also [9]. Point sets satisfying a certain geometrical property exhibited by the higher order digital nets and sequences are called $(t, \alpha, \beta, n, m, s)$-nets, which include both digital $(t, \alpha, \beta, n \times m, s)$-nets, [7], and (t, m, s)-nets, [20, 21], as special cases. One motivation for studying the geometrical properties of higher order digital nets and sequences lies in the conjecture that nondigital nets and sequences of better quality than their digital counterparts may exist [23]. Studying the geometrical properties of digital nets and sequences, we hope to find out what minimal properties point sets need to exhibit to still achieve optimal convergence rates when applied to numerical integration. This information can then be used for the construction of new higher order nondigital, that is non-linear, nets and sequences. Indeed, the results presented in this paper also turn out to be applicable to constructing new higher order nets and sequences.

In this paper, we firstly show how to characterize $(t, \alpha, \beta, n, m, s)$-nets using Weyl sums based on Walsh functions, in analogy to [17, Corollary 3], which provides the result for (t, m, s)-nets. This result also turns out to be useful for applications, which is the second contribution of the paper.

We study numerical integration in the Walsh space introduced in [7]. In particular, we show that if the function under consideration has square integrable

mixed partial derivatives of order α in each variable, the integration error resulting from approximating the integral with a quasi-Monte Carlo rule with a $(t, \alpha, \beta, n, m, s)$-net as quadrature points, converges at a rate of $N^{-(\alpha-1)}$ multiplied by a power of a $\log N$ factor. This bound is not optimal as one can obtain $N^{-\alpha}(\log N)^{\alpha s}$ with higher order digital nets [7] for example, but in Remark 2 we point out that for given concrete constructions optimal bounds may be obtained using further information about the construction.

Next, we study the randomization of $(t, \alpha, \beta, n, m, s)$-nets: Applying a random digital shift to a $(t, \alpha, \beta, n, m, s)$-net, the resulting point set is a $(t, \alpha, \beta, n, m, s)$- -net with probability one; using a random digital shift of depth n instead, see e.g., [13], we always obtain a $(t, \alpha, \beta, n, m, s)$-net. Randomizing $(t, \alpha, \beta, n, m, s)$-nets in this manner produces root mean-square integration errors converging at a rate of $N^{-(\alpha-\frac{1}{2})}$ multiplied by a power of a $\log N$ factor, for functions having square integrable mixed partial derivatives of order α in each variable.

We also generalize the $(u, u + v)$ construction, which is already used to construct (t, m, s)-nets [3] and digital $(t, \alpha, \beta, n \times m, s)$-nets [11], to the construction of $(t, \alpha, \beta, n, m, s)$-nets. Again, the characterization of $(t, \alpha, \beta, n, m, s)$-nets using Weyl sums turns out to be the appropriate tool to establish the result.

The main results of the paper are the following:

- Theorem 1, which shows that $(t, \alpha, \beta, n, m, s)$-nets can be characterized using Weyl sums based on Walsh functions.
- Theorem 2, which shows that $(t, \alpha, \beta, n, m, s)$-nets can achieve integration errors of order $N^{-(\alpha-1)}$ multiplied by a power of a $\log N$ factor.
- Theorem 3, which shows that randomly digitally shifted $(t, \alpha, \beta, n, m, s)$- -nets can achieve root mean-square integration errors of order $N^{-(\alpha-\frac{1}{2})}$ multiplied by a power of a $\log N$ factor.
- Theorem 4, which shows how to obtain new $(t, \alpha, \beta, n, m, s)$-nets using an analogue of the $(u, u + v)$-construction.

The paper is structured as follows: In Section 2, we provide the definition of $(t, \alpha, \beta, n, m, s)$-nets and state some of their properties, recall the definition of Walsh functions and Weyl sums and give the basic features of the function space under consideration. The characterization of $(t, \alpha, \beta, n, m, s)$-nets in terms of Weyl sums is given in Section 3. The application of the characterization to numerical integration is given in Section 4, randomized $(t, \alpha, \beta, n, m, s)$-nets are discussed in Section 5 and the characterization is used to establish the $(u, u+v)$- -construction for $(t, \alpha, \beta, n, m, s)$-nets in Section 6.

2. Basic definitions

In this section, we introduce $(t, \alpha, \beta, n, m, s)$-nets, Walsh functions, Weyl sums and the function space considered for numerical integration. In addition we also generalize the construction from [7, Section 4.4] (see also [6]).

Definition and construction of $(t, \alpha, \beta, n, m, s)$-**nets.** Before we can state the definition of $(t, \alpha, \beta, n, m, s)$-nets we need some notation.

Let $n, s \geq 1$, $b \geq 2$ be integers. For $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_s) \in \{0, \ldots, n\}^s$ let $|\boldsymbol{\nu}|_1 = \sum_{j=1}^s \nu_j$ and define $\boldsymbol{i_\nu} = (i_{1,1}, \ldots, i_{1,\nu_1}, \ldots, i_{s,1}, \ldots, i_{s,\nu_s})$ with integers $1 \leq i_{j,\nu_j} < \ldots < i_{j,1} \leq n$ in case $\nu_j > 0$ and $\{i_{j,1}, \ldots, i_{j,\nu_j}\} = \emptyset$ in case $\nu_j = 0$, for $j = 1, \ldots, s$. For given $\boldsymbol{\nu}$ and $\boldsymbol{i_\nu}$ let $\boldsymbol{a_\nu} \in \{0, \ldots, b-1\}^{|\boldsymbol{\nu}|_1}$, which we write as $\boldsymbol{a_\nu} = (a_{1,i_{1,1}}, \ldots, a_{1,i_{1,\nu_1}}, \ldots, a_{s,i_{s,1}}, \ldots, a_{s,i_{s,\nu_s}})$.

A *generalized elementary interval in base* b is a subset of $[0, 1)^s$ of the form

$$
J(\boldsymbol{i_\nu}, \boldsymbol{a_\nu}) \tag{1}
$$
$$
= \prod_{j=1}^s \bigcup_{\substack{a_{j,l}=0 \\ l \in \{1,\ldots,n\} \setminus \{i_{j,1},\ldots,i_{j,\nu_j}\}}}^{b-1} \left[\frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n}}{b^n}, \frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n}}{b^n} + \frac{1}{b^n} \right),
$$

where $\{i_{j,1}, \ldots, i_{j,\nu_j}\} = \emptyset$ in case $\nu_j = 0$ for $1 \leq j \leq s$.

We note that a generalized elementary interval is not always an elementary interval, but can be a union of several elementary intervals.

From [10, Lemmas 3.1 and 3.2] it is known that for $\boldsymbol{\nu} \in \{0, \ldots, n\}^s$ and $\boldsymbol{i_\nu}$ defined as above and fixed, the generalized elementary intervals $J(\boldsymbol{i_\nu}, \boldsymbol{a_\nu})$ for $\boldsymbol{a_\nu} \in \{0, \ldots, b-1\}^{|\boldsymbol{\nu}|_1}$ form a partition of $[0, 1)^s$ and the volume of $J(\boldsymbol{i_\nu}, \boldsymbol{a_\nu})$ is $b^{-|\boldsymbol{\nu}|_1}$.

We can now recall the definition of $(t, \alpha, \beta, n, m, s)$-nets which is based on [10, Definition 3.1].

DEFINITION 1. Let $n, m, s, \alpha \geq 1$ and $b \geq 2$ be integers, let $0 < \beta \leq 1$ be a real number and let $0 \leq t \leq \beta n$ be an integer. Let $\mathcal{P} = (\boldsymbol{x}_h)_{h=0}^{b^m-1} \subseteq [0, 1)^s$ be a point set in the s-dimensional unit cube. We say that $\mathcal{P}$ is a $(t, \alpha, \beta, n, m, s)$-net in base b, if for all integers $\nu_j \geq 0$ and $1 \leq i_{j,\nu_j} < \cdots < i_{j,1}$ satisfying

$$
\sum_{j=1}^s \sum_{l=1}^{\min(\nu_j,\alpha)} i_{j,l} \leq \beta n - t,
$$

where for $\nu_j = 0$ we set the empty sum $\sum_{l=1}^0 i_{j,l} = 0$, the generalized elementary interval $J(\boldsymbol{i_\nu}, \boldsymbol{a_\nu})$ contains exactly $b^{m-|\boldsymbol{\nu}|_1}$ points of $\mathcal{P}$ for each $\boldsymbol{a_\nu} \in \{0, \ldots, b-1\}^{|\boldsymbol{\nu}|_1}$.

Some remarks on the definition of $(t, \alpha, \beta, n, m, s)$-nets are in order (for more information see [10]).

REMARK 1.   1. We obtain the definition of a classical (t, m, s)-net (according to [20, 21]) from Definition 1 by setting $\alpha = \beta = 1$, $n = m$, and considering all $\nu_1, \ldots, \nu_s \geq 0$ so that $\sum_{j=1}^{s} \nu_j \leq m - t$, where we set $i_{j,k} = \nu_j - k + 1$ for $k = 1, \ldots, \nu_j$. Hence a $(t, 1, 1, m, m, s)$-net is a (t, m, s)-net.

2. Definition 1 says that for every generalized elementary interval $J(\boldsymbol{i_\nu}, \boldsymbol{a_\nu})$ of volume $b^{-|\boldsymbol{\nu}|_1}$ we have

$$\frac{|\{0 \leq h < b^m \, : \, \boldsymbol{x}_h \in J(\boldsymbol{i_\nu}, \boldsymbol{a_\nu})\}|}{b^m} - \lambda_s(J(\boldsymbol{i_\nu}, \boldsymbol{a_\nu})) = 0,$$

where λ_s denotes the s-dimensional Lebesgue measure.

For concrete constructions of $(t, \alpha, \beta, n, m, s)$-nets for various parameters see [7, Section 4.4], and also [10, 11] for bounds and further constructions of such nets. Most of these methods rely on the digital construction method, which is already well known for classical nets.

A method which does not necessarily use the digital construction scheme, but relies on classical (t, m, s)-nets instead, is as follows: for a fixed $d \in \mathbb{N}$, let $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{b^m-1}\}$ form a (t', m, sd)-net in base b. Let $\boldsymbol{x}_h = (x_{h,1}, \ldots, x_{h,sd})$, $x_{h,j} = \xi_{h,j,1}b^{-1} + \xi_{h,j,2}b^{-2} + \cdots$ for $h = 0, \ldots, b^m - 1$ and $1 \leq j \leq sd$. Then we construct a point set $\boldsymbol{y}_h = (y_{h,1}, \ldots, y_{h,s})$, $h = 0, \ldots, b^m - 1$, by

$$y_{h,j} = \sum_{i=1}^{m} \sum_{k=1}^{d} \xi_{h,(j-1)d+k,i} b^{-k-(i-1)d},$$

for any $1 \leq j \leq s$. It has been shown in [2] that for every $\alpha \geq 1$, the point set $\{\boldsymbol{y}_0, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_{b^m-1}\}$ forms a $(t, \alpha, \min(1, \frac{\alpha}{d}), dm, m, s)$-net in base b with

$$t = \min(d, \alpha) \min\left(m, t' + \left\lfloor \frac{s(d-1)}{2} \right\rfloor\right).$$

We remark that in Section 6, we will show how to combine two $(t, \alpha, \beta, n, m, s)$-nets to form another one using the $(u, u+v)$-construction.

Walsh functions and Weyl sums. In this subsection, we recall the concept of Weyl sums based on Walsh functions, see e.g., [17]; it turns out, see Section 3, that $(t, \alpha, \beta, n, m, s)$-nets can be characterized using Weyl sums.

Let, in the following, $\mathbb{N}_0$ denote the set of non-negative and $\mathbb{N}$ the set of positive integers and fix $b \in \mathbb{N}$, $b \geq 2$. Each $k \in \mathbb{N}_0$ has a unique b-adic representation $k = \sum_{l=0}^{a} \kappa_l b^l$, $\kappa_l \in \{0, \ldots, b-1\}$, where $\kappa_a \neq 0$. Each $x \in [0, 1)$ has a b-adic representation $x = \sum_{l=1}^{\infty} \xi_l b^{-l}$, $\xi_l \in \{0, \ldots, b-1\}$, which is unique in the sense

that infinitely many of the ξ_l must differ from $b - 1$. We define the k-th Walsh function in base b, $\mathrm{wal}_k : [0, 1) \to \mathbb{C}$, by

$$\mathrm{wal}_k(x) := \exp\left(\frac{2\pi\mathtt{i}}{b}\left(\xi_1\kappa_0 + \cdots + \xi_{a+1}\kappa_a\right)\right).$$

For dimension $s \geq 2$ and vectors $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ and $\boldsymbol{x} = (x_1, \ldots, x_s) \in [0, 1)^s$, we define $\mathrm{wal}_{\boldsymbol{k}} : [0, 1)^s \to \mathbb{C}$ by

$$\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) := \prod_{j=1}^{s} \mathrm{wal}_{k_j}(x_j).$$

It follows from the definition above that Walsh functions are piecewise constant functions. For more information on Walsh functions, see e.g., [5, 31]. Walsh functions were used for the first time in [18] to analyze (t, m, s)-nets.

We can now recall the concept of a Weyl sum.

DEFINITION 2. For a point set $\mathcal{P} = (\boldsymbol{x}_h)_{h=0}^{N-1} \in [0, 1)^s$, $N \in \mathbb{N}$, let

$$S_N(f, \mathcal{P}) = \frac{1}{N}\sum_{h=0}^{N-1} f(\boldsymbol{x}_h).$$

If $f = \mathrm{wal}_{\boldsymbol{k}}$ for some $\boldsymbol{k} \in \mathbb{N}_0^s$, then $S_N(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P})$ is called a Weyl sum (based on Walsh functions).

The function space $W_{\alpha,s,\boldsymbol{\gamma}}$. The function space under consideration in this paper is the space $W_{\alpha,s,\boldsymbol{\gamma}} \subseteq L_2([0, 1]^s)$ as introduced in [7]. Here $\boldsymbol{\gamma} = (\gamma_j)_{j=1}^{\infty}$ is a sequence of positive, non-increasing weights, which are introduced to model the importance of different variables for our approximation problem, see [29]. Given a positive integer k with base b expansion $k = \kappa_1 b^{a_1-1} + \kappa_2 b^{a_2-1} + \cdots + \kappa_v b^{a_v-1}$, $1 \leq a_v < \ldots < a_1$, $v \geq 1$, $\kappa_1, \ldots, \kappa_v \in \{1, \ldots, b-1\}$, we define
$$\mu_\alpha(k) := a_1 + \cdots + a_{\min(v,\alpha)}. \tag{2}$$

Furthermore, $\mu_\alpha(0) := 0$ and for $\boldsymbol{k} \in \mathbb{N}_0^s$, $\boldsymbol{k} = (k_1, \ldots, k_s)$,

$$\mu_\alpha(\boldsymbol{k}) = \sum_{j=1}^{s} \mu_\alpha(k_j).$$

For $k \in \mathbb{N}_0$ and a weight $\gamma > 0$, we define a function

$$r_{\alpha,\gamma}(k) := \begin{cases} 1 & \text{if } k = 0, \\ \gamma b^{-\mu_\alpha(k)} & \text{otherwise.} \end{cases}$$

If we consider a vector $\boldsymbol{k} \in \mathbb{N}_0^s$, $\boldsymbol{k} = (k_1, \ldots, k_s)$, we set

$$r_{\alpha,s,\boldsymbol{\gamma}}(\boldsymbol{k}) := \prod_{j=1}^{s} r_{\alpha,\gamma_j}(k_j), \tag{3}$$

where $\boldsymbol{\gamma} = (\gamma_j)_{j=1}^{\infty}$ is the sequence of positive, non-increasing weights introduced above.

In this paper, we study integration errors resulting from the approximation of an integral based on $(t, \alpha, \beta, n, m, s)$-nets by considering the Walsh series of the integrand f; we remark that this approach has also been used when studying integration errors resulting from the application of digital and higher order digital nets, see e.g., [7, 12]. In particular, for $f \in L_2([0,1]^s)$, the Walsh series of f is given by

$$f(\boldsymbol{x}) \sim \sum_{\boldsymbol{k} \in \mathbb{N}_0^s} \widehat{f}(\boldsymbol{k}) \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}), \tag{4}$$

where the Walsh coefficients $\widehat{f}(\boldsymbol{k})$ are given by

$$\widehat{f}(\boldsymbol{k}) = \int_{[0,1]^s} f(\boldsymbol{x}) \overline{\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})} \, \mathrm{d}\boldsymbol{x}.$$

In general, the Walsh series given in Equation (4) need not converge to f, however, for the space of Walsh series $W_{\alpha,s,\boldsymbol{\gamma}}$, which we define in the following, it does converge absolutely, see also [7].

The space $W_{\alpha,s,\boldsymbol{\gamma}}$ consists of all Walsh series $f = \sum_{\boldsymbol{k} \in \mathbb{N}_0^s} \widehat{f}(\boldsymbol{k}) \mathrm{wal}_{\boldsymbol{k}}$ for which the norm

$$\|f\|_{W_{\alpha,s,\boldsymbol{\gamma}}} := \sup_{\boldsymbol{k} \in \mathbb{N}_0^s} \frac{|\widehat{f}(\boldsymbol{k})|}{r_{\alpha,s,\boldsymbol{\gamma}}(\boldsymbol{k})},$$

is finite. It follows immediately that for any $f \in W_{\alpha,s,\boldsymbol{\gamma}}$, and any $\boldsymbol{k} \in \mathbb{N}_0^s$,

$$|\widehat{f}(\boldsymbol{k})| \leq \|f\|_{W_{\alpha,s,\boldsymbol{\gamma}}} \, r_{\alpha,s,\boldsymbol{\gamma}}(\boldsymbol{k}). \tag{5}$$

For $\alpha \geq 2$, the following property was shown in [7]: Let $f : [0,1]^s \to \mathbb{R}$ be such that all mixed partial derivatives up to order α in each variable are square integrable, then $f \in W_{\alpha,s,\boldsymbol{\gamma}}$, where $\boldsymbol{\gamma} = (\gamma_j)_{j=1}^{\infty}$ is the sequence of positive, non-increasing weights introduced above. Furthermore, an inequality using a Sobolev type norm and the norm (5) was shown, see also [6, 8]. Consequently, the results we are going to establish in the following for functions in $W_{\alpha,s,\boldsymbol{\gamma}}$ also apply automatically to smooth functions. The assumption $\alpha > 1$ is needed to ensure that the sum of the absolute values of the Walsh coefficients converges, the case $\alpha = 1$ requires a different analysis, which was carried out in [12] for numerical integration.

3. Characterization of higher order nets using Weyl sums

In this section, we characterize $(t, \alpha, \beta, n, m, s)$-nets using Weyl sums. Our results generalize [17, Lemmas 1 and 2 and Corollary 3].

LEMMA 1. *Let $\mathcal{P} = (\boldsymbol{x}_h)_{h=0}^{b^m-1}$ be a $(t, \alpha, \beta, n, m, s)$-net in base $b \geq 2$, where $\alpha \geq 2$ is an integer, β a real number such that $0 < \beta \leq 1$ and $n, m, s \in \mathbb{N}$. Then for all $\boldsymbol{k} \in \mathbb{N}_0^s$ satisfying $0 < \mu_\alpha(\boldsymbol{k}) \leq \beta n - t$ we have*

$$S_{b^m}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}) = 0.$$

P r o o f. Let $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$, be such that $0 < \mu_\alpha(\boldsymbol{k}) \leq \beta n - t$ (hence $\boldsymbol{k} \neq \boldsymbol{0}$) and for $k_j \neq 0$ let

$$k_j = \kappa_{j,1} b^{i_{j,1}-1} + \cdots + \kappa_{j,\nu_j} b^{i_{j,\nu_j}-1},$$

with $\kappa_{j,l} \in \{1, \ldots, b-1\}$, be the b-adic expansion of k_j, $1 \leq j \leq s$. Then for j with $k_j \neq 0$ and $x = \sum_{l=1}^\infty \xi_l b^{-l} \in [0, 1)$ we have

$$\mathrm{wal}_{k_j}(x) = \exp\left(\frac{2\pi\mathrm{i}}{b}\left(\kappa_{j,1}\xi_{i_{j,1}} + \cdots + \kappa_{j,\nu_j}\xi_{i_{j,\nu_j}}\right)\right).$$

Hence, if we set $\boldsymbol{i}_{\boldsymbol{\nu}} = (i_{1,1}, \ldots, i_{1,\nu_1}, \ldots, i_{s,1}, \ldots, i_{s,\nu_s})$, which only depends on $\boldsymbol{k}$, then $\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})$ is constant on generalized elementary intervals $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ of the form given in Equation (1). Furthermore we denote the value of $\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})$ on $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ by $c_{\boldsymbol{a}_{\boldsymbol{\nu}}}$. As $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$, $\boldsymbol{a}_{\boldsymbol{\nu}} \in \{0, \ldots, b-1\}^{|\boldsymbol{\nu}|_1}$ is a partition of $[0,1)^s$ we obtain

$$\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) = \sum_{\boldsymbol{a}_{\boldsymbol{\nu}} \in \{0,\ldots,b-1\}^{|\boldsymbol{\nu}|_1}} c_{\boldsymbol{a}_{\boldsymbol{\nu}}} \mathbf{1}_{J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})}(\boldsymbol{x}),$$

where $\mathbf{1}_{J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})}$ denotes the characteristic function of $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$.

For $\boldsymbol{k} \neq \boldsymbol{0}$ we have $\int_{[0,1]^s} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = 0$, and hence it follows that

$$\sum_{\boldsymbol{a}_{\boldsymbol{\nu}} \in \{0,\ldots,b-1\}^{|\boldsymbol{\nu}|_1}} c_{\boldsymbol{a}_{\boldsymbol{\nu}}} = 0,$$

as the volume of $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ depends only on $\boldsymbol{\nu}$. Consequently,

$$
\begin{aligned}
S_{b^m}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}) &= \sum_{\boldsymbol{a}_{\boldsymbol{\nu}} \in \{0,\ldots,b-1\}^{|\boldsymbol{\nu}|_1}} c_{\boldsymbol{a}_{\boldsymbol{\nu}}} S_{b^m}\left(\mathbf{1}_{J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})}, \mathcal{P}\right) \\
&= \sum_{\boldsymbol{a}_{\boldsymbol{\nu}} \in \{0,\ldots,b-1\}^{|\boldsymbol{\nu}|_1}} c_{\boldsymbol{a}_{\boldsymbol{\nu}}} S_{b^m}\left(\mathbf{1}_{J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})} - \lambda_s\left(J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})\right), \mathcal{P}\right).
\end{aligned}
$$

As $J(\boldsymbol{i_\nu}, \boldsymbol{a_\nu})$ is a generalized elementary interval of volume $b^{-|\boldsymbol{\nu}|_1}$ for which by assumption

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j, \alpha)} i_{j,l} = \mu_\alpha(\boldsymbol{k}) \leq \beta n - t,$$

it follows that $J(\boldsymbol{i_\nu}, \boldsymbol{a_\nu})$ contains $b^{m-|\boldsymbol{\nu}|_1}$ points of $\mathcal{P}$ and hence

$$S_{b^m} \left(\mathbf{1}_{J(\boldsymbol{i_\nu}, \boldsymbol{a_\nu})} - \lambda_s \left(J(\boldsymbol{i_\nu}, \boldsymbol{a_\nu}) \right), \mathcal{P} \right) = \frac{1}{b^m} \left(b^{m-|\boldsymbol{\nu}|_1} - b^m \lambda_s \left(J(\boldsymbol{i_\nu}, \boldsymbol{a_\nu}) \right) \right) = 0$$

as desired. $\qquad\square$

To establish the converse, we need the following lemma, which generalizes [16, Lemma 3(i)] and which can be proven along the same lines as [16, Remark (iv), Lemma 2(i) and Lemma 3(i)].

LEMMA 2. *For given $\boldsymbol{\nu}$, $\boldsymbol{i_\nu}$ and $\boldsymbol{a_\nu}$ let*

$$J(\boldsymbol{i_\nu}, \boldsymbol{a_\nu}) = \prod_{j=1}^{s} \bigcup_{\substack{a_{j,l}=0 \\ l \in \{1,\ldots,n\} \setminus \{i_{j,1},\ldots,i_{j,\nu_j}\}}}^{b-1} \left[\frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n}}{b^n}, \frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n}}{b^n} + \frac{1}{b^n} \right)$$

and let $f(\boldsymbol{x}) = \mathbf{1}_{J(\boldsymbol{i_\nu}, \boldsymbol{a_\nu})}(\boldsymbol{x}) - \lambda_s(J(\boldsymbol{i_\nu}, \boldsymbol{a_\nu}))$. Define

$$\Delta_{\boldsymbol{i_\nu}} := \left\{ \boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s : k_j = \kappa_{j,1} b^{i_{j,1}-1} + \cdots + \kappa_{j,\nu_j} b^{i_{j,\nu_j}-1}; \right.$$
$$\left. \kappa_{j,1}, \ldots, \kappa_{j,\nu_j} \in \{1, \ldots, b-1\} \text{ if } \nu_j > 0 \text{ and } k_j = 0 \text{ for } \nu_j = 0 \right\}.$$

Then for all $\boldsymbol{k} \notin \Delta_{\boldsymbol{i_\nu}}$ we have $|\widehat{f}(\boldsymbol{k})| = 0$.

The following lemma generalizes [17, Lemma 2].

LEMMA 3. *Let $\mathcal{P} = (\boldsymbol{x}_h)_{h=0}^{b^m-1}$ be a finite sequence of b^m points in the s-dimensional unit cube $[0,1)^s$ and suppose that for each $\boldsymbol{k} \in \mathbb{N}_0^s$ satisfying $0 < \mu_\alpha(\boldsymbol{k}) \leq \beta n - t$ we have*
$$S_{b^m}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}) = 0.$$
Then $\mathcal{P}$ is a $(t, \alpha, \beta, n, m, s)$-net in base b.

P r o o f. Suppose that $J(\boldsymbol{i_\nu}, \boldsymbol{a_\nu})$ is an arbitrary generalized elementary interval of the form given in Equation (1). We define $f(\boldsymbol{x}) = \mathbf{1}_{J(\boldsymbol{i_\nu}, \boldsymbol{a_\nu})}(\boldsymbol{x}) - \lambda_s(J(\boldsymbol{i_\nu}, \boldsymbol{a_\nu}))$. In order to show that $\mathcal{P}$ is a $(t, \alpha, \beta, n, m, s)$-net in base b, it suffices to prove that $S_{b^m}(f, \mathcal{P}) = 0$. If $\widehat{\mathbf{1}}_{J(\boldsymbol{i_\nu}, \boldsymbol{a_\nu})}(\boldsymbol{k})$ denotes the $\boldsymbol{k}$-th Walsh coefficient of $\mathbf{1}_{J(\boldsymbol{i_\nu}, \boldsymbol{a_\nu})}$, then for all $\boldsymbol{x} \in [0,1)^s$ we have

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \Delta_{\boldsymbol{i_\nu}}} \widehat{\mathbf{1}}_{J(\boldsymbol{i_\nu}, \boldsymbol{a_\nu})}(\boldsymbol{k}) \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}). \tag{6}$$

Equation (6) holds pointwise, as due to Lemma 2 only a finite number of Walsh coefficients of $f(\boldsymbol{x})$ are non-zero, implying that $f \in W_{\alpha,s,\boldsymbol{\gamma}}$; consequently, the equality follows from [7]. Hence, we obtain

$$S_{b^m}(f,\mathcal{P}) = \sum_{\boldsymbol{k} \in \Delta_{\boldsymbol{i_\nu}}} \widehat{\boldsymbol{1}}_{J(\boldsymbol{i_\nu},\boldsymbol{a_\nu})}(\boldsymbol{k}) S_{b^m}(\mathrm{wal}_{\boldsymbol{k}},\mathcal{P}).$$

But $\boldsymbol{k} \in \Delta_{\boldsymbol{i_\nu}}$ implies that $\mu_\alpha(\boldsymbol{k}) = \sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j,\alpha)} i_{j,l} \le \beta n - t$, hence

$$S_{b^m}(\mathrm{wal}_{\boldsymbol{k}},\mathcal{P}) = 0.$$

This implies that $S_{b^m}(f,\mathcal{P}) = 0$. □

Combining Lemma 1 and Lemma 3 we obtain the following characterization of (t,α,β,n,m,s)-nets in terms of Weyl sums (for the Walsh function system).

THEOREM 1. *Let $\mathcal{P} = (\boldsymbol{x}_h)_{h=0}^{b^m-1}$ be a finite sequence of b^m points in the s-dimensional unit cube $[0,1)^s$. Then $\mathcal{P}$ is a (t,α,β,n,m,s)-net in base b if and only if for all $\boldsymbol{k} \in \mathbb{N}_0^s$ satisfying $0 < \mu_\alpha(\boldsymbol{k}) \le \beta n - t$ we have*

$$S_{b^m}(\mathrm{wal}_{\boldsymbol{k}},\mathcal{P}) = 0.$$

4. Application to numerical integration

In this section, we establish that (t,α,β,n,m,s)-nets can exploit the smoothness α of a function $f \in W_{\alpha,s,\boldsymbol{\gamma}}$. We need to introduce some notation: Let $\mathcal{S} = \{1,\ldots,s\}$. For $\boldsymbol{k} = (k_1,\ldots,k_s) \in \mathbb{N}_0^s$ and for $\emptyset \ne \mathfrak{u} \subseteq \mathcal{S}$ let $\boldsymbol{k}_{\mathfrak{u}}$ be the vector in $\mathbb{N}_0^{|\mathfrak{u}|}$ which consists of all components of $\boldsymbol{k}$ whose indices belong to $\mathfrak{u}$. Furthermore let $(\boldsymbol{k}_{\mathfrak{u}}, \boldsymbol{0})$ be the vector $\boldsymbol{k}$ with all components whose indices are not in $\mathfrak{u}$ replaced by 0. With this notation we have $\mu_\alpha(\boldsymbol{k}_{\mathfrak{u}}) = \mu_\alpha(\boldsymbol{k}_{\mathfrak{u}}, \boldsymbol{0})$. For a sequence $\boldsymbol{\gamma} = (\gamma_j)_{j \ge 1}$ we write $\boldsymbol{\gamma}_{\mathfrak{u}} = \prod_{j \in \mathfrak{u}} \gamma_j$.

We need the following lemma.

LEMMA 4. *Let $(\boldsymbol{x}_h)_{h=0}^{b^m-1}$ be a (t,α,β,n,m,s)-net in base b and let $f \in W_{\alpha,s,\boldsymbol{\gamma}}$, then*

$$\left| \int_{[0,1]^s} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} - \frac{1}{b^m} \sum_{h=0}^{b^m-1} f(\boldsymbol{x}_h) \right| \le \|f\|_{W_{\alpha,s,\boldsymbol{\gamma}}} \sum_{\emptyset \ne \mathfrak{u} \subseteq \mathcal{S}} \boldsymbol{\gamma}_{\mathfrak{u}} \sum_{\substack{\boldsymbol{k}_{\mathfrak{u}} \in \mathbb{N}^{|\mathfrak{u}|} \\ \mu_\alpha(\boldsymbol{k}_{\mathfrak{u}}) > \beta n - t}} b^{-\mu_\alpha(\boldsymbol{k}_{\mathfrak{u}})}.$$

(7)

P r o o f. For $f \in W_{\alpha,s,\boldsymbol{\gamma}}$ and $(\boldsymbol{x}_h)_{h=0}^{b^m-1}$ a (t,α,β,n,m,s)-net, we can write

$$\left| \int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \frac{1}{b^m} \sum_{h=0}^{b^m-1} f(\boldsymbol{x}_h) \right| = \left| \widehat{f}(\boldsymbol{0}) - \frac{1}{b^m} \sum_{h=0}^{b^m-1} \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} \widehat{f}(\boldsymbol{k}) \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h) \right|$$

$$= \left| \widehat{f}(\boldsymbol{0}) - \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} \widehat{f}(\boldsymbol{k}) \frac{1}{b^m} \sum_{h=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h) \right| = \left| \sum_{\boldsymbol{k}\in\mathbb{N}_0^s\setminus\{\boldsymbol{0}\}} \widehat{f}(\boldsymbol{k}) \frac{1}{b^m} \sum_{h=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h) \right|$$

$$= \left| \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^s\setminus\{\boldsymbol{0}\} \\ \mu_\alpha(\boldsymbol{k})>\beta n-t}} \widehat{f}(\boldsymbol{k}) \frac{1}{b^m} \sum_{h=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h) \right|, \tag{8}$$

where we used Lemma 1. Using the triangle inequality it now follows that

$$\left| \int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \frac{1}{b^m} \sum_{h=0}^{b^m-1} f(\boldsymbol{x}_h) \right| \leq \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^s\setminus\{\boldsymbol{0}\} \\ \mu_\alpha(\boldsymbol{k})>\beta n-t}} \left| \widehat{f}(\boldsymbol{k}) \right|$$

$$\leq \|f\|_{W_{\alpha,s,\boldsymbol{\gamma}}} \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^s\setminus\{\boldsymbol{0}\} \\ \mu_\alpha(\boldsymbol{k})>\beta n-t}} r_{\alpha,s,\boldsymbol{\gamma}}(\boldsymbol{k}) = \|f\|_{W_{\alpha,s,\boldsymbol{\gamma}}} \sum_{\emptyset\neq\mathfrak{u}\subseteq\mathcal{S}} \gamma_{\mathfrak{u}} \sum_{\substack{\boldsymbol{k}_{\mathfrak{u}}\in\mathbb{N}^{|\mathfrak{u}|} \\ \mu_\alpha(\boldsymbol{k}_{\mathfrak{u}})>\beta n-t}} b^{-\mu_\alpha(\boldsymbol{k}_{\mathfrak{u}})}$$

as desired. □

REMARK 2. Comparing Equation (7) to [7, Equation (5.1)], we note that in Equation (7), the second sum runs over all $\boldsymbol{k}_{\mathfrak{u}} \in \mathbb{N}^{|\mathfrak{u}|}$ for which $\mu_\alpha(\boldsymbol{k}_{\mathfrak{u}}) > \beta n - t$, whereas in [7, Equation (5.1)], the corresponding sum is over all $\boldsymbol{k}_{\mathfrak{u}}$ in the dual space corresponding to the set $\mathfrak{u}$. We obtain this estimate as we estimate the absolute value of the character sum $b^{-m} \sum_{h=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h)$ in (8) by 1. Given concrete constructions, better estimates of this sum may be obtained, as is the case for digital nets and sequences.

To establish the main result of this section, we need the following lemma.

LEMMA 5. *Let $l \geq 1$ and $\alpha \geq 2$ be integers. Then*

$$\sum_{\substack{k\in\mathbb{N} \\ \mu_\alpha(k)=l}} 1 \leq 2\binom{l+\alpha-1}{\alpha-1}(b-1)^\alpha b^{\lfloor l/\alpha\rfloor}.$$

P r o o f. For $k \in \mathbb{N}$ let ν_k denote the number of non-zero digits in the base b representation of k. We represent $k \in \mathbb{N}$ as follows

$$k = \kappa_1 b^{a_1-1} + \cdots + \kappa_{\nu_k} b^{a_{\nu_k}-1},$$

where $\kappa_1, \ldots, \kappa_{\nu_k} \in \{1, \ldots, b-1\}$ and $a_1 > \ldots > a_{\nu_k} \geq 1$. We firstly consider those $k \in \mathbb{N}$ for which $\nu_k \leq \alpha$. In that case, we put a bound on the number of k for which $\mu_\alpha(k) = a_1 + \cdots + a_{\nu_k} = l$. Then we have

$$|\{(a_1, \ldots, a_{\nu_k}) \,:\, a_1 + \cdots + a_{\nu_k} = l, a_1 > \ldots > a_{\nu_k} \geq 1\}|$$
$$\leq |\{(a_1, \ldots, a_{\nu_k}) \,:\, a_1 + \cdots + a_{\nu_k} = l, a_1 \geq 0, \ldots, a_{\nu_k} \geq 0\}|$$
$$\leq |\{(a_1, \ldots, a_\alpha) \,:\, a_1 + \cdots + a_\alpha = l, a_1 \geq 0, \ldots, a_\alpha \geq 0\}|$$
$$= \binom{l + \alpha - 1}{\alpha - 1},$$

where the final equality employs a well-known combinatorial result, see also [7, Lemma 5.2]. The coefficients $\kappa_1, \ldots, \kappa_{\nu_k}$ take values in the set $\{1, \ldots, b-1\}$, hence there are $(b-1)^{\nu_k} \leq (b-1)^\alpha$ possibilities, hence

$$\sum_{\substack{k \in \mathbb{N} \\ \mu_\alpha(k) = l, \nu_k \leq \alpha}} 1 \;\leq\; (b-1)^\alpha \binom{l + \alpha - 1}{\alpha - 1}.$$

We now consider those k for which $\nu_k > \alpha$. Then

$$k = \kappa_1 b^{a_1 - 1} + \cdots + \kappa_\alpha b^{a_\alpha - 1} + \kappa_{\alpha+1} b^{a_{\alpha+1} - 1} + \cdots + \kappa_{\nu_k} b^{a_{\nu_k} - 1},$$

and we put a bound on the number of k for which $\mu_\alpha(k) = a_1 + \cdots + a_\alpha = l$. Now,

$$|\{(a_1, \ldots, a_\alpha) \,:\, a_1 + \cdots + a_\alpha = l, a_1 > \cdots > a_\alpha \geq 1\}|$$
$$\leq |\{(a_1, \ldots, a_\alpha) \,:\, a_1 + \cdots + a_\alpha = l, a_1 \geq 0, \ldots, a_\alpha \geq 0\}|$$
$$= \binom{l + \alpha - 1}{\alpha - 1}.$$

Regarding the coefficients, it is clear that $\kappa_1, \ldots, \kappa_{\nu_k} \in \{1, \ldots, b-1\}$, so the first α coefficients, $\kappa_1, \ldots, \kappa_\alpha$ can assume $(b-1)^\alpha$ different values. Regarding the sum

$$\kappa_{\alpha+1} b^{a_{\alpha+1} - 1} + \cdots + \kappa_{\nu_k} b^{a_{\nu_k} - 1}, \tag{9}$$

where $\kappa_{\alpha+1}, \ldots, \kappa_{\nu_k} \in \{1, \ldots, b-1\}$ and $a_{\alpha+1} > \ldots > a_{\nu_k} \geq 1$, it is clear that the number of different values that the sum in Equation (9) can assume is bounded by $b^{a_\alpha - 1}$. But by assumption, $a_\alpha + \cdots + a_1 = l$, hence $a_\alpha \leq \lfloor l/\alpha \rfloor$, so we conclude that

$$\sum_{\substack{k \in \mathbb{N} \\ \mu_\alpha(k) = l, \nu_k > \alpha}} 1 \;\leq\; (b-1)^\alpha b^{\lfloor l/\alpha \rfloor} \binom{l + \alpha - 1}{\alpha - 1}$$

and the result follows by summing up the two cases. $\qquad\square$

Also, we will use the following lemma, which appeared as [19, Lemma 2.18].

LEMMA 6. *For any $b > 1$ and integers $i, j_0 \geq 0$, we have*

$$\sum_{j=j_0}^{\infty} \binom{j+i-1}{i-1} b^{-j} \leq b^{-j_0} \binom{j_0+i-1}{i-1} \left(1 - \frac{1}{b}\right)^{-i}.$$

The next theorem establishes that $(t, \alpha, \beta, n, m, s)$-nets can exploit the smoothness α of a function $f \in W_{\alpha,s,\boldsymbol{\gamma}}$.

THEOREM 2. *Let $(\boldsymbol{x}_h)_{h=0}^{b^m-1}$ be a $(t, \alpha, \beta, n, m, s)$-net in base b and let $f \in W_{\alpha,s,\boldsymbol{\gamma}}$. Then*

$$\left| \int_{[0,1]^s} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} - \frac{1}{b^m} \sum_{h=0}^{b^m-1} f(\boldsymbol{x}_h) \right| \leq \|f\|_{W_{\alpha,s,\boldsymbol{\gamma}}}$$

$$\times \, b^{-(1-1/\alpha)(\lfloor \beta n - t \rfloor + 1)} \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{S}} \boldsymbol{\gamma}_{\mathfrak{u}} \left(\frac{b^{1-1/\alpha}(b-1)}{(b^{1-1/\alpha}-1)} \right)^{\alpha|\mathfrak{u}|} \frac{(\lfloor \beta n - t \rfloor + \alpha|\mathfrak{u}|)!}{(|\mathfrak{u}|-1)!(\lfloor \beta n - t \rfloor + 1)!}.$$

Before proving Theorem 2, we present the following remark, which deals with an important special case of the result presented in Theorem 2.

REMARK 3. For $\beta n = \alpha m$, we obtain a convergence rate of the integration error of $N^{-(\alpha-1)}$ multiplied by a power of a $\log N$ factor. This rate, although not optimal, see [6, 7], does establish that $(t, \alpha, \beta, n, m, s)$-nets can exploit the smoothness of functions lying in $W_{\alpha,s,\boldsymbol{\gamma}}$. This was not possible with the classical concept of (t, m, s)-nets. Intuitively, the superior convergence rate is achieved because $(t, \alpha, \beta, n, m, s)$-nets place the integration nodes more carefully in the unit cube than (t, m, s)-nets, see [9] for an expository paper, which also provides pictorial illustrations.

We provide the proof of Theorem 2.

Proof. Lemma 4 established that

$$\left| \int_{[0,1]^s} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} - \frac{1}{b^m} \sum_{h=0}^{b^m-1} f(\boldsymbol{x}_h) \right| \leq \|f\|_{W_{\alpha,s,\boldsymbol{\gamma}}} \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{S}} \boldsymbol{\gamma}_{\mathfrak{u}} \sum_{\substack{\boldsymbol{k}_{\mathfrak{u}} \in \mathbb{N}^{|\mathfrak{u}|} \\ \mu_\alpha(\boldsymbol{k}_{\mathfrak{u}}) > \beta n - t}} b^{-\mu_\alpha(\boldsymbol{k}_{\mathfrak{u}})}.$$

(10)

For a given $\emptyset \neq \mathfrak{u} \subseteq \mathcal{S}$, $|\mathfrak{u}| \geq 2$, we rewrite

$$\sum_{\substack{\boldsymbol{k}_{\mathfrak{u}} \in \mathbb{N}^{|\mathfrak{u}|} \\ \mu_\alpha(\boldsymbol{k}_{\mathfrak{u}}) > \beta n - t}} b^{-\mu_\alpha(\boldsymbol{k}_{\mathfrak{u}})} = \sum_{l=\lfloor \beta n - t \rfloor + 1}^{\infty} b^{-l} \sum_{\substack{\boldsymbol{k}_{\mathfrak{u}} \in \mathbb{N}^{|\mathfrak{u}|} \\ \mu_\alpha(\boldsymbol{k}_{\mathfrak{u}})=l}} 1.$$

Using Lemma 5 we obtain

$$
\sum_{\substack{\boldsymbol{k}_\mathfrak{u} \in \mathbb{N}^{|\mathfrak{u}|} \\ \mu_\alpha(\boldsymbol{k}_\mathfrak{u})=l}} 1 \;=\; \sum_{l_1+\cdots+l_{|\mathfrak{u}|}=l} \prod_{j=1}^{|\mathfrak{u}|} \sum_{\substack{k_j \in \mathbb{N} \\ \mu_\alpha(k_j)=l_j}} 1
$$

$$
\leq \sum_{l_1+\cdots+l_{|\mathfrak{u}|}=l} \prod_{j=1}^{|\mathfrak{u}|} \left[2\binom{l_j+\alpha-1}{\alpha-1}(b-1)^\alpha b^{\lfloor l_j/\alpha \rfloor} \right]
$$

$$
\leq 2^{|\mathfrak{u}|}(b-1)^{\alpha|\mathfrak{u}|}b^{l/\alpha} \sum_{l_1+\cdots+l_{|\mathfrak{u}|}=l} \prod_{j=1}^{|\mathfrak{u}|} \binom{l_j+\alpha-1}{\alpha-1}.
$$

For any $1 \leq j \leq |\mathfrak{u}|$ we have $\binom{l_j+\alpha-1}{\alpha-1} \leq (1+l_j)^{\alpha-1}$. Since $l_1,\ldots,l_{|\mathfrak{u}|} \geq 1$ and $l_1+\cdots+l_{|\mathfrak{u}|}=l$, for $|\mathfrak{u}| \geq 2$ we have $1+l_j \leq l$ and therefore $\binom{l_j+\alpha-1}{\alpha-1} \leq l^{\alpha-1}$. If $|\mathfrak{u}|=1$, then $l_1=l$ and $\binom{l+\alpha-1}{\alpha-1} \leq l^{\alpha-1}$. Hence we obtain

$$
2^{|\mathfrak{u}|}(b-1)^{\alpha|\mathfrak{u}|}b^{l/\alpha} \sum_{l_1+\cdots+l_{|\mathfrak{u}|}=l} \prod_{j=1}^{|\mathfrak{u}|} \binom{l_j+\alpha-1}{\alpha-1}
$$

$$
\leq 2^{|\mathfrak{u}|}(b-1)^{\alpha|\mathfrak{u}|}b^{l/\alpha} \sum_{l_1+\cdots+l_{|\mathfrak{u}|}=l} l^{(\alpha-1)|\mathfrak{u}|}
$$

$$
\leq 2^{|\mathfrak{u}|}(b-1)^{\alpha|\mathfrak{u}|}b^{l/\alpha}l^{(\alpha-1)|\mathfrak{u}|}\binom{l+|\mathfrak{u}|-1}{|\mathfrak{u}|-1}.
$$

Hence

$$
\sum_{l=\lfloor \beta n-t \rfloor+1}^{\infty} b^{-l} \sum_{\substack{\boldsymbol{k}_\mathfrak{u} \in \mathbb{N}^{|\mathfrak{u}|} \\ \mu_\alpha(\boldsymbol{k}_\mathfrak{u})=l}} 1 \;\leq\; 2^{|\mathfrak{u}|}(b-1)^{\alpha|\mathfrak{u}|} \sum_{l=\lfloor \beta n-t \rfloor+1}^{\infty} b^{-l}b^{l/\alpha}l^{(\alpha-1)|\mathfrak{u}|}\binom{l+|\mathfrak{u}|-1}{|\mathfrak{u}|-1}.
$$

Invoking the inequality $l^{(\alpha-1)|\mathfrak{u}|}\binom{l+|\mathfrak{u}|-1}{|\mathfrak{u}|-1} \leq \binom{l+\alpha|\mathfrak{u}|-1}{\alpha|\mathfrak{u}|-1}\frac{(\alpha|\mathfrak{u}|-1)!}{(|\mathfrak{u}|-1)!}$ we get

$$
2^{|\mathfrak{u}|}(b-1)^{\alpha|\mathfrak{u}|} \sum_{l=\lfloor \beta n-t \rfloor+1}^{\infty} b^{-l}b^{l/\alpha}l^{(\alpha-1)|\mathfrak{u}|}\binom{l+|\mathfrak{u}|-1}{|\mathfrak{u}|-1}
$$

$$
\leq 2^{|\mathfrak{u}|}(b-1)^{\alpha|\mathfrak{u}|}\frac{(\alpha|\mathfrak{u}|-1)!}{(|\mathfrak{u}|-1)!} \sum_{l=\lfloor \beta n-t \rfloor+1}^{\infty} b^{-(1-1/\alpha)l}\binom{l+\alpha|\mathfrak{u}|-1}{\alpha|\mathfrak{u}|-1}
$$

$$
\leq 2^{|\mathfrak{u}|}\left(\frac{b^{1-1/\alpha}(b-1)}{(b^{1-1/\alpha}-1)}\right)^{\alpha|\mathfrak{u}|}\frac{(\alpha|\mathfrak{u}|-1)!}{(|\mathfrak{u}|-1)!}b^{-(1-1/\alpha)(\lfloor \beta n-t \rfloor+1)}\binom{\lfloor \beta n-t \rfloor+\alpha|\mathfrak{u}|}{\alpha|\mathfrak{u}|-1},
$$

where we used Lemma 6. Consequently

$$\sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{S}} \gamma_{\mathfrak{u}} \sum_{\substack{\boldsymbol{k}_{\mathfrak{u}} \in \mathbb{N}^{|\mathfrak{u}|} \\ \mu_{\alpha}(\boldsymbol{k}_{\mathfrak{u}}) > \beta n - t}} b^{-\mu_{\alpha}(\boldsymbol{k}_{\mathfrak{u}})}$$

$$\leq \quad b^{-(1-1/\alpha)(\lfloor \beta n - t \rfloor + 1)}$$
$$\times \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{S}} \gamma_{\mathfrak{u}} 2^{|\mathfrak{u}|} \left(\frac{b^{1-1/\alpha}(b-1)}{(b^{1-1/\alpha}-1)} \right)^{\alpha|\mathfrak{u}|} \frac{(\alpha|\mathfrak{u}|-1)!}{(|\mathfrak{u}|-1)!} \binom{\lfloor \beta n - t \rfloor + \alpha|\mathfrak{u}|}{\alpha|\mathfrak{u}|-1}$$

$$= \quad b^{-(1-1/\alpha)(\lfloor \beta n - t \rfloor + 1)}$$
$$\times \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{S}} \gamma_{\mathfrak{u}} 2^{|\mathfrak{u}|} \left(\frac{b^{1-1/\alpha}(b-1)}{(b^{1-1/\alpha}-1)} \right)^{\alpha|\mathfrak{u}|} \frac{(\lfloor \beta n - t \rfloor + \alpha|\mathfrak{u}|)!}{(|\mathfrak{u}|-1)!(\lfloor \beta n - t \rfloor + 1)!},$$

which establishes the result. $\qquad\square$

5. Randomization of higher order nets

In this section, we will discuss the randomization of higher order nets and apply the randomized point sets to numerical integration. In particular, we consider randomizations using a digital shift, see e.g., [12, 13], and a digital shift of depth n, [13]. We let $\mathcal{P} = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{b^m-1}\}$ be a $(t, \alpha, \beta, n, m, s)$-net in base b, $\boldsymbol{x}_h = (x_{h,1}, \ldots, x_{h,s})$ for $0 \leq h < b^m$, and we assume that the b-adic expansion of $x_{h,j}$ is given by $x_{h,j} = \frac{\xi_{h,j,1}}{b} + \cdots + \frac{\xi_{h,j,n}}{b^n} + \frac{\xi_{h,j,n+1}}{b^{n+1}} + \ldots$ for $0 \leq h < b^m$ and $1 \leq j \leq s$. Also, let $\boldsymbol{\Delta} = (\Delta_1, \ldots, \Delta_s)$, where Δ_j, $1 \leq j \leq s$, are uniformly distributed in $[0, 1)$ and mutually independent. We also consider the b-adic expansion of each coordinate of $\boldsymbol{\Delta}$, i.e., $\Delta_j = \frac{\Delta_{j,1}}{b} + \frac{\Delta_{j,2}}{b^2} + \ldots$ for $1 \leq j \leq s$.

For the remainder of the paper, we use $\oplus$ to denote the digitwise addition modulo b, i.e., for $x, y \in [0, 1)$ with base b expansions $x = \sum_{l=1}^{\infty} \xi_l b^{-l}$ and $y = \sum_{l=1}^{\infty} \eta_l b^{-l}$, we define

$$x \oplus y = \sum_{l=1}^{\infty} \zeta_l b^{-l},$$

where $\zeta_l \in \{0, \ldots, b-1\}$ is given by $\zeta_l \equiv \xi_l + \eta_l \pmod{b}$. Let $\ominus$ denote the digitwise subtraction modulo b (for short we use $\ominus x := 0 \ominus x$). In the same fashion we also define the digitwise addition and digitwise subtraction for nonnegative integers based on the b-adic expansion. For vectors in $[0, 1)^s$ or $\mathbb{N}_0^s$, the operations $\oplus$ and $\ominus$ are carried out componentwise. Throughout the paper, we always use the same base b for the operations $\oplus$ and $\ominus$ as is used for the Walsh functions and the $(t, \alpha, \beta, n, m, s)$-nets.

Regarding the digital shift in base b, the randomly digitally shifted point set $\mathcal{P}_{\boldsymbol{\Delta}} = \{\boldsymbol{z}_0, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_{b^m-1}\}$ is given by

$$\boldsymbol{z}_h = \boldsymbol{x}_h \oplus \boldsymbol{\Delta} = (z_{h,1}, \ldots, z_{h,s}), \ \ 0 \leq h < b^m,$$

where $\oplus$ is carried out componentwise, where for $0 \leq h < b^m$ and $1 \leq j \leq s$

$$z_{h,j} := \frac{\xi_{h,j,1} \oplus \Delta_{j,1}}{b} + \cdots + \frac{\xi_{h,j,n} \oplus \Delta_{j,n}}{b^n} + \frac{\xi_{h,j,n+1} \oplus \Delta_{j,n+1}}{b^{n+1}} + \cdots$$

Regarding the digital shift in base b of depth n, we choose digits $\Delta_{j,l}$ for $1 \leq j \leq s$, $1 \leq l \leq n$ uniformly distributed on $\{0, 1, \ldots, b-1\}$ and mutually independent and also choose $\delta_{h,j}$ for $0 \leq h < b^m$, $1 \leq j \leq s$ uniformly distributed on $[0, b^{-n})$ and mutually independent. Consequently, recalling the digital expansion of $x_{h,j}$ we define

$$z_{h,j,l} = \xi_{h,j,l} + \delta_{j,l} \pmod{b}$$

for $0 \leq h < b^m$, $1 \leq j \leq s$ and $1 \leq l \leq n$, and finally set

$$z_{h,j} = \frac{z_{h,j,1}}{b} + \cdots + \frac{z_{h,j,n}}{b^n} + \delta_{h,j}, \ \ 0 \leq h < b^m, 1 \leq j \leq s,$$

to obtain the point set $\mathcal{P}_{\boldsymbol{\Delta}, \boldsymbol{\delta}} = \{\boldsymbol{z}_0, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_{b^m-1}\}$.

The next proposition establishes that each point in $\mathcal{P}_{\boldsymbol{\Delta}}$, $\mathcal{P}_{\boldsymbol{\Delta}, \boldsymbol{\delta}}$ is uniformly distributed in $[0,1)^s$, which is useful, as it means that estimators based on $\mathcal{P}_{\boldsymbol{\Delta}}$ or $\mathcal{P}_{\boldsymbol{\Delta}, \boldsymbol{\delta}}$ will be unbiased.

PROPOSITION 1. *Let $\mathcal{P}$ be a $(t, \alpha, \beta, n, m, s)$-net in base b and $\mathcal{P}_{\boldsymbol{\Delta}}$ and $\mathcal{P}_{\boldsymbol{\Delta}, \boldsymbol{\delta}}$ be defined as above. Then each point in $\mathcal{P}_{\boldsymbol{\Delta}}$ and $\mathcal{P}_{\boldsymbol{\Delta}, \boldsymbol{\delta}}$ is uniformly distributed in $[0,1)^s$.*

P r o o f. The proof follows immediately from [24, Proposition 3.1]. □

Using Theorem 1 one can show that a digital shift preserves the net property.

PROPOSITION 2. *Let $\mathcal{P}$ be a $(t, \alpha, \beta, n, m, s)$-net in base b and let $\mathcal{P}_{\boldsymbol{\Delta}}$ as well as $\mathcal{P}_{\boldsymbol{\Delta}, \boldsymbol{\delta}}$ be defined as above. Then $\mathcal{P}_{\boldsymbol{\Delta}}$ is a $(t, \alpha, \beta, n, m, s)$-net in base b with probability one and $\mathcal{P}_{\boldsymbol{\Delta}, \boldsymbol{\delta}}$ is a $(t, \alpha, \beta, n, m, s)$-net in base b.*

P r o o f. Using Theorem 1, we need to show that $S_{b^m}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}_{\boldsymbol{\Delta}}) = 0$, $\forall \boldsymbol{k} \in \mathbb{N}_0^s$, so that $0 < \mu_\alpha(\boldsymbol{k}) \leq \beta n - t$. Clearly, for $\boldsymbol{k} \in \mathbb{N}_0^s$, so that $0 < \mu_\alpha(\boldsymbol{k}) \leq \beta n - t$, we have

$$S_{b^m}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}_{\boldsymbol{\Delta}}) = \frac{1}{b^m} \sum_{h=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{z}_h) = \frac{1}{b^m} \sum_{h=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h) \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{\Delta}) = 0, \quad (11)$$

as $\mathcal{P}$ is a $(t, \alpha, \beta, n, m, s)$-net in base b. Equation (11) holds with probability one, as it holds only if infinitely many digits in the expansion of each coordinate of $\boldsymbol{z}_h$ are different from $b - 1$. This, however, occurs with probability one.

The result for $\mathcal{P}_{\boldsymbol{\Delta},\boldsymbol{\delta}}$ is shown in the same way, but we do not need the condition "with probability one", as the digital shift is only applied to the first n digits. $\quad\square$

Finally, we discuss numerical integration based on $\mathcal{P}_{\boldsymbol{\Delta}}$ and $\mathcal{P}_{\boldsymbol{\Delta},\boldsymbol{\delta}}$. Of course, we can easily obtain information on

$$\left| S_{b^m}(f,\widetilde{\mathcal{P}}) - \int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \right|, \qquad \widetilde{\mathcal{P}} \in \{\mathcal{P}_{\boldsymbol{\Delta}}, \mathcal{P}_{\boldsymbol{\Delta},\boldsymbol{\delta}}\}, \quad f \in W_{\alpha,s,\boldsymbol{\gamma}},$$

using Proposition 2 and Theorem 2. However, studying the root mean-square error is interesting, as we can improve on the convergence rate from Theorem 2.

THEOREM 3. *Let $f \in W_{\alpha,s,\boldsymbol{\gamma}}$, let $\mathcal{P}$ be a (t,α,β,n,m,s)-net in base b and let $\mathcal{P}_{\boldsymbol{\Delta}} = \{\boldsymbol{z}_0, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_{b^m-1}\}$ be defined as above. Then*

$$\mathbb{E}\left[\left| \frac{1}{b^m} \sum_{h=0}^{b^m-1} f(\boldsymbol{z}_h) - \int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \right|^2 \right] \leq \|f\|_{W_{\alpha,s,\boldsymbol{\gamma}}}^2 \, b^{-\frac{2\alpha-1}{\alpha}(\lfloor \beta n - t \rfloor + 1)}$$

$$\times \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{S}} \gamma_{\mathfrak{u}}^2 2^{|\mathfrak{u}|} \left(\frac{b^{\frac{2\alpha-1}{\alpha}}(b-1)}{b^{\frac{2\alpha-1}{\alpha}} - 1} \right)^{\alpha|\mathfrak{u}|} \frac{(\lfloor \beta n - t \rfloor + \alpha|\mathfrak{u}|)!}{(|\mathfrak{u}|-1)!(\lfloor \beta n - t \rfloor + 1)!}.$$

P r o o f. Arguing as in Lemma 4, we get

$$\left| \frac{1}{b^m} \sum_{h=0}^{b^m-1} f(\boldsymbol{z}_h) - \int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \right| = \left| \sum_{\boldsymbol{k} \in \mathbb{N}_0^s \setminus \{\boldsymbol{0}\}} \widehat{f}(\boldsymbol{k}) S_{b^m}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}_{\boldsymbol{\Delta}}) \right|.$$

Hence

$$\left| \frac{1}{b^m} \sum_{h=0}^{b^m-1} f(\boldsymbol{z}_h) - \int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \right|^2 = \sum_{\boldsymbol{k} \in \mathbb{N}_0^s \setminus \{\boldsymbol{0}\}} |\widehat{f}(\boldsymbol{k})|^2 |S_{b^m}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}_{\boldsymbol{\Delta}})|^2$$

$$+ \sum_{\boldsymbol{k},\boldsymbol{l} \in \mathbb{N}_0^s \setminus \{\boldsymbol{0}\}, \boldsymbol{k} \neq \boldsymbol{l}} \widehat{f}(\boldsymbol{k}) \overline{\widehat{f}(\boldsymbol{l})} S_{b^m}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}_{\boldsymbol{\Delta}}) \overline{S_{b^m}(\mathrm{wal}_{\boldsymbol{l}}, \mathcal{P}_{\boldsymbol{\Delta}})}.$$

Clearly,

$$|S_{b^m}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}_{\boldsymbol{\Delta}})|^2 = \frac{1}{b^{2m}} \sum_{i,h=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{z}_h \ominus \boldsymbol{z}_i)$$

and

$$S_{b^m}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}_{\boldsymbol{\Delta}}) \overline{S_{b^m}(\mathrm{wal}_{\boldsymbol{l}}, \mathcal{P}_{\boldsymbol{\Delta}})} = \frac{1}{b^{2m}} \sum_{i,h=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{z}_h) \overline{\mathrm{wal}_{\boldsymbol{l}}(\boldsymbol{z}_i)}.$$

Hence, using [6, Lemma 6.1], see also [13, Lemma 3], and Theorem 1

$$
\mathbb{E}\left[\left|\frac{1}{b^m}\sum_{h=0}^{b^m-1}f(\boldsymbol{z}_h)-\int_{[0,1]^s}f(\boldsymbol{x})d\boldsymbol{x}\right|^2\right]
$$

$$
=\mathbb{E}\left[\sum_{\boldsymbol{k}\in\mathbb{N}_0^s\setminus\{\boldsymbol{0}\}}|\widehat{f}(\boldsymbol{k})|^2\frac{1}{b^{2m}}\sum_{i,h=0}^{b^m-1}\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{z}_h)\overline{\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{z}_i)}\right]
$$

$$
=\sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^s\setminus\{\boldsymbol{0}\}\\ \mu_\alpha(\boldsymbol{k})>\beta n-t}}|\widehat{f}(\boldsymbol{k})|^2\frac{1}{b^{2m}}\sum_{i,h=0}^{b^m-1}\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h\ominus\boldsymbol{x}_i) \qquad (12)
$$

$$
\leq\sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^s\setminus\{\boldsymbol{0}\}\\ \mu_\alpha(\boldsymbol{k})>\beta n-t}}|\widehat{f}(\boldsymbol{k})|^2
$$

$$
\leq\|f\|_{W_{\alpha,s,\boldsymbol{\gamma}}}^2\sum_{\emptyset\neq\mathfrak{u}\subseteq\mathcal{S}}\gamma_\mathfrak{u}^2\sum_{\substack{\boldsymbol{k}_\mathfrak{u}\in\mathbb{N}^{|\mathfrak{u}|}\\ \mu_\alpha(\boldsymbol{k}_\mathfrak{u})>\beta n-t}}b^{-2\mu_\alpha(\boldsymbol{k}_\mathfrak{u})}.
$$

The proof can now be completed in the same way as the proof of Theorem 2. $\quad\square$

We point out that Equation (12) is reminiscent of the weighted b-adic diaphony introduced in [15], however, we remark that the definition of $r_{\alpha,s,\boldsymbol{\gamma}}(\boldsymbol{k})$ used in this paper, see Equation (3), differs from the definition of the corresponding function used in [15].

REMARK 4. Similar to Remark 3, we point out that setting $\beta n=\alpha m$ results in a convergence rate of $N^{-(\alpha-\frac{1}{2})}(\log N)^{\frac{s\alpha}{2}}$ for the root mean-square integration error, improving on the convergence rate given in Theorem 2. For the point set $\mathcal{P}_{\boldsymbol{\Delta},\boldsymbol{\delta}}$ the same bound as in Theorem 3 can be obtained using the same argument but employing [13, Lemma 3] instead of [6, Lemma 6.1].

6. The $(u,u+v)$ construction

In this section, we will generalize the $(u,u+v)$-construction from coding theory, which seems to stem from [30], to (t,α,β,n,m,s)-nets. We remark that the $(u,u+v)$-construction has already been used to construct (t,m,s)-nets, see [3], and recently to construct higher order digital nets, see [11]. As in Sections 4 and 5, the main tool in proving the result is Theorem 1. We now outline the $(u,u+v)$-construction.

Assume we are given a $(t_1, \alpha, \beta_1, n_1, m_1, s_1)$-net $\mathcal{P}_1$ denoted by $(\boldsymbol{x}_i)_{i=0}^{b^{m_1}-1}$ and a $(t_2, \alpha, \beta_2, n_2, m_2, s_2)$-net $\mathcal{P}_2$ denoted by $(\boldsymbol{y}_h)_{h=0}^{b^{m_2}-1}$, where we assume $s_1 \leq s_2$. W.l.o.g. we may assume that $\boldsymbol{x}_i = (x_{i,1}, \ldots, x_{i,s_1})$ with $x_{i,j} = \xi_{i,j,1}/b + \cdots + \xi_{i,j,n_1}/b^{n_1}$ and $\boldsymbol{y}_h = (y_{h,1}, \ldots, y_{h,s_2})$ with $y_{h,j} = \eta_{h,j,1}/b + \cdots + \eta_{h,j,n_2}/b^{n_2}$ (if there are digits $\xi_{i,j,r} \neq 0$ for $r > n_1$ or $\eta_{h,j,r} \neq 0$ for $r > n_2$ we can slightly change $\mathcal{P}_1, \mathcal{P}_2$ by setting $\xi_{i,j,r} = 0$ for $r > n_1$ and $\eta_{h,j,r} = 0$ for $r > n_2$, without changing the $(t_w, \alpha, \beta_w, n_w, m_w, s_w)$-net property of $\mathcal{P}_w$, $w = 1, 2$).

We now define a new point set $\mathcal{P} = (\boldsymbol{z}_h)_{h=0}^{b^{m_1+m_2}-1}$, $\boldsymbol{z}_h = (z_{h,1}, \ldots, z_{h,s_1+s_2})$, consisting of $b^{m_1+m_2}$ points in $[0,1)^{s_1+s_2}$ as follows: first we set

$$\ell := \min(2\beta_1 n_1 - 2t_1 + 1, \beta_2 n_2 - t_2).$$

We recall that the addition modulo b is denoted by $\oplus$ and the subtraction modulo b by $\ominus$ (for short we use $\ominus x := 0 \ominus x$).

- For $j = 1, \ldots, s_1$, $h = 0, \ldots, b^{m_2} - 1$ and $i = 0, \ldots, b^{m_1} - 1$ we set

$$
\begin{aligned}
z_{hb^{m_1}+i,j} &= \frac{\xi_{i,j,1} \ominus \eta_{h,j,1}}{b} + \cdots + \frac{\xi_{i,j,\min(\ell,n_1)} \ominus \eta_{h,j,\min(\ell,n_1)}}{b^{\min(\ell,n_1)}} \\
&\quad + \left(\frac{\xi_{i,j,\ell+1}}{b^{\ell+1}} + \cdots + \frac{\xi_{i,j,n_1}}{b^{n_1}} \right) \mathbf{1}_{n_1 > \ell} \\
&\quad + \left(\frac{\ominus \eta_{h,j,n_1+1}}{b^{n_1+1}} + \cdots + \frac{\ominus \eta_{h,j,\ell}}{b^{\ell}} \right) \mathbf{1}_{n_1 < \ell}.
\end{aligned}
$$

- For $j = s_1 + 1, \ldots, s_1 + s_2$, $h = 0, \ldots, b^{m_2} - 1$ and $i = 0, \ldots, b^{m_1} - 1$ we set

$$z_{hb^{m_1}+i,j} = y_{h,j-s_1}.$$

Note that for every component of $\boldsymbol{z}_h$ at most the first $\max(n_1, n_2) \leq n_1 + n_2 =: n$ digits in its b-adic expansion are non-zero.

In the following we analyze the Weyl sum $S_{b^{m_1+m_2}}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P})$ for $\boldsymbol{k} \in \mathbb{N}_0^{s_1+s_2}$ satisfying $\mu_\alpha(\boldsymbol{k}) \leq \ell$. For this analysis we need to recall some notation: for vectors $\boldsymbol{k}, \boldsymbol{l} \in \mathbb{N}_0^s$, $\boldsymbol{k} = (k_1, \ldots, k_s)$, $\boldsymbol{l} = (l_1, \ldots, l_s)$, $\boldsymbol{k} \oplus \boldsymbol{l} := (k_1 \oplus l_1, k_2 \oplus l_2, \ldots, k_s \oplus l_s)$.

We embed a vector $\boldsymbol{u} \in \mathbb{N}_0^{s_1}$ into $\mathbb{N}_0^{s_2}$ by filling up the remaining components with zeros. This vector will be denoted by $(\boldsymbol{u}, \boldsymbol{0}) \in \mathbb{N}_0^{s_2}$. In the following we will represent a vector $\boldsymbol{k} \in \mathbb{N}_0^{s_1+s_2}$ in the form $\boldsymbol{k} = (\boldsymbol{u}, (\boldsymbol{u}, \boldsymbol{0}) \oplus \boldsymbol{v})$, where $\boldsymbol{u} \in \mathbb{N}_0^{s_1}$, $\boldsymbol{v} \in \mathbb{N}_0^{s_2}$, i.e., $\boldsymbol{k}$ is the concatenation of the two vectors

$$\boldsymbol{u} \in \mathbb{N}_0^{s_1} \quad \text{and} \quad (\boldsymbol{u}, \boldsymbol{0}) \oplus \boldsymbol{v} \in \mathbb{N}_0^{s_2}.$$

LEMMA 7. *For $\boldsymbol{k} \in \{0, \ldots, b^\ell - 1\}^{s_1+s_2}$ and for $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}$ given above we have*

$$S_{b^{m_1+m_2}}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}) = S_{b^{m_1}}(\mathrm{wal}_{\boldsymbol{u}}, \mathcal{P}_1) S_{b^{m_2}}(\mathrm{wal}_{\boldsymbol{v}}, \mathcal{P}_2).$$

P r o o f. For $\boldsymbol{y}_h \in [0,1)^{s_2}$ we denote the projection onto its first s_1 components by $\boldsymbol{y}_h^{(s_1)}$. Then we have

$$\frac{1}{b^{m_1+m_2}} \sum_{h'=0}^{b^{m_1+m_2}-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{z}_{h'}) = \frac{1}{b^{m_1+m_2}} \sum_{h=0}^{b^{m_2}-1} \sum_{i=0}^{b^{m_1}-1} \mathrm{wal}_{(\boldsymbol{u},(\boldsymbol{u},\boldsymbol{0})\oplus\boldsymbol{v})}(\boldsymbol{z}_{hb^{m_1}+i})$$

$$= \frac{1}{b^{m_1+m_2}} \sum_{h=0}^{b^{m_2}-1} \sum_{i=0}^{b^{m_1}-1} \mathrm{wal}_{\boldsymbol{u}}\left(\boldsymbol{x}_i \ominus \boldsymbol{y}_h^{(s_1)}\right) \mathrm{wal}_{(\boldsymbol{u},\boldsymbol{0})\oplus\boldsymbol{v}}(\boldsymbol{y}_h)$$

$$= \frac{1}{b^{m_1}} \sum_{i=0}^{b^{m_1}-1} \mathrm{wal}_{\boldsymbol{u}}(\boldsymbol{x}_i) \frac{1}{b^{m_2}} \sum_{h=0}^{b^{m_2}-1} \mathrm{wal}_{\boldsymbol{v}}(\boldsymbol{y}_h).$$

The last two equalities use the assumption that $\boldsymbol{k} \in \{0,\dots,b^\ell-1\}^{s_1+s_2}$, which means that for all components of $\boldsymbol{k}$ at most the first ℓ digits in their b-adic expansion are different from zero. $\square$

We need the following lemma, which is [1, Lemma 5].

LEMMA 8. *For $\alpha \geq 2$, $\boldsymbol{k}, \boldsymbol{l} \in \mathbb{N}_0^s$ we have $\mu_\alpha(\boldsymbol{k} \oplus \boldsymbol{l}) \geq \mu_\alpha(\boldsymbol{k}) - \mu_\alpha(\boldsymbol{l})$.*

The following theorem establishes the main result of this section.

THEOREM 4. *Let $b \in \mathbb{N}$, $b \geq 2$, let $\mathcal{P}_1$ be a $(t_1, \alpha, \beta_1, n_1, m_1, s_1)$-net in base b, and $\mathcal{P}_2$ be a $(t_2, \alpha, \beta_2, n_2, m_2, s_2)$-net in base b. Then $\mathcal{P}$ defined as above is a $(t, \alpha, \beta, n, m, s)$-net in base b, where $n = n_1 + n_2$, $m = m_1 + m_2$, $s = s_1 + s_2$ and*

$$\beta = \min(\beta_1, \beta_2), \quad t = \beta n - \ell.$$

P r o o f. We will use Theorem 1 to establish the result, i.e., we need to show that for all $\boldsymbol{k} \in \mathbb{N}_0^{s_1+s_2}$ satisfying $0 < \mu_\alpha(\boldsymbol{k}) \leq \beta n - t$ we have

$$S_{b^{m_1+m_2}}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}) = 0.$$

For $\boldsymbol{k} \in \mathbb{N}_0^{s_1+s_2}$ satisfying $0 < \mu_\alpha(\boldsymbol{k}) \leq \beta n - t = \ell$ we necessarily have that $\boldsymbol{k} \in \{0,\dots,b^\ell-1\}^{s_1+s_2}$. Hence we may use Lemma 7 which states that

$$S_{b^{m_1+m_2}}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}) = S_{b^{m_1}}(\mathrm{wal}_{\boldsymbol{u}}, \mathcal{P}_1) S_{b^{m_2}}(\mathrm{wal}_{\boldsymbol{v}}, \mathcal{P}_2).$$

We proceed in a manner very similar to the proof of [25, Theorem 5.3] and distinguish three cases.

Case 1: We firstly assume that $\boldsymbol{v} \neq \boldsymbol{0}$ and $\mu_\alpha(\boldsymbol{k}) \leq \beta n - t$. We want to show that $0 < \mu_\alpha(\boldsymbol{v}) \leq \beta_2 n_2 - t_2$, in which case we obtain $S_{b^{m_2}}(\mathrm{wal}_{\boldsymbol{v}}, \mathcal{P}_2) = 0$ by Theorem 1. As $\boldsymbol{v} \neq \boldsymbol{0}$ we have $\mu_\alpha(\boldsymbol{v}) > 0$. Also, using Lemma 8,

$$\mu_\alpha(\boldsymbol{v}) \leq \mu_\alpha((\boldsymbol{u},\boldsymbol{0}) \oplus \boldsymbol{v}) + \mu_\alpha(\boldsymbol{u}) = \mu_\alpha(\boldsymbol{k}) \leq \beta n - t \leq \beta_2 n_2 - t_2.$$

Case 2: We now assume that $\boldsymbol{v} = \boldsymbol{0}$, $\boldsymbol{u} \neq \boldsymbol{0}$ and $0 < \mu_\alpha(\boldsymbol{k}) \leq \beta n - t$. We want to show that $0 < \mu_\alpha(\boldsymbol{u}) \leq \beta_1 n_1 - t_1$, in which case we obtain $S_{b^{m_1}}(\mathrm{wal}_{\boldsymbol{u}}, \mathcal{P}_1) = 0$ by Theorem 1. As $\boldsymbol{u} \neq \boldsymbol{0}$ we have $\mu_\alpha(\boldsymbol{u}) > 0$. Also,

$$2(\beta_1 n_1 - t_1) + 1 \geq \beta n - t \geq \mu_\alpha(\boldsymbol{k}) = \mu_\alpha((\boldsymbol{u}, \boldsymbol{0}) \oplus \boldsymbol{v}) + \mu_\alpha(\boldsymbol{u}) = 2\mu_\alpha(\boldsymbol{u}).$$

Hence $\mu_\alpha(\boldsymbol{u}) \leq \beta_1 n_1 - t_1$, as $\mu_\alpha(\boldsymbol{u})$ is an integer.

Case 3: We now assume that $\boldsymbol{v} = \boldsymbol{0}$ and $\boldsymbol{u} = \boldsymbol{0}$ and $0 < \mu_\alpha(\boldsymbol{k}) \leq \beta n - t$. However, as $\boldsymbol{v} = 0$ and $\boldsymbol{u} = 0$, it follows that $\mu_\alpha(\boldsymbol{k}) = 0$ hence this case need not be considered.

Thus we have $S_{b^{m_1+m_2}}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}) = 0$ whenever $0 < \mu_\alpha(\boldsymbol{k}) \leq \beta n - t$ and this completes the proof. $\qquad\square$

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